Hengyan Li, Xin Zhao, and Weiping Yan*

# Bifurcation of time-periodic solutions for the incompressible flow of nematic liquid crystals in three dimension

https://doi.org/10.1515/anona-2020-0052
Received September 24, 2019; accepted October 14, 2019.

**Abstract:** This paper is devoted to the study of the dynamical behavior for the 3D incompressible flow of liquid crystals. We prove that this system under smooth external forces possesses time dependent periodic solutions, bifurcating from a steady solution.

**Keywords:** Nematic liquid crystals, periodic solution, bifurcation

**MSC:** 37N10, 35B10, 70K50

## 1 Introduction and Main Results

We consider the 3D incompressible flow of liquid crystals under external time-independent force

$$\frac{\partial U}{\partial t} + U \cdot \nabla U - \mu \triangle U + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d) + f_\alpha, \tag{1}$$

$$\frac{\partial d}{\partial t} + U \cdot \nabla d = y(\triangle d - f(d)) + h_\alpha, \tag{2}$$

$$\nabla \cdot U = 0, \tag{3}$$

where $U \in \mathbf{R}^3$ denotes the velocity, $d \in \mathbf{R}^3$ the director field for the averaged macroscopic molecular orientations, $P \in \mathbf{R}$ the pressure arising from the incompressibility; and they all depend on the spatial variable $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ and the time variable $t > 0$. The positive constants $\mu$, $\lambda$, $y$ stand for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively; $f_\alpha$ and $h_\alpha$ are external time independent forces. The symbol $\nabla d \odot \nabla d$ denotes a matrix whose $ij$th entry is $< \partial_{x_i} d, \partial_{x_j} d >$, and it is easy to see that

$$\nabla d \odot \nabla d = (\nabla d)^T \nabla d,$$

$$\nabla \cdot (\nabla d \odot \nabla d) = \nabla(\frac{|\nabla d|^2}{2}) + (\nabla d)^T \triangle d, \tag{4}$$

where $(\nabla d)^T$ denotes the transpose of the $3 \times 3$ matrix $\nabla d$. In (2), $f(d)$ is the penalty function which will be assumed to be

$$f(d) = |\nabla d|^2 d. \tag{5}$$

**Hengyan Li,** School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, PR China, E-mail: hengyanli989@126.com
**Xin Zhao,** School of Mathematics and Big Data, Huizhou University, Huizhou, P.R. China, E-mail: xinzhaomath@126.com
***Corresponding Author: Weiping Yan,** School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China, E-mail: yanwp@xmu.edu.cn

One of the most common liquid crystal phases is the nematic, where the molecules have no positional order, but they have long-range orientational order. For more details of physics, we refer the readers to the two books of de Gennes-Prost [7] and Chandrasekhar [2]. Ericksen and Leslie cf.[6, 14] established the hydrodynamic theory of liquid crystals in 1960s. The Ericksen-Lislie theory describes the liquid crystal flow, including the velocity vector $u$ and direction vector $d$ of the fluid. Since the general Ericksen-Leslie system is very complicated, we only consider a simplified model (1)-(3) of the Ericksen-Leslie system, but still retains most of the essential features. One can see [16–18, 20] for more discussions on the relations of the two models. Both the Ericksen-Leslie system and the simplified one (1)-(3) describe the time evolution of liquid crystal materials under the influence of both the velocity field $u$ and the director field $d$. Hence, a natural question of the existence of time-periodic solution arises when (1)-(3) under the effect of the external forces.

Since the Ericksen-Leslie system (1)-(3) with $|u| = 1$ is complicated, Lin and Liu [18, 19] proposed to investigate an approximation model of the Ericksen-Leslie system by Ginzburg-Landau functionals. In order to relax the constraint $|u| = 1$ for the functional $\int |\nabla u|^2 dx$, Lin and Liu [18, 19] considered Ginzburg-Landau functionals

$$\int_{\Omega} [|\nabla d|^2 + \frac{1}{2\epsilon^2}(1 - |d|^2)^2] dx,$$

for any function $d \in \mathbf{H}^1(\Omega; \mathbf{R}^3)$ with a parameter $\epsilon > 0$. They obtained the global existence of weak solutions with large initial data and the global existence of classical solutions was also obtained if the coefficient $\mu$ is large enough in three dimensional spaces. Hu and Wang [12] prove the existence and uniqueness of the global strong solution with small initial data are established. Meanwhile, they obtained that when the strong solution exists, all the global weak solutions constructed in [18] must be equal to the unique strong solution. Hong [11] proved that the global existence of regular solutions to the Ericksen-Leslie system in $\mathbf{R}^2$ with initial data except for at a finite number of singular times. Li and Yan [15] showed this system admits a stable smooth steady solutions by assumption of existence of it.

Since the work of Sattinger [29], Iudovich [24] and Iooss [21] in 1971, the bifurcation of stationary solutions into time periodic solutions (i.e. Hopf-bifurcation) of incompressible Navier-Stokes equation has attracted much attention, see [3, 9, 13, 22, 23], etc. When the linearized operator possesses a continuous spectrum up to the imaginary axis and that a pair of imaginary eigenvalues crosses the imaginary axis, Melcher, A, et al. [26] proved Hopf-bifurcation for the vorticity formulation of the incompressible Navier-Stokes equations in $\mathbf{R}^3$. Their work is mainly motivated by the work of Brand, T, et al. [1] who studied the Hopf-bifurcation problem and its exchange of stability for a coupled reaction diffusion model in $\mathbf{R}^a$. We mention that Crandall and Rabinowitz [5] gave an abstract infinite-dimensional version of Hopf bifurcation theorem which has found many application. We refer the readers to [4, 27, 30, 32–35] corresponding Hopf-bifurcation result (bifurcating from viscous shock waves) has been established in.

In this paper, our aim is to establish the corresponding Hopf-bifurcation result for the three-dimensional incompressible flow of liquid crystals. But we can not directly use the method of dealing with Navier-Stokes equation to three-dimensional incompressible flow of liquid crystals because the presence of the velocity field and its interaction with the director field in the liquid crystals flow of large oscillation. A weighted Young theorem (see Lemma 6) is derived to deal with strong coupled between the velocity field and the director field.

We assume that $f_\alpha$ and $h_\alpha$ depend smoothly on some parameter $\alpha$, which can be chosen suitably so that $(u_\alpha(x) + u_c, d_\alpha(x) + d_c, p_\alpha(x))$ (the steady solution has certain smoothness property) is the solution of the three-dimensional steady incompressible flow of liquid crystals

$$U \cdot \nabla U - \mu \triangle U + \nabla P = -\lambda \nabla \cdot (\nabla d \odot \nabla d) + f_\alpha, \tag{6}$$

$$U \cdot \nabla d = y(\triangle d - f(d)) + h_\alpha, \tag{7}$$

$$\nabla \cdot U = 0, \tag{8}$$

with $u_c = (c_1, 0, 0)^T$, $d_c = (c_1, 0, 0)^T$ and

$$\lim_{|x| \to \infty} u_\alpha(x) = \mathbf{0}, \quad \lim_{|x| \to \infty} d_\alpha(x) = \mathbf{0},$$

where $\mathbf{0} = (0, 0, 0)^T$.

To seek the periodic solution, we linearize system (1)-(2) about the steady state $(u_\alpha, d_\alpha, p_\alpha)$ by writing

$$\begin{aligned} U(x,t) &= u(x,t) + u_\alpha(x), \\ d(x,t) &= z(x,t) + d_\alpha(x), \\ P &= p + p_\alpha. \end{aligned}$$

Then, the deviation $(u, z, p)$ from the stationary $(u_\alpha, d_\alpha, p_\alpha)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} - \mu\triangle u + c_1\partial_{x_1}u + u_\alpha\cdot\nabla u + u\cdot\nabla u_\alpha &+ u\cdot\nabla u + \nabla p = -\lambda\nabla(\frac{|\nabla z|^2}{2}) - \lambda\nabla(|\nabla z||\nabla d_\alpha|) \\ &- \lambda(\nabla z)^T\triangle(z + d_\alpha) - \lambda(\nabla d_\alpha)^T\triangle z, \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial z}{\partial t} - y\triangle z + c_1 y\partial_{x_1}z &+ u_\alpha\cdot\nabla z + u\cdot\nabla d_\alpha + u\cdot\nabla z = -y|\nabla z|^2 z - y|\nabla z|^2 d_\alpha \\ &-y|\nabla d_\alpha|^2 z - 2y|\nabla z||\nabla d_\alpha|(z + d_\alpha), \end{aligned} \tag{10}$$

Here, for general matrices $u = (u_{ij})_{i,j=1,2,3}$,

$$\nabla\cdot u = (\sum_{j=1}^{3}\partial_{x_1}u_{1j}, \sum_{j=1}^{3}\partial_{x_1}u_{2j}, \sum_{j=1}^{3}\partial_{x_1}u_{3j})^T.$$

We introduce a $3\times 3$ matrix

$$v = \nabla z, \quad v_\alpha = \nabla z_\alpha, \tag{11}$$

and take the gradient of (10) and notice (4)-(5) to rewrite (9)-(10) as

$$\begin{aligned} \frac{\partial u}{\partial t} - \mu\triangle u + c_1\partial_{x_1}u + u_\alpha\cdot\nabla u + u\cdot\nabla u_\alpha &+ u\cdot\nabla u + \nabla p = -\lambda\nabla(\frac{|v|^2}{2}) - \lambda\nabla(|v||\nabla d_\alpha|) \\ &+\lambda v^T\nabla(v + v_\alpha) + \lambda v_\alpha^T\nabla v, \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial v}{\partial t} - y\triangle v + c_1 y\partial_{x_1}v &+ u_\alpha\cdot\nabla v + v\nabla u_\alpha + u\cdot\nabla v_\alpha + v_\alpha\nabla u + u\cdot\nabla v + v\nabla u \\ &= -y\nabla(|v|^2 z + |v|^2 d_\alpha) - y\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha|(z + d_\alpha)), \end{aligned} \tag{13}$$

with incompressible condition

$$\nabla\cdot u = 0, \tag{14}$$

where we used, for all $i, j, k = 1, 2, 3$,

$$\frac{\partial}{\partial x_k}(u_j\frac{\partial d_i}{\partial x_j}) = \frac{\partial u_j}{\partial x_k}\frac{\partial d_i}{\partial x_j} + u_j\frac{\partial}{\partial x_j}(\frac{\partial d_i}{\partial x_k}) = (v\nabla u + u\cdot\nabla v)_{ik}.$$

In fact, by the incompressible condition (14), it follows that

$$\nabla\cdot(uv^T) = u\cdot\nabla u + u\nabla\cdot u = u\cdot\nabla u. \tag{15}$$

Thus using (14) and (15) to (12)-(13), we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} - \mu\triangle u + c_1\partial_{x_1}u &+ \nabla\cdot(u_\alpha u^T) + \nabla\cdot(uu_\alpha^T) + \nabla\cdot(uu^T) + \nabla p \\ &= -\lambda\nabla(\frac{|v|^2}{2}) - \lambda\nabla(|v||\nabla d_\alpha|) + \lambda v^T\nabla(v + v_\alpha) + \lambda v_\alpha^T\nabla v, \end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial v}{\partial t} - y\triangle v + c_1 y\partial_{x_1} v &+ u_\alpha \cdot \nabla v + v\nabla u_\alpha + u \cdot \nabla v_\alpha + v_\alpha \nabla u + u \cdot \nabla v + v\nabla u \\ &= -y\nabla(|v|^2 z + |v|^2 d_\alpha) - y\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha|(z + d_\alpha)). \end{aligned} \tag{17}$$

The vorticity associated with velocity field $u$ of the fluid is defined by $\omega = \nabla \times u$. Then, using

$$\nabla \times \nabla \cdot (uu^T) = \nabla \cdot (\omega u^T - u\omega^T),$$

we can rewrite system (16) as

$$\begin{aligned}\frac{\partial \omega}{\partial t} - \mu\triangle\omega + c_1\partial_{x_1}\omega &+ \nabla \cdot (\omega_\alpha u^T - u_\alpha \omega^T) + \nabla \cdot (\omega u_\alpha^T - u\omega_\alpha^T) + \nabla \cdot (\omega u^T - u\omega^T) \\ &= -\lambda\nabla(\frac{\nabla \times |v|^2}{2}) - \lambda\nabla(\nabla \times (|v||\nabla d_\alpha|)) \\ &\quad +\lambda\nabla \times (v^T\nabla(v + v_\alpha)) + \lambda\nabla \times (v_\alpha^T \nabla v). \end{aligned} \tag{18}$$

Note that the space of divergence free vector fields is invariant under the evolution (18). We can assume that

$$\nabla \cdot \omega = 0.$$

Moreover, we can reconstruct the velocity $u$ from the vorticity $\omega$ by solving the equation

$$\nabla \times u = \omega, \quad \nabla \cdot \omega = 0.$$

The velocity field $u$ is defined in terms of the vorticity via the Biot-Savart law

$$u(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(x-y)^\perp \times \omega(y)}{|x-y|^3} dy, \quad x \in \mathbf{R}^3. \tag{19}$$

Denote $\varphi = (\omega, v)^T$. Then, we can write system (17)-(18) as the evolution equation form

$$\frac{d\varphi}{dt} + \mathcal{N}\varphi + G(\varphi) = F(\varphi), \tag{20}$$

where

$$\mathcal{N} = \begin{pmatrix} -\mu\triangle + c_1\partial_{x_1} & 0 \\ 0 & -y\triangle + c_1\partial_{x_1} \end{pmatrix},$$

and

$$G(\varphi) = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}, \quad F(\varphi) = \begin{pmatrix} g^3 \\ g^4 \end{pmatrix}$$

with

$$\begin{aligned} g^1 &= \nabla \cdot (\omega_\alpha u^T - u_\alpha \omega^T) + \nabla \cdot (\omega u_\alpha^T - u\omega_\alpha^T) + \lambda\nabla(\nabla \times (|v||\nabla d_\alpha|)) - \lambda\nabla \times (v^T\nabla v_\alpha) \\ &\quad -\lambda\nabla \times (v_\alpha^T\nabla v), \\ g^2 &= u_\alpha \cdot \nabla v + v\nabla u_\alpha + u \cdot \nabla v_\alpha + v_\alpha\nabla u + y\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha| d_\alpha), \\ g^3 &= -\nabla \cdot (\omega u^T - u\omega^T) - \lambda\nabla(\frac{\nabla \times |v|^2}{2}) + \lambda\nabla \times (v^T\nabla v), \\ g^4 &= -u \cdot \nabla v - v\nabla u - y\nabla(|v|^2 z + |v|^2 d_\alpha) - 2y\nabla(|v||\nabla d_\alpha| z). \end{aligned}$$

For convenience, we denote the Fourier coefficient of operators $\mathcal{N}$ and $\mathcal{G}$ by $\widehat{\mathcal{N}}$ and $\widehat{\mathcal{G}}$, respective. To overcome the essential spectrum of operator $-(\widehat{\mathcal{N}}+\widehat{G})$ up to the imaginary axis, we need the following assumption:

(**H1**) For any $\alpha \in [\alpha_c - \alpha_0, \alpha_c + \alpha_0]$, 0 is not an eigenvalue of $\widehat{\mathcal{N}} + \widehat{G}$.

(**H2**)For $\alpha = \alpha_c$, the operator $-(\widehat{\mathcal{N}} + \widehat{G})$ has two pair eigenvalues $(\lambda_0^+, \mu_0^+)$ and $(\lambda_0^-, \mu_0^-)$ satisfying

$$\lambda_0^{\pm}(\alpha_c) = \mu_0^{\pm}(\alpha_c) = \pm i\xi_0 \neq 0, \quad for\ \xi_0 > 0,$$
$$\frac{d}{d\alpha}\mathbf{Re}(\lambda_0^{\pm}(\alpha))\mid_{\alpha=\alpha_c}, \frac{d}{d\alpha}\mathbf{Re}(\mu_0^{\pm}(\alpha))\mid_{\alpha=\alpha_c} > 0.$$

(**H3**) The rest eigenvalue of $-(\widehat{\mathcal{N}} + \widehat{G})$ is strictly bounded away from the imaginary axis in the left half plane for all $\alpha \in [\alpha_c - \alpha_0, \alpha_c + \alpha_0]$.

Under the generic assumption the cubic coefficient terms $a_1, a_2 \neq 0$ in (64)-(65), Hopf-bifurcation result about 3D incompressible flow of liquid crystals is stated:

**Theorem 1.** *Assume that (**H1**)-(**H3**) hold. Then system (1)-(3) admits a one dimensional family of small time-periodic solutions, i.e.*

$$U(x,t) = U(x, t + 2\pi/\xi_1), \quad d(x,t) = d(x, t + 2\pi/\xi_2)$$

*with $\alpha = \alpha_c + \epsilon$, $\epsilon \in (0, \alpha_0)$, and positive frequencies $\xi_1$ and $\xi_2$. Moreover,*

$$\xi_1 = \xi_0 + \mathcal{O}(\epsilon), \quad \xi_2 = \xi_0 + \mathcal{O}(\epsilon),$$

*and*

$$\|U(x,t)\|_{\boldsymbol{C}_b^0(\boldsymbol{R}^3\times[0,2\pi/\xi_1])} = \mathcal{O}(\epsilon), \quad \|d(x,t)\|_{\boldsymbol{C}_b^1(\boldsymbol{R}^3\times[0,2\pi/\xi_2])} = \mathcal{O}(\epsilon).$$

Above result also holds in a three dimensional torus $\mathbf{T}^3$ and a bounded domain.

This paper is organized as follows. In section 2, we introduce some notations and preliminaries. In section 3, The main proof of Theorem 1 is carried out by using Lyapunov-Schmidt method.

## 2 Preliminary and Some notations

We start this section by introducing some notations. Consider the following standard Sobolev space, spatially weighted Lebesgue space

$$\mathbf{W}_\kappa^q := \{u : \|u\|_\kappa^q := \sum_{|\alpha|\le\kappa} \|D^\alpha u\|_{\mathbf{L}^q}^q < \infty\},$$
$$\mathbf{L}_s^p := \{u : \|u\|_s^p := \int_{\mathbf{R}^3} \rho^s(x) u^p(x) dx < \infty\},$$

where weighted function $\rho(x) = \sqrt{1+|x|^2}$. The Fourier transform is a continuous mapping from $\mathbf{L}_s^p$ into $\mathbf{W}_\kappa^q$. Especially, when $p = 2$, the Fourier transform is an isomorphism between $\mathbf{H}^p$ and $\mathbf{L}_p^2$ with $\|u\|_{\mathbf{L}_p^2} = \|\rho^p u\|_{\mathbf{L}^2}$.

To investigate periodic solutions of system (1)-(2), we also introduce the space

$$\mathbf{X} := \{u = (u_n)_{n\in\mathbf{Z}} : \|u\|_{\mathbf{X}} < \infty\}$$

and weighted space

$$\mathcal{L}_s^p = \mathbf{L}_s^p \times \mathbf{L}_s^p, \quad \mathcal{H}^m = \mathbf{H}^m \times \mathbf{H}^m, \quad \mathcal{X} = \mathbf{X} \times \mathbf{X},$$

with norms

$$\|u\|_{\mathbf{X}} = \sum_{n\in\mathbf{Z}} \|u_n\|_{\mathbf{H}^p}, \quad \|\varphi\|_{\mathcal{X}} := \|u\|_{\mathbf{X}} + \|v\|_{\mathbf{X}},$$
$$\|\hat{\varphi}\|_{\mathcal{L}_s^p} := \|\hat{u}\|_{\mathbf{L}_s^p} + \|\hat{v}\|_{\mathbf{L}_s^p}, \quad \|\hat{\varphi}\|_{\mathcal{H}^m} := \|u\|_{\mathbf{H}^m} + \|v\|_{\mathbf{H}^m}.$$

for $\varphi = (u, v)^T \in \mathcal{L}_s^p$ or $\mathfrak{X}$, respectively.

In this paper, we consider the following form of time-periodic solution

$$\omega = \omega(x, t/\xi_1), \quad v = v(x, t/\xi_2),$$

where $\xi_1, \xi_2 \in \mathbf{R}^+$ denote the corresponding frequencies.

Thus we need to find $2\pi$ time periodic solutions of

$$\Xi \frac{d\varphi}{dt} + \mathcal{N}\varphi + G(\varphi) = F(\varphi), \tag{21}$$

where

$$\Xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} -\mu\triangle + c_1\partial_{x_1} & 0 \\ 0 & -y\triangle + c_1\partial_{x_1} \end{pmatrix},$$

and

$$G(\varphi) = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}, \quad F(\varphi) = \begin{pmatrix} g^3 \\ g^4 \end{pmatrix}$$

with

$$\begin{aligned} g^1 &= \nabla\cdot(\omega_\alpha u^T - u_\alpha \omega^T) + \nabla\cdot(\omega u_\alpha^T - u\omega_\alpha^T) + \lambda\nabla(\nabla\times(|v||\nabla d_\alpha|)) - \lambda\nabla\times(v^T\nabla v_\alpha) \\ &\quad -\lambda\nabla\times(v_\alpha^T\nabla v), \end{aligned}$$

$$g^2 = u_\alpha\cdot\nabla v + v\nabla u_\alpha + u\cdot\nabla v_\alpha + v_\alpha\nabla u + y\nabla(|\nabla d_\alpha|^2 z + 2|v||\nabla d_\alpha| d_\alpha),$$

$$g^3 = -\nabla\cdot(\omega u^T - u\omega^T) - \lambda\nabla\left(\frac{\nabla\times|v|^2}{2}\right) + \lambda\nabla\times(v^T\nabla v), \tag{22}$$

$$g^4 = -u\cdot\nabla v - v\nabla u - y\nabla(|v|^2 z + |v|^2 d_\alpha) - 2y\nabla(|v||\nabla d_\alpha| z). \tag{23}$$

By the classical result in [10], we know that the essential spectrum of the operator $\mathcal{N} + G$ is relatively compact perturbation of $\mathcal{N}$ which has the essential spectrum

$$essspec(\widehat{\mathcal{N}}) = \{\lambda \in \mathcal{C}^2 : \lambda = (-\mu|y|^2 + icy_1, -y|y|^2 + icy_1), \ y \in \mathbf{R}^3\}.$$

Moreover, the spectra of $\mathcal{N} + G$ and $\mathcal{N}$ only differ by isolated eigenvalues of finite multiplicity. Above spectrum properties are critical to prove our main result.

For convenience, we can rewrite (21) as

$$\xi_1\omega_t = M_1\omega + g^3(\omega, u, v), \tag{24}$$

$$\xi_2 v_t = M_2 v + g^4(\omega, u, v), \tag{25}$$

where $g^3$ and $g^4$ defined in (22)-(23),

$$\begin{aligned} M_1 &= \overline{M_1} + g^1 = \mu\triangle + c_1\partial_{x_1} + g^1, \\ M_2 &= \overline{M_2} + g^2 = y\triangle + c_1\partial_{x_1} + g^2. \end{aligned}$$

We make the ansatz

$$\omega(x, t) = \sum_{n\in\mathbf{Z}} \omega_n(x) e^{int}, \quad v(x, t) = \sum_{n\in\mathbf{Z}} v_n(x) e^{int}$$

to (24)-(25), we obtain

$$(in\xi_1 - M_1)\omega_n = g_n^3(\omega, u, v), \tag{26}$$
$$(in\xi_2 - M_2)v_n = g_n^4(\omega, u, v), \tag{27}$$

where

$$g^3(\omega, u, v)(x, t) = \sum_{n\in\mathbf{Z}} g_n^3(\omega, u, v)e^{int},$$
$$g^4(\omega, u, v)(x, t) = \sum_{n\in\mathbf{Z}} g_n^4(\omega, u, v)e^{int}.$$

Note that we are interested in real valued solution only. We will always suppose that $(\omega_n, v_n) = (\omega_{-n}, v_{-n})$ for $n \in \mathbf{Z}$. These series are uniformly convergent on $\mathbf{R}^3 \times [0, 2\pi]$ in the spaces which we have chosen. More precisely, we have the following results:

**Lemma 1.** *A linear operator $\boldsymbol{J} : \mathcal{X} \longrightarrow \boldsymbol{C}_b^0(\boldsymbol{R}^3 \times [0, \pi], \boldsymbol{C}^2)$ is defined by*

$$(\boldsymbol{J}u)(x, t) = \tilde{u}(x, t) := \sum_{n\in\boldsymbol{Z}} u_n(x)e^{int}, \quad u = (u_n)_{n\in\boldsymbol{Z}} \in \mathcal{X}.$$

*Then $\boldsymbol{J}$ is bounded.*

The counterpart to multiplication $uv$ in physical space is given by the convolution $(\sum_{k\in\mathbf{Z}} u_{n-k}v_k)_{n\in\mathbf{Z}}$, since

$$uv = \sum_{l\in\mathbf{Z}} u_l(x)e^{ilt} \sum_{j\in\mathbf{Z}} v_j(x)e^{ijt} = \sum_{n\in\mathbf{Z}} \left( \sum_{k\in\mathbf{Z}} u_{n-k}(x)v_k(x) \right) e^{int}.$$

**Lemma 2.** *For $u = (u_n)_{n\in\boldsymbol{Z}}$, $v = (v_n)_{n\in\boldsymbol{Z}} \in \boldsymbol{X}$, the convolution $u \star v \in \boldsymbol{X}$ is defined by*

$$(u \star v)_n = \sum_{k\in\boldsymbol{Z}} u_{n-k}v_k, \quad n \in \boldsymbol{Z}.$$

*Then there exists $C > 0$ such that*

$$\|u \star v\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}.$$

**Lemma 3.** *Let a linear operator $M_i : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ be defined component-wise as $(M_i u)_n = M_{in}u_n$ for $u = (u_n)_{n\in\boldsymbol{Z}}$. Then*

$$\|M_i u\|_{\boldsymbol{X}} = (\|M_{i0}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} + \sup_{n\in\boldsymbol{Z}\setminus\{0\}} \|M_i\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m})\|u\|_{\boldsymbol{X}}, \quad \text{for } i = 1, 2.$$

The proof of above three Lemmas are standard, so we omit it.

For any bounded analytic semigroup $A_\sigma^\vartheta$, the following result holds.

**Lemma 4.** *[28] For every $0 < \vartheta < 1$ and $p > 1$ there exists a constant $M > 0$ such that for all $t > 0$ one has*

$$\|A_\sigma^\vartheta e^{A_\sigma t}\|_{\boldsymbol{L}^p \longrightarrow \boldsymbol{L}^p} \le \frac{M}{t^\vartheta}.$$

The proof of following result can be found in [8] for bounded domain and [28] for $\mathbf{R}^n$.

**Lemma 5.** *For every $\frac{1}{2} < \vartheta < 1$ and $p > 1$ there exists a constant $C > 0$ such that*

$$\|A_\sigma^{-\vartheta} f\|_{\boldsymbol{L}^p} \le C\|f\|_{\boldsymbol{W}^{-2\vartheta,p}}.$$

The following result shows a weighted Young theorem.

**Lemma 6.** *There exists a positive constant C such that*

$$\|\omega \star u\|_{\boldsymbol{L}^2_m} \le C\|\omega\|_{\boldsymbol{L}^2_m}\|u\|_{\boldsymbol{L}^2_m}.$$

*Proof.* It is easy to check that

$$\rho(x) \le \rho(x-y)\rho(y), \quad \forall x, y \in \mathbf{R}^3, \tag{28}$$

where we take the weighted function as

$$\rho(x) = (1+|x|^2)^{\frac{1}{2}}.$$

Then, There exist positive constants $s_1$, $s_2$, $s$ such that $s_1 + s_2 = m + s$, with $s_1, s_2, s < m$. using Young inequality and (28), we derive

$$\begin{aligned}
\|\omega \star u\|^2_{\mathbf{L}^2_m} &= \int_{\mathbf{R}^3} \rho^{2m}(\omega \star u)^2(x)dx \\
&= \int_{\mathbf{R}^3} \left( \int_{\mathbf{R}^3} \omega(x-y)u(y)\rho^m(x)dy \right)^2 dx \\
&= \int_{\mathbf{R}^3} \rho^{-2s}(x) \left( \int_{\mathbf{R}^3} \rho^{s_1}(x-y)\omega(x-y)\rho^{s_2}(y)u(y)dy \right)^2 dx \\
&\le \int_{\mathbf{R}^3} \rho^{-2s}(x) \left( \int_{\mathbf{R}^3} \rho^{2s_1}(z)\omega^2(z)dz \right) \left( \int_{\mathbf{R}^3} \rho^{2s_2}(y)u^2(y)dy \right) dx \\
&\le C\|\omega\|^2_{\mathbf{L}^2_{s_1}}\|u\|^2_{\mathbf{L}^2_{s_2}} \le C\|\omega\|^2_{\mathbf{L}^2_m}\|u\|^2_{\mathbf{L}^2_m}.
\end{aligned}$$

This completes the proof. □

# 3 Proof of Theorem 1

In this section, we will give the detail of proof of Theorem 1. By (**H2**) and (**H3**), we know that the operator $M_i$ has two eigenvalues $\lambda_0^{\pm}(\beta)$ and all other eigenvalues of $M_i$ are strictly bounded away from the imaginary axis in the left half plane. Thus we construct a $M_i$-invariant projections $\mathbf{P}_{\pm 1,c}$ by

$$P_{1,c}\omega = (\psi^{+,\star}, \omega)_{\mathbf{L}^2}\psi^+, \quad P_{-1,c}\omega = (\psi^{-,\star}, \omega)_{\mathbf{L}^2}\psi^-, \tag{29}$$

$$P_{1,c}v = (\psi^{+,\star}, v)_{\mathbf{L}^2}\psi^+, \quad P_{-1,c}v = (\psi^{-,\star}, v)_{\mathbf{L}^2}\psi^-, \tag{30}$$

where $\psi^{\pm}$ denotes the associated normalized eigenfunctions, $\psi^{\pm 1,\star}$ denotes the associated normalized eigenfunctions of the adjoint operator $M_i^{\star}$. The bounded "stable" part of the projection is $\mathbf{P}_{\pm 1,s} = I - \mathbf{P}_{\pm 1,c}$, we also know that $\mathbf{P}_{\pm,c}M_i = M_i\mathbf{P}_{\pm,c}$ and $\mathbf{P}_{\pm,s}M_i = M_i\mathbf{P}_{\pm,s}$. Thus we can split $\omega_{\pm 1}$ and $v_{\pm 1}$ as

$$\begin{aligned}
&\omega_1 = \omega_{1,c} + \omega_{1,s}, \quad \omega_{-1} = \omega_{-1,c} + \omega_{-1,s}, \\
&v_1 = v_{1,c} + v_{1,s}, \quad v_{-1} = v_{-1,c} + v_{-1,s}
\end{aligned}$$

with

$$\begin{aligned}
&\omega_{\pm 1,c} = \mathbf{P}_{\pm 1,c}\omega_1, \quad \omega_{\pm 1,s} = \mathbf{P}_{\pm 1,s}\omega_1, \\
&v_{\pm 1,c} = \mathbf{P}_{\pm 1,c}v_1, \quad v_{\pm 1,s} = \mathbf{P}_{\pm 1,s}v_1.
\end{aligned}$$

Using above decompositions to (26)-(27), we have

$$(in\xi_1 - M_1)\omega_n = g_n^3(\omega, u, v), \quad n = \pm 2, \pm 3, \ldots, \tag{31}$$
$$(in\xi_2 - M_2)v_n = g_n^4(u, v), \quad n = \pm 2, \pm 3, \ldots, \tag{32}$$
$$M_1\omega_0 = g_0^3(\omega, u, v), \quad n = 0, \tag{33}$$
$$M_2 v_0 = g_0^4(u, v), \quad n = 0, \tag{34}$$
$$(\pm i\xi_1 - M_1)\omega_{\pm 1,s} = \mathbf{P}_{\pm 1,s} g_{\pm 1}^3(\omega, u, v), \tag{35}$$
$$(\pm i\xi_2 - M_2)v_{\pm 1,s} = \mathbf{P}_{\pm 1,s} g_{\pm 1}^4(u, v), \tag{36}$$
$$(\pm i\xi_1 - M_1)\omega_{\pm 1,c} = \mathbf{P}_{\pm 1,c} g_{\pm 1}^3(\omega, u, v), \tag{37}$$
$$(\pm i\xi_2 - M_2)v_{\pm 1,c} = \mathbf{P}_{\pm 1,c} g_{\pm 1}^4(u, v). \tag{38}$$

The organization of proof of Theorem 1 is that we first solve the equations (33)-(34). Then using the fixed point theorem to solve equations (31)-(32) and (35)-(36) which is nontrivial due to the nonlinear term $g_n^3(\omega, u, v)$ and $g_n^4(u, v)$. At last, we employ the implicit function theorem to solve equation (37)-(38). The process of solving equation (37)-(38) is inspired by the classical Hopf-Bifurcation result [25].

Rewrite (31)-(38) as

$$(in\Xi + \mathcal{N} + G)\varphi_n = F_n(\varphi, u), \quad n = \pm 2, \pm 3, \ldots, \tag{39}$$
$$(\mathcal{N} + G)\varphi_0 = F_0(\varphi, u), \quad n = 0, \tag{40}$$
$$(\pm i\Xi + \mathcal{N} + G)\varphi_{\pm 1,s} = \mathbf{P}_{\pm 1,s} F_{\pm 1}(\varphi, u), \tag{41}$$
$$(\pm i\Xi + \mathcal{N} + G)\varphi_{\pm 1,c} = \mathbf{P}_{\pm 1,c} F_{\pm 1}(\varphi, u). \tag{42}$$

Now we first solve the equation (40). The linear operator $\mathcal{N}$ has essential spectrum up to the imaginary axis, it can be be inverted in the following sense.

**Lemma 7.** *For $j = 1, 2$ and $f = (f^1, f^2)^T \in (\mathcal{H}^{m-1} \cap \mathcal{L}^1)$, the equation*

$$\mathcal{N}\varphi = \partial_j f$$

*admits a unique solution $\varphi = \mathcal{N}^{-1}\partial_j f \in \mathcal{H}^m$. Moreover,*

$$\|\varphi\|_{\mathcal{H}^m} \le C\|f\|_{\mathcal{H}^{m-1}\cap\mathcal{L}^1}.$$

*Proof.* Define a smooth cut-off function $\chi$ taking its value in $[0, 1]$ as

$$\chi(y) := \begin{cases} 1, & |y| \le 1, \\ 0, & |y| \ge 2. \end{cases}$$

We denote

$$(\hat{f}_1^1, \hat{f}_1^2) = (\hat{f}^1\chi, \hat{f}^2\chi) \ \text{and} \ (\hat{f}_2^1, \hat{f}_2^2) = (\hat{f}^1(1-\chi), \hat{f}^2(1-\chi))$$

with $\hat{f} = (f^1, f^2) = (\hat{f}_1^1 + \hat{f}_2^1, \hat{f}_1^2 + \hat{f}_2^2)$. Then

$$\hat{\omega}_1(y) = \frac{iy_j\hat{f}_1^1}{in\xi_1 - \mu|y|^2 - ic_1y_1} \ \text{and} \ \hat{\omega}_2(y) = \frac{iy_j\hat{f}_2^1}{in\xi_1 - \mu|y|^2 - ic_1y_1},$$
$$\hat{v}_1(y) = \frac{iy_j\hat{f}_1^2}{in\xi_2 - y|y|^2 - ic_1y_1} \ \text{and} \ \hat{v}_2(y) = \frac{iy_j\hat{f}_2^2}{in\xi_2 - y|y|^2 - ic_1y_1}.$$

Note that $(\omega, v) = (\omega_1 + \omega_2, v_1 + v_2)$. Moreover, it holds

$$\begin{aligned} \|\omega_1\|_{\mathbf{H}^m}^2 &= \|\hat{\omega}_1\|_{\mathbf{L}_m^2}^2 = \int_{\mathbf{R}^2} \frac{|y_j|^2|\hat{f}\chi(y)|^2}{|in\xi_1 - \mu|y|^2 - ic_1y_1|^2}\rho^{2m}(y)dy \\ &\le C\|f\|_{\mathbf{L}^1}^2 \int_{|y|\le 2} \frac{|y_j|^2}{r^4 + c^2y_1^2}dy \\ &\le C\|f\|_{\mathbf{L}^1}^2, \end{aligned}$$

and

$$\begin{aligned}\|\omega_2\|^2_{\mathbf{H}^m} &= \|\hat{\omega}_2\|^2_{\mathbf{L}^2_m} = \int\limits_{\mathbf{R}^2} \frac{|y_j|^2|\hat{f}(1-\chi(y))|^2}{|in\xi_1-\mu|y|^2-ic_1y_1|^2}\rho^{2m}(y)dy \\ &\le C\int\limits_{\mathbf{R}^2}|\hat{f}(y)|^2\rho^{(2m-1)}(y)dy \\ &\le C\|f\|^2_{\mathbf{H}^{m-1}}.\end{aligned}$$

By the same process, we can obtain

$$\begin{aligned}\|v_1\|^2_{\mathbf{H}^m} &= \|\hat{v}_1\|^2_{\mathbf{L}^2_m} \le C\|f\|^2_{\mathbf{L}^1}, \\ \|v_2\|^2_{\mathbf{H}^m} &= \|\hat{v}_2\|^2_{\mathbf{L}^2_m} \le C\|f\|^2_{\mathbf{H}^{m-1}}.\end{aligned}$$

This completes the proof. □

This Lemma tells us that $\widehat{\mathcal{N}}(iy_i, iy_i)^T\cdot$ is bounded compact operator in from $\mathbf{L}^2_m\times\mathbf{L}^2_m$ to itself. Furthermore, the spectra of $\widehat{\mathcal{N}}+\widehat{G}$ and $\widehat{\mathcal{N}}$ only differ by isolated eigenvalues of finite multiplicity (see the book of Henry [10] p.136).

The following Lemma gives the solvable of the equation (40).

**Lemma 8.** *Assume that (**H1**)-(**H3**) holds. Then the equation (40) has a unique solution*

$$\varphi_0 = (\mathcal{N}+G)^{-1}F_0(\varphi,u). \tag{43}$$

*Moreover,*

$$\|\varphi_0\|_{\mathcal{H}^m} \le C\|y_j^{-1}I_{2\times2}\widehat{F_0(\varphi,u)}\|_{\mathcal{L}^2_m}.$$

*Proof.* Since the operator $\widehat{\mathcal{N}}^{-1}\widehat{G}:\mathcal{L}^2_m\longrightarrow\mathcal{L}^2_m$ is compact, the operator $I+\widehat{\mathcal{N}}^{-1}\widehat{G}$ is Fredholm with index 0. If $(I+\widehat{\mathcal{N}}^{-1}\widehat{G})\hat{\varphi}=0$ had a nontrivial solution, then $(\widehat{\mathcal{N}}+\widehat{G})\hat{\varphi}=\widehat{\mathcal{N}}(I+\widehat{\mathcal{N}}^{-1}\widehat{G})\hat{\varphi}=0$ would also have a nontrivial solution. This would contradict (**H1**). Hence the Fredholm property implies that the existence of $(I+\widehat{\mathcal{N}}^{-1}\widehat{G})^{-1}:\mathcal{L}^2_m\longrightarrow\mathcal{L}^2_m$. Then we have

$$\widehat{\mathcal{N}}(I+\widehat{\mathcal{N}}^{-1}\widehat{G})\hat{\varphi} = iy_jI_{2\times2}\hat{f},$$

where $I_{2\times2}$ is the unit matrix.

Thus, by Lemma 7, we obtain

$$\begin{aligned}\|\varphi\|_{\mathcal{H}^m} &= \|\hat{\varphi}\|_{\mathcal{L}^2_m} \\ &\le \|(I+\widehat{\mathcal{N}}^{-1}\widehat{G})^{-1}\|_{\mathcal{L}^2_m\longrightarrow\mathcal{L}^2_m}\|\widehat{\mathcal{N}}^{-1}iy_jI_{2\times2}\hat{f}\|_{\mathcal{L}^2_m} \\ &\le C\|\hat{f}\|_{\mathcal{L}^2_m}.\end{aligned}$$

This completes the proof. □

**Lemma 9.** *There exist a constant $C>0$ such that*

$$\|u\|_{\boldsymbol{H}^m}\le C\|\omega\|_{\boldsymbol{H}^m},\ \ \|\partial_{x_i}u\|_{\boldsymbol{H}^m}\le C\|\omega\|_{\boldsymbol{H}^m}. \tag{44}$$

*Proof.* The related equation of the velocity $u$ and the vorticity $\omega$ is

$$\begin{aligned}\nabla\times u &= \omega, \\ \nabla\cdot u &= 0,\ \ \nabla\cdot\omega=0.\end{aligned}$$

This leads in Fourier space to

$$\begin{pmatrix} 0 & -iy_3 & iy_2 \\ iy_3 & 0 & -iy_1 \\ -iy_2 & iy_1 & 0 \\ iy_1 & iy_2 & iy_3 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = \begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \\ 0 \end{pmatrix}.$$

We can get

$$\hat{N}\hat{\omega} = -\frac{1}{|y|^2} \begin{pmatrix} 0 & iy_3 & -iy_2 & iy_1 \\ -iy_3 & 0 & iy_1 & iy_2 \\ iy_2 & -iy_1 & 0 & iy_3 \end{pmatrix} \begin{pmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = \hat{u}.$$

Using Hölder's inequality, for $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $p_1, p_2 > 1$, $s_1 + s_2 = 2m$ and $s_1, s_2 > 0$, we have

$$\begin{aligned} \|u\|^2_{\mathbf{H}^m} = \|\hat{u}\|^2_{\mathbf{L}^2_m} &\le C(\|\chi_{|y|\le 1}\hat{N}\|^2_{\mathbf{L}^{2p_1}_{\frac{s_1}{2}}}\|\hat{\omega}\|^2_{\mathbf{L}^{2p_2}_{\frac{s_2}{2}}} + \|\chi_{|y|\ge 1}\hat{N}\|^2_{\mathbf{L}^\infty}\|\hat{\omega}\|^2_{\mathbf{L}^2_m}) \\ &\le C(\|\chi_{|y|\le 1}\hat{N}\|^2_{\mathbf{L}^{2p_1}_{\frac{s_1}{2}}} + \|\chi_{|y|\ge 1}\hat{N}\|^2_{\mathbf{L}^\infty})\|\hat{\omega}\|^2_{\mathbf{L}^2_m} \\ &\le C\|\hat{\omega}\|^2_{\mathbf{L}^2_m} = C\|\omega\|^2_{\mathbf{H}^m}, \end{aligned}$$

where we use the weighted function $\rho(y) = |y|(1+|y|)^{\frac{1}{2}}$, the boundedness of $\|\chi_{|y|\ge 1}\frac{iy_i}{|y|^2}\|^2_{\mathbf{L}^\infty}$ and

$$\begin{aligned} \|\chi_{|y|\le 1}\frac{iy_i}{|y|^2}\|^2_{\mathbf{L}^{2p_1}_{\frac{s_1}{2}}} &= \int_{|y|\le 1} |\frac{iy_i}{|y|^2}|^{2p_1}\rho^{p_1 s}dy \\ &= \int_{|y|\le 1} |\frac{iy_i}{|y|^2}|^{2p_1}|y|^{p_1 s}(1+|y|)^{\frac{p_1 s}{2}}dy \le C\int_0^1 \frac{\varrho^{2p_1}}{\varrho^{4p_1}}\varrho^{p_1 s}(1+\varrho)^{p_1 s}\varrho^2 d\varrho \\ &= C\int_0^1 \varrho^{p_1 s - 2p_1 + 2}(1+\varrho)^{p_1 s}\varrho^2 d\varrho \le \infty, \ \ for\ p_1 s - 2p_1 + 2 > 0. \end{aligned}$$

The second estimate in (44) is followed by

$$\|\partial_{x_i} u\|_{\mathbf{H}^m} = \|iy_i\hat{u}\rho^m\|_{\mathbf{L}^2} \le \|iy_i\hat{N}\|_{\mathbf{L}^\infty}\|\hat{\omega}\|_{\mathbf{L}^2_m} \le C\|\omega\|_{\mathbf{H}^m}.$$

This completes the proof. □

From the form of the nonlinear terms $g^3$ and $g^4$, it is critical to estimate the term as $uv$ and $u^2$. For convenience, we derive some estimates about the nonlinear term $N^1(\varphi) = \varphi^2$ and $N^2(\varphi, \psi) = \varphi\psi$. This proof is similar with Lemma 4 in [1], so we omit it.

**Lemma 10.** *Define $N^1 : \mathfrak{X} \longrightarrow \mathfrak{X}$ by $N^1(\varphi)_n = N^1_n(\boldsymbol{J}\varphi)$ and $N^2 : \mathfrak{X}\times\mathfrak{X} \longrightarrow \mathfrak{X}$ by $N^2(\varphi)_n = N^2_n(\boldsymbol{J}\varphi, \boldsymbol{J}\psi)$ for $\varphi, \psi \in \mathfrak{X}$. Then there exists $C > 0$ such that*

$$\|N^1(\varphi)\|_{\mathfrak{X}} \le C\|\varphi\|^2_{\mathfrak{X}}, \ \ \|N^2(\varphi,\psi)\|_{\mathfrak{X}} \le C\|\psi\|_{\mathfrak{X}}\|\varphi\|_{\mathfrak{X}} \tag{45}$$

*for $\varphi, \psi \in \mathfrak{X}$ with $\|\varphi\|_{\mathfrak{X}} \le 1$ and $\|\psi\|_{\mathfrak{X}} \le 1$. Moreover, there exists $C > 0$ such that*

$$\|N^1(\varphi^1) - N^1(\varphi^2)\|_{\mathfrak{X}} \le C(\|\varphi^1\|_{\mathfrak{X}} + \|\varphi^2\|_{\mathfrak{X}})\|\varphi^1 - \varphi^2\|_{\mathfrak{X}}, \tag{46}$$

$$\begin{aligned} \|N^2(\varphi^1,\psi^1) - N^2(\varphi^2,\psi^2)\|_{\mathfrak{X}} &\le C\left(\|\varphi^1\|_{\mathfrak{X}} + \|\varphi^2\|_{\mathfrak{X}} + \|\psi^1\|_{\mathfrak{X}} + \|\psi^2\|_{\mathfrak{X}}\right) \\ &\times \left(\|\varphi^1 - \varphi^2\|_{\mathfrak{X}} + \|\psi^1 - \psi^2\|_{\mathfrak{X}}\right), \end{aligned} \tag{47}$$

*for $\varphi^1, \varphi^2, \psi^1, \psi^2 \in \mathfrak{X}$ with $\|\varphi^1\|_{\mathfrak{X}}, \|\varphi^2\|_{\mathfrak{X}}, \|\psi^1\|_{\mathfrak{X}}, \|\psi^2\|_{\mathfrak{X}} \le 1$.*

Then we have the following result.

**Lemma 11.** *Assume that $\xi_i$ close enough to $\xi_0$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|(in\Xi + \mathcal{N})^{-1}\|_{\mathcal{X}\longrightarrow\mathcal{X}} \le C,\\
&\|(in\Xi + \mathcal{N} - G)^{-1}\|_{\mathcal{X}\longrightarrow\mathcal{X}} \le C,\\
&\|(in\Xi + \mathcal{N} - G)^{-1}\boldsymbol{P}_{\pm1,s}\|_{\mathcal{X}\longrightarrow\mathcal{X}} \le C,
\end{aligned}$$

*for $n \neq 0$.*

*Proof.* We observe that the solution $\varphi$ of the equation $(in\Xi + \mathcal{N})\varphi = f$ is given by

$$\widehat{\varphi}(y) = \begin{pmatrix} in\xi_1 + \mu|y|^2 - ic_1y_1 & 0\\ 0 & in\xi_2 + y|y|^2 - ic_1y_1 \end{pmatrix}^{-1} \widehat{f}(y), \quad y \in \mathcal{R}^3.$$

For $\delta_1 = \frac{\mu^2\xi^2}{\xi_1^2+4c_1^2}$ and $\delta_2 = \frac{y^2\xi_2^2}{\xi_2^2+4c_1^2}$, we have

$$\begin{aligned}
|in\xi_1 + \nu|y|^2 - ic_1y_1|^2 &= \mu^2|y|^4 + (c_1y_1 + n\xi_1)^2 \ge \frac{\omega^2}{4c_1^2}\chi_{|y|\le\frac{\omega}{2c_1}} + \delta_1^2(1+|y|^2)\chi_{|y|\ge\frac{\omega}{2c_1}},\\
|in\xi_2 + y|y|^2 - ic_1y_1|^2 &= y^2|y|^4 + (c_1y_1 + n\xi_2)^2 \ge \frac{\omega^2}{4c_1^2}\chi_{|y|\le\frac{\omega}{2c_1}} + \delta_2^2(1+|y|^2)\chi_{|y|\ge\frac{\omega}{2c_1}}.
\end{aligned}$$

It follows for $f \in \mathcal{H}^m$ that $\widehat{\varphi} \in \mathcal{L}^2_{m+2}$, thus $\varphi \in \mathcal{H}^{m+2}$.

Let $\hat{f} \in \mathcal{L}^2_{m+2} \subset \mathcal{L}^2_m$, $\widehat{\omega} = \rho(y,\epsilon)\hat{\omega}$ and $\widehat{\nu} = \rho(y,\epsilon)\hat{\nu}$ with $\rho(x,\epsilon) = \sqrt{1+\epsilon|x|^2}$. Note that $\varphi$ is a solution of the equation $(in\Xi + \mathcal{N})\varphi = f$. By a direct computation, we have

$$(in\Xi + \widehat{\mathcal{N}}\widehat{\bar{\varphi}}) + \epsilon L(y,\epsilon)\widehat{\bar{\varphi}} = \hat{g},$$

where $\bar{\varphi} = (\hat{\omega}, \hat{\nu})^T$, $\hat{g} = \rho(y,\epsilon)\hat{f}$ and

$$\epsilon L(y,\epsilon) = \begin{pmatrix} (in\xi_1 + \mu|y|^2 - icy_1)(1-\rho^{-1}(y,\epsilon)) & 0\\ 0 & (in\xi_2 + y|y|^2 - icy_1)(1-\rho^{-1}(y,\epsilon)) \end{pmatrix}$$

Here we use the fact that $\mathcal{N}$ is elliptic of order of 2. Hence it derives from the form of $\rho(y,\epsilon) = \sqrt{1+\epsilon|y|^2}$ that

$$L(y,\epsilon) \sim \frac{\epsilon|y|^4}{1+\epsilon|y|^2+\sqrt{1+\epsilon|y|^2}} \begin{pmatrix} \mu & 0\\ 0 & y \end{pmatrix}.$$

Using a Neumann series, it derives from the boundness of the operator $L : \mathcal{L}^2_{m+2} \longrightarrow \mathcal{L}^2_m$ that

$$(in\Xi + \widehat{\mathcal{N}}) + \epsilon L : \mathcal{L}^2_{m+2} \longrightarrow \mathcal{L}^2_m$$

is invertible with a bounded inverse, for sufficient small $\epsilon > 0$. This implies that $\bar{\varphi} \in \mathcal{L}^2_{m+2}$, i.e., $\varphi \in \mathcal{H}^{m+2}$. Moreover, we have

$$\begin{aligned}
\|\varphi\|_{\mathcal{H}^{m+2}} &= \|\hat{\varphi}\|_{\mathcal{L}^2_{m+2}} = \|\widehat{\bar{\varphi}}\|_{\mathcal{L}^2_{m+2}}\\
&\le C\|\hat{g}\|_{\mathcal{L}^2_m} = C\|f\|_{\mathcal{H}^{m+2}}.
\end{aligned}$$

Above result shows that $(in\Xi+\mathcal{N})^{-1} : \mathcal{H}^m \longrightarrow \mathcal{H}^{m+2}$ is bounded. But we only need this operator to be bounded $\mathcal{X} \longrightarrow \mathcal{X}$. This implies that the spectrum of $\mathcal{N}$ in $\mathcal{X}$ well separated from $in\Xi$ for $n \neq 0$ and $\epsilon > 0$ sufficient small. In a similar manner to prove the first inequality, the rest two inequalities can be obtained, so we omit it. This completes the proof. □

By the same proof in Lemma 11, we obtain the following result.

**Lemma 12.** *Assume that $\xi_i$ close enough to $\xi_0$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\|(in\xi_i - \overline{M_i})^{-1}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C, \quad \|(in\xi_i - \overline{M_i})^{-1}\nabla^j \cdot \|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C,$$
$$\|(in\xi_i - M_i)^{-1}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C, \quad \|(in\xi_i - M_i)^{-1}\nabla^j \cdot \|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C,$$
$$\|(in\xi_i - M_i)^{-1}\boldsymbol{P}_{\pm1,s}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C, \quad \|(in\xi_i - M_i)^{-1}\nabla^j \cdot \boldsymbol{P}_{\pm1,s}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \le C,$$

*for $n \neq 0$ and $j = 1, 2$.*

Thus by Lemmas 11-12, we can obtain the solution of equations (39) and (41) as

$$\varphi_n = (in\Xi + \mathcal{N})^{-1}F_n(\varphi, u), \quad n = \pm2, \pm3, \ldots,$$
$$\varphi_{\pm1,s} = (\pm i\Xi + \mathcal{N})^{-1}\mathbf{P}_{\pm1,s}F_{\pm1}(\varphi, u),$$

i.e.

$$\omega_n = (in\xi_1 - M_1)^{-1}g_n^3(\omega, u, v), \quad n = \pm2, \pm3, \ldots, \tag{48}$$
$$v_n = (in\xi_2 - M_2)^{-1}g_n^4(\omega, u, v), \quad n = \pm2, \pm3, \ldots, \tag{49}$$
$$\omega_{\pm1,s} = (\pm i\xi_1 - M_1)^{-1}\mathbf{P}_{\pm1,s}g_{\pm1}^3(\omega, u, v), \tag{50}$$
$$v_{\pm1,s} = (\pm i\xi_2 - M_2)^{-1}\mathbf{P}_{\pm1,s}g_{\pm1}^4(\omega, u, v). \tag{51}$$

The following Lemma shows the solvable of equations (48)-(51).

**Lemma 13.** *Assume that there exist $\sigma_1, \sigma_2 > 0$ such that for all $\xi_1, \xi_2 > 0$ with $|\xi_1 - \xi_0|, |\xi_2 - \xi_0| \le \sigma_1$ and all $\omega_{\pm1,c}, v_{\pm1,c} \in \boldsymbol{H}^m$ with $\|\omega_{\pm1,c}\|_{\boldsymbol{H}^m}, \|v_{\pm1,c}\|_{\boldsymbol{H}^m} \le \sigma_2$. Then equations (48)-(51) has a unique solution $(\tilde{\omega}, \tilde{v}) = \Phi(\omega_c, v_c) \in \mathcal{X}$, where*

$$\omega_c = (\omega_{-1,c}, \omega_{1,c}), \quad v_c = (v_{-1,c}, v_{1,c}),$$
$$\tilde{\omega} = (\ldots, \omega_{-2}, \omega_{-1,c} + \omega_{-1,s}, \omega_0, \omega_{1,c} + \omega_{1,s}, \omega_2, \ldots),$$
$$\tilde{v} = (\ldots, v_{-2}, v_{-1,c} + v_{-1,s}, v_0, v_{1,c} + v_{1,s}, v_2, \ldots).$$

*Moreover, there exits $C > 0$ such that*

$$\Phi(0, 0) = (0, 0), \quad \|\tilde{\omega} - \omega_c\|_{\boldsymbol{X}} \le C(\|\omega_{-1,c}\|_{\boldsymbol{H}^m}^2 + \|\omega_{1,c}\|_{\boldsymbol{H}^m}^2), \tag{52}$$
$$\|\tilde{v} - v_c\|_{\boldsymbol{X}} \le C(\|v_{-1,c}\|_{\boldsymbol{H}^m}^2 + \|v_{1,c}\|_{\boldsymbol{H}^m}^2), \tag{53}$$

*with*

$$\tilde{\omega} - \omega_c := (\ldots, , 0, \omega_{-1,c}, 0, \omega_{1,c}, 0, \ldots),$$
$$\tilde{v} - v_c := (\ldots, , 0, v_{-1,c}, 0, v_{1,c}, 0, \ldots).$$

*Proof.* For fixed $\xi_1, \xi_2 > 0$ so close to $\xi_0$ and given $\omega_{\pm1,c}, v_{\pm1,c} \in \mathbf{H}^m$ with

$$\|\omega_{\pm1,c}\|_{\mathbf{H}^m}, \|v_{\pm1,c}\|_{\mathbf{H}^m} \le \sigma_2.$$

Define the operator

$$\begin{aligned}
\Gamma : (\tilde{\omega}^\star, \tilde{v}^\star) &\longmapsto (\tilde{\omega}, \tilde{v}) \\
&= (\tilde{\omega}^\star + (\ldots, 0, \omega_{-1,c}, 0, \omega_{1,c}, 0, \ldots), \tilde{v}^\star + (\ldots, 0, v_{-1,c}, 0, v_{1,c}, 0, \ldots)) \\
&\longmapsto (\omega, v) \longmapsto (\tilde{\omega}^{\star\star}, \tilde{v}^{\star\star}) = \textit{right hand side of } (48) - (51),
\end{aligned}$$

where $(\omega, v) = (\mathbf{J}\tilde{\omega}, \mathbf{J}\tilde{v})$ are defined in Lemma 1 and

$$\begin{aligned}
(\tilde{\omega}^\star, \tilde{v}^\star) &= ((\ldots, \omega_{-2}, \omega_{-1,s}, \omega_0, \omega_{1,s}, \omega_2, \ldots), (\ldots, v_{-2}, v_{-1,s}, v_0, v_{1,s}, v_2, \ldots)), \\
(\tilde{\omega}, \tilde{v}) &= (\tilde{\omega}^\star + \omega_c, \tilde{v}^\star + v_c) \\
&= (\tilde{\omega}^\star + (\ldots, 0, \omega_{-1,c}, 0, \omega_{1,c}, 0, \ldots), \tilde{v}^\star + (\ldots, 0, v_{-1,c}, 0, v_{1,c}, 0, \ldots)).
\end{aligned}$$

By Lemma 2, Young inequality, (11) and the form of nonlinear terms $g^3$ and $g^4$ in (22)-(23), we derive

$$\begin{aligned}\|g^3\|_{\mathbf{H}^{m-2}} &\le C(\|\nabla\cdot(\omega u^T - u\omega^T)\|_{\mathbf{H}^{m-2}} + \|\nabla(\frac{\nabla\times|v|^2}{2})\|_{\mathbf{H}^{m-2}} + \|\nabla\times(v^T\nabla v)\|_{\mathbf{H}^{m-2}})\\ &\le C(\|\omega\|_{\mathbf{H}^{m-1}}\|u\|_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^m})\\ &\le C(\|\omega\|^2_{\mathbf{H}^{m-1}} + \|u\|^2_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^m}),\end{aligned}$$

$$\begin{aligned}\|g^4\|_{\mathbf{H}^{m-2}} &\le C\left(\|u\cdot\nabla v\|_{\mathbf{H}^{m-2}} + \|v\nabla u\|_{\mathbf{H}^{m-2}} + \|\nabla(|v|^2 z + |v|^2 d_\alpha)\|_{\mathbf{H}^{m-2}} + \|\nabla(|v||\nabla d_\alpha|z)\|_{\mathbf{H}^{m-2}}\right)\\ &\le C\left(\|u\|_{\mathbf{H}^{m-1}}\|v\|_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^{m-1}}\|z\|_{\mathbf{H}^{m-1}} + \|v\|_{\mathbf{H}^m}\|z\|_{\mathbf{H}^m}\right)\\ &\le C\left(\|u\|^2_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^{m-1}} + \|z\|^2_{\mathbf{H}^{m-1}} + \|v\|^4_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^m} + \|z\|^2_{\mathbf{H}^m}\right)\\ &\le C\left(\|u\|^2_{\mathbf{H}^{m-1}} + \|v\|^2_{\mathbf{H}^{m-1}} + \|v\|^4_{\mathbf{H}^{m-1}}\right).\end{aligned}$$

Now we prove the operator $\Gamma$ is a self-map of a sufficiently small ball in $\mathcal{X}$. Using Lemma 9 and Lemma 12, we have

$$\begin{aligned}\|\tilde\omega^{\star\star}\|_{\mathbf{X}} &\le C\sup\{\|(in\xi_1 - M_1)^{-1}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}, \|(\pm i\xi_1 - M_1)^{-1}\mathbf{P}_{\pm1,s}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m},\\ &\quad \|(in\xi_1 - M_1)^{-1}\nabla^j\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}, \|(\pm i\xi_1 - M_1)^{-1}\nabla^j\mathbf{P}_{\pm1,s}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}\\ &\quad : n\in\mathbf{Z}\setminus\{-1,1\}\}\times(\|\tilde\omega\|^2_{\mathbf{X}} + \|u\|^2_{\mathbf{X}} + \|\tilde v\|^2_{\mathbf{X}})\\ &\le C(\|\tilde\omega^\star\|^2_{\mathbf{X}} + \|\omega_{-1,c}\|^2_{\mathbf{H}^m} + \|\omega_{1,c}\|^2_{\mathbf{H}^m} + \|\tilde v^\star\|^2_{\mathbf{X}} + \|v_{-1,c}\|^2_{\mathbf{H}^m} + \|v_{1,c}\|^2_{\mathbf{H}^m})\\ &\le C(\|\tilde\omega^\star\|^2_{\mathbf{X}} + \|\tilde v^\star\|^2_{\mathbf{X}} + \sigma_2^2),\end{aligned} \tag{54}$$

$$\begin{aligned}\|\tilde v^{\star\star}\|_{\mathbf{X}} &\le C\sup\{\|(in\xi_2 - M_2)^{-1}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}, \|(\pm i\xi_2 - M_2)^{-1}\mathbf{P}_{\pm1,s}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m},\\ &\quad \|(in\xi_2 - M_2)^{-1}\nabla^j\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}, \|(\pm i\xi_2 - M_2)^{-1}\nabla^j\mathbf{P}_{\pm1,s}\|_{\mathbf{H}^m\longrightarrow\mathbf{H}^m}\\ &\quad : n\in\mathbf{Z}\setminus\{-1,1\}\}\times(\|\tilde u\|^2_{\mathbf{X}} + \|\tilde v\|^2_{\mathbf{X}} + \|\tilde v\|^4_{\mathbf{X}})\\ &\le C(\|\tilde\omega^\star\|^2_{\mathbf{X}} + \|\omega_{-1,c}\|^2_{\mathbf{H}^m} + \|\omega_{1,c}\|^2_{\mathbf{H}^m} + \|\tilde v^\star\|^2_{\mathbf{X}} + \|v_{-1,c}\|^2_{\mathbf{H}^m} + \|v_{1,c}\|^2_{\mathbf{H}^m})\\ &\le C(\|\tilde\omega^\star\|^2_{\mathbf{X}} + \|\tilde v^\star\|^2_{\mathbf{X}} + \sigma_2^2).\end{aligned} \tag{55}$$

Thus, for $\sigma_2 \le \frac{1}{\sqrt{2C}}$ and $(\tilde\omega^\star, \tilde v^\star)\in\mathcal{X}$ with $\|(\tilde\omega^\star,\tilde v^\star)\|_{\mathcal{X}} \le \frac{1}{\sqrt{2C}}$, we have

$$\begin{aligned}\|\Gamma(\tilde\omega^\star,\tilde v^\star)\|_{\mathcal{X}} &= \|\tilde\omega^{\star\star}\|_{\mathcal{X}} + \|\tilde v^{\star\star}\|_{\mathbf{X}}\\ &\le C\left((\|\tilde\omega^\star\|_{\mathbf{X}} + \|\tilde v^\star\|_{\mathbf{X}})^2 + \sigma_2^2\right) \le 1,\end{aligned}$$

which implies that for sufficient small $\sigma_2 > 0$, $\Gamma$ maps the $\|\cdot\|_{\mathcal{X}}$ ball of radius $r = 1$. Hence, we obtain a unique fixed point $(\tilde\omega^\star, \tilde v^\star)\in\mathcal{X}$ of $\Gamma$, which means that equations (48)-(51) has solution of $(\tilde\omega, \tilde v) = (\tilde\omega^\star + \omega_c, \tilde v^\star + v_c)$. Moreover, if $(\omega_{\pm1,c}, v_{\pm1,c}) = (0,0)$, then $\Phi(0,0) = (0,0)$. Next we prove the second inequality in (52). Note that

$$(\tilde\omega^\star,\tilde v^\star) = \Gamma(\tilde\omega^\star,\tilde v^\star) = (\tilde\omega^{\star\star},\tilde v^{\star\star}),$$

which combine with (54)-(55), we derive

$$\begin{aligned}\|\tilde\omega - \omega_c\|_{\mathbf{X}} &= \|\tilde\omega^\star\|_{\mathbf{X}} = \|\tilde\omega^{\star\star}\|_{\mathbf{X}}\\ &\le C(\|\tilde\omega^\star\|^2_{\mathbf{X}} + \|\omega_{-1,c}\|^2_{\mathcal{H}^m} + \|\omega_{1,c}\|^2_{\mathcal{H}^m}),\\ \|\tilde v - v_c\|_{\mathbf{X}} &= \|\tilde v^\star\|_{\mathbf{X}} = \|\tilde v^{\star\star}\|_{\mathbf{X}}\\ &\le C(\|\tilde v^\star\|^2_{\mathbf{X}} + \|v_{-1,c}\|^2_{\mathcal{H}^m} + \|v_{-1,c}\|^2_{\mathcal{H}^m}).\end{aligned}$$

Thus we deduce that for sufficient small ball $\mathbf{B}_r(0)\subset\mathbf{B}_1(0)$,

$$\|\tilde\omega - \omega_c\|_{\mathbf{X}} \le C(\|\omega_{-1,c}\|^2_{\mathbf{H}^m} + \|\omega_{1,c}\|^2_{\mathbf{H}^m}), \tag{56}$$

$$\|\tilde v - v_c\|_{\mathbf{X}} \le C(\|v_{-1,c}\|^2_{\mathbf{H}^m} + \|v_{1,c}\|^2_{\mathbf{H}^m}), \tag{57}$$

where

$$\tilde{\omega} - \omega_c := (\ldots, , 0, \omega_{-1,c}, 0, \omega_{1,c}, 0, \ldots), \quad \tilde{\nu} - \nu_c := (\ldots, , 0, \nu_{-1,c}, 0, \nu_{1,c}, 0, \ldots).$$

This completes the proof. □

**Proof of Theorem 1** To prove Theorem 1, the rest remains to analyze equations (37)-(38). We restate equations:

$$\begin{aligned} (\pm i\xi_1 - M_1)\omega_{\pm 1,c} &= \mathbf{P}_{\pm 1,c} g^3_{\pm 1}(\omega, u, v), \\ (\pm i\xi_2 - M_2)v_{\pm 1,c} &= \mathbf{P}_{\pm 1,c} g^4_{\pm 1}(u, v). \end{aligned}$$

It follows from $(\omega_{-1}, v_{-1}) = (\overline{\omega_1}, \overline{v_1})$ and $(g^3_{\pm 1}, g^4_{\pm 1}) = (\overline{g^3_{\pm 1}}, \overline{g^4_{\pm 1}})$ that the "–" equation is the complex conjugate of the "+" equation. By Lemma 1, we can denote $(\omega, v) = (\mathbf{J}\tilde{\omega}, \mathbf{J}\tilde{v})$ by means of

$$(\tilde{\omega}, \tilde{v}) = \Phi(\omega_c, v_c) = \Phi((\overline{\omega_{1,c}}, \omega_{1,c}), (\overline{v_{1,c}}, v_{1,c})).$$

Our target is to find $(\xi_1, \beta)$ and $(\xi_2, \beta)$ close to $(\xi_0, \beta_c)$ and a nontrivial solution $(\omega_{1,c}, v_{1,c}) = (\omega_{1,c}, v_{1,c})(x)$ of

$$-i\xi_1\omega_{1,c} + M_1\omega_{1,c} + \mathbf{P}_{1,c} g^3_1(\mathbf{J}\Phi(\overline{\omega_{1,c}}, \omega_{1,c}, \overline{v_{1,c}}, v_{1,c})) = 0, \tag{58}$$

$$-i\xi_2 v_{1,c} + M_2 v_{1,c} + \mathbf{P}_{1,c} g^4_1(\mathbf{J}\Phi(\overline{\omega_{1,c}}, \omega_{1,c}, \overline{v_{1,c}}, v_{1,c})) = 0. \tag{59}$$

Due to $\omega_{1,c}, v_{1,c} \in \mathrm{C}\psi^+$ and $(M_1\psi^+, M_2\psi^+) = (\lambda^+_0(\beta)\psi^+, \mu^+_0(\beta)\psi^+)$, we can write

$$\omega_{1,c} = \eta\psi^+, \quad v_{1,c} = \delta\psi^+.$$

Then by (58)-(59), we obtain

$$-i\xi_1\eta\psi^+ + \lambda^+_0(\beta)\eta\psi^+ + \mathbf{P}_{1,c} g^3_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)) = 0, \tag{60}$$

$$-i\xi_2\delta\psi^+ + \mu^+_0(\beta)\delta\psi^+ + \mathbf{P}_{1,c} g^4_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)) = 0, \tag{61}$$

for some $\eta, \delta \in \mathrm{C} \setminus \{0\}$.

To be simple, we introduce $(p_{1,c}, \theta_{1,c})$ by

$$(\mathbf{P}_{1,c}\omega, \mathbf{P}_{1,c}v) = (p_{1,c}(\omega)\psi^+, \theta_{1,c}(v)\psi^+).$$

Then equations (60)-(61) can be written as

$$-i\xi_1\eta + \lambda^+_0(\beta)\eta + g^3(\beta, \eta, \delta) = 0, \quad \textit{for some } \eta \in \mathrm{C}, \tag{62}$$

$$-i\xi_2\delta + \mu^+_0(\beta)\delta + g^4(\beta, \eta, \delta) = 0, \quad \textit{for some } \delta \in \mathrm{C}, \tag{63}$$

where the cubic coefficient $\mu \neq 0$ in

$$g^3(\beta, \eta, \delta) := p_{1,c}\left(g^3_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)\right), \tag{64}$$

$$g^4(\beta, \eta, \delta) := \theta_{1,c}\left(g^4_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)\right). \tag{65}$$

Note that

$$|p_{1,c}(\omega)| \leq C\|\mathbf{P}_{1,c}\omega\|_{\mathbf{H}^m} \leq C\|\omega\|_{\mathbf{H}^m}, \tag{66}$$

$$|p_{1,c}(v)| \leq C\|\mathbf{P}_{1,c}v\|_{\mathbf{H}^m} \leq C\|v\|_{\mathbf{H}^m}. \tag{67}$$

So by (64)-(65), (66)-(67), we derive

$$\begin{aligned} |p_{1,c}\left(g^3_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)\right)| &\leq C\|g^3_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)\|_{\mathbf{H}^m} \\ &\leq C\|\Phi(\overline{\eta\psi^+}, \eta\psi^+, \overline{\delta\psi^+}, \delta\psi^+)\|_{\mathfrak{X}} \\ &\leq C(\|\overline{\omega_{1,c}}\|^2_{\mathbf{H}^m} + \|\omega_{1,c}\|^2_{\mathbf{H}^m} + \|\overline{v_{1,c}}\|^2_{\mathbf{H}^m} + \|v_{1,c}\|^2_{\mathbf{H}^m}) \\ &\leq C(\|\eta\psi^+\|^2_{\mathbf{H}^m} + \|\delta\psi^+\|^2_{\mathbf{H}^m}) \\ &\leq C(|\eta|^2 + |\delta|^2), \end{aligned}$$

$$\begin{aligned}|\theta_{1,c}\left(g_1^4(\mathbf{J}\Phi(\overline{\eta\psi^+},\eta\psi^+,\overline{\delta\psi^+},\delta\psi^+)\right)| &\le C\|g_1^4(\mathbf{J}\Phi(\overline{\eta\psi^+},\eta\psi^+,\overline{\delta\psi^+},\delta\psi^+)\|_{\mathbf{H}^m}\\ &\le C\|\Phi(\overline{\eta\psi^+},\eta\psi^+,\overline{\delta\psi^+},\delta\psi^+)\|_{\mathfrak{X}}\\ &\le C(\|\overline{\omega_{1,c}}\|_{\mathbf{H}^m}^2+\|\omega_{1,c}\|_{\mathbf{H}^m}^2+\|\overline{\nu_{1,c}}\|_{\mathbf{H}^m}^2+\|\nu_{1,c}\|_{\mathbf{H}^m}^2\\ &\quad+\|\overline{\nu_{1,c}}\|_{\mathbf{H}^m}^4+\|\nu_{1,c}\|_{\mathbf{H}^m}^4)\\ &\le C(\|\eta\psi^+\|_{\mathbf{H}^m}^2+\|\delta\psi^+\|_{\mathbf{H}^m}^2+\|\delta\psi^+\|_{\mathbf{H}^m}^4)\\ &\le C(|\eta|^2+|\delta|^2+|\delta|^4),\end{aligned}$$

where we use the notation

$$(\tilde{\omega},\tilde{\nu})=\Phi(\omega_c,\nu_c)=\Phi(\overline{\eta\psi^+},\eta\psi^+,\overline{\delta\psi^+},\delta\psi^+).$$

Inspired by the classical Hopf-Bifurcation result [25], if we exclude the zero solution, we can employ the implicit function theorem to find real value solutions (i.e. find $(y_1,y_2)=(\eta,\delta)\in\mathbf{R}^2$) of equations (62)-(63). Hence, we define the complex-valued smooth function

$$Y^1(y_1,y_2;\varrho,\beta):=\begin{cases}-i(\xi_0+\varrho)+\lambda_0^+(\beta_c+\epsilon)+y_1^{-1}g^3(\beta_c+\epsilon,y_1,y_2), & y_1\neq 0,\\ -i(\xi_0+\varrho)+\lambda_0^+(\beta_c+\epsilon), & y_1=0,\end{cases}$$

$$Y^2(y_1,y_2;\varrho,\beta):=\begin{cases}-i(\xi_0+\varrho)+\mu_0^+(\beta_c+\epsilon)+y_2^{-1}g^4(\beta_c+\epsilon,y_1,y_2), & y_2\neq 0,\\ -i(\xi_0+\varrho)+\mu_0^+(\beta_c+\epsilon), & y_2=0.\end{cases}$$

It follows from $(\lambda_0^+(\beta_c),\mu_0^+(\beta_c))=(i\xi_0,i\xi_0)$ that $(Y^1(0,0,0,0),Y^2(0,0,0,0))=(0,0)$. Moreover, by assumption (H2) the Jacobi Matrix

$$\mathbf{D}_{\rho,\epsilon}Y^1(y_1,y_2;\varrho,\epsilon)|_{y_1=y_2=\varrho=\epsilon=0}=\begin{pmatrix}0 & \frac{d}{d\beta}\mathbf{Re}\lambda_0^+(\beta)|_{\beta=\beta_c}\\ -1 & \frac{d}{d\beta}\mathbf{Im}\lambda_0^+(\beta)|_{\beta=\beta_c},\end{pmatrix}$$

$$\mathbf{D}_{\rho,\epsilon}Y^2(y_1,y_2;\varrho,\epsilon)|_{y_1=y_2=\varrho=\epsilon=0}=\begin{pmatrix}0 & \frac{d}{d\beta}\mathbf{Re}\mu_0^+(\beta)|_{\beta=\beta_c}\\ -1 & \frac{d}{d\beta}\mathbf{Im}\mu_0^+(\beta)|_{\beta=\beta_c}\end{pmatrix}$$

with respect to $\rho,\epsilon$ has

$$\begin{aligned}det\mathbf{D}_{\rho,\epsilon}Y^1(y_1,y_2;\varrho,\epsilon)|_{y_1=y_2=\varrho=\epsilon=0}&=\frac{d}{d\beta}\mathbf{Re}\lambda_0^+(\beta)|_{\beta=\beta_c}>0,\\ det\mathbf{D}_{\rho,\epsilon}Y^2(y,y_2;\varrho,\epsilon)|_{y_1=y_2=\varrho=\epsilon=0}&=\frac{d}{d\beta}\mathbf{Re}\mu_0^+(\beta)|_{\beta=\beta_c}>0.\end{aligned}$$

Thus, for sufficient small $y_1,y_2>0$, we find a function $y_1\mapsto(\varrho(y_1),\epsilon(y_1))$ and $y_2\mapsto(\varrho(y_2),\epsilon(y_2))$ with $\varrho(0)=\epsilon(0)=0$ such that

$$\begin{aligned}&-i(\xi_0+\varrho(y_1))+\lambda_0^+(\beta_c+\epsilon(y_1))-y_1^{-1}g^3(\beta_c+\epsilon(y_1),y_1,\beta_c+\epsilon(y_2),y_2)=0,\\ &-i(\xi_0+\varrho(y_2))+\mu_0^+(\beta_c+\epsilon(y_2))-y_2^{-1}g^4(\beta_c+\epsilon(y_1),y_1,\beta_c+\epsilon(y_2),y_2)=0\end{aligned}$$

Note that the degree of nonlinearity. Then it follows from differentiating this equation that $\epsilon^i\neq 0$ for some first $i$. Hence, the function $y_1\mapsto\epsilon(y_1)$ and $y_1\mapsto\epsilon(y_2)$ are locally invertible, and have $\epsilon\mapsto y_1(\epsilon)$ and $\epsilon\mapsto y_2(\epsilon)$. It implies that the following equation holds

$$\begin{aligned}-\quad & i(\xi_0+\varrho(y_1(\epsilon)))y_1(\epsilon)+\lambda_0^+(\beta_c+\epsilon)y_1(\epsilon)-g^3(\beta_c+\epsilon,y_1(\epsilon),y_2(\epsilon))=0,\\ -\quad & i(\xi_0+\varrho(y_2(\epsilon)))y_2(\epsilon)+\mu_0^+(\beta_c+\epsilon)y_2(\epsilon)-g^4(\beta_c+\epsilon,y_1(\epsilon),y_2(\epsilon))=0,\end{aligned}$$

for sufficient small $\epsilon>0$.

Therefore we obtain the desired solutions of (58)-(59) by setting $(\xi_1,\xi_2)=(\xi_0+\varrho(y_1(\epsilon)),\xi_0+\varrho(y_2(\epsilon)))$, $\beta=\beta_c+\epsilon$ and $(\omega_{1,c},\nu_{1,c})=(y_1(\epsilon)\psi^+_{\beta_c+\epsilon},y_2(\epsilon)\psi^+_{\beta_c+\epsilon})(x)$. This result combining with Lemma 8, Lemma 13 and (19) give the proof of Theorem 1.

**Acknowledgement:** This work is supported by NSFC No 11771359, and the Fundamental Research Funds for the Central Universities (Grant No. 20720190070, No.201709000061 and No. 20720180009). The second author is supported by Huizhou University Professor Doctor Launch Project Grant, No. 20187B037.

# References

[1] T. Brand, M. Kunze, G. Schneider, T. Seelbach, *Hopf bifurcation and exchange of stability in diffusive media*, Arch. Rational Mech. Anal., 171 (2004), pp. 263-296.
[2] S. Chandrasekhar, Liquid Crystals. 2nd ed., Cambridge: Cambridge University Press (1992).
[3] Z.M. Chen, W.G. Price, *Time dependent periodic Navier-Stokes flows on a two-dimensional torus*, Comm. Math. Phys., 179 (1996), pp. 577-597.
[4] M. Cencelj and D. Repovs, V.D. Rădulescu, Double phase problems with variable growth, Nonlinear Analysis 177 (2018) 270-287.
[5] M.G. Crandall, P. Rabinowitz, *The Hopf Bifurcation theorem in infinite dimensions*, Arch. Rational Mech. Anal., 67 (1977), pp. 53-72.
[6] J. Ericksen, *Equilibrium Theory of Liquid Crystals*, Academic Press, New York (1976).
[7] P.G. de Gennes, J. Prost, *The Physics of Liquid Crystals*, New York: Oxford University Press (1993).
[8] Y. Giga, *The Stokes operator in $L_r$ spaces*, Proc. Japan Acad. Ser. A Math Sci., 57 (1981), pp. 85-89.
[9] R. Glowinski, G. Guidoboni, *Hopf bifurcation in viscous incompressible flow down an inclined plane: a numerical approach*, J. math. fluid mech., 10 (2008), pp. 434-454.
[10] D. Henry, *Geometric theory of semilinear parabolic equations*, New York: Springer-Verlag (1981).
[11] M.C. Hong, *Global existence of solutions of the simplified Ericksen–Leslie system in dimension two*, Calc. Var. PDE., 40 (2011), 15-36
[12] X.P. Hu, D.H. Wang, *Global solution to the three-dimensional incompressible flow of liquid crystals*, Commun. Math. Phys., 296 (2010), pp. 861–C880.
[13] D.D. Joseph, D.H. Sattinger, Bifurcating time periodic solutions and their stability. Arch. Rational Mech. Anal. 45, 75-109 (1972)
[14] F. Leslie, *Theory of flow phenomemum in liquid crystal*, In: Brown (ed.) Advances in Liquid Crystals, vol. 4, pp. 1–81. Academic Press, New York (1979)
[15] H.Y. Li, W.P. Yan, *Nonlinear stability for the three dimensional incompressible flow of nematic liquid crystals*, Appl. Math. Lett., 39 (2015), pp. 42-46.
[16] F.H. Lin, *Nonlinear theory of defects in nematic liquid crystals: phase transition and flow phenomena*, Commun. Pure. Appl. Math., 42 (1989), pp. 789-814.
[17] F.H. Lin, *Mathematics theory of liquid crystals*, In: Applied Mathematics at the Turn of the Century, Lecture Notes of the 1993 Summer School, Universidad Complutense de Madrid: Editorial Complutense (1995).
[18] F.H. Lin, C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Commun. Pure. Appl. Math., 48 (1995), pp. 501-537.
[19] F.H. Lin, C. Liu, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Disc. Cont. Dyn. Sys., 2 (1996), pp. 1-22.
[20] C. Liu, *Walkington, N.J.: Approximation of liquid crystal flow*, SIAM J. Numer. Anal., 37 (2000), pp. 725-741.
[21] G. Iooss, *Bifurcation des solutions périodiques de certains problèmes dévolution*, C.R. Acad. Sci. Paris., 273 (1971), série A, pp. 624-627.
[22] G. Iooss, *Bifurcation of a periodic solution of the Navier-Stokes equations into an invariant torus*, Arch. Rational Mech. Anal., 58 (1975), pp. 35-56.
[23] G. Iooss, A. Mielke, *Bifurcating time-periodic solutions of Navier-Stokes equations in infinite cylinders*, J. Nonlinear. Sci., 1 (1991), pp. 107-146.
[24] V.I. Iudovich, *Apprearance of auto-oscillations in a fluid*, Prikl. Mat. Mek. 35 (1971), pp. 638-655.
[25] J.E. Marsden, M. Mccracken, *The Hopf bifurcation and its applications*, Berlin-New York: Springer (1976).
[26] A. Melcher, G. Schneider, H. Uecker, *A hopf-bifurcation theorem for the vorticity formulation of the Navier-Stokes equations in $\boldsymbol{R}^3$*, Commun.Partial Differential Eqs., 33 (2008), pp. 772-783.
[27] P. Mironescu, V.D. Rădulescu, A bifurcation problem associated to a convex, asymptotically linear function, C.R. Acad. Sci. Paris, Ser. I 316 (1993), 667-672.
[28] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York, Springer-Verlag (1983).
[29] D.H. Sattinger, *Bifurcation of periodic solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., 41 (1971), pp. 66-80.
[30] B. Sandstede, A. Scheel, *Hopf Bifurcation From Viscous Shock Waves*, SIAM J. Math. Anal., 39 (2008), pp. 2033-2052.

[31] R. Temam, *Some developments on Navier-Stokes equations in the second half of the* 20*th century*, Development of mathematics 1950-2000, Basel, Birkhäuser: 2000, pp. 1049-1106

[32] B. Texier, K. Zumbrun, *Hopf bifurcation of viscous shock waves in compressible gas dynamics and MHD*, Arch. Rational Mech. Anal., 190 (2008), pp. 107-140.

[33] N. Papageorgiou, V.D. Rădulescu, Bifurcation of positive solutions for nonlinear nonhomogeneous Robin and Neumann problems with competing nonlinearities, Discrete and Continuous Dynamical Systems 35 (2015) 5003-5036.

[34] M. Xiang, B. Zhang, V.D. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, Advances in Nonlinear Analysis 9 (2020) 690-709.

[35] W.P. Yan, *Hopf-bifurcation theorem and stability for the magneto-hydrodynamics equations*, Topol. Methods. Nonlinear Anal., 46 (2015), pp. 471-493.