Abdelouahed El Khalil, Mohamed Laghzal*, My Driss Morchid Alaoui, and Abdelfattah Touzani

# Eigenvalues for a class of singular problems involving $p(x)$-Biharmonic operator and $q(x)$-Hardy potential

https://doi.org/10.1515/anona-2020-0042
Received March 22, 2019; accepted July 26, 2019.

**Abstract:** In this paper, we consider the nonlinear eigenvalue problem:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} \text{ in } \Omega, \\ u \in W_0^{2,p(x)}(\Omega), \end{cases}$$

where $\Omega$ is a regular bounded domain of $\mathbb{R}^N$, $\delta(x) = dist(x, \partial\Omega)$ the distance function from the boundary $\partial\Omega$, $\lambda$ is a positive real number, and functions $p(\cdot)$, $q(\cdot)$ are supposed to be continuous on $\overline{\Omega}$ satisfying

$$1 < \min_{\overline{\Omega}} q \leq \max_{\overline{\Omega}} q < \min_{\overline{\Omega}} p \leq \max_{\overline{\Omega}} p < \frac{N}{2} \text{ and } \max_{\overline{\Omega}} q < p_2^\star := \frac{Np(x)}{N-2p(x)}$$

for any $x \in \overline{\Omega}$. We prove the existence of at least one non-decreasing sequence of positive eigenvalues. Moreover, we prove that $\sup \Lambda = +\infty$, where $\Lambda$ is the spectrum of the problem. Furthermore, we give a proof of positivity of $\inf \Lambda > 0$ provided that Hardy-Rellich inequality holds.

**Keywords:** $p(x)$-biharmonic operator; nonlinear eigenvalue problems; variational methods; Ljusternik-Schnirelman; Hardy-Rellich inequality

**MSC:** 58E05, 35J35, 35J60, 47J10.

## 1 Introduction

During the last two decades the field of variable exponent Lebesgue spaces $L^{p(.)}$ or Sobolev $W^{m,p(.)}$ have received a strong deal of attention by exploiting the well-known techniques of Orlicz spaces dated back to the work of Orlicz [13] in 1931. Motivated by large applications involving partial differential equations governed by operators may vary with respect to the spacial variable $x$ itself, this filed becomes today a research area called variable exponent analysis. This area is necessary in nonlinear electrorheological fluids and elastic mechanics. In that context, we refer the reader to Ruzicka [18], Zhikov [23] and the references therein. Our results in the present paper are partially inspired from the work of Harjulehto et al. [9] where the authors

**Abdelouahed El Khalil,** Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 90950, 11623 Riyadh, KSA, E-mail: alakhalil@imamu.edu.sa
***Corresponding Author: Mohamed Laghzal,** Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, P.O. Box 1796 Atlas Fez, Morocco, E-mail: mohamed.laghzal@usmba.ac.ma
**My Driss Morchid Alaoui,** Department of Mathematics Moulay Ismail University of Meknes Faculty of Sciences and Technologies BP 509, Boutalamine, 52000 Errachidia, Morocco, E-mail: morchid_driss@yahoo.fr
**Abdelfattah Touzani,** Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, P.O. Box 1796 Atlas Fez, Morocco, E-mail: Abdelfattah.touzani@usmba.ac.ma

surveyed the differential equations governed by $p(.)$-Lapalcian with non-standard growth and compared the results on the existence and the regularity.

In particular, the equations treated here contain also Hardy potential term, which make the analysis more delicate and quite interesting. Namely, will deal with the following nonlinear eigenvalue problem involving the $p(x)$-Biharmonic and Hardy type potential:

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} \text{ in } \Omega, \\ u \in W_0^{2,p(x)}(\Omega), \end{cases} \tag{1.1}$$

where $\Omega$ is a regular bounded domain in $\mathbb{R}^N$, we denote by $\delta(x) := dist(x, \partial\Omega)$ the distance from the boundary $\partial\Omega$, $\lambda$ is a positive real number playing the role of an eigenvalue, and the exponent functions $p(\cdot)$, $q(\cdot)$ are continuous on $\overline{\Omega}$, $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the fourth order $p(x)$-biharmonic operator.

Recently, El Khalil et al. proved the existence of a sequence of positive eigenvalues for problem (1.1) only for the constant case $q(x) = p(x) = p$ in [6].

As far as we are aware, nonlinear eigenvalue problems like (1.1) involving the nonhomogenous Hardy inequality in variable exponent Sobolev space have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction.

In this way, inspired by the works of [12, 15, 16], we attempt to expend the results of [6] to the space of variable exponent. More precisely, we establish the existence of at least one non-decreasing sequence of nonnegative eigenvalues of this problem such that $\sup \Lambda = \infty$, where $\Lambda$ is the spectrum of (1.1). We prove that $\inf \Lambda > 0$ when the $q(.)$-Hardy-Rellich inequality holds, as a characterization of the positivity of the principal eigenvalue of problem (1.1).

In this paper we assume the following conditions:

**H(p,q)** $1 < q^- \leq q^+ < p^- \leq p^+ < \frac{N}{2}$ and $q^+ < p_2^\star(x)$, for any $x \in \overline{\Omega}$,

where

$$q^- := \min_{\overline{\Omega}} q, \quad q^+ := \max_{\overline{\Omega}} q \text{ and } p_2^\star(x) := \frac{Np(x)}{N - 2p(x)}, \text{ for any } x \in \overline{\Omega}$$

The rest of this paper is structured as follows. Section 2 states some preliminary properties to establish our results presented in Section 4. Section 3 we establish and prove an axillary inequality of Hardy type in variable exponent Lebesgue spaces, and give a necessary and sufficient condition for Hardy's inequality to be true. In Section 4 we prove the existence of at least one non-decreasing sequence of nonnegative eigenvalues to problem (1.1) under appropriate hypotheses. Section 5 discusses the positivity of the infimum of the spectrum of our eigenvalue problem.

## 2 Space $W^{2,p(\cdot)}(\Omega)$ and useful results

We introduce some basic properties of Lebesgue-Sobolev spaces with variable exponent $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$. For details, we refer to the book [4] and the references therein. To handle better theses spaces we define first, for each fixed exponent function $p(.)$, the modular functional $\rho_{p(x)}$ defined on $L^{p(\cdot)}(\Omega)$ by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)}\, dx.$$

Set

$$C_1^+(\Omega) := \{h : h \in C(\overline{\Omega}) \text{ and } h(x) > 1, \forall x \in \overline{\Omega}\}$$

The generalized Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \rho_{p(x)}(u) < \infty\right\}.$$

We endow it with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{\mu > 0 : \rho_{p(x)}(\frac{u}{\mu}) \leq 1\right\}.$$

The following proposition assembles the most important topological features of our variable exponent space setting that we need to overcome most of difficulties related to our objectives.

**Proposition 2.1** ([21]). *The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(.)})$ is separable, uniformly convex, reflexive and its conjugate dual space is $L^{p'(\cdot)}(\Omega)$ where $p'(\cdot)$ is the conjugate function of $p(\cdot)$, i.e.,*

$$p'(x) = \frac{p(x)}{p(x)-1} \quad \textit{for all} \ \ x \in \Omega.$$

*For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ we have the "equivalent" Hölder's inequality*

$$\left|\int_\Omega u(x)v(x)\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{p'(\cdot)}.$$

The Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$ is defined as

$$W^{m,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), \quad |\alpha| \leq m\right\}.$$

Equipped with the norm

$$\|u\|_{m,p(x)} = \sum_{|\alpha|\leq m} |D^\alpha u|_{p(x)}$$

$W^{m,p(\cdot)}(\Omega)$ has the same topological features as $L^{p(\cdot)}(\Omega)$. For more details, we refer the reader to [7, 8, 11] and [20]. We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

We define also the functional space

$$\mathcal{L}_{\div}^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \frac{1}{p(x)}|u(x)|^{p(x)}dx < \infty\right\}$$

and we equip it with the norm

$$|u|_{p(.)} = \inf\left\{\mu > 0 : \int_\Omega \frac{1}{p(x)}\left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$$

so that $\mathcal{L}_{\div}^{p(\cdot)}(\Omega)$ is a Banach space having similar topological properties to those of variable exponent Lebesgue spaces.

**Proposition 2.2** ([8, Theorem 1.3]). *Let $u_n,\ u \in L^{p(\cdot)}$, we have*
(1) $|u|_{p(\cdot)} = a \Leftrightarrow \rho_{p(x)}(\frac{u}{a}) = 1$ *for* $u \neq 0$ *and* $a > 0$.
(2) $|u|_{p(\cdot)} < (=; > 1) \Leftrightarrow \rho_{p(x)}(u) < (=; > 1)$.
(3) $|u_n| \to 0(resp \to +\infty) \Leftrightarrow \rho_{p(x)}(u_n) \to 0, (resp \to +\infty)$.
(4) *the following statements are equivalent.*
*4(i)* $\lim_{n\to+\infty}|u_n - u|_{p(\cdot)} = 0$,
*4(ii)*$\lim_{n\to+\infty}\rho_{p(x)}(u_n - u) = 0$,
*4(iii)*$u_n \to u$ *in measure in* $\Omega$ *and* $\lim_{n\to+\infty}\rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

Return back to our problem (1.1), solutions are considered in weak sense, namely in $W_0^{2,p(\cdot)}(\Omega)$ endowed with the norm

$$\|u\| := \inf\left\{\mu > 0 : \ \rho_{p(x)}(\frac{\Delta u(x)}{\mu}) \leq 1\right\}.$$

**Remark 2.3.** *According to [21] the norm $\|.\|_{2,p(\cdot)}$ is equivalent to the norm $\|\Delta.\|_{p(\cdot)}$ in the space $W_0^{2,p(\cdot)}(\Omega)$, for the Lipschitz boundary and the exponent $p(\cdot)$ is in a class functions for which $\frac{1}{p(\cdot)}$ is globally log-Hölder continuous. Consequently, the norms $\|.\|_{2,p(\cdot)}$, $\|.\|$ and $\|\Delta.\|_{p(\cdot)}$ are equivalent. See also [2].*

**Remark 2.4.** *In view of Proposition 2.2, for every $u \in L^{p(.)}(\Omega)$, then*

$$\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} \le \int_\Omega |\Delta u|^{p(x)}\, dx \le \max\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\}. \tag{2.1}$$

**Theorem 2.5** ([1, Theorem 3.2]). *Let $p, q \in C_1^+(\Omega)$. Assume $p(x) < \frac{N}{2}$ and $q(x) < p_2^\star(x)$. Then there is a continuous and compact embedding of $W_0^{2,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$.*

**Definition 2.6.** *A function $u \in W_0^{2,p(\cdot)}(\Omega)$ is said to be a weak solution of* (1.1) *if*

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)\, dx = \lambda \int_\Omega \frac{|u(x)|^{q(x)-2}u(x)}{\delta(x)^{2q(x)}} v(x)\, dx \quad \text{for all} \quad v \in W_0^{2,p(\cdot)}(\Omega).$$

*We point out that in the case $u$ is nontrivial we say that $\lambda$ is an eigenvalue of* (1.1) *corresponding to the eigenfunction $u$.*

We give a direct characterization of $\lambda_k$ involving a mini-max argument over sets of genus greater than $k$. We set

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx, \text{ subject to } \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\, dx = 1\right\}. \tag{2.2}$$

The value defined in (2.2) can be written as the Rayleigh quotient

$$\lambda_1 = \inf \frac{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)} dx}{\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx}, \tag{2.3}$$

where the infimum is taken over $W_0^{2,p(\cdot)}(\Omega) \setminus \{0\}$.

# 3 Improved $q(.)$-Hardy-Rellich inequality

Famous Hardy-Rellich inequality states that for all $u$ in the usual Hilbert space $H_0^2(\Omega)$,

$$\int_\Omega |\Delta u|^2\, dx \ge \frac{N^2(N-4)^2}{16}\int_\Omega \frac{|u|^2}{\delta(x)^4}\, dx, \quad N \ge 5. \tag{3.1}$$

The constant $\frac{N^2(N-4)^2}{16}$ is optimal, but it is never achieved in any domain $\Omega \in \mathbb{R}^N$. This inequality was first proved by Rellich [17] in $H_0^2(\Omega)$ and it was extended to functions in $H^2(\Omega) \cap H_0^1(\Omega)$ by Doled et al [5]. Davis and Hinz [3] generalized (3.1) and proved that for any $p \in (1, \frac{N}{2})$,

$$\int_\Omega |\Delta u|^p\, dx \ge \left(\frac{N(p-1)(N-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx, \tag{3.2}$$

whenever $u \in C_c^\infty(\Omega)$. The inequality (3.2) was proved in [10], for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for $1 < p < \frac{N}{2}$, where $p$ is constant. Owen [14] proved for $p = 2$ and $\Omega$ is bounded convex open set that

$$\int_\Omega |\Delta u|^2\, dx \ge \frac{9}{16}\int_\Omega \frac{|u|^2}{\delta(x)^4}\, dx,$$

whenever $u \in C_c^\infty(\Omega)$.

**Lemma 3.1.** *Assume that **H(p,q)** holds. Then there exists a positive constant C such that the q(.)-Hardy-Rellich inequality*

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \geq C\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}dx \tag{3.3}$$

*holds for all $u \in W_0^{2,p(\cdot)}(\Omega)$ in one of the following cases:*
(a) $|u| \leq \delta(x)^2$ *and* $|\triangle u| \geq 1$;
(b) $|u| \geq \delta(x)^2$ *and* $|\triangle u| \leq 1$.

*Proof.* Let $u \in W_0^{2,p(\cdot)}(\Omega)$ such that $1 < q^- \leq q^+ < p^- \leq p^+ < \frac{N}{2}$.
From (a) we have $|\triangle u| \geq 1$. Thus

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq \int_\Omega |\Delta u|^{q^-}\,dx. \tag{3.4}$$

In view of (3.2), we deduce that

$$\int_\Omega |\Delta u|^{q^-}\,dx \geq C^-\int_\Omega \frac{|u|^{q^-}}{\delta(x)^{2q^-}}\,dx,$$

where $C^- = \left(\frac{N(q^- -1)(N-2q^-)}{(q^-)^2}\right)^{q^-}$. This and (3.4) permit us to write

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq q^-C^-\int_\Omega \frac{1}{q^-}\frac{|u|^{q^-}}{\delta(x)^{2q^-}}\,dx.$$

Since $|u| \leq \delta(x)^2$, we obtain

$$\left(\frac{|u|}{\delta(x)^2}\right)^{q^-} \geq \left(\frac{|u|}{\delta(x)^2}\right)^{q(x)}.$$

Therefore

$$q^-C^-\int_\Omega \frac{1}{q^-}\left(\frac{|u|}{\delta(x)^2}\right)^{q^-}\,dx \geq q^-C^-\int_\Omega \frac{1}{q(x)}\left(\frac{|u|}{\delta(x)^2}\right)^{q(x)}\,dx.$$

Therefore

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq q^-C^-\int_\Omega \frac{1}{q(x)}\left(\frac{|u|}{\delta(x)^2}\right)^{q(x)}\,dx.$$

That is

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \geq \frac{q^-}{p^+}C^-\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx.$$

Hence we conclude that

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \geq C_1\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx,$$

where $C_1 = \frac{q^-}{p^+}C^-$.
From (b) w have $|\triangle u| \leq 1$. Thus in this case

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq \int_\Omega |\Delta u|^{p^+}\,dx. \tag{3.5}$$

Using again (3.2), we have

$$\int_\Omega |\Delta u|^{p^+}\,dx \geq C^+\int_\Omega \frac{|u|^{p^+}}{\delta(x)^{2p^+}}\,dx,$$

where $C^+ = \left(\frac{N(p^+ -1)(N-2p^+)}{(p^+)^2}\right)^{p^+}$. This and (3.5) give

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq q^-C^+\int_\Omega \frac{1}{q^-}\frac{|u|^{p^+}}{\delta(x)^{2p^+}}\,dx.$$

Since $|u| \geq \delta(x)^2$, we get

$$\Big(\frac{|u|}{\delta(x)^2}\Big)^{p^+} \geq \Big(\frac{|u|}{\delta(x)^2}\Big)^{q(x)}.$$

Thus

$$q^- C^+ \int_\Omega \frac{1}{q^-}\Big(\frac{|u|}{\delta(x)^2}\Big)^{p^+}\,dx \geq q^- C^+ \int_\Omega \frac{1}{q(x)}\Big(\frac{|u|}{\delta(x)^2}\Big)^{q(x)}\,dx.$$

Therefore

$$\int_\Omega \frac{p^+}{p(x)}|\Delta u|^{p(x)}\,dx \geq q^- C^+ \int_\Omega \frac{1}{q(x)}\Big(\frac{|u|}{\delta(x)^2}\Big)^{q(x)}\,dx.$$

Then

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \geq \frac{q^-}{p^+} C^+ \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx.$$

Hence we deduce that

$$\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \geq C_2 \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx,$$

where $C_2 = \frac{q^-}{p^+}C^+$. The result follows. □

**Remark 3.2.** *In view of* (3.3), $\lambda_1 > 0$ *if and only if the $q(.)$-Hardy-Rellich inequality holds in $\Omega$. Moreover, $\lambda_1$ is exactly the best constant in the inequality* (3.3). *This makes our problem* (1.1) *naturally well defined.*

**Definition 3.3.** *Let $X$ be a real reflexive Banach space and let $X^\star$ stand for its dual with respect to the pairing $\langle\cdot,\cdot\rangle$. We shall deal with mappings $T$ acting from $X$ into $X^\star$. The strong convergence in $X$ (and in $X^\star$) is denoted by $\to$ and the weak convergence by $\rightharpoonup$. $T$ is said to belong to the class $(S^+)$ if for any sequence $u_n$ in $X$ converging weakly to $u \in X$ and* $\limsup_{n\to+\infty}\langle Tu_n, u_n - u\rangle \leq 0$, *$u_n$ converges strongly to $u$ in $X$. We write $T \in (S^+)$.*

Define $\Phi(u), \varphi(u) : W_0^{2,p(\cdot)}(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx \quad \text{and} \quad \varphi(u) = \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx,$$

and set $\mathcal{H} = \{u \in W_0^{2,p(\cdot)}(\Omega);\ \varphi(u) = 1\}$.

**Lemma 3.4.** *We have the following statements:*
(i) *$\Phi$ and $\varphi$ are even, and of class $C^1$ on $W_0^{2,p(\cdot)}(\Omega)$.*
(ii) *$\mathcal{H}$ is a closed $C^1$-manifold.*

*Proof.* It is clear that $\varphi$ and $\Phi$ are even and of class $C^1$ on $W_0^{2,p(\cdot)}(\Omega)$ and $\mathcal{H} = \varphi^{-1}\{1\}$. In view of (3.3) $\mathcal{H}$ is closed. The derivative operator $\varphi'$ satisfies $\varphi'(u) \neq 0\ \forall u \in \mathcal{H}$ (i.e., $\varphi'(u)$ is onto for all $u \in \mathcal{H}$). Hence $\varphi$ is a submersion, which proves that $\mathcal{H}$ is a $C^1$-manifold. □

The operator $T := \Phi' : W_0^{2,p(\cdot)}(\Omega) \to W^{-2,p'(\cdot)}(\Omega)$ defined as

$$\langle T(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx \quad \text{for any } u, v \in W_0^{2,p(\cdot)}(\Omega)$$

satisfies the assertions of the following lemma.

**Lemma 3.5.** *The following statements hold:*
(1) *$T$ is continuous, bounded and strictly monotone.*
(2) *$T$ is of $(S_+)$ type.*
(3) *$T$ is a homeomorphism.*

*Proof.* (1) Since $T$ is the Fréchet derivative of $\Phi$, it follows that $T$ is continuous and bounded.

Using the following elementary inequalities, which hold for any three real $x$, $y$ and $\gamma$.

$$|x-y|^{\gamma} \le 2^{\gamma}(|x|^{\gamma-2}x-|y|^{\gamma-2}y)(x-y) \quad \text{if } \gamma \ge 2,$$
$$|x-y|^{2} \le \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x-|y|^{\gamma-2}y)(x-y) \quad \text{if } 1<\gamma<2,$$

we obtain for all $u, v \in W_0^{2,p(\cdot)}(\Omega)$ such that $u \ne v$,

$$\langle T(u)-T(v), u-v\rangle > 0,$$

which means that $T$ is strictly monotone.

(2) Let $(u_n)_n$ be a sequence of $W_0^{2,p(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{2,p(\cdot)}(\Omega) \text{ and } \limsup_{n\to+\infty}\langle T(u_n), u_n-u\rangle \le 0.$$

In view of the monotonicity of $T$, we have

$$\langle T(u_n)-T(u), u_n-u\rangle \ge 0, \tag{3.6}$$

and since $u_n \rightharpoonup u$ weakly in $W_0^{2,p(\cdot)}(\Omega)$, it follows that

$$\limsup_{n\to+\infty}\langle T(u_n)-T(u), u_n-u\rangle = 0. \tag{3.7}$$

Thanks to the above inequalities,

$$\begin{cases} \displaystyle\int_{\{x\in\Omega:p(x)\ge 2\}} |\Delta u_n-\Delta u|^{p(x)}\,dx \le 2^{(p^- -2)}\int_\Omega A(u_n,u)\,dx, \\ \displaystyle\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n-\Delta u|^{p(x)}dx \le (p^+-1)\int_\Omega (A(u_n,u))^{\frac{p(x)}{2}}(B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx, \end{cases}$$

where

$$\begin{cases} A(u_n,u) = (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n-\Delta u), \\ B(u_n,u) = (|\Delta u_n|+|\Delta u|)^{2-p(x)}. \end{cases}$$

$$\int_\Omega A(u_n,u)\,dx = \langle T(u_n)-T(u), u_n-u\rangle,$$

we can consider

$$0 \le \int_\Omega A(u_n,u)\,dx < 1$$

and we distinguish two cases.

First, if $\int_\Omega A(u_n,u)\,dx = 0$, then $A(u_n,u)=0$, since $A(u_n,u)\ge 0$ a.e. in $\Omega$.

Second, if $0 < \int_\Omega A(u_n,u)\,dx < 1$, then

$$t^{p(x)} := \Bigg(\int_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\Bigg)^{-1}$$

is positive and by applying Young's inequality we deduce that

$$\int_{\{x\in\Omega:1<p(x)<2\}} \big[t(A(u_n,u))^{\frac{p(x)}{2}}\big](B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx$$
$$\le \int_{\{x\in\Omega:1<p(x)<2\}} \Big(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\Big)\,dx.$$

Now, by the fact that $\frac{2}{p(x)} < 2$, we have

$$\int\limits_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\right)\,dx \le \int\limits_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)t^2 + (B(u_n,u))^{p(x)}\right)\,dx$$
$$\le 1 + \int\limits_{\{x\in\Omega:1<p(x)<2\}} (B(u_n,u))^{p(x)}\,dx.$$

Hence

$$\int\limits_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \le \left(\int\limits_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\right)^{\frac{1}{2}} \left(1 + \int\limits_{\Omega} (B(u_n,u))^{p(x)}\,dx\right).$$

Since $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, we have

$$\int\limits_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \to 0 \quad \text{as} \quad n\to\infty.$$

(3) Note that the strict monotonicity of $T$ implies that $T$ is into an operator.

Moreover, $T$ is a coercive operator. Indeed, from (2.1) and since $p^- - 1 > 0$, for each $u \in W_0^{2,p(\cdot)}(\Omega)$ such that $\|u\| \ge 1$, we have

$$\frac{\langle T(u),u\rangle}{\|u\|} = \frac{\Phi'(u)}{\|u\|} \ge \|u\|^{p^- -1} \to \infty \quad \text{as} \quad \|u\|\to\infty.$$

Finally, thanks to the Minty-Browder Theorem [22], the operator $T$ is surjective and admits an inverse mapping.

To complete the proof of (3), it suffices then to show the continuity of $T^{-1}$. Indeed, let $(f_n)_n$ be a sequence of $W^{-2,p'(\cdot)}(\Omega)$ such that $f_n \to f$ in $W^{-2,p'(\cdot)}(\Omega)$. Let $u_n$ and $u$ in $W_0^{2,p(\cdot)}(\Omega)$ such that

$$T^{-1}(f_n) = u_n \quad \text{and} \quad T^{-1}(f) = u.$$

By the coercivity of $T$, we deduce that the sequence $(u_n)_n$ is bounded in the reflexive space $W_0^{2,p(\cdot)}(\Omega)$. For a subsequence, if necessary, we have $u_n \rightharpoonup \widehat{u}$ in $W_0^{2,p(\cdot)}(\Omega)$ for a some $\widehat{u}$. Then

$$\lim_{n\to+\infty}\langle T(u_n) - T(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty}\langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the assertion (2) and the continuity of $T$ that

$$u_n \to \widehat{u} \text{ in } W_0^{2,p(x)}(\Omega) \text{ and } T(u_n) \to T(\widehat{u}) = T(u) \text{ in } W^{-2,p'(x)}(\Omega).$$

Further, since $T$ is an into operator, we conclude that $u \equiv \widehat{u}$.

□

We shall use The following results to prove our theorem related to the existence.

**Lemma 3.6.** *We have the following statements:*
(i) *$\varphi'$ is completely continuous.*
(ii) *The functional $\Phi$ satisfies the Palais-Smale condition on $\mathcal{H}$, i.e., for $\{u_n\} \subset \mathcal{H}$, if $\{\Phi(u_n)\}_n$ is bounded and*

$$\alpha_n = \Phi'(u_n) - \beta_n\varphi'(u_n) \to 0 \quad as \quad n\to+\infty, \tag{3.8}$$

*where*

$$\beta_n = \frac{\langle \Phi'(u_n), u_n\rangle}{\langle \varphi'(u_n), u_n\rangle},$$

*then $\{u_n\}_{n\ge1}$ has a convergent subsequence in $W_0^{2,p(\cdot)}(\Omega)$.*

*Proof.* (i) First let us prove that $\varphi'$ is well defined. Let $u, v \in W_0^{2,p(\cdot)}(\Omega)$. We have

$$\langle \varphi'(u), v\rangle = \int_\Omega \frac{|u|^{q(x)-2}u}{\delta(x)^{2q(x)}} v\,dx.$$

Thus

$$|\langle \varphi'(u), v\rangle| \leq \int_{\{x\in\Omega:\delta(x)>1\}} \frac{|u|^{q(x)-1}}{\delta(x)^{2q(x)}} v\,dx + \int_{\{x\in\Omega:\delta(x)\leq 1\}} \frac{|u|^{q(x)-1}}{\delta(x)^{2q(x)}} v\,dx.$$

Therefore

$$|\langle \varphi'(u), v\rangle| \leq \int_{\{x\in\Omega:\delta(x)>1\}} |u|^{q(x)-1} v\,dx + \int_{\{x\in\Omega:\delta(x)\leq 1\}} \frac{1}{\delta(x)^2}\frac{|u|^{q(x)-1}}{\delta(x)^{2(q(x)-1)}} v\,dx.$$

By applying Hölder's inequality, we obtain

$$|\langle \varphi'(u), v\rangle| \leq 2\left(|u|_{r(x)}^{q(x)-1}|v|_{q(x)} + \left|\frac{u}{\delta(x)^2}\right|_{r(x)}^{q(x)-1}\left|\frac{v}{\delta(x)^2}\right|_{q(x)}\right).$$

where $r(x) = \frac{q(x)}{q(x)-1}$.

This and (3.3) yield

$$|\langle \varphi'(u), v\rangle| \leq 2\left(|u|_{r(x)}^{q(x)-1}|v|_{q(x)} + \frac{1}{C^2}|\Delta u|_{r(x)}^{q(x)-1}|\Delta v|_{q(x)}\right).$$

Then

$$|\langle \varphi'(u), v\rangle| \leq 2\left(k_1\|u\|^{q(x)-1}\|v\| + \frac{k_2}{C^2}\|u\|^{q(x)-1}\|v\|\right),$$

where $k_1$ is a constant given by the embedding of $W_0^{2,p(\cdot)}(\Omega)$ in $L^{q(\cdot)}(\Omega)$ and $k_2$ is given by the equivalence of the norm $|\Delta .|_{p(\cdot)}$ and $\|.\|$. Hence

$$\|\varphi'(u)\|_\star \leq 2\left(k_1 + \frac{k_2}{C^2}\right)\|u\|^{q(x)-1},$$

where $\|\cdot\|_\star$ is the dual norm associated with $\|.\|$.

For the complete continuity of $\varphi'$, we argue as follow. Let $(u_n)_n \subset W_0^{2,p(\cdot)}(\Omega)$ be a bounded sequence and $u_n \rightharpoonup u$ (weakly) in $W_0^{2,p(\cdot)}(\Omega)$. Due to the $q(\cdot)$-Hardy inequality (3.3) $u_n \rightharpoonup u$ in $L^{q(\cdot)}(\Omega; \frac{1}{q(\cdot)})$ and due to the fact that the embedding $W_0^{2,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, $u_n$ converges strongly to $u$ in $L^{q(\cdot)}(\Omega)$. Consequently, there exists a positive function $g \in L^{q(\cdot)}(\Omega)$ such that

$$| u | \leq g \quad \text{a.e. in } \Omega.$$

Since $g \in L^{q(\cdot)-1}(\Omega)$, it follows from the Dominated Convergence Theorem that

$$| u_n |^{q(x)-2} u_n \to | u |^{q(x)-2} u \quad \text{in} \quad L^{q'(\cdot)}(\Omega).$$

That is,

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in} \quad L^{q'(\cdot)}(\Omega).$$

Recall that the embedding

$$L^{q'(\cdot)}(\Omega) \hookrightarrow W^{-2,p'(\cdot)}(\Omega)$$

is compact. Thus

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in} \quad W^{-2,p'(\cdot)}(\Omega).$$

This proves the assertion (i).

(ii) By the definition of $\Phi$ we have that $\rho_{p(x)}(\Delta u_n)$ is bounded in $\mathbb{R}$. Thus, without loss of generality, we can assume that $u_n$ converges weakly in $W_0^{2,p(\cdot)}(\Omega)$ for some functions $u \in W_0^{2,p(\cdot)}(\Omega)$ and $\rho_{p(x)}(\Delta u_n) \to \ell$.

For the rest we distinguish two cases. If $\ell = 0$, then $u_n$ converges strongly to 0 in $W_0^{2,p(\cdot)}(\Omega)$. Otherwise, let us prove that

$$\limsup_{n\to\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \leq 0.$$

Indeed, notice that

$$\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle = \rho_{p(x)}(\Delta u_n) - \langle \Delta^2_{p(\cdot)} u_n, u\rangle.$$

Applying $\alpha_n$ of (3.8) to $u$, we deduce that

$$\theta_n = \langle \Delta^2_{p(\cdot)} u_n, u\rangle - \beta_n \langle \varphi'(u_n), u\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore

$$\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle = \rho_{p(x)}(\Delta u_n) - \theta_n - \left(\frac{\langle \Phi'(u_n), u_n\rangle}{\langle \varphi'(u_n), u_n\rangle}\right)\langle \varphi'(u_n), u\rangle.$$

That is,

$$\langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle = \frac{\rho_{p(x)}(\Delta u_n)}{\langle \varphi'(u_n), u_n\rangle}\Big(\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u_n), u\rangle\Big) - \theta_n.$$

On the other hand, from Lemma 3.6, $\varphi'$ is completely continuous. Thus

$$\varphi'(u_n) \to \varphi'(u) \quad \text{and} \quad \langle \varphi'(u_n), u_n\rangle \to \langle \varphi'(u), u\rangle.$$

Then

$$|\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u_n), u\rangle| \leq |\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u), u\rangle| + |\langle \varphi'(u_n), u\rangle - \langle \varphi'(u), u\rangle|.$$

It follows that

$$|\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u_n), u\rangle| \leq |\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u), u\rangle| + \|\varphi'(u_n) - \varphi'(u)\|_\star \|u\|.$$

This implies that

$$\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u_n), u\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Combining with the above equalities, we obtain

$$\limsup_{n\to+\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \leq \frac{\ell}{\langle \varphi'(u), u\rangle} \limsup_{n\to\infty} (\langle \varphi'(u_n), u_n\rangle - \langle \varphi'(u_n), u\rangle).$$

We deduce

$$\limsup_{n\to\infty} \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle \leq 0. \tag{3.10}$$

On the other hand,

$$\langle \Phi'(u_n), u_n - u\rangle = \langle \Delta^2_{p(\cdot)} u_n, u_n - u\rangle.$$

According to (3.10), we conclude that

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \leq 0. \tag{3.11}$$

In view of Lemma 3.5, $u_n \to u$ strongly in $W_0^{2,p(\cdot)}(\Omega)$. This achieves the proof of Lemma 3.6. □

# 4 Existence of infinitely many eigenvalue sequences

Set

$$\Gamma_j = \left\{K \subset \mathcal{H} : K \text{ is symmetric, compact and } \gamma(K) \geq j\right\},$$

where $\gamma(K) = j$ is the Krasnoselskii genus of the set $K$, i.e., the smallest integer $j$, such that there exists an odd continuous map from $K$ to $\mathbb{R}^j \setminus \{0\}$.

Now, let us establish some useful properties of the Krasnoselskii genus proved by Szulkin [19].

**Lemma 4.1.** *Let $X$ be a real Banach space and $A$, $B$ be symmetric subsets of $X \setminus \{0\}$ which are closed in $X$. Then*
(a) *If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \leq \gamma(B)$.*
(b) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
(d) *If $\gamma(B) < +\infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*
(e) *If $A$ is compact, then $\gamma(A) < +\infty$ and there exists a neighborhood $N$ of $A$, $N$ is a symmetric subset of $X \setminus \{0\}$, closed in $X$ such that $\gamma(N) = \gamma(A)$.*
(f) *If $N$ is a symmetric and bounded neighborhood of the origin in $\mathbb{R}^k$ and if $A$ is homeomorphic to the boundary of $N$ by an odd homeomorphism, then $\gamma(A) = k$.*
(g) *If $X_0$ is a subspace of $X$ of codimension $k$ and if $\gamma(A) > k$ then $A \cap X_0 \neq \phi$.*

We now state our first main theorem of this paper using the Ljusternik-Schnirelmann theory.

**Theorem 4.2.** *For any integer $j \in \mathbb{N}^\star$,*

$$\lambda_j = \inf_{K \in \Gamma_j} \max_{u \in K} \Phi(u)$$

*is a critical value of $\Phi$ restricted on $\mathcal{H}$. More precisely, there exists $u_j \in K$ such that*

$$\lambda_j = \Phi(u_j) = \sup_{u \in K} \Phi(u),$$

*and $u_j$ is a solution of* (2.6) *associated to the positive eigenvalue $\lambda_j$. Moreover,*

$$\lambda_j \to \infty, \quad as \;\; j \to \infty.$$

*Proof.* We only need to prove that for any $j \in \mathbb{N}^\star$, $\Gamma_j \neq \emptyset$ and the last assertion. Indeed, let $j \in \mathbb{N}$ be given and let $x_1 \in \Omega$ and $r_1 > 0$ be small enough such that $\overline{B(x_1, r_1)} \subset \Omega$ and $meas(\overline{B(x_1, r_1)}) < \frac{meas(\Omega)}{2}$. First, we take $u_1 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(u_1) = \overline{B(x_1, r_1)}$. Put $B_1 := \Omega \setminus \overline{B(x_1, r_1)}$, then $meas(B_1) > \frac{meas(\Omega)}{2}$. Let $x_2 \in B_1$ and $r_2 > 0$ such that $\overline{B(x_2, r_2)} \subset B_1$ and $meas(\overline{B(x_1, r_1)}) < \frac{meas(B_1)}{2}$.
Next, we take $u_2 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(u_2) = \overline{B(x_2, r_2)}$.

Continuing the process described above we can construct by recurrence a sequence of functions $u_1, u_2, ...., u_j \in C_0^\infty(\Omega)$ such that $supp(u_i) \cap supp(u_j) = \emptyset$ if $i \neq j$ and $meas(supp(u_i)) > 0$ for $i \in \{1, 2, ..., j\}$.
Let $E_j = Span\{u_1, u_2, ..., u_j\}$ be the vector subspace of $C_0^\infty(\Omega)$ Spanned by $\{u_1, u_2, ..., u_j\}$. Then, $dim\ E_j = j$ and note that the map

$$w \mapsto |w| := \left\{\alpha > 0 : \int_\Omega \left|\frac{w(x)}{\alpha}\right|^{p(x)} dx\right\},$$

defines a norm on $E_j$. Putting $S_j := \{E_j : |v| = 1\}$ the unit sphere of $E_j$. Let us introduce the functional $g : \mathbb{R}^+ \times E_j \longrightarrow \mathbb{R}$ by $g(s, u) = \varphi(su)$. It is clear that $g(0, u) = 0$ and $g(s, u)$ is non decreasing with respect to $s$. More, for $s > 1$ we have

$$g(s, u) \geq s^{q^-} \varphi(u),$$

and so $\lim_{s \to +\infty} g(s, u) = +\infty$. Therefore, for every $u \in S_j$ fixed, there is a unique value $s = s(u) > 0$ such that $g(s(u), u) = 1$.
On the other hand, since

$$\frac{\partial g}{\partial s}(s(u), u) = \int_\Omega (s(u))^{q(x)-1} \frac{|u|^{q(x)}}{\delta^{2q(x)}}\, dx \geq \frac{q^-}{s(u)} g(s(u), u) = \frac{q^-}{s(u)} > 0.$$

The implicit function theorem implies that the map $u \mapsto s(u)$ is continuous and even by uniqueness.
Now, take the compact $K_j := \mathcal{H} \cap E_j$. Since the map $h : S_j \longrightarrow K_j$ defined by $h(u) = s(u).u$ is continuous and odd, it follows by the property of genus that $\gamma(K_j) \geq j$. This completes the proof of first part of the theorem.

Now, we claim that

$$\lambda_j \to \infty \quad \text{as} \quad j \to \infty.$$

Since $W_0^{2,p(\cdot)}(\Omega)$ is separable, there exists $(e_k, e_n^\star)_{k,n}$ a bi-orthogonal system such that $e_k \in W_0^{2,p(\cdot)}(\Omega)$ and $e_n^\star \in W^{-2,p'(\cdot)}(\Omega)$, the $(e_k)_k$ are linearly dense in $W_0^{2,p(\cdot)}(\Omega)$ and the $(e_n^\star)_n$ are total for the dual $W^{-2,p'(\cdot)}(\Omega))$. For $k \in \mathbb{N}^\star$, set

$$F_k = \operatorname{span}\{e_1, \dots, e_k\} \quad \text{and} \quad F_k^\perp = \operatorname{span}\{e_{k+1}, e_{k+2}, \dots\}.$$

By $(g)$ of Lemma 4.1, we have for any $K \in \Gamma_k$, $K \cap F_{k-1}^\perp \neq \emptyset$. Thus

$$t_k = \inf_{K\in\Gamma_k} \sup_{u\in K\cap F_{k-1}^\perp} \Phi(u) \to \infty \quad \text{as} \quad k \to \infty.$$

Indeed, if not, for large $k$ there exists $u_k \in F_{k-1}^\perp$ with $\|u_k\|_{p(\cdot)} = 1$ such that $t_k \leq \Phi(u_k) \leq M$ for some $M > 0$ independent of $k$. Thus $\|\Delta u_k\|_{p(\cdot)} \leq M$. This implies that $(u_k)_k$ is bounded in $W_0^{2,p(\cdot)}(\Omega)$. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in $W_0^{2,p(\cdot)}(\Omega)$ and strongly in $L^{p(\cdot)}(\Omega)$. By our choice of $F_{k-1}^\perp$, we have $u_k \rightharpoonup 0$ weakly in $W_0^{2,p(\cdot)}(\Omega)$ because $\langle e_n^\star, e_k\rangle = 0$, for any $k > n$. This contradicts the fact that $\|u_k\|_{p(\cdot)} = 1$ for all $k$. Since $\lambda_k \geq t_k$, the claim is proved. □

# 5 The infimum of the eigenvalues

Now, we give the following lemma which will be used in Theorem 5.2 which is the second main result of this paper.

**Lemma 5.1.** *$\lambda_1 = 0 \Leftrightarrow R(u) = 0$, where*

$$R(u) = R_{p(\cdot)}(u) = \inf_{u\in W_0^{2,p(\cdot)}(\Omega), u\not\equiv 0} \frac{\int_\Omega |\Delta u|^{p(x)} dx}{\int_\Omega \frac{|u|^{q(x)}}{\delta(x)^{2q(x)}} dx}.$$

*Proof.* It is clear that we have the bounds

$$\frac{q^-}{p^+} R(u) \leq \lambda_1 \leq \frac{p^+}{q^-} R(u),$$

then from this it follows that $\lambda_1 = 0 \Leftrightarrow R(u) = 0$. □

**Theorem 5.2.** *If there are an open subset $U \subset \Omega$ and a point $x_0 \in \Omega$ such that $p(x_0) < (\text{ or } >) p(x)$ for all $x \in \partial U$, then $\lambda_1 = 0$.*

*Proof.* We only deal with the case that $p(x_0) < p(x)$. The proof of the case that $p(x_0) > p(x)$ is similar. Denote for $O \subset \overline{\Omega}$ and $\delta > 0$, $B(O; \delta) = \{x \in \mathbb{R}^N : \operatorname{dist}(x; O) < \delta\}$. Without loss of generality, we may assume that $\overline{U} \subset \Omega$, then there is $\varepsilon_0 > 0$ such that

$$p(x_0) < p(x) - 4\varepsilon_0 \quad \text{for all} \quad x \in \partial U,$$

and there is $\varepsilon_1 > 0$ such that

$$p(x_0) < p(x) - 2\varepsilon_0 \quad \text{for all} \quad x \in B(\partial U, \varepsilon_1), \tag{5.1}$$

where $B(\partial U, \varepsilon_1) = \{x : \exists y \in \partial U \text{ such that } |x - y| < \varepsilon_1\} \subset \Omega$, and there is $\varepsilon_2 > 0$ such that $B(x_0, \varepsilon_2) \subset U \setminus B(\partial U, \varepsilon_1)$, and

$$|p(x_0) - p(x)| < \varepsilon_0 \quad \text{for all} \quad x \in B(x_0, \varepsilon_2). \tag{5.2}$$

We can find a function $u_0 \in C_0^\infty(\Omega)$ such that $|\Delta(u_0(x))| \leq C$, $0 \leq u_0 \leq 1$ and

$$u_0(x) = \begin{cases} 1 & \text{if } x \in U \setminus B(\partial U, \varepsilon_1), \\ 0 & \text{if } x \notin U \cup B(\partial U, \varepsilon_1). \end{cases}$$

Then, for $t > 0$ small enough such that

$$\frac{\int_\Omega \frac{|tu_0|^{p(x)}}{\delta(x)^{2p(x)}}\,dx}{\int_\Omega \frac{|tu_0|^{q(x)}}{\delta(x)^{2q(x)}}\,dx} < 1.$$

Thus

$$\begin{aligned} R(tu_0) = \frac{\int_\Omega |\Delta(tu_0(x))|^{p(x)}\,dx}{\int_\Omega \frac{|tu_0|^{q(x)}}{\delta(x)^{2q(x)}}\,dx} &= \left(\frac{\int_\Omega |\Delta(tu_0(x))|^{p(x)}\,dx}{\int_\Omega \frac{|tu_0|^{p(x)}}{\delta(x)^{2p(x)}}\,dx}\right)\left(\frac{\int_\Omega \frac{|tu_0|^{p(x)}}{\delta(x)^{2p(x)}}\,dx}{\int_\Omega \frac{|tu_0|^{q(x)}}{\delta(x)^{2q(x)}}\,dx}\right) \\ &\leq \frac{\int_{B(x_0,\varepsilon_2)} |\Delta(tu_0(x))|^{p(x)}\,dx}{\int_{B(x_0,\varepsilon_2)} \frac{|tu_0|^{p(x)}}{\delta(x)^{2p(x)}}\,dx} \\ &\leq \frac{C_1}{C_2} t^{p(\xi_1)-p(\xi_2)}, \end{aligned}$$

where $C_1 = \int_{B(\partial U,\varepsilon_1)} |\Delta u_0(x)|^{p(x)}\,dx$ and $C_2 = \int_{B(x_0,\varepsilon_2)} \frac{|u_0|^{p(x)}}{\delta(x)^{2p(x)}}\,dx$ are positive constants independent of $t$ with $\xi_1 \in \overline{B(\partial U, \varepsilon_1)}$ and $\xi_2 \in \overline{B(x_0, \varepsilon_2)}$.

Using (5.1) and (5.2), we get $|p(\xi_1) - p(\xi_2)| > \varepsilon_0$. Therefore,

$$R(tu_0) \leq \frac{C_1}{C_2} t^{\varepsilon_0} \quad \text{for all} \quad t \in (0; 1).$$

When $t \to 0^+$, we obtain $R(u) = 0$, because $\lim_{t\to 0^+} \frac{C_1}{C_2} t^{\varepsilon_0} = 0$ and in view of Lemma 5.1, we deduce $\lambda_1 = 0$. This completes the proof. □

**Corollary 5.3.** *If $\Omega$ satisfies the $q(.)$-Hardy-Rellich inequality, then we have the following statements:*
(i) $\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx\right.$, *where* $u \in W_0^{2,p(\cdot)}(\Omega)$ *and* $\left.\int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx = 1\right\}$;
(ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$;
(iii) $\lambda_1 = \inf \Lambda$ *(i.e., $\lambda_1$ is the smallest eigenvalue in the spectrum of* (1.1)*).*

*Proof.* (i) For $u \in \mathcal{H}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, $\Phi$ is even and

$$\Phi(u) = \max_{K_1} \Phi \geq \inf_{K\in\Gamma_1} \max_{u\in K} \Phi(u).$$

Thus

$$\inf_{u\in\mathcal{H}} \Phi(u) \geq \inf_{K\in\Gamma_1} \max_{u\in K} \Phi(u) = \lambda_1.$$

On the other hand, for all $K \in \Gamma_1$ and $u \in K$, we have

$$\sup_{u\in K} \Phi \geq \Phi(u) \geq \inf_{u\in\mathcal{H}} \Phi(u).$$

It follows that

$$\inf_{K\in\Gamma_1} \max_K \Phi = \lambda_1 \geq \inf_{u\in\mathcal{H}} \Phi(u).$$

Then

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx, \text{ where } u \in W_0^{2,p(\cdot)}(\Omega) \text{ and } \int_\Omega \frac{1}{q(x)}\frac{|u|^{q(x)}}{\delta(x)^{2q(x)}}\,dx = 1\right\}.$$

(ii) For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of the definition of $\lambda_i$, $i \in \mathbb{N}^\star$, we get $\lambda_i \geq \lambda_j$. As regards $\lambda_n \to \infty$, it has been proved in Theorem 4.2.

(iii) Let $\lambda \in \Lambda$. Thus there exists $u_\lambda$ an eigenfunction of $\lambda$ such that

$$\int_\Omega \frac{1}{q(x)} \frac{|u_\lambda|^{q(x)}}{\delta(x)^{2q(x)}}\, dx = 1.$$

Therefore

$$\Delta^2_{p(x)} u_\lambda = \lambda \frac{|u_\lambda|^{q(x)-2} u_\lambda}{\delta(x)^{2q(x)}} \quad \text{in } \Omega.$$

Then

$$\int_\Omega \frac{1}{p(x)} |\Delta u_\lambda|^{p(x)}\, dx = \lambda \int_\Omega \frac{1}{q(x)} \frac{|u_\lambda|^{q(x)}}{2q(x)^{2q(x)}}\, dx.$$

In view of the characterization of $\lambda_1$ in (2.2), we conclude that

$$\lambda = \frac{\int_\Omega \frac{1}{p(x)} |\Delta u_\lambda|^{p(x)}\, dx}{\int_\Omega \frac{1}{q(x)} \frac{|u_\lambda|^{q(x)}}{2q(x)^{2q(x)}}\, dx} = \int_\Omega \frac{1}{p(x)} |\Delta u_\lambda|^{p(x)}\, dx \geq \lambda_1.$$

This implies that $\lambda_1 = \inf \Lambda$. □

## References

[1] A. Ayoujil, A., El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent, *Nonlinear Anal. Theory Methods.***71**(2009),4916-4926.
[2] F. Colasuonno and P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.* **74** (2011), no. 17, 5962–5974.
[3] E. B. Davies and A. M. Hinz, Explicit constants for Rellich inequalities in $L_p(\Omega)$, *Math. Z.* **227** (1998), no. 3, 511–523.
[4] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
[5] J. W. Dold, V. A. Galaktionov, A. A. Lacey and J. L. Vázquez, Rate of approach to a singular steady state in quasilinear reaction-diffusion equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **26** (1998), no. 4, 663–687.
[6] A. El khalil, M. D. Morchid Alaoui and A. Touzani, On the spectrum of the $p$-biharmonic operator involving $p$-Hardy's inequality, *Appl. Math. (Warsaw)* **41** (2014), no. 2-3, 239–246.
[7] X. L. Fan and X. Fan, A Knobloch-type result for $p(t)$-Laplacian systems, *J. Math. Anal. Appl.* **282** (2003), no. 2, 453–464.
[8] X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), no. 2, 424–446.
[9] P. Harjulehto, P. Hästö, V. Lê, Út and M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.* **72** (2010), no. 12, 4551–4574.
[10] É. Mitidieri, A simple approach to Hardy inequalities (in Russian), translated from *Mat. Zametki* **67** (2000), no. 4, 563–572; *Math. Notes* **67** (2000), no. 3-4, 479–486.
[11] M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, *Nonlinear Anal.* **67** (2007), no. 5, 1419–1425.
[12] M. Mihăilescu, V. Rădulescu; A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc, A 462 (2006) 2073, 2625–2641.
[13] W. Orlicz, b̈er konjugierte exponentenfolgen, *Studia Mathematica***3**, (1931), 200-211.
[14] M. P. Owen, The Hardy–Rellich inequality for polyharmonic operators, *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), no. 4, 825–839.
[15] V. Rădulescu; Nonlinear elliptic equations with variable exponent, Old and new, Nonlinear Anal.121 (2015), 336–369.
[16] V. Rădulescu, D.D. Repov's; Partial Differential Equations with Variable Equations, Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.
[17] F. Rellich, in: Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 243–250. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.
[18] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.
[19] A. Szulkin, Ljusternik-Schnirelmann theory on $C^1$-manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5** (1988), no. 2, 119–139.
[20] J. H. Yao, Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators, *Nonlinear Anal.* **68** (2008), no. 5, 1271–1283.

[21] A. Zang and Y. Fu, Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces, *Nonlinear Anal.* **69** (2008), no. 10, 3629–3636.

[22] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. II/B. Nonlinear monotone operators*, Translated from the German by the author and Leo F. Boron, Springer-Verlag, New York, 1990.

[23] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, 675–710.