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Critical growth elliptic problems involving Hardy-Littlewood-Sobolev critical exponent in non-contractible domains

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Abstract: The paper is concerned with the existence and multiplicity of positive solutions of the nonhomogeneous Choquard equation over an annular type bounded domain. Precisely, we consider the following equation

$$-\Delta u = \left(\int\limits_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\right)|u|^{2^*_\mu-2}u + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth bounded annular domain in $\mathbb{R}^N (N \geq 3)$, $2^*_\mu = \frac{2N-\mu}{N-2}$, $f \in L^\infty(\Omega)$ and $f \geq 0$. We prove the existence of four positive solutions of the above problem using the Lusternik-Schnirelmann theory and varitaional methods, when the inner hole of the annulus is sufficiently small.

Keywords: Hardy-Littlewood-Sobolev inequality, critical problems, non-contractible domains

MSC: 35A15, 35J60, 35J20

1 Introduction

In the pioneering work, Tarantello [31] studied the nonhomogeneous elliptic equation

$$-\Delta u = |u|^{2^*-2}u + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. If $f \in H^{-1}$ then it is shown that there exists at least two solutions of (1.1) by using variational methods. Cao and Zhou [9] proved the existence of two positive solutions of the following nonhomogeneous elliptic equation

$$-\Delta u = f(x, u(x)) + h \text{ in } \mathbb{R}^N \tag{1.2}$$

where $f(x, u)$ is a Carathéodory function with subcritical grotwh at ∞. Further, many researchers investigated (1.1) and (1.2) for the existence and multiplicity of solutions. For details, we refer [10, 11, 20, 21, 33] and references therein. Recently, Gao and Yang [30] proved the existence of two positive solutions of the nonhomogeneous Choquard equation involving Hardy-Littlewood-Sobolev critical exponent using the splitting Nehari manifold method of Tarantello [31].

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The existence, uniqueness, and multiplicity of positive solutions of the nonlocal elliptic equation, precisely the Choquard equation both for mathematical analysis and in perspective of physical models has recently gained significant attention amongst researchers. As an instance, in 1954 Pekar [28] proposed the equation

$$-\Delta u + u = \left(\frac{1}{|x|} \star |u|^2\right) u \text{ in } \mathbb{R}^3 \tag{1.3}$$

to study the quantum theory of polaron. Later in 1976, Ph. Choquard [22] examined the steady state of one component plasma approximation in Hartee-Fock theory using (1.3). In [22], Leib proved the existence and uniqueness of the ground state of (1.3). The work of Moroz and Schaftingen enriches the literature of Choquard equations. In [25] authors studied the following Choquard equation

$$-\Delta u + Vu = \big(I_\alpha \star F(u)\big)\, F'(u), \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $\alpha \in (0, N)$, $N \geq 3$, I_α is the Riesz Potential and $F(u) \in C^1(\mathbb{R}, \mathbb{R})$ with sub critical growth. In this work authors established the existence of ground state soloutions of (1.4) and assuming some suitable growth conditions on F and V, they studied the properties like constant sign solutions and radial symmetry of the solution. Moreover, authors proved the Pohožaev identity and nonlocal Brezis-Kato type estimate. Interested readers are referred to [16, 24, 26, 27] and references therein for the study of Choquard equation on the unbounded domain.

Concerning the boundary value problems of Choquard equation, Gao and Yang [15] studied the Brezis-Nirenberg type existence results for the following critical equation

$$-\Delta u = \lambda h(u) + \left(\int\limits_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy\right) |u|^{2^*_\mu - 2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $\lambda > 0$, $0 < \mu < N$, $h(u) = u$, Ω is a smooth bounded domain in $\mathbb{R}^N$. Later in [14] authors proved the existence and multiplicity of positive solutions for convex and convex-concave type nonlinearities ($h(u) = u^q, 0 < q < 1$) using variational methods.

The geometry of the domain Ω plays an essential and significant role on the existence and multiplicity of the elliptic boundary value problems. Indeed, in [12], Coron proved the existence of a high energy positive solution of the problem

$$-\Delta u = |u|^{2^*-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$, precisely an annulus with a small hole. Later in [3], Bahri and Coron, proved that a positive solution always exists as long as the domain has non-trivial homology with $\mathbb{Z}_2$-coefficients. In [6], Benci and Cerami studied the following equation

$$-\varepsilon \Delta u + u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where $\varepsilon \in \mathbb{R}^+$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ and $f : \mathbb{R}_+ \to \mathbb{R}$ is a $C^{1,1}$ function. Here authors proved that there exists $\epsilon^\star > 0$ such that for all $\varepsilon \in (0, \epsilon^\star)$, (1.6) has $\operatorname{cat}(\Omega)+1$ solutions under some growth conditions on the function f. Since then, the study of existence and multiplicity of solutions of elliptic equations over non-contractible domain has been substantially studied, for instance, [4, 5, 13, 20, 29, 32] and references therein.

The existence of high energy solution of (1.5) is a much more delicate issue. In this spirit, recently Goel, Rădulescu and Sreenadh [19] studied the Coron problem for Choquard equations. Here authors proved the existence of a positive high energy solution for the problem (P_f) when $f(x) \equiv 0$ and Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$ satisfying the following condition

(A) There exists constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N \ : \ R_1 < |x| < R_2\} \subset \Omega, \qquad \{x \in \mathbb{R}^N \ : \ |x| < R_1\} \not\subseteq \overline{\Omega}.$$

In the light of above works, in this article, we study following problem

$$(P_f) \begin{cases} -\Delta u = \left(\int\limits_{\Omega} \frac{|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u^+|^{2^*_\mu - 2} u^+ + f, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $2^*_\mu = \frac{2N-\mu}{N-2}$, is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality (2.1) and $f \in \hat{F}$ with $\hat{F} := \{f \ : \ f \in L^\infty(\Omega),\ f \geq 0,\ f \not\equiv 0\}$. The domain $\Omega \subset \mathbb{R}^N (N \geq 3)$ satisfies the condition (A). Here we prove the existence of four solutions of the problem (P_f). To achieve this, we first seek the help of Nehari manifold associated with (P_f) to prove the existence of the first solution (say u_1). To proceed further, we prove many new estimates on the convolution terms involving the minimizers of best constant $S_{H,L}$ (see Lemma 4.1, 4.3 and 4.4). With the help of these estimates we prove that the minima of the functional over $\mathcal{N}_f$ is below the first critical level where the first critical level is

$$\mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Here $\mathcal{J}_f$ is the energy functional associated to (P_f) (defined in (2.3)). Moreover, $\mathcal{J}_f$ satisfies the Palais-Smale condition below the first critical level. Subsequently, we show the existence of the second and the third solution of (P_f), in $\mathcal{N}_f^-$ (a closed subset of the Nehari manifold) by using a well-known result of Ambrosetti [2](see Lemma 5.2) and assumption (A). To study the existence of the fourth solution, a high energy solution, we prove that the functional $\mathcal{J}_f$ satisfies the Palais-Smale condition between the first and the second critical levels, where the second critical level is

$$\inf_{u \in \mathcal{N}_f^-} \mathcal{J}_f(u) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

To prove the existence of fourth solution, we use the minmax Lemma (See Lemma 6.6). To the best of our knowledge, there is no work on the existence and multiplicity of solutions to Choquard equations (P_f) in non-contractible domains. With this introduction, we state our main result.

Theorem 1.1. *Assume $\mu < \min\{4, N\}$, $f \in L^\infty(\Omega)$ and $f \geq 0$ and Ω be a bounded domain satisfying the conditon (A). Then there exists $e^* > 0$ such that (P_f) has at least three positive solutions whenever $0 < \|f\|_{H^{-1}} < e^*$. Moreover, if R_1 is small enough then there exists $e^{**} > 0$ such that (P_f) has at least four positive solutions whenever $0 < \|f\|_{H^{-1}} < e^{**}$.*

The paper is organized as follows: In Section 2, we give the variational framework and preliminary results. In section 3, using the Nehari manifold technique, we prove the existence of the first solution. In section 4, we prove some crucial estimates of the minimizer of $S_{H,L}$(defined in (2.2)) and analyze the Palais-Smale sequences. In section 5, we prove the existence of the second and third solution. In section 6, we prove the existence of the fourth solution.

2 Variational framework and preliminary results

We start with the familiar Hardy-Littlewood-Sobolev Inequality which leads to the study of nonlocal Choquard equation using variational methods.

Proposition 2.1. *[23](**Hardy-Littlewood-Sobolev Inequality**) Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, r, \mu, N)$ independent of f and h such that*

$$\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\, dxdy \leq C(t, r, \mu, N) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)}. \tag{2.1}$$

If $t = r = 2N/(2N-\mu)$, then

$$C(t, r, \mu, N) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{\mu}{2})} \right\}^{-1+\frac{\mu}{N}}.$$

Equality holds in (2.1) *if and only if* $f \equiv (constant)h$ *and*

$$h(x) = A(y^2 + |x-a|^2)^{(2N-\mu)/2},$$

for some $A \in \mathbb{C}$, $0 \neq y \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$. □

The best constant for the embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ (where $2^* = \frac{2N}{N-2}$)is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\}.$$

Consequently, we define

$$S_{H,L} = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dxdy = 1 \right\}. \tag{2.2}$$

Lemma 2.2. *[15] The constant* $S_{H,L}$ *defined in* (2.2) *is achieved if and only if*

$$u = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}}$$

where $C > 0$ *is a fixed constant ,* $a \in \mathbb{R}^N$ *and* $b \in (0, \infty)$ *are parameters. Moreover,*

$$S = S_{H,L}\,(C(N,\mu))^{\frac{N-2}{2N-\mu}}.$$

Lemma 2.3. *[15] For* $N \geq 3$ *and* $0 < \mu < N$. *Then*

$$\|.\|_{NL} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|.|^{2^*_\mu}|.|^{2^*_\mu}}{|x-y|^\mu} \, dxdy \right)^{\frac{1}{2.2^*_\mu}}$$

defines a norm on $L^{2^*}(\mathbb{R}^N)$.

The energy functional $\mathcal{J}_f : H_0^1(\Omega) \to \mathbb{R}$ associated with the problem (P_f) is

$$\mathcal{J}_f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2.2^*_\mu}\int_\Omega\int_\Omega \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} \, dxdy - \int_\Omega fu \, dx, \tag{2.3}$$

where $u^+ = \max(u, 0)$. By using Hardy-Littlewood-Sobolev inequality (2.1), we have

$$\left(\int_\Omega\int_\Omega \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu}}{|x-y|^\mu} \, dxdy \right)^{\frac{1}{2^*_\mu}} \leq C(N,\mu)^{\frac{2N-\mu}{N-2}} |u|^2_{2^*}.$$

It is not difficult to show that the functional $\mathcal{J}_f \in C^1(H_0^1(\Omega), \mathbb{R})$ and moreover, if $\mu < \min\{4, N\}$ then $\mathcal{J}_f \in C^2(H_0^1(\Omega), \mathbb{R})$.

Definition 2.4. *A function* $u \in H_0^1(\Omega)$ *is called a weak solution of the problem* (P_f) *if for all* $v \in H_0^1(\Omega)$ *the following holds*

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega\int_\Omega \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu - 1}v(y)}{|x-y|^\mu} \, dxdy - \int_\Omega fv \, dx = 0.$$

Definition 2.5. *For* $c \in \mathbb{R}$, $\{u_n\}$ *is a* $(PS)_c$ *sequence in* $H_0^1(\Omega)$ *for* $\mathcal{J}_f$ *if* $\mathcal{J}_f = c + o(1)$ *and* $\mathcal{J}'_f(u_n) = o(1)$ *strongly in* H^{-1} *as* $n \to \infty$. *We say* $\mathcal{J}_f$ *satisfies the* $(PS)_c$ *condition in* $H_0^1(\Omega)$ *if every* $(PS)_c$ *sequence in* $H_0^1(\Omega)$ *has a convergent subsequence.*

Since $\mathcal{J}_f$ is not bounded below on $H_0^1(\Omega)$, it is worth to consider the Nehari manifold

$$\mathcal{N}_f := \{u \in H_0^1(\Omega) \setminus \{0\} \mid u^+ \not\equiv 0 \text{ and } \langle \mathcal{J}_f'(u), u\rangle = 0\},$$

where $\langle\ ,\ \rangle$ denotes the usual duality. We define

$$\Upsilon_f = \inf_{u\in\mathcal{N}_f} \mathcal{J}_f(u).$$

Note that when $f(x) \equiv 0$, $\Upsilon_0(\Omega)$ is independednt of Ω and $\Upsilon_0(\Omega) := \Upsilon_0 = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$.

Notations: Throughout the paper we will use the notation $\mathcal{J}_0 = \mathcal{J}$, $\mathcal{N}_0 = \mathcal{N}$, $\|.\| = \|.\|_{H_0^1(\Omega)}$

$$a(u) = \int_\Omega |\nabla u|^2\, dx \text{ and } b(u) = \int_\Omega\int_\Omega \frac{(u^+(x))^{2^*_\mu}(u^+(y))^{2^*_\mu}}{|x-y|^\mu}\, dxdy.$$

An easy consequence of (2.1) gives $\mathcal{J}_f$ is coercive and bounded below on $\mathcal{N}_f$.

Proposition 2.6. *For any $u, v \in H_0^1(\Omega)$, we have*

$$\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\, dxdy \le \left(\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\, dxdy\right)^{\frac{1}{2}} \left(\int_\Omega\int_\Omega \frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\, dxdy\right)^{\frac{1}{2}}.$$

Proof. For details of the proof, see [17, Lemma 2.3]. □

Lemma 2.7. *For each $u \in H_0^1(\Omega)$, there exists a unique $t > 0$ such that $tu \in \mathcal{N}$. Moreover, there holds* $\Upsilon_0 \le \left(\frac{N-\mu+2}{2(2N-\mu)}\right)\left(\frac{a(u)^{2^*_\mu}}{b(u)}\right)^{\frac{1}{2^*_\mu-1}}$.

Proof. Let $m_u(t) = \frac{t^2}{2}a(u) - \frac{t^{2.2^*_\mu}}{2.2^*_\mu}b(u)$ then on solving $m_u'(t) = 0$, we get unique $t(u) = \left(\frac{a(u)}{b(u)}\right)^{\frac{1}{2(2^*_\mu-1)}}$ such that $t(u)u \in \mathcal{N}$. From the definition of Υ_0, we have

$$\Upsilon_0 \le \mathcal{J}(t(u)u) = \left(\frac{1}{2} - \frac{1}{2.2^*_\mu}\right)\left(\frac{a(u)}{b(u)}\right)^{\frac{1}{2^*_\mu-1}} a(u) = \left(\frac{N-\mu+2}{2(2N-\mu)}\right)\left(\frac{a(u)^{2^*_\mu}}{b(u)}\right)^{\frac{1}{2^*_\mu-1}}.$$

□

Remark 2.8. *We remark that by [15, Lemma 1.3], $S_{H,L}$ is never achieved on bounded domain. Therefore if u is a solution of the following equation*

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy\right)|u|^{2^*_\mu-2}u \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

then $\mathcal{J}(u) > \Upsilon_0 = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$.

Lemma 2.9. *A sequence $\{u_n\}$ is a $(PS)_{\Upsilon_0}$- sequence for $\mathcal{J}$ in $H_0^1(\Omega)$ if and only if $\mathcal{J}(u_n) = \Upsilon_0 + o_n(1)$ and $a(u_n) = b(u_n) + o_n(1)$.*

Proof. Clearly, any $(PS)_{\Upsilon_0}$- sequence satisfies $a(u_n) = b(u_n) + o_n(1)$ and $\mathcal{J}(u_n) = \Upsilon_0 + o_n(1)$. Conversely, let $\mathcal{J}(u_n) = \Upsilon_0 + o_n(1)$ and $a(u_n) = b(u_n) + o_n(1)$ then $\Upsilon_0 = \mathcal{J}(u_n) = \frac{N-\mu+2}{2(2N-\mu)} b(u_n) + o_n(1)$ and hence we have

$$b(u_n) = D\Upsilon_0 + o_n(1) \text{ where } D = \frac{2(2N-\mu)}{N-\mu+2}. \tag{2.4}$$

Define $T_n(\psi) = \int\limits_{\Omega}\int\limits_{\Omega} \frac{(u_n^+(x))^{2^*_\mu}(u_n^+(y))^{2^*_\mu-1}\psi(y)}{|x-y|^\mu}\,dxdy$ for $\psi \in H_0^1(\Omega)$ and $n = 1,2,\cdots$.

Claim: $\|T_n\|_{H^{-1}} = (D\Upsilon_0)^{\frac{1}{2}} + o_n(1)$.

Let $\psi \in H_0^1(\Omega)$ such that $\|\psi\| = 1$ then by Lemma 2.7, we know that there exists a $t > 0$ such that $a(t\psi) = b(t\psi)$. Therefore, $t = \|\psi\|_{NL}^{-\frac{2^*_\mu}{2^*_\mu-1}}$ and $\Upsilon_0 \leq \frac{1}{D}\|\psi\|_{NL}^{-\frac{2.2^*_\mu}{2^*_\mu-1}}$. This implies,

$$\|\psi\|_{NL} \leq \left(\frac{1}{D\Upsilon_0}\right)^{\frac{2^*_\mu-1}{2.2^*_\mu}}. \tag{2.5}$$

Taking into account (2.4), (2.5), Proposition 2.6 and employing Hölder's inequality, for each n, we have

$$\begin{aligned}|T_n(\psi)| &\leq \left(\int\limits_{\Omega}\int\limits_{\Omega}\frac{(u_n^+(x))^{2^*_\mu}(u_n^+(y))^{2^*_\mu}}{|x-y|^\mu}\,dxdy\right)^{\frac{2.2^*_\mu-1}{2.2^*_\mu}}\left(\int\limits_{\Omega}\int\limits_{\Omega}\frac{|\psi(x)|^{2^*_\mu}|\psi(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy\right)^{\frac{1}{2.2^*_\mu}}\\
&= b(u_n)^{\frac{2.2^*_\mu-1}{2.2^*_\mu}}\|\psi\|_{NL}\\
&\leq \left(\frac{1}{D\Upsilon_0}\right)^{\frac{2^*_\mu-1}{2.2^*_\mu}}(D\Upsilon_0+o_n(1))^{\frac{2.2^*_\mu-1}{2.2^*_\mu}} = (D\Upsilon_0)^{\frac{1}{2}} + o_n(1) \text{ as } n\to\infty.\end{aligned}$$

So, we get $\|T_n\|_{H^{-1}} \leq (D\Upsilon_0)^{\frac{1}{2}} + o_n(1)$. Moreover, $T_n\left(\frac{u_n}{\|u_n\|}\right) = (b(u_n))^{\frac{1}{2}} = (D\Upsilon_0)^{\frac{1}{2}} + o_n(1)$. This implies $\|T_n\|_{H^{-1}} = (D\Upsilon_0)^{\frac{1}{2}} + o_n(1)$. Hence the proof of claim follows. Now, by Riesz representation theorem, for each n, there exists $v_n \in H_0^1(\Omega)$ such that

$$T_n(\psi) = \langle v_n, \psi\rangle = \int\limits_{\Omega}\nabla v_n\cdot\nabla\psi\,dx \text{ and } \|v_n\| = \|T_n\|_{H^{-1}} = (D\Upsilon_0)^{\frac{1}{2}} + o_n(1).$$

Thus, $\langle v_n, u_n\rangle = T_n(u_n) = b(u_n) = D\Upsilon_0 + o_n(1)$. Hence,

$$\begin{aligned}\|u_n - v_n\|^2 &= \|u_n\|^2 - 2\langle u_n, v_n\rangle + \|v_n\|^2\\
&= D\Upsilon_0 - 2D\Upsilon_0 + D\Upsilon_0 + o_n(1) = o_n(1) \text{ as } n\to\infty.\end{aligned}$$

For any $\psi \in H_0^1(\Omega)$ with $\|\psi\| = 1$, we have

$$\langle \mathcal{J}'(u_n), \psi\rangle = \int\limits_{\Omega}\nabla u_n\cdot\nabla\psi\,dx - T_n(\psi) = \langle u_n,\psi\rangle - \langle v_n,\psi\rangle = \langle u_n - v_n, \psi\rangle.$$

Therefore, $\|\mathcal{J}'(u_n)\|_{H^{-1}} \leq \|u_n - v_n\| = o_n(1)$. It implies $\mathcal{J}'(u_n) \to 0$ in H^{-1}. □

Clearly, $\mathcal{N}_f$ contains every non zero solution of (P_f) and we know that the Nehari manifold is closely related to the behavior of the fibering maps $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ defined as $\phi_u(t) = \mathcal{J}_f(tu)$. It is easy to see that $tu \in \mathcal{N}_f$ if and only if $\phi_u'(t) = 0$ and elements of $\mathcal{N}_f$ correspond to stationary points of the fibering maps. It is natural to divide $\mathcal{N}_f$ into the following sets

$$\mathcal{N}_f^+ =: \{u \in \mathcal{N}_f | \phi_u''(1) > 0\},\ \mathcal{N}_f^- =: \{u \in \mathcal{N}_f | \phi_u''(1) < 0\}, \text{ and } \mathcal{N}_f^0 =: \{u \in \mathcal{N}_f | \phi_u''(1) = 0\}.$$

We also denote the infimum over $\mathcal{N}_f^+$ and $\mathcal{N}_f^-$ as

$$\Upsilon_f^+ = \inf_{u\in\mathcal{N}_f^+}\mathcal{J}_f(u) \qquad \Upsilon_f^- = \inf_{u\in\mathcal{N}_f^-}\mathcal{J}_f(u).$$

3 Existence of First Solution

In this section we prove the existence of first solution by showing the existence of minimizer for $\mathcal{J}_f$ over the Nehari manifold $\mathcal{N}_f$. First we state some Lemmas whose proof can be found in [30]. We further prove some properties of the manifold $\mathcal{N}_f^+$.

Lemma 3.1. *If $f \in \hat{F}$ and $\|f\|_{H^{-1}} < e_{00} := C_{N,\mu} S_{H,L}^{\frac{2^*_\mu}{2.2^*_\mu-2}}$ where $C_{N,\mu} = \left(\frac{1}{2.2^*_\mu-1}\right)^{\frac{2.2^*_\mu-1}{2.2^*_\mu-2}}(2.2^*_\mu - 2)$ then $\alpha_0 := \inf\limits_{u\in E}\left\{C_{N,\mu}\|u\|^{\frac{2.2^*_\mu-1}{2^*_\mu-1}} - \int\limits_\Omega fu\,dx\right\}$ is acheived, where*

$$E := \left\{u \in H_0^1(\Omega) : \int\limits_\Omega\int\limits_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy = 1\right\}.$$

Proof. Proof follows from [30, Lemma 4.1]. Since we consider $\lambda = 0$ in equation (4.1) of [30], our result holds for all $N \geq 3$. □

Lemma 3.2. *For every $u \in \mathcal{N}_f$, $u \not\equiv 0$ we have $a(u) - (2.2^*_\mu - 1)b(u) \neq 0$. In particular, $\mathcal{N}_f^0 = \{0\}$.*

Lemma 3.3. *For each $u \in H_0^1(\Omega)$ with $u^+ \not\equiv 0$ the following holds:*

(a) There exists a unique $t^- = t^-(u) > 0$ such that $t^- u \in \mathcal{N}_f^-(\Omega)$. In particular,

$$t^- > \left(\frac{a(u)}{(2.2^*_\mu - 1)b(u)}\right)^{\frac{1}{2.2^*_\mu-2}} := t_{max}$$

and $\mathcal{J}_f(t^- u) = \max\limits_{t\geq t_{max}} \mathcal{J}_f(tu)$.

(b) If $\int\limits_\Omega fu > 0$, then there exists unique $t^+ \in (0, t_{max})$ such that $t^+ u \in \mathcal{N}_f^+(\Omega)$ and

$$\mathcal{J}_f(t^+ u) = \min_{0<t\leq t^-} \mathcal{J}_f(tu).$$

(c) $t^-(u)$ is a continuous function.

(d) $\mathcal{N}_f^- = \{u \in H_0^1(\Omega)\setminus\{0\} \mid u^+ \not\equiv 0$ and $\frac{1}{\|u\|} t^-(\frac{u}{\|u\|}) = 1\}$.

Lemma 3.4. *For each $u \in \mathcal{N}_f^+(\Omega)$, we have $\int_\Omega fu\,dx > 0$ and $\mathcal{J}_f(u) < 0$. In particular, $\Upsilon_f(\Omega) \leq \Upsilon_f^+(\Omega) < 0$.*

Lemma 3.5. *Let $u \in \mathcal{N}_f(\Omega)$ be such that $\mathcal{J}_f(u) = \min\limits_{w\in\mathcal{N}_f(\Omega)} \mathcal{J}_f(w) = \Upsilon_f(\Omega)$ then $\int_\Omega fu\,dx > 0$ and u is a solution of (P_f).*

Lemma 3.6. *$\mathcal{J}_f$ has Palais-Smale sequences at each of the levels $\Upsilon_f(\Omega)$, $\Upsilon_f^+(\Omega)$ and $\Upsilon_f^-(\Omega)$.*

Lemma 3.7. *Let $\{u_n\} \in \mathcal{N}_f$ be a $(PS)_{\Upsilon_f(\Omega)}$ sequence for $\mathcal{J}_f$, then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a non-zero $u_1 \in H_0^1(\Omega)$ such that $u_n \to u_1$ strongly in $H_0^1(\Omega)$. Moreover, $u_1 \in \mathcal{N}_f$ and is a solution to (P_f).*

Proof. $\mathcal{J}_f$ is bounded below and coercive implies $\{u_n\}$ is bounded in $H_0^1(\Omega)$. So, there exists a subsequence still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u_1$ weakly in $H_0^1(\Omega)$. By [19, Lemma 4.2], we have $\mathcal{J}_f'(u_1) = 0$. In particular, $u_1 \in \mathcal{N}_f$ and $\mathcal{J}_f(u_1) = \left(\frac{1}{2} - \frac{1}{2.2^*_\mu}\right)a(u_1) - \left(1 - \frac{1}{2.2^*_\mu}\right)\int\limits_\Omega fu_1\,dx$. Now, using the fact that a is weakly lower semi continuous we have

$$\Upsilon_f(\Omega) \leq \mathcal{J}_f(u_1) \leq \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2.2^*_\mu}\right)a(u_n) - \lim_{n\to\infty}\left(1 - \frac{1}{2.2^*_\mu}\right)\int\limits_\Omega fu_n\,dx = \Upsilon_f(\Omega).$$

Consequently, we have $\Upsilon_f(\Omega) = \mathcal{J}_f(u_1)$. Let $w_n = u_n - u_1$ then by [19, Lemma 4.1], [15, Lemma 2.2] and the fact that $\mathcal{J}_f'(u_1) = 0$, we obtain $\mathcal{J}_f(w_n) = \mathcal{J}_f(u_n) - \mathcal{J}_f(u_1) = o_n(1)$ and $\langle \mathcal{J}_f'(w_n), \phi\rangle = \langle \mathcal{J}_f'(u_n), \phi\rangle - \langle \mathcal{J}_f'(u_1), \phi\rangle + o_n(1) = o_n(1)$. Therefore, $\langle \mathcal{J}_f'(w_n), w_n\rangle = o_n(1)$. It implies $\mathcal{J}_f(w_n) = \left(\frac{1}{2} - \frac{1}{2.2^\star_\mu}\right) a(w_n) - \int_\Omega f w_n \, dx = o_n(1)$ and since $\int_\Omega f w_n \, dx = o_n(1)$, we get $a(w_n) = o_n(1)$. Hence $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Lemma 3.8. *If u be a solution of (P_f) then $u \in C^2(\overline{\Omega})$. Moreover, u is a positive solution.*

Proof. Let u be a solution of (P_f) and $G(x,u) = \left(\int_\Omega \frac{|u^+(y)|^{2^\star_\mu}}{|x-y|^\mu} dy\right) |u^+|^{2^\star_\mu - 2} u + f$. By using same assertions and arguments as in [25, Proposition 3.1 and Theorem 2], we have $\left(\int_\Omega \frac{|u^+(y)|^{2^\star_\mu}}{|x-y|^\mu} dy\right) \in L^\infty(\Omega)$ and since $f \in \hat{F}$, we have $|G(x,u)| \le C(1 + |u|^{2^*-1})$. Then by the standard elliptic regularity $u \in C^2(\overline{\Omega})$. Since $f \ge 0$, we get $u \ge 0$ and by using strong maximum principle, u is a positive solution of (P_f). □

Lemma 3.9. *Let $\mu < \min\{4, N\}$ and $k_0 = \left(\frac{1}{2.2^\star_\mu - 1}\right)^{\frac{1}{2(2^\star_\mu - 1)}} S_{H,L}^{\frac{2^\star_\mu}{2(2^\star_\mu - 1)}}$ and $f \in \hat{F}$, $\|f\|_{H^{-1}} \le e_{00}$ (where e_{00} is defined in Lemma 3.1) then*

1. *$\mathcal{N}_f^+(\Omega) \subset B_{k_0}(0) := \{u \in H_0^1(\Omega) \mid \|u\| < k_0\}$.*
2. *$\mathcal{J}_f$ is strictly convex in $B_{k_0}(0)$.*

Proof.

1. Let $u \in \mathcal{N}_f^+(\Omega)$ then $\phi_u'(1) = 0$ and $\phi_u''(1) > 0$. That is, $a(u) = b(u) + \int_\Omega fu \, dx$ and $a(u) > (2.2^\star_\mu - 1) b(u)$. Therefore, $a(u) = b(u) + \int_\Omega fu \, dx < \frac{1}{(2.2^\star_\mu - 1)} a(u) + \int_\Omega fu \, dx$. It implies $\left(1 - \frac{1}{(2.2^\star_\mu - 1)}\right) a(u) \le \|f\|_{H^{-1}} \|u\|$. So,
$$
\begin{aligned}
\|u\| &\le \frac{(2.2^\star_\mu - 1)}{2(2^\star_\mu - 1)} \|f\|_{H^{-1}} \\
&\le \frac{(2.2^\star_\mu - 1)}{2(2^\star_\mu - 1)} C_{N,\mu} S_{H,L}^{\frac{2^\star_\mu}{2.2^\star_\mu - 2}} = \left(\frac{1}{2.2^\star_\mu - 1}\right)^{\frac{1}{2(2^\star_\mu - 1)}} S_{H,L}^{\frac{2^\star_\mu}{2(2^\star_\mu - 1)}} = k_0.
\end{aligned}
$$

2. By using Hölders inequality and equation (2.2), we have
$$
\begin{aligned}
\int_\Omega \int_\Omega \frac{(u^+(x))^{2^\star_\mu - 1} (u^+(y))^{2^\star_\mu - 1} z(x) z(y)}{|x-y|^\mu} dx dy &\le b(u)^{\frac{2^\star_\mu - 1}{2^\star_\mu}} \|z\|_{NL}^2 \\
&\le S_{H,L}^{-(2^\star_\mu - 1)} a(u)^{(2^\star_\mu - 1)} S_{H,L}^{-1} a(z) \\
&= S_{H,L}^{-2^\star_\mu} a(u)^{(2^\star_\mu - 1)} a(z).
\end{aligned} \tag{3.1}
$$
Again using Hölders inequality, Proposition 2.6 and (2.2), we have
$$
\int_\Omega \int_\Omega \frac{(u^+(x))^{2^\star_\mu} (u^+(y))^{2^\star_\mu - 2} z^2(y)}{|x-y|^\mu} dx dy \le b(u)^{\frac{2^\star_\mu - 1}{2^\star_\mu}} \|z\|_{NL}^2 \le S_{H,L}^{-2^\star_\mu} a(u)^{(2^\star_\mu - 1)} a(z). \tag{3.2}
$$

From equations (3.1), (3.2) and definition of $\mathcal{J}_f''(u)(z,z)$, we get

$$\begin{aligned}\mathcal{J}_f''(u)(z,z) =& a(z) - 2^*_\mu \int_\Omega\int_\Omega \frac{(u^+(x))^{2^*_\mu-1}(u^+(y))^{2^*_\mu-1}z(x)z(y)}{|x-y|^\mu}\,dxdy\\ &- (2^*_\mu-1)\int_\Omega\int_\Omega \frac{(u^+(x))^{2^*_\mu}(u^+(y))^{2^*_\mu-2}z^2(y)}{|x-y|^\mu}\,dxdy\\ &\geq a(z)\left(1 - 2^*_\mu S_{H,L}^{-2^*_\mu} a(u)^{(2^*_\mu-1)} - (2^*_\mu-1)S_{H,L}^{-2^*_\mu} a(u)^{(2^*_\mu-1)}\right)\\ &= a(z)\left(1-(2.2^*_\mu-1)S_{H,L}^{-2^*_\mu}a(u)^{(2^*_\mu-1)}\right)\\ &> a(z)\left(1-\frac{(2.2^*_\mu-1)}{(2.2^*_\mu-1)}\right)=0\end{aligned}$$

for $u \in B_{k_0}(0)\setminus\{0\}$. Then $\mathcal{J}_f''(u)$ is positive definite for $u \in B_{k_0}(0)$ and $\mathcal{J}_f(u)$ is strictly convex on $B_{k_0}(0)$.□

Lemma 3.10. *It holds that $u_1 \in \mathcal{N}_f^+$ and $\mathcal{J}_f(u_1) = \Upsilon_f^+(\Omega) = \Upsilon_f(\Omega)$. Moreover, u_1 is the unique critical point of $\mathcal{J}_f$ in $B_{k_0}(0)$ and u_1 is a local minimum of $\mathcal{J}_f$ in $H_0^1(\Omega)$.*

Proof. Using the proof of [30, Theorem 1.3], we have $\int_\Omega fu_1\,dx > 0$. Now if $u_1 \in \mathcal{N}_f^-$ then there exists a unique $t^-(u_1) = 1 > t_{max} > t^+(u_1) > 0$ such that $t^+(u_1)u_1 \in \mathcal{N}_f^+$ then by Lemma 3.3 (b) we have

$$\Upsilon_f(\Omega) \leq \Upsilon_f^+(\Omega) \leq \mathcal{J}_f(t^+(u_1)u_1) < \mathcal{J}_f(t^-(u_1)u_1) = \mathcal{J}_f(u_1) = \Upsilon_f(\Omega).$$

which is a contradiction. It implies $u_1 \in \mathcal{N}_f^+$ and $\Upsilon_f^+(\Omega) \leq \mathcal{J}_f(u_1) = \Upsilon_f(\Omega) \leq \Upsilon_f^+(\Omega)$ that is, $\mathcal{J}_f(u_1) = \Upsilon_f(\Omega) = \Upsilon_f^+(\Omega)$. Using Lemma 3.5 and Lemma 3.9, we get u_1 is the unique critical point of $\mathcal{J}_f$ in $B_{k_0}(0)$ and the proof of local minimum follows from [30, Lemma 3.2]. □

Lemma 3.11. *Let $\mu < \min\{4,\ N\}$ and $u \in H_0^1(\Omega)$ be a critical point of $\mathcal{J}_f$ then either $u \in \mathcal{N}_f^-$ or $u = u_1$.*

Proof. If $u \in H_0^1(\Omega)$ be a critical point of $\mathcal{J}_f$ then $u \in \mathcal{N}_f = \mathcal{N}_f^+ \cup \mathcal{N}_f^-$. Now using the fact that $\mathcal{N}_f^+ \cap \mathcal{N}_f^- = \emptyset$ and $\mathcal{N}_f^+ \subset B_{k_0}(0)$ we have either $u \in \mathcal{N}_f^-$ or $u = u_1$. □

4 Asymptotic estimates and Palais-Smale Analysis

In this section we shall prove that the functional $\mathcal{J}_f$ satisfies Palais-Smale condition strictly below the first critical level and (strictly) between the first and second critical levels. To start with, we shall prove several new estimates on the nonlinearity.

It is known from Lemma 2.2 that the best constant $S_{H,L}$ is achieved by the function

$$u(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}}(C(N,\mu))^{\frac{2-N}{2(N-\mu+2)}}\frac{(N(N-2))^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}},$$

which is a solution of the problem $-\Delta u = (|x|^{-\mu} \star |u|^{2^*_\mu})|u|^{2^*_\mu-1}$ in $\mathbb{R}^N$ with

$$\int_{\mathbb{R}^N}|\nabla u|^2\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

We may assume $R_1 = \rho$, $R_2 = 1/\rho$ for $\rho \in (0, \frac{1}{2})$. Now, define $\upsilon_\rho \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \upsilon_\rho(x) \le 1$ for all $x \in \mathbb{R}^N$, radially symmetric and

$$\upsilon_\rho(x) = \begin{cases} 0 & 0 < |x| < \dfrac{3\rho}{2}, \\ 1 & 2\rho \le |x| \le \dfrac{1}{2\rho}, \\ 0 & |x| \ge \dfrac{3}{4\rho}, \end{cases}$$

and

$$u_\sigma^\epsilon(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \frac{(N(N-2)\epsilon^2)^{\frac{N-2}{4}}}{\left(\epsilon^2 + |x-(1-\epsilon)\sigma|^2\right)^{\frac{N-2}{2}}},$$

where $\sigma \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, $0 < \epsilon \le 1$. Set

$$g_\rho^{\epsilon,\sigma}(x) := \upsilon_\rho(x) u_\sigma^\epsilon(x) \in H_0^1(\Omega). \tag{4.1}$$

Lemma 4.1. *(i)* $a(g_\rho^{\epsilon,\sigma}) = b(g_\rho^{\epsilon,\sigma}) = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o_\epsilon(1)$ *uniformly in* σ *as* $\epsilon \to 0$.

(ii) $\mathcal{J}(g_\rho^{\epsilon,\sigma}) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o_\epsilon(1)$ *uniformly in* σ *as* $\epsilon \to 0$.

(iii) $g_\rho^{\epsilon,\sigma} \rightharpoonup 0$ *weakly in* $H_0^1(\Omega)$ *uniformly in* σ *as* $\epsilon \to 0$.

Proof.

(i) Observe the fact that there exist constants $d_1,\ d_2 > 0$ such that

$$d_1 < |x-(1-\epsilon)\sigma| < d_2 \text{ for all } x \in B_{2\rho} \text{ whenever } \epsilon < 1-2\rho. \tag{4.2}$$

$$\begin{aligned} \|\nabla g_\rho^{\epsilon,\sigma}\|_{L^2(\mathbb{R}^N)} - \|\nabla u_\epsilon^\sigma\|_{L^2(\mathbb{R}^N)} &\le \int\limits_{(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}) \cup B_{2\rho}} |\nabla u_\epsilon^\sigma|^2\,dx + \rho^{-2} \int\limits_{B_{2\rho}} |u_\epsilon^\sigma|^2\,dx \\ &\quad + \rho^2 \int\limits_{B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} |u_\epsilon^\sigma|^2\,dx \\ &\le C\epsilon^{N-2} \int\limits_{(\mathbb{R}^N \setminus B_{\frac{1}{2\rho}}) \cup B_{2\rho}} \frac{|x-(1-\epsilon)\sigma|^2}{|x-(1-\epsilon)\sigma|^{2N}}\,dx \\ &\quad + C\epsilon^{N-2} \int\limits_{B_{2\rho} \cup B_{\frac{3}{4\rho}} \setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2(N-2)}} = O(\epsilon^{N-2}). \end{aligned}$$

Thus, $\|\nabla g_\rho^{\epsilon,\sigma}\|_{L^2(\mathbb{R}^N)} = \|\nabla u_\epsilon^\sigma\|_{L^2(\mathbb{R}^N)} + o_\epsilon(1) = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o_\epsilon(1)$.

Next we will prove that $b(g_\rho^{\epsilon,\sigma}) = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o_\epsilon(1)$ uniformly in σ as $\epsilon \to 0$. For this consider

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g_\rho^{\epsilon,\sigma}(x)|^{2_\mu^*}|g_\rho^{\epsilon,\sigma}(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy\\
&\quad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(|v_\rho(x)|^{2_\mu^*}|v_\rho(y)|^{2_\mu^*}-1)|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy\\
&\quad\le C\left(\int_{B_{2\rho}}\int_{B_{2\rho}} + \int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}}\int_{B_{2\rho}} + \int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}}\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\right.\\
&\qquad\left. + \int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\int_{B_{2\rho}} + \int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\right)\frac{|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy,\\
&\quad= C\sum_{i=1}^{i=5} J_i,
\end{aligned}
\tag{4.3}
$$

Let $\xi_\epsilon(x) = \frac{\epsilon^N}{(\epsilon^2+|x-(1-\epsilon)\sigma|^2)^N}$ then taking into account the definition of u_ϵ^σ, (4.2) and Hardy-Littlewood-Sobolev inequality, we have the following estimates:

$$
\begin{aligned}
J_1 &\le C(N,\mu)\left(\int_{B_{2\rho}} S^{\frac{-N(N-\mu)}{2(N-\mu+2)}} C(N,\mu)^{\frac{-N}{(N-\mu+2)}}(N(N-2))^{\frac{N}{2}}\xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{N}}\\
&\le C\epsilon^{2N-\mu}\left(\int_{B_{2\rho}}\frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{N}} \le C\epsilon^{2N-\mu}\left(\int_{B_{2\rho}} dx\right)^{\frac{2N-\mu}{N}} = O(\epsilon^{2N-\mu}),\\
J_2 &\le C\left(\int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{2N}}\left(\int_{B_{2\rho}}\xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{2N}}\\
&\le C\epsilon^{\frac{2N-\mu}{2}}\left(\int_{B_{2\rho}}\frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{2N}} = O(\epsilon^{\frac{2N-\mu}{2}}),\\
J_3 &\le C\left(\int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \xi_\epsilon(x)\right)^{\frac{2N-\mu}{2N}}\left(\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{2N}}\\
&\le C\epsilon^{\frac{2N-\mu}{2}}\left(\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}}\frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{2N}} = O(\epsilon^{\frac{2N-\mu}{2}}),
\end{aligned}
$$

$$J_4 \leq C\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{2N}}\left(\int\limits_{B_{2\rho}} \xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{2N}}$$

$$\leq C\epsilon^{2N-\mu}\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\int\limits_{B_{2\rho}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{2N}} = O(\epsilon^{2N-\mu}),$$

$$J_5 \leq C\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \xi_\epsilon(x)\,dx\right)^{\frac{2N-\mu}{N}} \leq C\epsilon^{2N-\mu}\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{N}}$$

$$= O(\epsilon^{2N-\mu}).$$

Therefore, $b(g_\rho^{\epsilon,\sigma}) - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy \to 0$ as $\epsilon \to 0$ that is, $b(g_\rho^{\epsilon,\sigma}) \to S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ as $\epsilon \to 0$ and completes the proof of (i).

(ii) Result follows from the definition of $\mathcal{J}$ and by (i).

(iii) Assume by contradiction, $g_\rho^{\epsilon,\sigma} \rightharpoonup g_1 \not\equiv 0$ weakly in $H_0^1(\Omega)$ then $g_\rho^{\epsilon,\sigma} \to g_1$ strongly in $L^2(\Omega)$. Then by using the inequality $r^{2(N-2)} + s^{2(N-2)} \leq (r^2+s^2)^{N-2}$ for all $r,\ s \geq 0$, we have

$$0 \leq \int\limits_\Omega |g_\rho^{\epsilon,\sigma}|^2\,dx \leq C \int\limits_{\frac{3\rho}{2}\leq|x|\leq\frac{3}{4\rho}} \frac{\epsilon^{N-2}}{(\epsilon^2+|x-(1-\epsilon)\sigma|^2)^{N-2}}\,dx$$

$$= C \int\limits_{\frac{3\rho}{2}\leq|y+(1-\epsilon)\sigma|\leq\frac{3}{4\rho}} \frac{\epsilon^{N-2}}{\epsilon^{2(N-2)}+|y|^{2(N-2)}}\,dy$$

$$\leq C \int\limits_0^{\frac{3}{4\rho}+(1-\epsilon)} \frac{\epsilon^{N-2}r^{N-1}}{\epsilon^{2(N-2)}+r^{2(N-2)}}\,dy \to 0.$$

It yields a contradiction. Hence results follows. □

Lemma 4.2. *Let $\sigma \in \mathbb{S}^{N-1}$ and $\epsilon \in (0,1)$, then the following holds:*

(i) $\lim\limits_{\rho\to 0} \sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]} \|\nabla(g_\rho^{\epsilon,\sigma} - u_\epsilon^\sigma)\|_{L^2(\mathbb{R}^N)}^2 = 0.$

(ii) $\lim\limits_{\rho\to 0} \sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]} \|g_\rho^{\epsilon,\sigma}\|_{NL}^{2.2_\mu^*} = \|u_\epsilon^\sigma\|_{NL}^{2.2_\mu^*}.$

Proof.

(i) Consider

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla g_\rho^{\epsilon,\sigma} - \nabla u_\epsilon^\sigma|^2 dx &\le 2\int_{\mathbb{R}^N} |u_\epsilon^\sigma(x)\nabla v_\rho(x)|^2\, dx + 2\int_{\mathbb{R}^N} |\nabla u_\epsilon^\sigma(x) v_\rho(x) - \nabla u_\epsilon^\sigma(x)|^2\, dx \\
&\le C\left(\rho^{-2}\int_{B_{2\rho}} |u_\epsilon^\sigma(x)|^2\, dx + \int_{B_{2\rho}} |\nabla u_\epsilon^\sigma(x)|^2\, dx\right) \\
&\quad + C\left(\rho^2 \int_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} |u_\epsilon^\sigma(x)|^2\, dx + \int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} |\nabla u_\epsilon^\sigma(x)|^2\, dx\right).
\end{aligned}
\tag{4.4}
$$

From the definition of u_ϵ^σ, we have the following estimates

$$
\begin{aligned}
\rho^{-2}\int_{B_{2\rho}} |u_\epsilon^\sigma(x)|^2\, dx &\le C\rho^{-2}\int_{B_{2\rho}} dx \le C\rho^{N-2}, \\
\int_{B_{2\rho}} |\nabla u_\epsilon^\sigma(x)|^2\, dx &\le C\int_{B_{2\rho}} |x - t\sigma|\, dx \le C\int_{B_{2\rho}} dx \le C\rho^N, \\
\rho^2 \int_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} |u_\epsilon^\sigma(x)|^2\, dx &\le C\rho^2 \int_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} \frac{1}{|x|^{2N-4}}\, dx \le C\rho^{N-2}, \\
\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} |\nabla u_\epsilon^\sigma(x)|^2\, dx &\le C\int_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \frac{1}{|x|^{2N-2}}\, dx \le C\rho^{N-2}.
\end{aligned}
$$

Therefore, from above estimates and (4.4), we obtain desired result.

(ii) Consider

$$
\begin{aligned}
\|g_\rho^{\epsilon,\sigma}\|_{NL}^{2.2_\mu^*} - \|u_\epsilon^\sigma\|_{NL}^{2.2_\mu^*} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_\rho^{2_\mu^*}(x) v_\rho^{2_\mu^*}(y) - 1)|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\, dxdy \\
&\le C\sum_{i=1}^{5} J_i,
\end{aligned}
$$

where J_i are defined in (4.3). Using the Hardy-Littlewood-Sobolev inequality and the definition of ξ_ϵ, we have the following estimates:

$$
\begin{aligned}
J_1 &\le C(N,\mu)\left(\int_{B_{2\rho}} \xi_\epsilon(x)\, dx\right)^{\frac{2N-\mu}{N}} \le C\left(\int_{B_{2\rho}} dx\right)^{\frac{2N-\mu}{N}} \le C\rho^{2N-\mu}, \\
J_2 &\le C(N,\mu)\left(\int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \xi_\epsilon(x)\, dx\right)^{\frac{2N-\mu}{2N}} \left(\int_{B_{2\rho}} \xi_\epsilon(x)\, dx\right)^{\frac{2N-\mu}{2N}} \\
&\le C\left(\int_{B_{2\rho}} dx\right)^{\frac{2N-\mu}{N}} \le C\rho^{\frac{2N-\mu}{2}},
\end{aligned}
$$

$$J_3 \leq C(N,\mu)\left(\int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \xi_\epsilon(x)dx\right)^{\frac{2N-\mu}{2N}}\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \xi_\epsilon(x)dx\right)^{\frac{2N-\mu}{2N}}$$
$$\leq C\left(\int\limits_{\mathbb{R}^N\setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2N}}\right)^{\frac{2N-\mu}{2N}}$$
$$= \left(\int\limits_{|y+(1-\epsilon)\sigma|\geq\frac{1}{2\rho}} \frac{dy}{|y|^{2N}}\right)^{\frac{2N-\mu}{2N}} \leq \left(\int\limits_{|y|\geq\frac{1}{2\rho}-1} \frac{dy}{|y|^{2N}}\right)^{\frac{2N-\mu}{2N}} \leq C\left(\frac{(2\rho)^N}{1-(2\rho)^N}\right)^{\frac{2N-\mu}{2N}},$$

Now using the same estimates as above we can easily obtain

$$J_4 \leq C\rho^{\frac{2N-\mu}{2}} \text{ and } J_5 \leq C\left(\frac{(2\rho)^N}{1-(2\rho)^N}\right)^{\frac{2N-\mu}{N}}.$$

Hence $\sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]}\left(\|g_\rho^{\epsilon,\sigma}\|_{NL}^{2.2_\mu^*} - \|u_\epsilon^\sigma\|_{NL}^{2.2_\mu^*}\right) \to 0$ as $\rho \to 0$ and completes the proof. □

Lemma 4.3. *The following asymptic estimates hold:*

(i) $a(g_\rho^{\epsilon,\sigma}) \leq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N-2})$.

(ii) $b(g_\rho^{\epsilon,\sigma}) \leq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N})$.

(iii) $b(g_\rho^{\epsilon,\sigma}) \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - O(\epsilon^{\frac{2N-\mu}{2}})$.

Proof. Part (i) follows from Lemma 4.1 (i). For part (ii) we will first estimate the integral $\int\limits_\Omega |g_\rho^{\epsilon,\sigma}|^{2^*}\,dx$. Since

$$\int\limits_\Omega |g_\rho^{\epsilon,\sigma}|^{2^*}\,dx \leq C\int\limits_{B_{\frac{3}{4\rho}}\setminus B_{\frac{3\rho}{2}}} |u_\epsilon^\sigma|^{2^*}\,dx \leq \int\limits_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} |u_\epsilon^\sigma|^{2^*}\,dx + \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{\frac{3\rho}{2}}} |u_\epsilon^\sigma|^{2^*}\,dx$$

and

$$\int\limits_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} |u_\epsilon^\sigma|^{2^*}\,dx \leq C\epsilon^N \int\limits_{B_{\frac{3}{4\rho}}\setminus B_{\frac{1}{2\rho}}} \frac{dx}{|x-(1-\epsilon)\sigma|^{2N}} = O(\epsilon^N),$$
$$\int\limits_{B_{\frac{1}{2\rho}}\setminus B_{\frac{3\rho}{2}}} |u_\epsilon^\sigma|^{2^*}\,dx \leq \int\limits_{\mathbb{R}^N} |u_\epsilon^\sigma|^{2^*}\,dx = S^{\frac{N}{N-\mu+2}}C(N,\mu)^{\frac{-N}{N-\mu+2}}.$$

It implies $\int_\Omega |g_\rho^{\epsilon,\sigma}|^{2^*}\,dx \le S^{\frac{N}{N-\mu+2}} C(N,\mu)^{\frac{-N}{N-\mu+2}} + O(\epsilon^N)$ and now using this and Hardy-Littlewood-Sobolev inequality we have

$$
\begin{aligned}
b(g_\rho^{\epsilon,\sigma}) &= \int_\Omega\int_\Omega \frac{|g_\rho^{\epsilon,\sigma}(x)|^{2_\mu^*}|g_\rho^{\epsilon,\sigma}(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy \\
&\le C(N,\mu)\left(\int_\Omega |g_\rho^{\epsilon,\sigma}|^{2^*}\,dx\right)^{\frac{2N-\mu}{N}} \\
&\le C(N,\mu)\left(S^{\frac{N}{N-\mu+2}} C(N,\mu)^{\frac{-N}{N-\mu+2}} + O(\epsilon^N)\right)^{\frac{2N-\mu}{N}} \le S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^N).
\end{aligned}
$$

This proves part (ii). Now to prove part (iii), consider

$$
\begin{aligned}
b(g_\rho^{\epsilon,\sigma}) &= \int_\Omega\int_\Omega \frac{|g_\rho^{\epsilon,\sigma}(x)|^{2_\mu^*}|g_\rho^{\epsilon,\sigma}(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy \\
&\ge \int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}}\int_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \frac{|g_\rho^{\epsilon,\sigma}(x)|^{2_\mu^*}|g_\rho^{\epsilon,\sigma}(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\epsilon^\sigma(x)|^{2_\mu^*}|u_\epsilon^\sigma(y)|^{2_\mu^*}}{|x-y|^\mu}\,dxdy - \sum_{i=1}^{i=5} J_i,
\end{aligned}
$$

where J_i are defined in equation (4.3). Using the proof of Lemma 4.1(i) and the fact that $\|u_\epsilon^\sigma\|_{NL}^{2.2_\mu^*} = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o_\epsilon(1)$, we obtain the required result. □

Now we will give a Lemma which is taken from [18]. For the sake of completeness, we provide a complete proof.

Lemma 4.4. *If* $\mu < \min\{4, N\}$ *then*

$$
\begin{aligned}
b(u_1 + tg_\rho^{\epsilon,\sigma}) \ge b(u_1) + b(tg_\rho^{\epsilon,\sigma}) + \widehat{C}t^{2.2_\mu^*-1}\int_\Omega\int_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}u_1(y)}{|x-y|^\mu}\,dxdy \\
+ 2.2_\mu^* t\int_\Omega\int_\Omega \frac{(u_1(x))^{2_\mu^*}(u_1(y))^{2_\mu^*-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy - O(\epsilon^{(\frac{2N-\mu}{4})\Theta})\ \textit{for all}\ \Theta < 1,
\end{aligned}
$$

where u_1 is the local minimum obtained in Lemma 3.10.

Proof. We will divide the proof in two cases:
Case 1: $2_\mu^* > 3$.
It is easy to see that there exists $\widehat{A} > 0$ such that

$$(a+b)^p \ge a^p + b^p + pa^{p-1}b + \widehat{A}ab^{p-1} \text{ for all } a,\ b \ge 0 \text{ and } p > 3,$$

which implies that

$$
\begin{aligned}
b(u_1 + tg_\rho^{\epsilon,\sigma}) \ge b(u_1) + b(tg_\rho^{\epsilon,\sigma}) + \widehat{C}t^{2.2_\mu^*-1}\int_\Omega\int_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}u_1(y)}{|x-y|^\mu}\,dxdy \\
+ 2.2_\mu^* t\int_\Omega\int_\Omega \frac{(u_1(x))^{2_\mu^*}(u_1(y))^{2_\mu^*-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy,\ \text{where } \widehat{C} = \min\{\widehat{A}, 2.2_\mu^*\}.
\end{aligned}
$$

Case 2: $2 < 2^\star_\mu \leq 3$.
We recall the inequality from [7, Lemma 4]: there exist C(depending on $2^\star_\mu$) such that, for all $a, b \geq 0$,

$$(a+b)^{2^\star_\mu} \geq \begin{cases} a^{2^\star_\mu} + b^{2^\star_\mu} + 2^\star_\mu a^{2^\star_\mu - 1} b + 2^\star_\mu a b^{2^\star_\mu - 1} - C a b^{2^\star_\mu - 1} & \text{if } a \geq b, \\ a^{2^\star_\mu} + b^{2^\star_\mu} + 2^\star_\mu a^{2^\star_\mu - 1} b + 2^\star_\mu a b^{2^\star_\mu - 1} - C a^{2^\star_\mu - 1} b & \text{if } a \leq b, \end{cases} \tag{4.5}$$

Consider $\Omega \times \Omega = O_1 \cup O_2 \cup O_3 \cup O_4$, where

$$\begin{aligned} O_1 &= \{(x,y) \in \Omega \times \Omega \mid u_1(x) \geq tg_\rho^{\epsilon,\sigma}(x) \text{ and } u_1(y) \geq tg_\rho^{\epsilon,\sigma}(y)\}, \\ O_2 &= \{(x,y) \in \Omega \times \Omega \mid u_1(x) \geq tg_\rho^{\epsilon,\sigma}(x) \text{ and } u_1(y) < tg_\rho^{\epsilon,\sigma}(y)\}, \\ O_3 &= \{(x,y) \in \Omega \times \Omega \mid u_1(x) < tg_\rho^{\epsilon,\sigma}(x) \text{ and } u_1(y) \geq tg_\rho^{\epsilon,\sigma}(y)\}, \\ O_4 &= \{(x,y) \in \Omega \times \Omega \mid u_1(x) < tg_\rho^{\epsilon,\sigma}(x) \text{ and } u_1(y) < tg_\rho^{\epsilon,\sigma}(y)\}. \end{aligned}$$

Also, define the $b(u)_{|O_i} = \int\int_{O_i} \frac{(u(x))^{2^\star_\mu}(u(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy$, for all $u \in H^1_0(\Omega)$ and $i = 1, 2, 3, 4$.

Subcase 1: when $(x,y) \in O_1$.
Employing (4.5), we have the following inequality:

$$\begin{aligned} b(u_1 + tg_\rho^{\epsilon,\sigma})_{|O_1} \geq (b(u_1) + b(tg_\rho^{\epsilon,\sigma}))_{|O_1} &+ 2.2^\star_\mu t^{2.2^\star_\mu - 1} \int\int_{O_1} \frac{(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}(g_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1} u_1(y)}{|x-y|^\mu}\,dxdy \\ &+ 2.2^\star_\mu t \iint_{O_1} \frac{(u_1(x))^{2^\star_\mu}(u_1(y))^{2^\star_\mu - 1} g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy - A^1_\epsilon, \end{aligned}$$

where A^1_ϵ is sum of eight non-negative integrals and each integral has an upper bound of the form $C\int\int_{O_1} \frac{u_1(x)(tg_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu - 1}(u_1(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy$ or $C\int\int_{O_1} \frac{u_1(y)(tg_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1}(u_1(x))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy$. Write $(tg_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu - 1} = (tg_\rho^{\epsilon,\sigma}(x))^r.(tg_\rho^{\epsilon,\sigma}(x))^s$ with $2^\star_\mu - 1 = r + s$, $0 < s < \frac{2^\star_\mu}{2}$. Then utilizing the definition of O_1, $u_1 \in L^\infty(\Omega)$ and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \int\int_{O_1} \frac{u_1(x)(tg_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu - 1}(u_1(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy &\leq C\int\int_{O_1} \frac{(u_1(x))^{1+r}(tg_\rho^{\epsilon,\sigma}(x))^s(u_1(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy \\ &\leq C\int_\Omega\int_\Omega \frac{(tg_\rho^{\epsilon,\sigma}(x))^s(u_1(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy \\ &\leq C\int_\Omega\int_\Omega \frac{\epsilon^{\frac{s(N-2)}{2}}}{|x-y|^\mu|x-(1-\epsilon)\sigma|^{s(N-2)}}\,dxdy \\ &\leq C\epsilon^{\frac{s(N-2)}{2}}\left(\int_\Omega \frac{dx}{|x-(1-\epsilon)\sigma|^{\frac{s(2N)(N-2)}{2N-\mu}}}\right)^{\frac{2N-\mu}{2N}} \\ &\leq C\epsilon^{\frac{s(N-2)}{2}}\left(\int_\Omega \frac{dx}{|x-(1-\epsilon)\sigma|^{\frac{s(2N)(N-2)}{2N-\mu}}}\right)^{\frac{2N-\mu}{2N}}. \end{aligned}$$

By the choice of s we have $\int_\Omega \frac{dx}{|x-(1-\epsilon)\sigma|^{\frac{s(2N)(N-2)}{2N-\mu}}} < \infty$. As a result, we get

$$\int\int_{O_1} \frac{u_1(x)(tg_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu - 1}(u_1(y))^{2^\star_\mu}}{|x-y|^\mu}\,dxdy \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta}) \text{ for all } \Theta < 1.$$

In a similar manner, we have

$$C\int\int_{O_1} \frac{u_1(y)(tg_\rho^{\epsilon,\sigma}(y))^{2^*_\mu-1}(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta}) \text{ for all } \Theta < 1.$$

Subcase 2: when $(x,y) \in O_2$.
Once again using (4.5), we have the following inequality:

$$\begin{aligned} b(u_1 + tg_\rho^{\epsilon,\sigma})_{|O_2} \geq [b(u_1) + b(tg_\rho^{\epsilon,\sigma})]|_{O_2} &+ 2.2^*_\mu t^{2.2^*_\mu-1}\iint_{O_2} \frac{(g_\rho^{\epsilon,\sigma}(x))^{2^*_\mu}(g_\rho^{\epsilon,\sigma}(y))^{2^*_\mu-1}u_1(y)}{|x-y|^\mu}dxdy \\ &+ 2.2^*_\mu t\iint_{O_2} \frac{(u_1(x))^{2^*_\mu}(u_1(y))^{2^*_\mu-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy - A_\epsilon^2, \end{aligned}$$

where A_ϵ^2 is sum of eight non-negative integrals and each integral has an upper bound of the form $C\int\int_{O_2} \frac{u_1(x)(tg_\rho^{\epsilon,\sigma}(x))^{2^*_\mu-1}(g_\rho^{\epsilon,\sigma}(y))^{2^*_\mu}}{|x-y|^\mu}\,dxdy$ or $C\int\int_{O_2} \frac{(u_1(y))^{2^*_\mu-1}(tg_\rho^{\epsilon,\sigma}(y))(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy$. By the similar estimates as in Subcase 1, definition of O_2, the fact that $tg_\rho^{\epsilon,\sigma} \in H_0^1(\Omega)$ and regularity of u_1, we have

$$\int\int_{O_2} \frac{u_1(x)(tg_\rho^{\epsilon,\sigma}(x))^{2^*_\mu-1}(g_\rho^{\epsilon,\sigma}(y))^{2^*_\mu}}{|x-y|^\mu}\,dxdy \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta}) \text{ for all } \Theta < 1.$$

Write $(u_1(y))^{2^*_\mu-1} = (u_1(y))^r.(u_1(y))^s$ with $2^*_\mu - 1 = r + s,\ 0 < 1 + s < \frac{2^*_\mu}{2}$. Then utilizing the definition of O_2, $u_1 \in L^\infty(\Omega)$ and Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} \int\int_{O_2} \frac{(u_1(y))^{2^*_\mu-1}(tg_\rho^{\epsilon,\sigma}(y))(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy &\leq \int\int_{O_2} \frac{(u_1(y))^r(tg_\rho^{\epsilon,\sigma}(y))^{1+s}(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy \\ &\leq C\int_\Omega\int_\Omega \frac{(tg_\rho^{\epsilon,\sigma}(y))^{1+s}(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy \\ &\leq C\int_\Omega\int_\Omega \frac{\epsilon^{\frac{(1+s)(N-2)}{2}}}{|x-y|^\mu|y-(1-\epsilon)\sigma|^{(1+s)(N-2)}}\,dxdy \\ &\leq C\epsilon^{\frac{(1+s)(N-2)}{2}}\left(\int_\Omega \frac{dy}{|y-(1-\epsilon)\sigma|^{\frac{(1+s)(2N)(N-2)}{2N-\mu}}}\right)^{\frac{2N-\mu}{2N}} \\ &\leq C\epsilon^{\frac{(1+s)(N-2)}{2}}\left(\int_\Omega \frac{dy}{|y-(1-\epsilon)\sigma|^{\frac{(1+s)(2N)(N-2)}{2N-\mu}}}\right)^{\frac{2N-\mu}{2N}}. \end{aligned}$$

By the choice of s we have $\displaystyle\int_\Omega \frac{dx}{|x-(1-\epsilon)\sigma|^{\frac{(1+s)(2N)(N-2)}{2N-\mu}}} < \infty$. Hence we obtain

$$\int\int_{O_2} \frac{(u_1(y))^{2^*_\mu-1}(tg_\rho^{\epsilon,\sigma}(y))(u_1(x))^{2^*_\mu}}{|x-y|^\mu}\,dxdy \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta}) \text{ for all } \Theta < 1.$$

Subcase 3: when $(x,y) \in O_3$.
Using (4.5), we have

$$b(u_1 + tg_\rho^{\epsilon,\sigma})_{|O_3} \geq (b(u_1) + b(tg_\rho^{\epsilon,\sigma}))|_{O_3} + 2.2^\star_\mu t^{2.2^\star_\mu - 1} \iint\limits_{O_3} \frac{(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}(g_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1} u_1(y)}{|x-y|^\mu} dxdy$$
$$+ 2.2^\star_\mu t \iint\limits_{O_3} \frac{(u_1(x))^{2^\star_\mu}(u_1(y))^{2^\star_\mu - 1} g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu} dxdy - A_\epsilon^3,$$

where A_ϵ^3 is sum of eight non-negative integrals and each integral has an upper bound of the form $C \iint_{O_3} \frac{(u_1(x))^{2^\star_\mu - 1}(tg_\rho^{\epsilon,\sigma}(x))(u_1(y))^{2^\star_\mu}}{|x-y|^\mu} dxdy$ or $C \iint_{O_3} \frac{u_1(y)(tg_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1}(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}}{|x-y|^\mu} dxdy$. By the similar estimates as in Subcase 1, Subcase 2, definition of O_3 and regularity of u_1, we get $A_\epsilon^3 \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta})$ for all $\Theta < 1$.
Subcase 4: when $(x,y) \in O_4$.
Using (4.5), we have

$$b(u_1 + tg_\rho^{\epsilon,\sigma})_{|O_4} \geq (b(u_1) + b(tg_\rho^{\epsilon,\sigma}))|_{O_4} + 2.2^\star_\mu t^{2.2^\star_\mu - 1} \iint\limits_{O_4} \frac{(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}(g_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1} u_1(y)}{|x-y|^\mu} dxdy$$
$$+ 2.2^\star_\mu t \iint\limits_{O_4} \frac{(u_1(x))^{2^\star_\mu}(u_1(y))^{2^\star_\mu - 1} g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu} dxdy - A_\epsilon^4,$$

where A_ϵ^4 is sum of eight non-negative integrals and each integral has an upper bound of the form $C \int\int_{O_4} \frac{(u_1(x))^{2^\star_\mu - 1}(tg_\rho^{\epsilon,\sigma}(x))(tg_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu}}{|x-y|^\mu} dxdy$ or $C \int\int_{O_4} \frac{u_1(y)(tg_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1}(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}}{|x-y|^\mu} dxdy$. By the similar estimates as in Subcase 2, we have

$$A_\epsilon^4 \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta}) \text{ for all } \Theta < 1.$$

From all subcases we obtain $A_\epsilon^i \leq O(\epsilon^{(\frac{2N-\mu}{4})\Theta})$ for all $\Theta < 1$ and $i = 1, 2, 3, 4$. Combining all sub cases we conclude Case 2. From Case 1 and Case 2 we have the required result. □

Proposition 4.5. *Let $\mu < \min\{4, N\}$ then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ we have*

$$\sup_{t\geq 0} \mathcal{J}_f(u_1 + tg_\rho^{\epsilon,\sigma}) < \mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \text{ uniformly in } \sigma \in \mathbb{S}^{N-1},$$

where u_1 is the local minimum in Lemma 3.10.

Proof. By Lemma 3.8, $u \in L^\infty(\Omega)$ and $u > 0$ in Ω. This implies

$$b(u_1 + tg_\rho^{\epsilon,\sigma}) = \int\limits_\Omega \int\limits_\Omega \frac{(u_1 + tg_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}(u_1 + tg_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu}}{|x-y|^\mu} dxdy.$$

Claim 1: There exists a $R_0 > 0$ such that

$$I = \int\limits_\Omega \int\limits_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2^\star_\mu}(g_\rho^{\epsilon,\sigma}(y))^{2^\star_\mu - 1} u_1(y)}{|x-y|^\mu} dxdy \geq \widehat{C} R_0 \epsilon^{\frac{N-2}{2}}.$$

Clearly,

$$\begin{aligned} I &\geq \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}u_1(y)}{|x-y|^\mu}\,dxdy \\ &\geq C \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \frac{(u_\epsilon^\sigma(x))^{2_\mu^*}(u_\epsilon^\sigma(y))^{2_\mu^*-1}}{|x-y|^\mu}\,dxdy \\ &\geq C \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \int\limits_{B_{\frac{1}{2\rho}}\setminus B_{2\rho}} \frac{\epsilon^{\frac{3N}{2}+1-\mu}\,dxdy}{|x-y|^\mu(\epsilon^2+|x-(1-\epsilon)\sigma|^2)^{\frac{2N-\mu}{2}}(\epsilon^2+|y-(1-\epsilon)\sigma|^2)^{\frac{N-\mu+2}{2}}}. \end{aligned}$$

For any $\epsilon < 1-2\rho$ there exists $c > 0$ such that $1-\epsilon > c > 2\rho$ so we get

$$\begin{aligned} I &\geq C\epsilon^{\frac{3N}{2}+1-\mu} \int\limits_{B_c}\int\limits_{B_c} \frac{dzdw}{|z-w|^\mu(\epsilon^2+|z|^2)^{\frac{2N-\mu}{2}}(\epsilon^2+|w|^2)^{\frac{N-\mu+2}{2}}} \\ &\geq C\epsilon^{\frac{N-2}{2}} \int\limits_{B_c}\int\limits_{B_c} \frac{dzdw}{|z-w|^\mu(1+|z|^2)^{\frac{2N-\mu}{2}}(1+|w|^2)^{\frac{N-\mu+2}{2}}} = O(\epsilon^{\frac{N-2}{2}}). \end{aligned}$$

This proves the claim 1. Now using Lemma 4.4, we have

$$\begin{aligned} \mathcal{J}_f(u_1+tg_\rho^{\epsilon,\sigma}) \leq{}& \frac{1}{2}a(u_1)+\frac{1}{2}a(tg_\rho^{\epsilon,\sigma})+t\langle u_1, g_\rho^{\epsilon,\sigma}\rangle_{H_0^1(\Omega)} - \frac{1}{2.2_\mu^*}b(u_1) - \frac{1}{2.2_\mu^*}b(tg_\rho^{\epsilon,\sigma}) \\ &- \widehat{C}t^{2.2_\mu^*-1}\int\limits_\Omega\int\limits_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}u_1(y)}{|x-y|^\mu}\,dxdy - \int\limits_\Omega fu_1\,dx \\ &- t\int\limits_\Omega fg_\rho^{\epsilon,\sigma}\,dx - t\int\limits_\Omega\int\limits_\Omega \frac{(u_1(x))^{2_\mu^*}(u_1(y))^{2_\mu^*-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy + O(\epsilon^{(\frac{2N-\mu}{4})\Theta}). \end{aligned}$$

for all $\Theta < 1$. Taking $\Theta = \frac{2}{2_\mu^*}$, we have

$$\begin{aligned} \mathcal{J}_f(u_1+tg_\rho^{\epsilon,\sigma}) \leq{}& \frac{1}{2}a(u_1)+\frac{1}{2}a(tg_\rho^{\epsilon,\sigma})+t\langle u_1, g_\rho^{\epsilon,\sigma}\rangle_{H_0^1(\Omega)} - \frac{1}{2.2_\mu^*}b(u_1) - \frac{1}{2.2_\mu^*}b(tg_\rho^{\epsilon,\sigma}) \\ &- \widehat{C}t^{2.2_\mu^*-1}\int\limits_\Omega\int\limits_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}u_1(y)}{|x-y|^\mu}\,dxdy - \int\limits_\Omega fu_1\,dx \\ &- t\int\limits_\Omega fg_\rho^{\epsilon,\sigma}\,dx - t\int\limits_\Omega\int\limits_\Omega \frac{(u_1(x))^{2_\mu^*}(u_1(y))^{2_\mu^*-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy + o(\epsilon^{\frac{N-2}{2}}). \end{aligned}$$

This on utilizing Lemma 4.3 and claim 1 gives

$$\begin{aligned}
\mathcal{J}_f(u_1 + tg_\rho^{\epsilon,\sigma}) \leq{}& \frac{1}{2}a(u_1) + \frac{1}{2}a(tg_\rho^{\epsilon,\sigma}) + t\langle u_1, g_\rho^{\epsilon,\sigma}\rangle_{H_0^1(\Omega)} - \frac{1}{2.2_\mu^*}b(u_1) - \frac{1}{2.2_\mu^*}b(tg_\rho^{\epsilon,\sigma}) \\
& - \widehat{C}t^{2.2_\mu^*-1}\int_\Omega\int_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}}{|x-y|^\mu}\,dxdy - \int_\Omega fu_1\,dx - t\int_\Omega fg_\rho^{\epsilon,\sigma}\,dx \\
& - t\int_\Omega\int_\Omega \frac{(u_1(x))^{2_\mu^*}(u_1(y))^{2_\mu^*-1}g_\rho^{\epsilon,\sigma}(y)}{|x-y|^\mu}\,dxdy + o(\epsilon^{\frac{N-2}{2}}) \\
={}& \mathcal{J}_f(u_1) + \mathcal{J}(tg_\rho^{\epsilon,\sigma}) - \widehat{C}t^{2.2_\mu^*-1}\int_\Omega\int_\Omega \frac{(g_\rho^{\epsilon,\sigma}(x))^{2_\mu^*}(g_\rho^{\epsilon,\sigma}(y))^{2_\mu^*-1}}{|x-y|^\mu}\,dxdy + o(\epsilon^{\frac{N-2}{2}}) \\
\leq{}& \mathcal{J}_f(u_1) + \frac{t^2}{2}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N-2})\right) - \frac{t^{2.2_\mu^*}}{2.2_\mu^*}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - O(\epsilon^{\frac{2N-\mu}{2}})\right) \\
& - t^{2.2_\mu^*-1}\widehat{C}R_0\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Now define $K(t) := \frac{t^2}{2}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N-2})\right) - \frac{t^{2.2_\mu^*}}{2.2_\mu^*}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - O(\epsilon^{\frac{2N-\mu}{2}})\right) - t^{2.2_\mu^*-1}\widehat{C}R_0\epsilon^{\frac{N-2}{2}}$ then $K(t) \to \infty$ as $t \to \infty$ and $\lim\limits_{t\to 0^+} K(t) > 0$ so there exists a $t_\epsilon > 0$ such that $\sup\limits_{t>0} K(t)$ is attained and $t_\epsilon < \left(\frac{S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N-2})}{S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - O(\epsilon^{\frac{2N-\mu}{2}})}\right)^{\frac{1}{2.2_\mu^*-2}} := S_{H,L}(\epsilon)$. Moreover there exists a $t_1 > 0$ such that for sufficiently small $\epsilon > 0$ we have $t_\epsilon > t_1$. Clearly the function

$$t \mapsto \frac{t^2}{2}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{N-2})\right) - \frac{t^{2.2_\mu^*}}{2.2_\mu^*}\left(S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - O(\epsilon^{\frac{2N-\mu}{2}})\right)$$

is an increasing function in $[0, S_{H,L}(\epsilon)]$. Therefore,

$$\begin{aligned}
\sup_{t\geq 0}\mathcal{J}_f(u_1 + tg_\rho^{\epsilon,\sigma}) \leq{}& \mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + O(\epsilon^{\min\{\frac{2N-\mu}{2},\, N-2\}}) \\
& - t_1^{2.2_\mu^*-1}\widehat{C}R_0\epsilon^{\frac{N-2}{2}} + o(\epsilon^{\frac{N-2}{2}}).
\end{aligned}$$

Hence there exits a $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ we have

$$\sup_{t\geq 0}\mathcal{J}_f(u_1 + tg_\rho^{\epsilon,\sigma}) < \mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \text{ uniformly in } \sigma \in \mathbb{S}^{N-1}.$$

□

Lemma 4.6. *The following holds:*

(i) $H_0^1(\Omega) \setminus \mathcal{N}_f^- = U_1 \cup U_2$, *where*

$$U_1 := \left\{u \in H_0^1(\Omega)\setminus\{0\} \;\middle|\; u^+ \not\equiv 0,\ \|u\| < t^-\left(\frac{u}{\|u\|}\right)\right\} \cup \{0\},$$
$$U_2 := \left\{u \in H_0^1(\Omega)\setminus\{0\} \;\middle|\; u^+ \not\equiv 0,\ \|u\| > t^-\left(\frac{u}{\|u\|}\right)\right\}.$$

(ii) $\mathcal{N}_f^+ \subset U_1$.

(iii) For each $0 < \epsilon \leq \epsilon_0$, *there exists* $t_0 > 1$ *and such that* $u_1 + t_0 g_\rho^{\epsilon,\sigma} \in U_2$.

(iv) For each $0 < \epsilon < \epsilon_0$, *there exists* $s_0 \subset (0, 1)$ *and such that* $u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma} \in \mathcal{N}_f^-$.

(v) $\Upsilon_f^- < \Upsilon_f + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$.

Proof.

(i) It holds by Lemma 3.3 (d).

(ii) Let $u \in \mathcal{N}_f^+$ then $t^+(u) = 1$ and $1 < t^+(u) < t_{max} < t^-(u) = \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right)$ that is, $\mathcal{N}_f^+ \subset U_1$.

(iii) First, we will show that there exists a constant $c > 0$ such that $0 < t^-\left(\frac{u_1+tg_\rho^{\epsilon,\sigma}}{\|u_1+tg_\rho^{\epsilon,\sigma}\|}\right) < c$ for all $t > 0$. On the contrary, let there exist a sequence $\{t_n\}$ such that $t_n \to \infty$ and $t^-\left(\frac{u_1+t_n g_\rho^{\epsilon,\sigma}}{\|u_1+t_n g_\rho^{\epsilon,\sigma}\|}\right) \to \infty$ as $n \to \infty$. Define $u_n := \frac{u_1+t_n g_\rho^{\epsilon,\sigma}}{\|u_1+t_n g_\rho^{\epsilon,\sigma}\|}$ so there exists $t^-(u_n)$ such that $t^-(u_n)u_n \in \mathcal{N}_f^-$. By dominated convergence theorem,

$$b(u_n) = \frac{b(u_1 + t_n g_\rho^{\epsilon,\sigma})}{\|u_1 + t_n g_\rho^{\epsilon,\sigma}\|^{2.2_\mu^*}} = \frac{b(\frac{u_1}{t_n} + g_\rho^{\epsilon,\sigma})}{\|\frac{u_1}{t_n} + g_\rho^{\epsilon,\sigma}\|^{2.2_\mu^*}} \to \frac{b(g_\rho^{\epsilon,\sigma})}{\|g_\rho^{\epsilon,\sigma}\|^{2.2_\mu^*}} \text{ as } n \to \infty.$$

Hence, $\mathcal{J}_f(t^-(u_n)u_n) \to -\infty$ as $n \to \infty$, contradicts the fact that $\mathcal{J}_f$ is bounded below on $\mathcal{N}_f$. Therefore, there exists $c > 0$ such that $0 < t^-\left(\frac{u_1+tg_\rho^{\epsilon,\sigma}}{\|u_1+tg_\rho^{\epsilon,\sigma}\|}\right) < c$ for all $t > 0$. Let $t_0 = \dfrac{|c^2 - \|u_1\|^2|^{\frac{1}{2}}}{\|g_\rho^{\epsilon,\sigma}\|} + 1$ then

$$\begin{aligned}\|u_1 + t_0 g_\rho^{\epsilon,\sigma}\|^2 &= \|u_1\|^2 + t_0^2\|g_\rho^{\epsilon,\sigma}\|^2 + 2t_0\langle u_1,\ g_\rho^{\epsilon,\sigma}\rangle \\ &\geq \|u_1\|^2 + |c^2 - \|u_1\|^2| \geq c^2 \geq \left(t^-\left(\frac{u_1 + tg_\rho^{\epsilon,\sigma}}{\|u_1 + tg_\rho^{\epsilon,\sigma}\|}\right)\right)^2.\end{aligned}$$

It implies that $u_1 + t_0 g_\rho^{\epsilon,\sigma} \in U_2$.

(iv) For each $0 < \epsilon < \epsilon_0$, define a path $\xi_\epsilon(s) = u_1 + st_0 g_\rho^{\epsilon,\sigma}$ for $s \in [0, 1]$. Then

$$\xi_\epsilon(0) = u_1 \quad \text{and} \quad \xi_\epsilon(1) = u_1 + t_0 g_\rho^{\epsilon,\sigma} \in U_2.$$

Since $\frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right)$ is a continuous function and $\xi_\epsilon([0, 1])$ is connected. So, there exists $s_0 \in [0, 1]$ such that $\xi_\epsilon(s_0) = u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma} \in \mathcal{N}_f^-$.

(v) Using part (iv) and Proposition 4.5. □

At this point we will state Global compactness Lemma for the functional $\mathcal{J}_f$ which is a version of Theorem 4.4 of [19].

Lemma 4.7. *Let $\{u_n\} \subset H_0^1(\Omega)$ be such that $\mathcal{J}_f(u_n) \to c$, $\mathcal{J}_f'(u_n) \to 0$. Then passing if necessary to a subsequence, there exists a solution $v_0 \in H_0^1(\Omega)$ of*

$$-\Delta u = \left(\int_\Omega \frac{|u^+(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right) |u^+|^{2_\mu^*-1} + f \text{ in } \Omega$$

and (possibly) $k \in \mathbb{N} \cup \{0\}$, non-trivial solutions $\{v_1, v_2, \ldots, v_k\}$ of

$$-\Delta u = (|x|^{-\mu} \star |u^+|^{2_\mu^*})|u^+|^{2_\mu^*-1} \text{ in } \mathbb{R}^N$$

with $v_i \in D^{1,2}(\mathbb{R}^N)$ and k sequences $\{y_n^i\}_n \subset \Omega$ and $\{\lambda_n^i\}_n \subset \mathbb{R}_+$ $i = 1, 2, \cdots k$, satisfying

$$\frac{1}{\lambda_n^i} dist(y_n^i, \partial\Omega) \to \infty, \text{ and } \|u_n - v_0 - \sum_{i=1}^k (\lambda_n^i)^{\frac{2-N}{2}} v_i((. - y_n^i)/\lambda_n^i)\|_{D^{1,2}(\mathbb{R}^N)} \to 0,\ n \to \infty,$$

$$\|u_n\|_{D^{1,2}(\mathbb{R}^N)}^2 \to \sum_{i=0}^k \|v_i\|_{D^{1,2}(\mathbb{R}^N)}^2, \text{ as } n \to \infty, \quad \mathcal{J}_f(v_0) + \sum_{i=1}^k \mathcal{J}_\infty(v_i) = c,$$

where $\mathcal{J}_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2.2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^+(x)|^{2_\mu^*}|u^+(y)|^{2_\mu^*}}{|x-y|^\mu}\, dxdy, \quad u \in D^{1,2}(\mathbb{R}^N)$.

Lemma 4.8. *(i) Let $\{u_n\}$ be a $(PS)_c$ sequence for $\mathcal{J}_f$ with $c < \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ then there exists a subsequence still denoted by $\{u_n\}$ and a nonzero $u^0 \in H_0^1(\Omega)$ such that $u_n \to u^0$ strongly in $H_0^1(\Omega)$ and $\mathcal{J}_f(u^0) = c$.*

(ii) Let $\{u_n\} \subset \mathcal{N}_f^-$ be a $(PS)_c$ sequence for $\mathcal{J}_f$ with

$$\Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < c < \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$$

then there exists subsequence still denoted by $\{u_n\}$ and a nonzero $u^0 \in \mathcal{N}_f^-$ such that $u_n \to u^0$ strongly in $H_0^1(\Omega)$ and $\mathcal{J}_f(u^0) = c$.

Proof. Proof of (i) follows from [30, Lemma 3.4]. To prove (ii), Let $\{u_n\}$ be a $(PS)_c$ sequence then by standard arguments $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $u^0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u^0$ in $H_0^1(\Omega)$ and $\mathcal{J}_f'(u^0) = 0$. Then by Lemma 3.11, we have either $u^0 \in \mathcal{N}_f^-$ or $u^0 = u_1$. Now using Lemma 4.7 we obtain

$$\Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \geq c = \mathcal{J}_f(u^0) + \sum_{i=1}^{k} \mathcal{J}_\infty(v_i) \geq \Upsilon_f(\Omega) + k\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

which on using Lemma 4.6(e), gives $k \leq 1$. By [19, corollary 3.3], we get v_1 is a constant multiple of Talenti function that is, $\mathcal{J}_\infty(v_1) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $k = 0$ then we are done and if $k = 1$ and $u^0 = u_1$, then

$$c = \mathcal{J}_f(u^0) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} = \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}},$$

a contradiction. If $k = 1$ and $u^0 \in \mathcal{N}_f^-$, we get

$$c = \mathcal{J}_f(u^0) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \geq \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}},$$

which is again a contradiction. Hence $k = 0$ and result follows. □

5 Existence of Second and third Solution

In this section we will show the existence of second and third solution of problem (P_f). To prove this, we shall show that for a sufficiently small $\delta > 0$,

$$cat\left(\left\{u \in \mathcal{N}_f^- \ : \ \mathcal{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta\right\}\right) \geq 2,$$

where $cat(X)$ is the category of the set X is defined in the Definition 5.1. And then employing Lemma 5.2, we conclude the existence of second and third solutions. We shall first gather some preliminaries.
For $c \in \mathbb{R}$, we define

$$b_c(u) = cb(u), \ \mathcal{I}_c(u) = \frac{1}{2}a(u) - \frac{1}{2.2_\mu^*} b_c(u), \ \mathcal{M}_c = \{u \in H_0^1(\Omega) \setminus \{0\} | \langle \mathcal{I}_c'(u), \ u\rangle = 0\}.$$

We denote

$$[\mathcal{J}_f \leq c] = \{u \in \mathcal{N}_f | \mathcal{J}_f(u) \leq c\}.$$

Definition 5.1. *(i) For a topological space X, we say that a non-empty, closed subset $Y \subset X$ is contractible to a point in X if and only if there exists a continuous mapping $\varrho : [0,1] \times Y \to X$ such that for some $x_0 \in X$, $\varrho(0,x) = x$ for all $x \in Y$ and $\varrho(1,x) = x_0$ for all $x \in Y$.*

(ii) We define

$$cat(X) = \min\{k \in \mathbb{N} \mid \text{there exists closed subsets } Y_1, Y_2, \cdots Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \cup_{j=1}^{k} Y_j = X\}$$

Lemma 5.2. *[2] Suppose that X is a Hilbert manifold and $G \in C^1(X, \mathbb{R})$. Assume that there are $c_1 \in \mathbb{R}$ and $k \in \mathbb{N}$, such that*

1. *G satisfies the Palais-Smale condition for energy level $c \leq c_1$;*
2. *$cat(\{x \in X \mid G(x) \leq c_1\}) \geq k$.*

Then G has at least k critical points in $\{x \in X \mid G(x) \leq c_1\}$. □

Lemma 5.3. *[1, Theorem 2.5] Let X be a topological space. Suppose that there are two continuous maps $\Phi : \mathbb{S}^{N-1} \to X$ and $\Psi : X \to \mathbb{S}^{N-1}$ such that $\Psi o \Phi$ is homotopic to the identity map of $\mathbb{S}^{N-1}$. Then $cat(X) \geq 2$.* □

Now we will proof a Lemma which will relate the functional $\mathcal{J}_f$ and $\mathcal{I}_c$. Note that for each $u \in H_0^1(\Omega)$ there exists a unique $t^- > 0$ and a unique $t^\star > 0$ such that $t^- u \in \mathcal{N}_f^-$ and $t^\star u \in \mathcal{N}$.

Lemma 5.4. *(i) For each $u \in \Sigma := \{u \in H_0^1(\Omega) | \quad \|u\| = 1\}$, there exists a unique $t_c(u) > 0$ such that $t_c(u)u \in \mathcal{M}_c$ and*

$$\max_{t \geq 0} \mathcal{I}_c(tu) = \mathcal{I}_c(t_c(u)u) = \frac{N-\mu+2}{2(2N-\mu)} \left(b_c(u)\right)^{-\frac{N-2}{N-\mu+2}}.$$

(ii) For each $u \in H_0^1(\Omega)$ with $u^+ \not\equiv 0$ and $0 < \omega < 1$, we have

$$(1-\omega)\mathcal{I}_{\frac{1}{1-\omega}}(u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \mathcal{J}_f(u) \leq (1+\omega)\mathcal{I}_{\frac{1}{1+\omega}}(u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2$$

(iii) For each $u \in \Sigma$ and $0 < \omega < 1$, we have

$$(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathcal{J}(t^\star u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \mathcal{J}_f(t^- u) \leq (1+\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathcal{J}(t^\star u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.$$

(iv) There exists $e_{11} > 0$ such that if $0 < \|f\|_{H^{-1}} < e_{11}$ then $\Upsilon_f^- > 0$.

Proof.

(i) For each $u \in \Sigma$, define $k(t) = \frac{1}{2}t^2 - \frac{t^{2.2_\mu^\star}}{2.2_\mu^\star} b_c(u)$, then if

$$t_c(u) = \left(\frac{1}{b_c(u)}\right)^{\frac{1}{2(2_\mu^\star - 1)}},$$

we obtain $k'(t_c(u)) = 0$ and $k''(t_c(u)) < 0$. Therefore, there exists a unique $t_c(u) > 0$ such that

$$\max_{t \geq 0} \mathcal{I}_c(tu) = \mathcal{I}_c(t_c(u)u) = \frac{N-\mu+2}{2(2N-\mu)} \left(b_c(u)\right)^{-\frac{N-2}{N-\mu+2}}.$$

(ii) For $0 < \omega < 1$, we have

$$\left|\int_\Omega fu\, dx\right| \leq \|f\|_{H^{-1}}\|u\| \leq \frac{\omega}{2}\|u\|^2 + \frac{1}{2\omega}\|f\|_{H^{-1}}^2,$$

and for $u \in H_0^1(\Omega)$ with $u^+ \not\equiv 0$ by the above inequality, we get

$$\frac{1-\omega}{2}\|u\|^2 - \frac{1}{2.2_\mu^\star} b(u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \mathcal{J}_f(u) \leq \frac{1+\omega}{2}\|u\|^2 - \frac{1}{2.2_\mu^\star} b(u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.$$

This implies that

$$(1-\omega)\mathcal{I}_{\frac{1}{1-\omega}}(u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \mathcal{J}_f(u) \leq (1+\omega)\mathcal{I}_{\frac{1}{1+\omega}}(u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.$$

(iii) Using part (ii), we obtain the following estimate for each $u \in \Sigma$ and $0 < \omega < 1$

$$(1-\omega)\mathcal{I}_{\frac{1}{1-\omega}}(t_{\frac{1}{1-\omega}}(u)u) - \frac{1}{2\omega}\|f\|^2_{H^{-1}} \le \mathcal{J}_f(t^-(u)u) \le (1+\omega)\mathcal{I}_{\frac{1}{1+\omega}}(t_{\frac{1}{1+\omega}}(u)u) + \frac{1}{2\omega}\|f\|^2_{H^{-1}}. \tag{5.1}$$

Using (5.1) in part (i) we get

$$\begin{aligned}\mathcal{I}_{\frac{1}{1-\omega}}(t_{\frac{1}{1-\omega}}(u)u) &= \frac{N-\mu+2}{2(2N-\mu)} b_{\frac{1}{1-\omega}}(u)^{-\frac{N-2}{N-\mu+2}}\\ &= (1-\omega)^{\frac{N-2}{N-\mu+2}} \frac{N-\mu+2}{2(2N-\mu)} b(u)^{-\frac{N-2}{N-\mu+2}} = (1-\omega)^{\frac{N-2}{N-\mu+2}} \mathcal{J}(t^\star u).\end{aligned}$$

Therefore, we get

$$(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathcal{J}(t^\star u) - \frac{1}{2\omega}\|f\|^2_{H^{-1}} \le \mathcal{J}_f(t^- u) \le (1+\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathcal{J}(t^\star u) - \frac{1}{2\omega}\|f\|^2_{H^{-1}}.$$

(iv) Combining part (iii) with the fact that $\Upsilon_0 = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} > 0$ contributes that

$$\begin{aligned}\Upsilon_f^-(\Omega) &> (1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \Upsilon_0 - \frac{1}{2\omega}\|f\|^2_{H^{-1}}\\ &= (1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \frac{1}{2\omega}\|f\|^2_{H^{-1}}.\end{aligned}$$

Thus, there exists $e_{11} > 0$ such that $\Upsilon_f^-(\Omega) > 0$ whenever $\|f\|_{H^{-1}} < e_{11}$ □

Lemma 5.5. *If Ω satisfies condition (A) then there exists a $\delta_0 > 0$ such that if $u \in \mathcal{N}$ with $\mathcal{J}(u) \le \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \delta_0$, then $\int_{\mathbb{R}^N} \frac{x}{|x|}|\nabla u|^2\, dx \ne 0$*

Proof. Let $\{u_n\} \in \mathcal{N}$ such that $\mathcal{J}(u_n) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o(1)$ and $\int_{\mathbb{R}^N} \frac{x}{|x|}|\nabla u_n|^2\, dx = 0$. Since $\{u_n\} \in \mathcal{N}$ therefore by Lemma 2.9, $\{u_n\}$ is a Palais-Smale sequence of $\mathcal{J}$ at level $\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. Now using [19, Theorem 4.4] and Remark 2.8, we have

$$\|u_n - (\lambda_n^1)^{\frac{2-N}{2}} v_1((. - y_n^1)/\lambda_n^1)\|_{D^{1,2}(\mathbb{R}^N)} \to 0,$$

where v_1 is a minimizer of $S_{H,L}$, $\lambda_n^1 \in \mathbb{R}^+$, $y_n^1 \in \overline{\Omega}$. Moreover, if $n \to \infty$ then $\lambda_n^1 \to 0$, $\frac{y_n^1}{|y_n^1|} \to y_0$ is the unit vector in $\mathbb{R}^N$. Thus we obtain

$$\begin{aligned}0 = \int_{\mathbb{R}^N} \frac{x}{|x|}|\nabla u_n|^2\, dx &= \int_{\mathbb{R}^N} \frac{x}{|x|}\left(|\nabla u_n|^2 - |\nabla(\lambda_n^1)^{\frac{2-N}{2}} v_1((. - y_n^1)/\lambda_n^1)|^2\right) dx\\ &\quad + \int_{\mathbb{R}^N} \frac{x}{|x|}|\nabla(\lambda_n^1)^{\frac{2-N}{2}} v_1((. - y_n^1)/\lambda_n^1)|^2\, dx\\ &= o_n(1) + \int_{\mathbb{R}^N} \frac{y_n^1 + \lambda_n^1 z}{|y_n^1 + \lambda_n^1 z|}|\nabla v_1(z)|^2\, dz\\ &= o_n(1) + y_0 S_{H,L}^{\frac{2N-\mu}{N-\mu+2}},\end{aligned}$$

as $n \to \infty$, which is not possible. □

For $0 < \epsilon \le \epsilon_0$ (defined in Proposition 4.5), define $H_\epsilon : \mathbb{S}^{N-1} \to H_0^1(\Omega)$ as

$$H_\epsilon(\sigma) = u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma},$$

where the function $u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma}$ defined in Lemma 4.6.

Lemma 5.6. *There exists a $\delta_\epsilon \in \mathbb{R}^+$ such that*

$$H_\epsilon(\mathbb{S}^{N-1}) \subset \left[\mathcal{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta_\epsilon\right].$$

Proof. Trivially, $H_\epsilon(\sigma) = u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma} \in \mathcal{N}_f^-$. So we only have to prove that $\mathcal{J}_f(u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma}) \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta_\epsilon$ for some $\delta_\epsilon > 0$. Since by Proposition 4.5,

$$\sup_{t\geq 0} \mathcal{J}_f(u_1 + t g_\rho^{\epsilon,\sigma}) < \mathcal{J}_f(u_1) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} = \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Hence there exists a $\delta_\epsilon > 0$ such that

$$\mathcal{J}_f(u_1 + s_0 t_0 g_\rho^{\epsilon,\sigma}) \leq \sup_{t\geq 0} \mathcal{J}_f(u_1 + t g_\rho^{\epsilon,\sigma}) \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta_\epsilon.$$

□

Lemma 5.7. *There exists a $e_{22} > 0$ such that $\|f\|_{H^{-1}} < e_{22}$ then for any*

$$u \in \left[\mathcal{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}\right] \text{ we have } \int_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx \neq 0.$$

Proof. Let $u \in \left[\mathcal{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}\right]$ then $\mathcal{J}_f(u) \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ and $u \in \mathcal{N}_f^-$, that is, $\frac{1}{\|u\|} t^- \left(\frac{u}{\|u\|}\right) = 1$. Since $\Upsilon_f(\Omega) < 0$ we have $\mathcal{J}_f(u) \leq \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. So for $\frac{u}{\|u\|} \in \Sigma$ there exits a $t^\star > 0$ such that $\frac{t^\star u}{\|u\|} \in \mathcal{N}$ which on using Lemma 5.4 (iii) implies

$$(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathcal{J}\left(t^\star \frac{u}{\|u\|}\right) - \frac{1}{2\omega} \|f\|_{H^{-1}}^2 \leq \mathcal{J}_f\left(t^- \frac{u}{\|u\|}\right) = \mathcal{J}_f(u).$$

Now using Lemma 3.4, we have

$$\begin{aligned}
\mathcal{J}\left(t^\star \frac{u}{\|u\|}\right) &\leq (1-\omega)^{-\frac{2N-\mu}{N-\mu+2}} \left(\mathcal{J}_f(u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2\right) \\
&\leq (1-\omega)^{-\frac{2N-\mu}{N-\mu+2}} \left(\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \frac{1}{2\omega}\|f\|_{H^{-1}}^2\right) \\
&= \left((1-\omega)^{-\frac{2N-\mu}{N-\mu+2}} - 1\right) \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \\
&\quad + \left(\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \frac{1}{2\omega(1-\omega)^{\frac{2N-\mu}{N-\mu+2}}}\|f\|_{H^{-1}}^2\right).
\end{aligned}$$

Choose $\omega_0 > 0$ such that for $0 < \omega < \omega_0$, we have $\left((1-\omega)^{-\frac{2N-\mu}{N-\mu+2}} - 1\right) \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < \frac{\delta_0}{2}$ where δ_0 is defined in Lemma 5.5. Now for $0 < \omega < \omega_0$ choose e_{22} such that if $\|f\|_{H^{-1}} < e_{22}$ then $\frac{1}{2\omega(1-\omega)^{\frac{2N-\mu}{N-\mu+2}}}\|f\|_{H^{-1}}^2 < \frac{\delta_0}{2}$. Therefore, we obtain

$$\mathcal{J}\left(t^\star \frac{u}{\|u\|}\right) \leq \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \delta_0$$

Using Lemma 5.5 we conclude the result. □

Define $G : [\mathcal{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}] \to \mathbb{S}^{N-1}$ by

$$G(u) = \frac{\int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx}{\left| \int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx \right|}.$$

Note that from Lemma 5.5, G is well defined.

Lemma 5.8. *For $0 < \epsilon < \epsilon_0$ and $\|f\|_{H^{-1}} < e_{22}$, the map*

$$G \ o \ H_\epsilon : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$$

is homotopic to the identity.

Proof. Define $\mathcal{K} := \left\{ u \in H_0^1(\Omega) \setminus \{0\} \middle| \int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx \neq 0 \right\}$ and $\overline{G} : \mathcal{K} \to \mathbb{S}^{N-1}$ by

$$\overline{G}(u) = \int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx \Big/ \left| \int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 \, dx \right|$$

as an extension of G. This on using Lemma 4.1 and Lemma 5.5, gives $\int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla g_\rho^{\epsilon,\sigma}|^2 \, dx \neq 0$ for sufficiently small ϵ. Thus, $\overline{G}(g_\rho^{\epsilon,\sigma})$ is well defined. Now let $y : [s_1, s_2] \to \mathbb{S}^{N-1}$ be a regular geodesic between $\overline{G}(g_\rho^{\epsilon,\sigma})$ and $\overline{G}(H_\epsilon(\sigma))$ such that $y(s_1) = \overline{G}(g_\rho^{\epsilon,\sigma})$ and $y(s_2) = \overline{G}(H_\epsilon(\sigma))$. Moreover, by a analogous argument as in Lemma 4.1, for $\delta_0 > 0$ there exists a $\epsilon_0 > 0$ such that

$$\mathcal{J}(g_\rho^{2(1-\lambda)\epsilon}) < \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \delta_0 \text{ for all } 0 < \epsilon < \epsilon_0 \text{ and } \sigma \in \mathbb{S}^{N-1}, \lambda \in [\frac{1}{2}, 1),$$

where δ_0 is defined in Lemma 5.5. Now define $\varsigma_\epsilon(\lambda, \sigma) : [0,1] \times \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ by

$$\varsigma_\epsilon(\lambda, \sigma) = \begin{cases} y(2\lambda(s_1 - s_2) + s_2) & \text{if } \lambda \in [0, \frac{1}{2}), \\ \overline{G}(g_\rho^{2(1-\lambda)\epsilon}) & \text{if } \lambda \in [\frac{1}{2}, 1), \\ \sigma & \text{if } \lambda = 1. \end{cases}$$

Clearly, ς_ϵ is well defined. We claim that $\lim\limits_{\lambda \to 1^-} \varsigma_\epsilon(\lambda, \sigma) = \sigma$ and $\lim\limits_{\lambda \to \frac{1}{2}^-} \varsigma_\epsilon(\lambda, \sigma) = \overline{G}(g_\rho^{\epsilon,\sigma})$.

(i) $\lim\limits_{\lambda \to 1^-} \varsigma_\epsilon(\lambda, \sigma) = \sigma$: Indeed

$$\int\limits_{\mathbb{R}^N} \frac{x}{|x|} |\nabla g_\rho^{2(1-\lambda)\epsilon}|^2 \, dx = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \sigma + o(1) \text{ as } \lambda \to 1^-$$

then $\lim\limits_{\lambda \to 1^-} \varsigma_\epsilon(\lambda, \sigma) = \sigma$.

(b) $\lim\limits_{\lambda \to \frac{1}{2}^-} \varsigma_\epsilon(\lambda, \sigma) = \overline{G}(g_\rho^{\epsilon,\sigma})$: Indeed

$$\lim_{\lambda \to \frac{1}{2}^-} \varsigma_\epsilon(\lambda, \sigma) = \lim_{\lambda \to \frac{1}{2}^-} y(2\lambda(s_1 - s_2) + s_2) = y(s_1) = \overline{G}(g_\rho^{\epsilon,\sigma}).$$

Hence, $\varsigma_\epsilon \in C([0,1] \times \mathbb{S}^{N-1}, \mathbb{S}^{N-1})$ and $\varsigma_\epsilon(0, \sigma) = \overline{G}(H_\epsilon(\sigma))$ and $\varsigma_\epsilon(1, \sigma) = \sigma$ for $\sigma \in \mathbb{S}^{N-1}$ provided $0 < \epsilon < \epsilon_0$ and $\|f\|_{H^{-1}} < e_{22}$. Thus the result follows. □

Proposition 5.9. *Let $e^{\star} := \min\{e_{00}, e_{11}, e_{22}\}$ where e_{00}; e_{11} and e_{22} defined in Lemma 3.1, Lemma 5.4 and Lemma 5.7 respectively. Let $\|f\|_{H^{-1}} < e^{\star}$ then $\mathfrak{J}_f$ has two critical points in*

$$\left[\mathfrak{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}\right].$$

Equivalently, (P_f) have another two different solutions which are different from u_1.

Proof. Using Lemma 5.8 and Lemma 5.3, we have

$$cat\left(\left[\mathfrak{J}_f \leq \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta_\epsilon\right]\right) \geq 2.$$

Now the proof follows from Lemma 4.8(i) and Lemma 5.2. □

6 Existence of Fourth solution

In this section we will prove the existence of high energy solution by using Brouwer's degree theory and minmax theorem given by Brezis and Nirenberg [8].

Let $\mathcal{V} := \left\{u \in H_0^1(\Omega) : \int\limits_\Omega \int\limits_\Omega \frac{|u^+(x)|^{2_\mu^*}|u^+(y)|^{2_\mu^*}}{|x-y|^\mu}\, dxdy = 1\right\}$, $h_\rho^{\epsilon,\sigma}(x) = \dfrac{g_\rho^{\epsilon,\sigma}(x)}{\|g_\rho^{\epsilon,\sigma}\|_{NL}}$ where $g_\rho^{\epsilon,\sigma}$ is defined in (4.1).

Lemma 6.1. $\|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)} \to S_{H,L}$ *as $\epsilon \to 0$ uniformly in $\sigma \in \mathbb{S}^{N-1}$.*

Proof. Proof follows from Lemma 4.1*(i)*. □

Lemma 6.2. *There exists a $\rho_0 > 0$ such that for $0 < \rho < \rho_0$,* $\sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]} \|h_\rho^{\epsilon,\sigma}\|^2 < 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L}$.

Proof. Since we know that $\|\nabla u_\epsilon^\sigma\|^2_{L^2(\mathbb{R}^N)} = \|u_\epsilon^\sigma\|_{NL}^{2.2_\mu^*} = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ and this on using Lemma 4.2 we get $\sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]} \|h_\rho^{\epsilon,\sigma}\|^2 \to S_{H,L}$ as $\rho \to 0$. So there exists a ρ_0 such that $0 < \rho < \rho_0$, we obtain $\sup\limits_{\sigma\in\mathbb{S}^{N-1},\epsilon\in(0,1]} \|h_\rho^{\epsilon,\sigma}\|^2 < 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L}$. □

Now for any $u \in H_0^1(\Omega)$, by extending it to be zero outside Ω, we define Barycenter mapping $\beta : \mathcal{V} \to \mathbb{R}^N$ as

$$\beta(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{x|u^+(x)|^{2_\mu^*}|u^+(y)|^{2_\mu^*}}{|x-y|^\mu}\, dxdy,$$

and also let $\mathcal{Q} := \{u \in \mathcal{V} : \beta(u) = 0\}$.

Lemma 6.3. *There holds* $\lim\limits_{\epsilon\to 0} \beta(h_\rho^{\epsilon,\sigma}) = \sigma$.

Proof. If there exists $\eta > 0$ and a sequence $\epsilon_n \to 0^+$ such that $|\beta(h_\rho^{\epsilon_n}) - \sigma| \geq \eta$. Then

$$\beta(h_\rho^{\epsilon_n}) = \frac{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{x|h_\rho^{\epsilon_n}(x)|^{2^*_\mu}|h_\rho^{\epsilon_n}(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy}{\|h_\rho^{\epsilon_n}\|_{NL}^{2.2^*_\mu}}$$

$$= \sigma + \frac{\epsilon_n \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{(z-\sigma)|\upsilon_\rho(\epsilon_n z + (1-\epsilon_n)\sigma)|^{2^*_\mu}|\upsilon_\rho(\epsilon_n w + (1-\epsilon_n)\sigma)|^{2^*_\mu}}{|z-w|^\mu[1+|z|^2]^{\frac{2N-\mu}{2}}[1+|w|^2]^{\frac{2N-\mu}{2}}}\,dzdw}{\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|\upsilon_\rho(\epsilon_n z + (1-\epsilon_n)\sigma)|^{2^*_\mu}|\upsilon_\rho(\epsilon_n w + (1-\epsilon_n)\sigma)|^{2^*_\mu}}{|z-w|^\mu[1+|z|^2]^{\frac{2N-\mu}{2}}[1+|w|^2]^{\frac{2N-\mu}{2}}}\,dzdw}$$

$$\leq \sigma + \epsilon_n \sup_{z\in supp(\upsilon_\rho)} |z-\sigma| \leq \sigma + C\epsilon_n, \text{ for some } C > 0.$$

It implies that $0 < \eta \leq |\beta(h_\rho^{\epsilon_n}) - \sigma| \leq C\epsilon_n \to 0^+$ as $\epsilon_n \to 0^+$, a contradiction. ☐

Lemma 6.4. *Let* $m_0 = \inf\limits_{u\in\Omega} \|u\|^2$ *then* $S_{H,L} < m_0$.

Proof. Obviously $S_{H,L} \leq m_0$, so let if possible, $S_{H,L} = \inf\limits_{u\in\Omega} \|u\|^2$ then there exists a sequence $\{v_n\} \in H_0^1(\Omega)$ such that $\|v_n\|_{NL} = 1,\ \beta(v_n) = 0,\ \|v_n\|^2 \to S_{H,L}$ as $n \to \infty$. Setting $w_n = S_{H,L}^{\frac{N-2}{2(N-\mu+2)}} v_n$ we get $\|w_n\|_{NL}^{2.2^*_\mu} = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ and $\|w_n\|^2 \to S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. Therefore, $\mathcal{J}(w_n) \to \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ and $\mathcal{J}'(w_n)(w_n) = o(1)$. Using Lemma 2.9, we obtain $\{w_n\}$ is a Palais-Smale sequence of $\mathcal{J}$ at level $\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. Subsequently, by [19, Theorem 4.4] and Remark 2.8, there exist sequences $y_n \in \Omega,\ \lambda_n \in \mathbb{R}^+$ such that $y_n \to y_0 \in \overline{\Omega}$ and $\lambda_n \to 0$, for the functions

$$v_n = S_{H,L}^{-\frac{N-2}{2(N-\mu+2)}} w_n, \text{ where } w_n = C\left(\frac{\lambda_n}{\lambda_n^2 + |x-y_n|^2}\right)^{\frac{N-2}{2}} \text{ for some } C > 0.$$

Thus if

$$C_1 = C\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{z}{|z-w|^\mu[1+|z|^2]^{\frac{2N-\mu}{2}}[1+|w|^2]^{\frac{2N-\mu}{2}}}\,dzdw \quad \text{and}$$

$$C_2 = C\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|z-w|^\mu[1+|z|^2]^{\frac{2N-\mu}{2}}[1+|w|^2]^{\frac{2N-\mu}{2}}}\,dzdw,$$

then

$$0 = \beta(v_n) = C\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{x|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy = \lambda_n C_1 + y_n C_2 \to C_2 y_0.$$

This is a contradiction. Hence $S_{H,L} < m_0$. ☐

Lemma 6.5. *There exists* $\epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$ *and* $|\sigma| = 1$ *we have*

$$S_{H,L} < \|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)} < \frac{m_0 + S_{H,L}}{2}.$$

Proof. Apparently $S_{H,L} \leq \|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)}$ and we know that $S_{H,L}$ is not attained on a bounded domain. Thus, $S_{H,L} < \|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)}$. Since $S_{H,L} < m_0$, there exists δ_0 such that $\frac{S_{H,L}}{2} + \delta_0 < \frac{m_0}{2}$ and from Lemma 6.1 we know that $\|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)} \to S_{H,L}$ as $\epsilon \to 0$. Therefore for $\delta_0 > 0$ there exists a $\epsilon_0 > 0$ such that $\|h_\rho^{\epsilon,\sigma}\|^2_{D^{1,2}(\mathbb{R}^N)} < S_{H,L} + \delta_0$ whenever $0 < \epsilon < \epsilon_0$. Hence we have the desired result. ☐

Now we will state the minimax lemma given by Brezis and Nirenberg [8].

Lemma 6.6. *Let Y be a Banach space and $\phi \in C^1(Y, \mathbb{R})$. Let A be a compact metric space, $A_0 \subset A$ be a closed set and $y \in C(A_0, Y)$. Define*

$$\Gamma = \{g \in C(A, Y) : g(s) = y(s) \text{ if } s \in A_0\}, \quad \overline{c} = \inf_{g\in\Gamma} \sup_{s\in A} \phi(g(s)), \quad \hat{c} = \sup_{y(A_0)} \phi.$$

If $\overline{c} > \hat{c}$ then there exists a sequence $\{u_n\} \in Y$ satisfying $\phi(u_n) \to \overline{c}$ and $\phi'(u_n) \to 0$.. Further, if ϕ satisfies $(PS)_{\overline{c}}$ condition then there exists $u_0 \in Y$ such that $\phi(u_0) = \overline{c}$ and $\phi'(u_0) = 0$. □

Let $r_0 = 1 - \epsilon_0$ and $\overline{B}_{r_0} = \{(1-\epsilon)\sigma \in \mathbb{R}^N : |(1-\epsilon)\sigma| \leq r_0, \ \sigma \in \mathbb{S}^{N-1}, \ 0 < \epsilon \leq 1\}$, where ϵ_0 is defined in Lemma 6.5. Then we set $F = \{q \in C(\overline{B}_{r_0}, \mathcal{V}); \ q_{|\partial\overline{B}_{r_0}} = h_\rho^{\epsilon,\sigma}\}$ and

$$\overline{c} = \inf_{q\in F} \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \|q((1-\epsilon)\sigma)\|^2, \qquad \hat{c} = \sup_{\partial\overline{B}_{r_0}} \|h_\rho^{\epsilon,\sigma}\|^2$$

Lemma 6.7. *For each $q \in F$, we have $q(\overline{B}_{r_0}) \cap \mathcal{Q} \neq \emptyset$.*

Proof. It is enough to show there exist $\tilde{e} > 0$ and $\tilde{\sigma} \in \mathbb{S}^{N-1}$ such that $\beta(q((1-\tilde{e})\tilde{\sigma})) = 0$. Define $\psi : \overline{B}_{r_0} \to \mathbb{R}^N$ by $\psi((1-\epsilon)\sigma) = \beta(q((1-\epsilon)\sigma))$. We claim that

$$d(\psi, \overline{B}_{r_0}, 0) = d(I, \overline{B}_{r_0}, 0) \neq 0, \text{ where } d \text{ is Brouwer's topological degree.}$$

If $(1-\epsilon)\sigma \in \partial\overline{B}_{r_0}$ then $q((1-\epsilon)\sigma) = h_\rho^{\epsilon,\sigma}$ which implies

$$\psi((1-\epsilon)\sigma) = \beta(q((1-\epsilon)\sigma)) = \beta(h_\rho^{\epsilon,\sigma}) = \sigma + o(1) \text{ as } \epsilon \to 0.$$

Now define the homotopy $\mathcal{H} : [0,1] \times \overline{B}_{r_0} \to \mathbb{R}^N$ by

$$\mathcal{H}(t, (1-\epsilon)\sigma) = (1-t)\psi((1-\epsilon)\sigma) + tI((1-\epsilon)\sigma)$$

then for $(1-\epsilon)\sigma \in \partial\overline{B}_{r_0}$ and $t \in [0,1]$ we have

$$\begin{aligned}\mathcal{H}(t, (1-\epsilon)\sigma) &= (1-t)\sigma + o(1) + t(1-\epsilon_0)\sigma \\ &= o(1) + (1-\epsilon_0 t)\sigma \neq 0, \qquad \text{as } \epsilon \to 0.\end{aligned}$$

So by Brouwer's degree theory, claim holds. It implies that there exist $\tilde{e} > 0$ and $\tilde{\sigma} \in \mathbb{S}^{N-1}$ such that $\psi((1-\tilde{e})\tilde{\sigma}) = 0$ that is, $\beta(q((1-\tilde{e})\tilde{\sigma})) = 0$. □

Using above Lemma we have $m_0 \leq \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \|q((1-\epsilon)\sigma)\|^2$ for all $q \in F$. Hence

$$m_0 \leq \inf_{q\in F} \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \|q((1-\epsilon)\sigma)\|^2 = \overline{c}.$$

Also, by the definition of $\overline{c}$, and Lemma 6.2, we have $\overline{c} < 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L}$ for $0 < \rho < \rho_0$. Combining all these and using Lemma 6.4 we have

$$S_{H,L} < m_0 \leq \overline{c} < 2^{\frac{N-\mu+2}{2N-\mu}} S_{H,L} \text{ for } \rho \text{ sufficiently small}. \tag{6.1}$$

In addition, from Lemma 6.5, we get

$$\hat{c} = \sup_{\partial\overline{B}_{r_0}} \|h_\rho^{\epsilon,\sigma}\|^2 < \frac{m_0 + S_{H,L}}{2} < m_0 \leq \overline{c}.$$

Now we define

$$\begin{aligned}&\breve{\mathcal{J}}_f(u) = \max_{t>0} \mathcal{J}_f(tu) : \mathcal{V} \to \mathbb{R}^N \quad \text{and} \quad \breve{\mathcal{J}}(u) = \max_{t>0} \mathcal{J}(tu) : \mathcal{V} \to \mathbb{R}^N, \\ &y_f = \inf_{q\in F} \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \breve{\mathcal{J}}_f(q((1-\epsilon)\sigma)) \quad \text{and} \quad y_0 = \inf_{q\in F} \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \breve{\mathcal{J}}(q((1-\epsilon)\sigma)).\end{aligned}$$

We remark that the conclusion of Lemma 5.4 (iii) holds true for $\breve{\mathcal{J}}_f$. Moreover, $\breve{\mathcal{J}}_f(u) = \max_{t>0} \mathcal{J}_f(tu) = \mathcal{J}_f(t^-(u)u)$, where $t^-(u)$ is defined in Lemma 3.3.

Lemma 6.8. *The following holds:*

(i) $\breve{\mathcal{J}}_f \in C^1(\mathcal{V}, \mathbb{R})$ *and* $\langle \breve{\mathcal{J}}_f'(u), h\rangle = t^-(u)\langle \mathcal{J}_f'(t^-(u)u), h\rangle$ *for all* $h \in T_u(\mathcal{V})$ *where*

$$T_u(\mathcal{V}) := \left\{ h \in H_0^1(\Omega) \middle| \int_\Omega \int_\Omega \frac{|u^+(x)|^{2^*_\mu}|u^+(y)|^{2^*_\mu-1}h(y)}{|x-y|^\mu} = 0 \right\}.$$

(ii) If $u \in \mathcal{V}$ *is a critical point of* $\breve{\mathcal{J}}_f$ *then* $t^-(u)u \in \mathcal{N}_f^-$ *is a critical point of* $\mathcal{J}_f$.

(iii) If $\{u_n\}_{n\in\mathbb{N}}$ *is a* $(PS)_c$ *sequence of* $\breve{\mathcal{J}}_f$ *then* $\{t^-(u_n)u_n\}_{n\in\mathbb{N}} \in \mathcal{N}_f^-$ *is a* $(PS)_c$ *sequence for* $\mathcal{J}_f$.

Proof. (i) For every $u \in H_0^1(\Omega)$, $t^-(u)u \in \mathcal{N}_f^-$ that is, $\langle \mathcal{J}_f'(t^-(u)u), u\rangle = 0$ and $\frac{d^2}{dt^2}\Big|_{t=t^-(u)} \mathcal{J}_f(tu) < 0$. Therefore, by implicit function theorem, we get $t^-(u) \in C^1(\mathcal{V}, (0,\infty))$. As a result, $\breve{\mathcal{J}}_f(u) = \mathcal{J}_f(t^-(u)u) \in C^1(\mathcal{V}, \mathbb{R})$ and for all $h \in T_u(\mathcal{V})$, we have

$$\langle \breve{\mathcal{J}}_f'(u), h\rangle = t^-(u)\langle \mathcal{J}_f'(t^-(u)u), h\rangle + \langle \mathcal{J}_f'(t^-(u)u), u\rangle\langle (t^-(u))', h\rangle = t^-(u)\langle \mathcal{J}_f'(t^-(u)u), h\rangle.$$

(ii) Combining the fact that $u \in \mathcal{V}$ is a critical point of $\breve{\mathcal{J}}_f$ and $\langle \mathcal{J}_f'(t^-(u)u), u\rangle = 0$, we get the desired result.
(iii) Let $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence of $\breve{\mathcal{J}}_f$, that is, $u_n \in \mathcal{V}$, $\breve{\mathcal{J}}_f(u_n) \to c$ and

$$\|\breve{\mathcal{J}}_f'(u)\|_{T^*_{u_n}(\mathcal{V})} = \sup\{|\langle \breve{\mathcal{J}}_f'(u_n), h\rangle| : h \in T_{u_n}(\mathcal{V}), \|h\| = 1\} \to 0 \text{ as } n \to \infty.$$

By Lemma 3.3 we have $t^-(u_n) > \left(\frac{\|u\|^2}{2.2^*_\mu-1}\right)^{\frac{1}{2.2^*_\mu-2}} > \left(\frac{S_{H,L}}{2.2^*_\mu-1}\right)^{\frac{1}{2.2^*_\mu-2}} > C$ for some $C > 0$. Since $H_0^1(\Omega) = R_{u_n} \oplus T_{u_n}(\mathcal{V})$ so $\langle \mathcal{J}_f'(u_n), v\rangle = \langle \mathcal{J}_f'(u_n), h_v\rangle$, where h_v is the projection of v in $T_{u_n}(\mathcal{V})$. Hence,

$$\begin{aligned}\|\mathcal{J}_f'(t^-(u_n)u_n)\| &= \sup_{v\in H_0^1(\Omega), \|v\|=1} |\langle \mathcal{J}_f'(t^-(u_n)u_n), v\rangle| \\ &= \sup_{v\in H_0^1(\Omega), \|v\|=1} |\langle \mathcal{J}_f'(t^-(u_n)u_n), h_v\rangle| \\ &= \sup_{v\in H_0^1(\Omega), \|v\|=1} \frac{1}{t^-(u_n)}|\langle \breve{\mathcal{J}}_f'(u_n), h_v\rangle| \le \frac{1}{C}\|\breve{\mathcal{J}}_f'(u)\|_{T^*_{u_n}(\mathcal{V})} \to 0.\end{aligned}$$

Clearly, $\mathcal{J}_f(t^-(u_n)u_n) \to c$. Therefore, $\{t^-(u_n)u_n\}_{n\in\mathbb{N}} \in \mathcal{N}_f^-$ is a $(PS)_c$ sequence for $\mathcal{J}_f$. □

Lemma 6.9. *If* $0 < \rho < \rho_0$, *then* $\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < y_0 < \frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$.

Proof. For $u \in \mathcal{V}$, solving $\mathcal{J}'(tu) = t\, a(u) - t^{2.2^*_\mu-1} = 0$ we get $t = 0$ and $t = (a(u))^{\frac{1}{2.2^*_\mu-2}}$. Therefore,

$$\breve{\mathcal{J}}(u) = \max_{t>0} \mathcal{J}(tu) = \frac{N-\mu+2}{2(2N-\mu)}\|u\|^{\frac{2(2N-\mu)}{N-\mu+2}}.$$

From the definition of $\bar{c}$, we obtain

$$y_0 = \frac{N-\mu+2}{2(2N-\mu)} \inf_{q\in F} \sup_{(1-\epsilon)\sigma\in\overline{B}_{r_0}} \|q((1-\epsilon)\sigma)\|^{\frac{2(2N-\mu)}{N-\mu+2}} = \frac{N-\mu+2}{2(2N-\mu)} \bar{c}^{\frac{2N-\mu}{N-\mu+2}}$$

which on using (6.1) yields the desired result. □

Lemma 6.10. $\breve{\mathcal{J}}_f(h_\rho^{\epsilon,\sigma}) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o(1)$ *as* $\epsilon \to 0$.

Proof. By Lemma 4.1, $h_\rho^{\epsilon,\sigma} \rightharpoonup 0$ in $H_0^1(\Omega)$ as $\epsilon \to 0$. On solving

$$\mathcal{J}_f'(th_\rho^{\epsilon,\sigma}) = t\, a(h_\rho^{\epsilon,\sigma}) - t^{2.2^*_\mu-1} - \int_\Omega f h_\rho^{\epsilon,\sigma}\, dx = 0,$$

we conclude $t_f = \|g_\rho^{\epsilon,\sigma}\|_{NL} + o(1)$. Hence again from the Lemma 4.1 we obtain

$$\breve{\mathcal{J}}_f(h_\rho^{\epsilon,\sigma}) = \max_{t>0} \mathcal{J}_f(th_\rho^{\epsilon,\sigma}) = \mathcal{J}_f(t_f h_\rho^{\epsilon,\sigma}) = \mathcal{J}_f(g_\rho^{\epsilon,\sigma}) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o(1) \text{ as } \epsilon \to 0.$$ □

Lemma 6.11. *There exists $e_0^\star > 0$ such that if $0 < \|f\|_{H^{-1}} < e_0^\star$,*

$$\Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < y_f < \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Proof. Analogous to the proof of Lemma 5.4(iii) we can have

$$(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathfrak{J}(t^\star u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \mathfrak{J}_f(t^- u) \leq (1+\omega)^{\frac{2N-\mu}{N-\mu+2}} \mathfrak{J}(t^\star u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.$$

Using the above inequality with the definition of $\breve{\mathfrak{J}}$ and $\breve{\mathfrak{J}}_f$, we get

$$(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} \breve{\mathfrak{J}}(u) - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \leq \breve{\mathfrak{J}}_f(u) \leq (1+\omega)^{\frac{2N-\mu}{N-\mu+2}} \breve{\mathfrak{J}}(u) + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.$$

For $\delta > 0$ there exists $e_1(\delta)$ such that if $\|f\|_{H^{-1}} < e_1(\delta)$ then

$$y_0 - \delta < y_f < y_0 + \delta. \tag{6.2}$$

Now from Lemma 5.4(iii) for each $0 < \omega < 1$, we have

$$\begin{aligned}(1-\omega)^{\frac{2N-\mu}{N-\mu+2}} &\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \frac{1}{2\omega}\|f\|_{H^{-1}}^2 \\ &\leq \Upsilon_f^-(\Omega) \leq (1+\omega)^{\frac{2N-\mu}{N-\mu+2}} \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \frac{1}{2\omega}\|f\|_{H^{-1}}^2.\end{aligned}$$

So for $\delta > 0$ there exists $e_2(\delta) > 0$ such that whenever $\|f\|_{H^{-1}} < e_2(\delta)$ then

$$\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta \leq \Upsilon_f^-(\Omega) \leq \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \delta.$$

It implies

$$\frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \delta \leq \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leq \frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \delta. \tag{6.3}$$

Moreover, from Lemma 6.9

$$\frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < y_0 < \frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Hence for fix small $0 < \epsilon < \min\left\{ \frac{\frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - y_0}{2}, y_0 - \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \right\}$ such that if $\|f\|_{H^{-1}} < e_0^\star = \min\{e_2(\epsilon),\ e_2(\epsilon)\}$ then using (6.2) and (6.3), we obtain

$$\begin{aligned}&\Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < y_0 - \epsilon \leq y_f \quad \text{and} \\ &y_f < y_0 + 2\epsilon - \epsilon < \frac{N-\mu+2}{2N-\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - \epsilon \leq \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.\end{aligned}$$

That is, $\Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} < y_f < \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. □

Proposition 6.12. *If $0 < \rho < \rho_0$, $0 < \|f\|_{H^{-1}} < e_0^\star$ (defined in Lemma 6.11) then there exists a critical point $u_4 \in \mathcal{N}_f^-$ of $\mathfrak{J}_f$ with $\mathfrak{J}_f(u_4) = y_f$.*

Proof. Let $c \in \left(\Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}},\ \Upsilon_f^-(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}\right)$ and $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence of $\breve{\mathfrak{J}}_f$. Then by Lemma 6.8, $\{t^-(u_n)u_n\}_{n\in\mathbb{N}} \in \mathcal{N}_f^-$ is a $(PS)_c$ sequence for $\mathfrak{J}_f$ which on using Lemma 4.8 gives that $\{u_n\}_{n\in\mathbb{N}}$ is compact. Moreover, from Lemma 6.10, $y_f > \breve{\mathfrak{J}}_f(h_\rho^{\epsilon,\sigma}) = \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + o(1)$ as ϵ sufficiently small. Using Lemma 6.6 we have y_f is a critical value of $\breve{\mathfrak{J}}_f$. Therefore, there exists $v_4 \in \mathcal{V}$ such that $\breve{\mathfrak{J}}_f(v_4) = y_f$ and $\breve{\mathfrak{J}}_f'(v_4) = 0$. Thus by Lemma 6.8, $u_4 := t^-(v_4)v_4 \in \mathcal{N}_f^-$ is a critical point of $\mathfrak{J}_f$ and $\mathfrak{J}_f(u_4) = y_f$. □

Proof of Theorem 1.1 : First note that by Lemma 3.8, we have all solutions of (P_f) are positive in Ω and from Lemma 3.7, we have $u_1 \in \mathcal{N}_f^+ \subset H_0^1(\Omega)$ such that $\mathcal{J}_f(u_1) = \Upsilon_f$ whenever $0 < \|f\|_{H^{-1}} < e_{00}$. By Proposition 5.9 we have two more critical point $u_2, u_3 \in \mathcal{N}_f^-$ of $\mathcal{J}_f$ such that in $\mathcal{J}_f(u_2), \mathcal{J}_f(u_3) < \Upsilon_f(\Omega) + \frac{N-\mu+2}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. Therefore we get three positive solutions of (P_f) whenever $0 < \|f\|_{H^{-1}} < e^*$ where e^* is defined in Proposition 5.9. Let $e^{**} = \min\{e^*, e_0^*\}$ then by Proposition 6.12, we get $u_4 \in \mathcal{N}_f^-$ $\mathcal{J}_f(u_4) = y_f$. □

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