M. Abdelwahed, N. Chorfi*, and R. Malek

Reconstruction of Tesla micro-valve using topological sensitivity analysis

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Abstract: In this paper, we deal with topology optimization attributed to the non stationary Navier-Stokes equations. We propose an approach where we analyze the sensitivity of a shape function relating to a perturbation of the flow domain. A numerical optimization algorithm based on topological gradient method is built and applied to the 2D Tesla micro valve reconstruction. Some numerical results confirm the efficiency of the proposed approach.

Keywords: Non-stationary Navier-Stokes equations, Tesla valve, Shape optimization, Topological sensitivity method

1 Introduction

Tesla valves are no-moving-part valves that utilize fluidic inertial forces to inhibit flow in the reverse direction. It was patented in 1920 by Nikola Tesla as a "Valvular conduit" [1] (see figure 1), and has since made the subject of various applications in micro-satellite [2], drug delivery [3], microbiology [4, 5] and hydrocephalus treatment in medicine [6, 7].

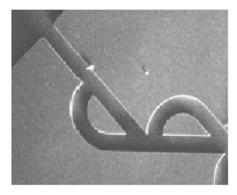


Fig. 1: A rotated scanning electron microscope photograph of a Tesla valve by Forster et al. [8]

The Tesla micro-valve performance is evaluated by the diodicity parameter (represents the ratio of the pressure drop in backward and forward direction) which evaluates the ability of allowing forward flow while

M. Abdelwahed, Department of Mathematics, College of Sciences, King Saud University, Saudi Arabia

^{*}Corresponding Author: N. Chorfi, Department of Mathematics, College of Sciences, King Saud University, Saudi Arabia, E-mail: nchorfi@ksu.edu.sa

R. Malek, FSM, Department of Mathematics, Monastir University, Tunisia

inhibiting the reverse one,

$$Di = \frac{\Delta p_{backward}}{\Delta p_{forward}}.$$

Different works have focused on the optimal shape of the tesla micro-valve. However, the majority of works concerns stationary Partial Differential Equations (PDE). Forster et al. [8] proved the possibility of using Tesla valves in micro-fluidics and determined experimentally the diodicity for Reynolds number (Re) \simeq 180. Truong et al. in [9] derived numerically the optimum geometry of Tesla valve for 100 < Re < 600 with better diodicity than [8]. Bardell et al [10] analyzed the mechanism of the diodicity and proposed a Tesla valve optimal design for low Re. In the case when Re = 100, they finished with Di = 1.4. Gamboa et al. [11] optimized the shape of Tesla valve for application with piezoactuated plenums. The obtained fluid domain related to Re = 100 is characterized by a diodicity number Di = 1.1. In 2008, Pingen et al. [12] used the Lattice Bolzmann Method for the optimization of a micro Tesla valve without any information on diodicity. The used objective function was the pressure drop between inlet and outlet. After that in 2010, Lin et al. [13] used a topology optimization technique based on the power dissipation energy [14] of forward flow as objective function and the diodicity was built into the model as a constraint. For Re = 100, they found a new design of Tesla valve given Di = 1.2. Next in 2015, Lin et al. [15] solved the Tesla valve topology optimization using the approach of material distribution with inverse diodicity as objective function and fluid volume fraction as the constraint.

Until recently, there were no investigations dealing with the non-stationary case. We propose in this paper a new reconstruction method using the sensitivity analysis approach [16–19] for a non stationary flow.

The principal results of this work concern both theoretical and numerical aspects associated with the Tesla micro-valve problem. The theoretical part is related to the analysis of the topological sensitivity for the non stationary Navier-Stokes equations. The numerical part concerns the 2D optimization of the Tesla micro-valve shape. The optimal shape is constructed by inserting obstacles in the considered initial domain. We build a simple and fast numerical reconstruction algorithm based on the topological gradient technique. The efficiency of the presented approach is confirmed by some numerical tests.

The paper is presented as following: Firstly we formulate the problem in section 2. Section 3 concerns the theoretical aspects. The numerical aspects are given in section 4. Finally section 5 includes Theorems proofs.

2 Problem formulation

Let $\Omega \subset \mathbb{R}^d$, d=2, 3 a bounded domain with regular boundary $\Gamma=\partial\Omega$. We consider the blood as an incompressible viscous fluid flow described by the non stationary Navier-Stokes equations [20]. The velocity w and the pressure p satisfy the following system:

$$\begin{cases} \frac{\partial w}{\partial t} + \nabla w \cdot w - \nu \Delta w + \nabla p &= G & \text{in} \quad \Omega \times]0, T[, \\ \text{div } w &= 0 & \text{in} \quad \Omega \times]0, T[, \\ w &= w_d & \text{on} \quad \Gamma \times]0, T[, \\ w(., 0) &= 0 & \text{in} \quad \Omega, \end{cases}$$
(1)

where v is the kinematic viscosity coefficient, G is the gravitational force, T is the computational time and w_d is a given Dirichlet boundary data. Because of the divergence free condition on w, w_d must necessarily satisfy the compatibility condition,

$$\int_{\Gamma} w_d(x, t).nds(x) = 0, \quad \text{a.e. } t \in]0, T[$$

where n is the unit outward normal vector along Γ .

Remark 2.1. Problem (1) has at least one solution (see [21](Ch.II,eq.(1.89)). If $|w|_{1,\Omega} < v/k$, with

$$k = \begin{cases} \frac{2\sqrt{2}}{3} meas(\Omega)^{1/6} & \text{if} \quad d = 3, \\ \frac{1}{2} meas(\Omega)^{1/2} & \text{if} \quad d = 2, \end{cases}$$

then problem (1) has a unique solution (see [17]).

The topological sensitivity method idea is to study the variation of a given shape function j relating to a perturbation in the fluid flow domain geometry.

In structural shape optimization case (respectively electromagnetism and fluid dynamics cases) a geometry perturbation means removing some material (respectively the insertion of an obstacle).

Let $\mathcal{O}_{z,\mathcal{E}} = z + \varepsilon \mathcal{O}$, a small obstacle inserted in Ω characterized by its center z, its size ε and its shape \mathcal{O} . $\mathbb O$ is a bounded domain of $\mathbb R^d$ containing the origin and $\partial \mathbb O$ (its boundary) is connected and piecewise $\mathbb C^1$.

The shape function variation is written

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) - j(\Omega) = \rho(\varepsilon)\delta j(z) + o(\rho(\varepsilon)), \ \forall z \in \Omega,$$

where

- $\varepsilon \mapsto \rho(\varepsilon)$, a positive scalar function going to zero with ε
- $z \mapsto \delta j(z)$, called the topological gradient, describes the shape function variation when an obstacle is inserted in z. It plays the role of descent direction in the algorithm of optimization.

To our knowledge, the majority of works leading with topological sensitivity method concern the stationary case such as Stokes problem [16, 18], quasi-Stokes [19], stationary Navier Stokes problem [17]. We extend this method to the nonlinear unsteady Navier Stokes flow. To overcome the difficulty due to the non linear operator and its associated adjoint problem we extend the perturbed velocity by zero in the inclusion which permits to use the adjoint method in the whole domain. For the time dependent term we will use the fundamental solution of the non stationary Stokes operator and decompose the velocity variation.

We define the time dependent shape function as:

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\varepsilon}}) = \int_{0}^{T} J_{\varepsilon}(w_{\varepsilon}(., t)) dt, \qquad (2)$$

where I_{ε} in $H^1(\Omega \setminus \overline{\mathbb{O}_{z,\varepsilon}})^d$ and w_{ε} is solution to

$$\begin{cases}
\frac{\partial w_{\varepsilon}}{\partial t} + \nabla w_{\varepsilon} \cdot w_{\varepsilon} - \nu \Delta w_{\varepsilon} + \nabla p_{\varepsilon} &= G & \text{in} & \Omega_{z, \varepsilon} \times]0, T[, \\
\text{div } w_{\varepsilon} &= 0 & \text{in} & \Omega_{z, \varepsilon} \times]0, T[, \\
w_{\varepsilon} &= w_{d} & \text{on} & \Gamma \times]0, T[, \\
w_{\varepsilon} &= 0 & \text{on} & \partial O_{z, \varepsilon} \times]0, T[, \\
w_{\varepsilon}(., 0) &= 0 & \text{in} & \Omega_{z, \varepsilon},
\end{cases} \tag{3}$$

with $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}$ is the perturbed domain. Note that if $\varepsilon = 0$ (without obstacle), (w_0, p_0) verify (1) and $\Omega_0 = \Omega$.

In the following, we will derive a general mathematical analysis for $I_{\mathcal{E}}$ satisfying the following assump-

Assumption (A)

- i) $\forall \varepsilon \geq 0, t \mapsto J_{\varepsilon}(w_{\varepsilon}(., t)) \in L^{1}(0, T).$
- *ii)* J_0 is differentiable in $H^1(\Omega)$ and we denote $DJ_0(w)$ its derivative.
- *iii)* $\exists \rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $\delta \mathcal{J} \in \mathbb{R}$ such that $\forall \epsilon \geq 0$

$$\int_{0}^{T} \left[J_{\varepsilon}(w_{\varepsilon}(.,t)) - J_{0}(w_{0}(.,t)) \right] dt = \int_{0}^{T} DJ_{0}(w_{0}(.,t)) \left(w_{\varepsilon}(.,t) - w_{0}(.,t) \right) dt + \rho(\varepsilon) \delta \mathcal{J} + o(\rho(\varepsilon)).$$

3 Main results

We deal in this section with the non stationary Navier-Stokes topological sensitivity relating to the domain perturbation. We consider the shape functions verifying the assumption (A).

3.1 Asymptotic behavior of the velocity variation

We first study the influence on the velocity $v_{\varepsilon} = w_{\varepsilon} - w_0$ of inserting a small obstacle $O_{z,\varepsilon}$ in Ω . From (1) and (3), it is straightforward to show that $(v_{\varepsilon}, p_{v_{\varepsilon}})$ satisfy the system

$$\begin{cases} \frac{\partial v_{\varepsilon}}{\partial t} + \nabla v_{\varepsilon} \cdot v_{\varepsilon} - v \Delta v_{\varepsilon} + \nabla v_{\varepsilon} \cdot w_{0} + \nabla w_{0} \cdot v_{\varepsilon} + \nabla p_{v_{\varepsilon}} &= 0 & \text{in} \quad \Omega_{z,\varepsilon} \times]0, T[, \\ \text{div } v_{\varepsilon} &= 0 & \text{in} \quad \Omega_{z,\varepsilon} \times]0, T[, \\ v_{\varepsilon} &= 0 & \text{on} \quad \Gamma \times]0, T[, \\ v_{\varepsilon} &= -w_{0} & \text{on} \quad \partial O_{z,\varepsilon} \times]0, T[, \\ v_{\varepsilon}(.,0) &= 0 & \text{in} \quad \Omega_{z,\varepsilon}. \end{cases}$$
(4)

We will distinguish in the following the 2D and 3D cases.

3.1.1 Three dimensional case

Theorem 3.1. There exists c > 0 independent of ε , such that

$$\parallel v_{\varepsilon}(x,t) - W(x,t) \parallel_{L^{2}(0,T; H^{1}(\Omega_{z,\varepsilon}))} \leq c\varepsilon,$$

where $W = (W^1, W^2, W^3) \in H^1(\Omega_{z, \varepsilon})^3$ is defined by

$$W^{j}(x,t) = U^{j}(\frac{x-z}{\varepsilon}).w_{0}(z,t), \ \forall (x,t) \in \mathbb{R}^{3} \backslash \overline{\mathbb{O}}_{\varepsilon} \times]0, \ T[,$$
 (5)

with U^{j} is solution of (exterior Stokes problem)

$$\begin{cases}
-\nu \Delta U^{j} + \nabla P^{j} = 0 & \text{in} \quad \mathbb{R}^{3} \backslash \overline{\mathbb{O}}, \\
\text{div } U^{j} = 0 & \text{in} \quad \mathbb{R}^{3} \backslash \overline{\mathbb{O}}, \\
U^{j} \longrightarrow 0 & \text{at} \quad \infty, \\
U^{j} = -e_{j} & \text{on} \quad \partial \mathbb{O},
\end{cases} \tag{6}$$

with $\{e_i\}_{i=1,2,3}$ is the \mathbb{R}^3 canonical basis.

We show by using a single layer potential (see [22]) that

$$U^{j}(y) = \int_{\partial \mathcal{O}} E(y-x) \, \eta_{j}(x) \, ds(x), \quad P^{j}(y) = \int_{\partial \mathcal{O}} \Pi(y-x) \, \eta_{j}(x) \, ds(x), \ \forall y \in \mathbb{R}^{3} \setminus \overline{\mathcal{O}}.$$

where

$$E(y) = \frac{1}{8\pi\nu r}(I + e_r e_r^T), \quad \Pi(y) = \frac{y}{4\pi r^3} \ \forall \, y \in \mathbb{R}^3.$$

with r = ||y||, $e_r = \frac{y}{r}$, e_r^T is the transpose of e_r and $\eta_j \in H^{-1/2}(\partial \mathcal{O})^3$ is a solution of the boundary integral equation

$$\int_{\partial \Omega} E(y-x) \, \eta_j(x) \, ds(x) = -e_j, \quad \forall \, y \in \partial \Omega. \tag{7}$$

Using Theorem 3.1 we obtain the following corollary.

Corollary 3.2. We have

$$v_{\varepsilon}(x, t) = W(x, t) + O(\varepsilon), \quad x \in \Omega_{z, \varepsilon}, t \in]0, T[.$$

3.1.2 Two dimensional case

Theorem 3.3. There exists c > 0 independent on ε , verifying

$$\| v_{\varepsilon}(x,t) - \frac{1}{\log(\varepsilon)} W(x,t) \|_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))} \le \frac{-c}{\log(\varepsilon)},$$

where

$$W(x, t) = 4\pi v \sum_{j=1}^{2} [E^{j}(x - z)w_{0}(z, t)]e_{j}, \quad \forall (x, t) \in \Omega_{z, \varepsilon} \times]0, T[,$$
 (8)

with $E^j(y) = E(y)e_j$, $1 \le j \le 2$, $\{e_j\}_{j=1,2}$ is the \mathbb{R}^2 canonical basis and

$$E(y) = \frac{1}{4\pi \nu} \left(-\log(r)I + e_r e_r^T \right), \ \Pi(y) = \frac{y}{2\pi r^2}, \quad \forall y \in \mathbb{R}^2,$$

represents the fundamental solution of the Stokes System in \mathbb{R}^2 with r = ||y|| and $e_r = \frac{y}{x}$.

Using Theorem 3.3 it follows the velocity estimation in the perturbed fluid flow domain.

Corollary 3.4. We have

$$v_{\varepsilon}(x,t) = W(x,t) + O(\frac{-1}{\log(\varepsilon)}), \quad x \in \Omega_{z,\varepsilon}, \ t > 0.$$

3.2 Asymptotic behavior of the shape function

The topological sensitivity analysis for the non stationary Navier-Stokes operator in three and two dimensional cases is given in this section. The presented results are satisfied by all shape functions *j* defined by (2) and J_{ε} verifies the Assumption (A).

3.2.1 Three dimensional case

Theorem 3.5. If J_{ε} satisfies the Assumption (A) with $\rho(\varepsilon) = \varepsilon$, then j defined by (2) verifies

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) = j(\Omega) + \varepsilon \left[\int_{0}^{T} w_{0}(z, t) . \mathcal{M}_{\mathcal{O}} u_{0}(z, t) dt + \delta \mathcal{J} \right] + o(\varepsilon),$$

where

the matrix M_{\odot} is given by

$$\mathcal{M}_{\mathcal{O}_{ij}} = \int_{\partial \mathcal{O}} \eta_j^i(y) ds(y), \ 1 \leq i, \ j \leq 3.$$

 u_0 is the solution to the adjoint problem

$$\begin{cases}
-\frac{\partial u_{0}}{\partial t} - \nabla u_{0} \cdot w_{0} + \nabla w_{0}^{T} \cdot u_{0} - \nu \Delta u_{0} + \nabla p_{u_{0}} &= -DJ_{0}(w_{0}) & \text{in} \quad \Omega \times]0, T[, \\
div u_{0} &= 0 & \text{in} \quad \Omega \times]0, T[, \\
u_{0} &= 0 & \text{on} \quad \Gamma \times]0, T[, \\
u_{0}(., T) &= 0 & \text{in} \quad \Omega.
\end{cases}$$
(9)

Corollary 3.6. If O = B(0, 1) (the unit ball), $\eta_j(y) = -\frac{3\nu}{2}e_j$, $\forall y \in \partial O$ and

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\,\varepsilon}}) = j(\Omega) + \varepsilon \left[\int_{0}^{T} 6\pi v w_{0}(z,\,t).u_{0}(z,\,t)\,dt + \delta \mathcal{J} \right] + o(\varepsilon).$$

3.2.2 Two dimensional case

Theorem 3.7. If J_{ε} satisfies the Assumption (A) then j defined by (2) verifies

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) = j(\Omega) + \frac{-1}{\log(\varepsilon)} \left[4\pi v \int_{0}^{T} w_0(z, t) u_0(z, t) dt + \delta \mathcal{J} \right] + o\left(\frac{-1}{\log(\varepsilon)}\right),$$

where u_0 is the adjoint state solution to the problem (9).

The proofs of Theorems 3.1, 3.3, 3.5 and 3.7 are relegated to section 5. The variation $\delta \mathcal{J}$ depends on the shape functions expressions. Some useful examples in numerical applications will be presented in section 3.3.

3.3 Shape function examples

3.3.1 First example

We define the shape function

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\,\varepsilon}}) = \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |w_{\varepsilon} - W_{d}(.,\,t)|^{2} dxdt$$

where $W_d \in L^1(0, T; H^1(\Omega))$ is a datum representing a desired fluid flow state.

This example concerns the L^2 -norm shape function that has been used in geometric control problems like the optimization of location of some obstacle in a tank to approximate an object flow W_d (see [16]).

Proposition 3.8. The function

$$J_{\varepsilon}(w) = \int_{\Omega_{z,\varepsilon}} |w - W_d(., t)|^2 dx, \quad \forall w \in H^1(\Omega_{z,\varepsilon}),$$

satisfies the assumption (A) with

$$DJ_0(w_0(., t))v = 2 \int_{\Omega} (w_0(., t) - W_d(., t)) v dx, \quad \forall v \in H^1(\Omega),$$

$$\delta J(z) = 0, \quad \forall z \in \Omega.$$

3.3.2 Second example

We define the shape function which corresponds to the dissipation energy minimization

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\varepsilon}}) = \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |\nabla w_{\varepsilon} - \nabla W_{d}(., t)|^{2} dx dt,$$

where $W_d \in L^1(0, T; H^2(\Omega))$ is a given datum. It was used in several optimization problems such as minimum drag problem [23], pipe bend design [10?], cavity example [24], reconstruction of Tesla valve [13].

Proposition 3.9. The function

$$J_{\varepsilon}(w) = \int_{\Omega_{z,\varepsilon}} v \left| \nabla w - \nabla W_d(., t) \right|^2 dx, \quad \forall w \in H^1(\Omega_{z,\varepsilon}),$$

satisfies the assumption (A) with

$$DJ_0(w_0(., t))v = 2v \int_{\Omega} (\nabla w_0(., t) - \nabla W_d(., t)) \nabla v \, dx, \quad \forall v \in H^1(\Omega),$$

$$\delta J(z) = \begin{cases} 4\pi v \int_{\Omega}^T |w_0(z, t)|^2 \, dt, & \forall z \in \Omega & \text{if } d = 2, \\ -\int_{\Omega} \left(\int_{\Omega} \eta(y) ds(y) \right) . w_0(z, t) \, dt, & \forall z \in \Omega & \text{if } d = 3. \end{cases}$$

4 Numerical results

In this section, we deal with some numerical applications to validate the obtained theoretical results given in section 3.

4.1 Validation of the asymptotic expansion

To establish the numerical validation of Theorem 3.7, we consider the variation relating to ε of

$$\Delta_z(\varepsilon) = j(\Omega \setminus \overline{\mathcal{O}_{z,\,\varepsilon}}) - j(\Omega) + \frac{1}{\log(\varepsilon)} \delta j(z),$$

where

$$\delta j(z) = 4\pi \nu \int_{0}^{T} w_0(z, t) u_0(z, t) dt + \delta \Im.$$
 (10)

We expect to prove numerically that $\Delta_z(\varepsilon)$ satisfies the previously derived theoretical estimate $\Delta_z(\varepsilon) = o(\frac{-1}{\log(\varepsilon)})$.

To this aim, we consider the following data:

- $-\Omega =]0, 1[\times]0, 1[$ is a square domain.
- The locations $z_{\varepsilon}^i = z_i + \varepsilon B(0, 1)$ of the considered obstacles are arbitrary chosen (see Table 1).
- The shape function *j* is defined by the semi-norm

$$j(\Omega_{z,\,\varepsilon}) = \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |\nabla w_{\varepsilon} - \nabla W_{d}|^{2} dx dt, \tag{11}$$

where W_d is a given velocity state.

Table 1: Location of obstacles

obstacle $z^i_{arepsilon}$	$z^1_{arepsilon}$	$z_arepsilon^2$	$z_{arepsilon}^3$	$z_{arepsilon}^4$
location z_i	$z_1 = (0.2, 0.8)$	$z_2 = (0.8, 0.2)$	$z_3 = (0.5, 0.5)$	$z_4 = (0.7, 0.7)$

In this case, the function $\Delta_{z_i}(\varepsilon)$ is defined by (see Theorem 3.7 and Proposition 3.9):

$$\Delta_{z_i}(\varepsilon) = j(\Omega \setminus \overline{\mathcal{O}_{\varepsilon}^i}) - j(\Omega) + \frac{4\pi\nu}{\log(\varepsilon)} \Big(\int_0^T w_0(z_i, t) u_0(z_i, t) dt + \int_0^T |w_0(z_i, t)|^2 dt \Big).$$

The validation algorithm uses the following steps:

The validation algorithm:

- Step 1:
 - compute the solution w_0 and the associated adjoint state u_0 in the domain Ω .
 - determine $j(\Omega)$ defined by (11).
- Step 2: For each obstacle $z_{\varepsilon}^{i} = z_{i} + \varepsilon B(0, 1), i = 1, ..., 4$:
 - determine the variation $\delta j(z_i)$ given in (10),
 - choose $\varepsilon_0^i = \max\{\varepsilon > 0$, such that $z_i + \varepsilon_0^i B(0, 1) \subset \Omega\}$,
 - compute an approximation of the function $\varepsilon \mapsto j(\Omega \setminus \overline{\mathcal{O}_{\varepsilon}^i}), \ \varepsilon \in]0, \ \varepsilon_0^i].$
- Step 3: Deduce numerically the function $\varepsilon \mapsto \log(|\Delta_{z^i}(\varepsilon)|), \ \varepsilon \in]0, \ \varepsilon_0^i].$

For each considered obstacle $\mathcal{O}_{\varepsilon}^i = z_i + \varepsilon B(0, 1)$, we plot in Figure 2 the variation of $\log(|\Delta_{z_i}(\varepsilon)|)$ relating to $\log(-\log(\varepsilon))$.

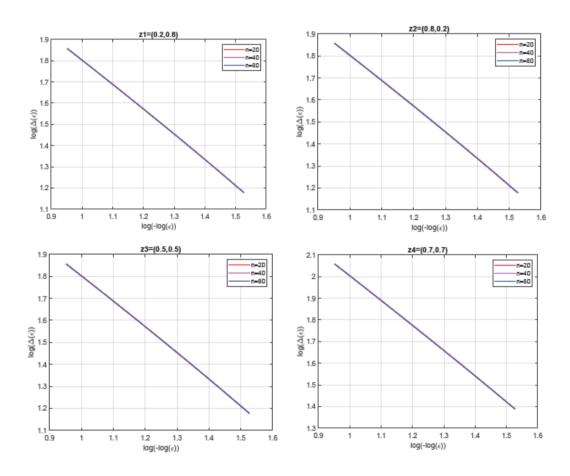


Fig. 2: Variation of $\log(|\Delta_{z_i}(\varepsilon)|)$ relating to $\log(-\log(\varepsilon))$.

We define β_i to describe the behavior of $\varepsilon \mapsto \Delta_{z_i}(\varepsilon)$ relating to $-\log(\varepsilon)$, i.e.

$$|\Delta_{z_i}(\varepsilon)| = O\left(\left(-\log(\varepsilon)\right)^{\beta_i}\right).$$

It corresponds to the slope of the line approximating the variation $\varepsilon \mapsto \log(|\Delta_{z_i}(\varepsilon)|)$ relating to $\log(-\log(\varepsilon))$ for each obstacle z_{ε}^{i} , i = 1, ..., 4.

From the plotted curves in Figure 2, one deduce the slopes β_i , i = 1, ..., 4 in table 2.

Table 2: The obtained slopes β_i of the lines associated with the obstacles $\mathcal{O}_{\varepsilon}^i$, $i=1,\ldots,4$.

The considered obstacles $\mathbb{O}^i_{arepsilon}$	$\mathcal{O}^1_{\varepsilon}$	$\mathcal{O}_{arepsilon}^2$	$\mathcal{O}_{arepsilon}^3$	$\mathcal{O}_{\varepsilon}^{4}$
The obtained slopes eta_i	-1.18	-1.187	-1.217	-1.163

We deduce that the numerical results confirm the behavior predicted by the theoretical estimate

$$\Delta_{z_i}(\varepsilon) = o(\frac{-1}{\log(\varepsilon)}).$$

4.2 The Tesla micro-valve application

The hydrocephalus treatment is a very important application in medicine. The problem is to optimize numerically the design of the 2D Tesla micro-valve at Re=100. To solve this problem we consider the objective function as the forward energy dissipation and the diodicity as a constraint. The optimal domain is constructed through the insertion of some obstacles in the initial one. The problem leads to optimize the location of obstacles.

4.2.1 Shape optimization problem

We define Ω as the pentagon [15] having one inclined inlet Γ_{in} and one horizontal outlet Γ_{out} (see Figure 3). The aim is to find the fluid flow optimal domain Ω^* which minimizes the dissipated energy by the forward

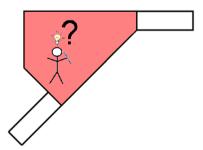


Fig. 3: Considered pentagon design domain

fluid flow and reproducing the original Tesla valve design given in Figure 1. This can be formulated:

$$\begin{cases}
& \text{Find } \Omega^* \text{ solution to} \\
& \min_{\Omega \subset \mathcal{D}_{ad}} j(\Omega),
\end{cases} \tag{12}$$

where

$$\mathcal{D}_{ad} = \{D \subset \Omega \text{ such that } \Gamma_{in} \subset \Gamma \cap \partial D, \ \Gamma_{out} \subset \Gamma \cap \partial D \text{ and } |D| \leq V_{desired} \},$$

with |.| and $V_{desired}$ represents respectively the Lebesgue measure and the target volume.

We recall that the performance of the Tesla valve is measured by diodicity *Di* which is known as the ratio of the pressure drop in backward direction to that in forward direction, which is equivalent to the ratio of dissipation of reverse and forward flows [15]:

$$Di = \frac{\Phi(w_r)}{\Phi(w_f)} \text{ with } \Phi(w) = \int_{\Omega} \left[\frac{v}{2} \sum_{i,j} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right)^2 \right],$$

with (w_f, p_f) and (w_r, p_r) are respectively the solution to the Navier Stokes system in the forward and the reverse flows. Then, diodicity can be maximized by minimizing forward dissipation while maximizing reverse dissipation. That is why our optimization problem is defined with the diodicity Di as a constraint; Di > 1. Using the above definitions, the optimization problem [15] for reconstructing Tesla valve can be expressed as

* Objective: power dissipation of forward flow

$$j(\Omega) = \Phi(w_f) = v \int_{0}^{T} \int_{\Omega} |\nabla w_f|^2 dx dt,$$

- Constraints
 - Volume fraction | $V_{desired}$ |< 0.8 | V_0 |.
 - Diodicity $Di \ge c > 1$.
 - Navier Stokes equations for forward and backward directions.

We use the obtained theoretical results in 3.2.2 to solve (12).

4.2.2 The topology optimization process

To obtain the optimal domain, an iterative process is applied to construct a sequence of geometries $(\Omega_k)_{k\geq 0}$ with $\Omega_0 = \Omega$ and $\Omega_{k+1} = \Omega_k \setminus \overline{\mathbb{O}_k}$ where \mathbb{O}_k is an obstacle inserted in Ω_k . To define the obstacle location and size, we find the function δj_k defined by (see Theorem 3.7)

$$\delta j_k(z) = 4\pi v \left[\int_0^T \left(w_k(z,t) u_k(z,t) + \left| w_k(z,t) \right|^2 \right) dt \right], \forall z \in \Omega_k, \tag{13}$$

where

– w_k represents the velocity, solution to the Navier-Stokes problem in Ω_k

$$\begin{cases}
\frac{\partial w_k}{\partial t} + \nabla w_k \cdot w_k - \nu \Delta w_k + \nabla p_k &= G & \text{in} \quad \Omega_k \times]0, T[, \\
\text{div } w_k &= 0 & \text{in} \quad \Omega_k \times]0, T[, \\
w_k &= w_d & \text{on} \quad \Gamma \times]0, T[, \\
w_k &= 0 & \text{on} \quad \Sigma_k \times]0, T[, \\
w_k(., 0) &= w^0 & \text{in} \quad \Omega_k.
\end{cases} \tag{14}$$

 $-u_k$ is the adjoint state, solution to

$$\begin{cases}
-\frac{\partial u_{k}}{\partial t} - \nabla u_{k} \cdot w_{k-1} + \nabla w_{k-1}^{T} \cdot u_{k} - \nu \Delta u_{k} + \nabla q_{k} &= -DJ_{0}(w_{k-1}) & \text{in} \quad \Omega_{k} \times]0, T[, \\
\text{div } u_{k} &= 0 & \text{in} \quad \Omega_{k} \times]0, T[, \\
u_{k} &= 0 & \text{on} \quad \Gamma \times]0, T[, \\
u_{k} &= 0 & \text{on} \quad \Sigma_{k} \times]0, T[, \\
u_{k}(., T) &= 0 & \text{in} \quad \Omega_{k},
\end{cases} \tag{15}$$

where $\Sigma_k = \partial(\bigcup_{l=0}^k \mathcal{O}_l)$ is the obstacle boundary inserted during the previous iterations. The optimization steps are summarized as:

The Algorithm:

- Initialization: Set $\Omega_0 = \Omega$, and k = 0
- 2. Repeat until $|\Omega_k| \leq V_{desired}$:
 - (a) The topological sensitivity function:
 - compute w_k , solution to the non stationary Stokes problem (14) in Ω_k ,
 - compute v_k , solution to the associated adjoint problem (15) in Ω_k ,
 - compute the term $\delta \mathcal{J}_k$ and deduce the function $\delta j_k(z)$, $\forall z \in \Omega_k$.
 - (b) The obstacle to be inserted:
 - determine $\rho_k^{\star} \in [0, 1]$ such that $j(\Omega_k \setminus \overline{O_{\rho_k^{\star}}^k}) \leq j(\Omega_k \setminus \overline{O_{\rho}^k}), \ \forall \rho \in [0, 1],$ set $\mathcal{O}_k = \left\{ x \in \Omega_k; \ \delta j_k(x) \leq \rho_k^{\star} \ \delta_{min}^k \right\}$, where $\delta_{min}^k = \min(\delta j_k(z))$.
 - The new domain: (c)
 - set $\Omega_{k+1} = \Omega_k \setminus \overline{\mathcal{O}_k}$,
 - $k \leftarrow k + 1$ and go to (2). (d)

The stopping criteria is defined by the natural optimality condition

$$\delta j^k(x) \ge 0, \ \forall x \in \Omega_k.$$

This algorithm is like a descent method where δj^k represents the descent direction and $|\mathcal{O}_k| = |\Omega_k \setminus \Omega_{k+1}|$ the step length. The parameter ρ_k^* is chosen to allow $\rho \longmapsto j(\Omega_k \setminus \overline{O_\rho^k})$ to decrease as much as possible. The computation of ρ_k^* in (b) can be viewed as line search step.

The numerical discretization of problems (14) and (15) is done by P1 – bubble/P1 finite element method [25]. The computation of the approximated solutions is achieved by the Uzawa's algorithm. The function δj^k is computed piecewise constant over elements.

Next, we will apply the proposed algorithm to reconstruct Tesla micro valve.

4.2.3 Reproducing the Tesla micro valve

We illustrate in this section the strengths of topology optimization method, namely the ability to find optimal design using only information on boundary conditions and constraints without the need of initial design.

The considered design domain is the pentagon domain (see Figure 3). This problem example has already been studied by S. Lin and al. in [15] in the steady state regime using projection method.

For the forward direction, the inlet boundary velocity has a parabolic behavior (Re=100 relating to the inlet dimension). At the outlet boundary, the pressure is taken constant and no-slip condition is considered on the walls. For the backward flow direction, we reverse these boundary conditions. Besides, we prescribe solid regions close to the inlets/outlets to minimize the boundary effect on the final design solution.

We illustrate the geometries obtained during the optimization process in Figure 4. The optimal domain is obtained after four iterations. It is nearly identical to literature [1, 15] (see Figure 5).

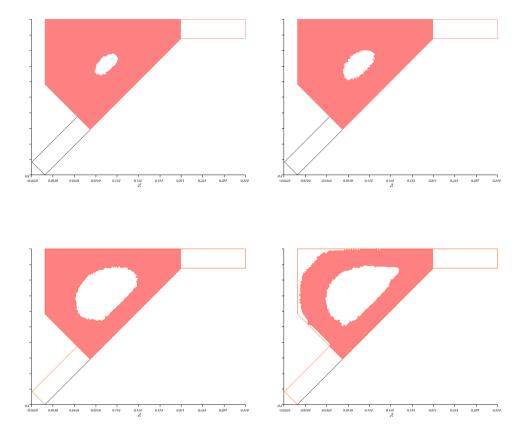


Fig. 4: Geometries obtained during the optimization process

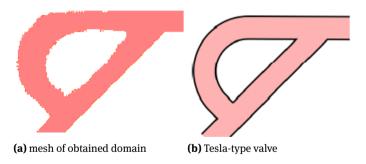
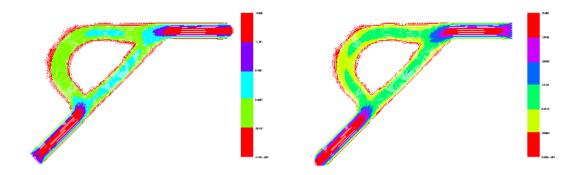


Fig. 5: mesh of obtained design (left) and reference Tesla valve (right)

4.2.4 Discussion

In the previous Figure, thanks to the topological gradient, we deduce an easy reconstruction of tesla valve. Now, we normalize the obtained tesla valve behavior by plotting the obtained forward and reverse flows respectively in figures 6(a) and 6(b). It is clear that the velocity field is strongly different for the two cases.



(a) Forward flow velocity field

(b) Backward flow velocity field

Fig. 6: Forward and backward flow velocity field

To study the obtained tesla valve performance, we calculate the diodicity. Using the energy view point expression of diodicity, the experimentally derived value is 1.137. In bibliography [26], the diodicity is well predicted using

$$Di \cong 1 + 4.78 * 10^{-5} * (N^{0.16}Re^{1.72})$$

with N is the number of tesla valves and Re is the Reynolds number. Based on this expression, we found $Di \cong 1.1316$ which ensures an agreement between the obtained diodicity and the experimental one.

5 Mathematical analysis

This section deals with the proofs of Theorems 3.1, 3.3, 3.5 and 3.7.

5.1 Proof of Theorem 3.1

Let *Q* be the pressure associated with the velocity *W*:

$$Q(x,t) = \frac{1}{\varepsilon} P(\frac{x-z}{\varepsilon}). w_0(z,t) = \frac{1}{\varepsilon} \sum_{i=1}^3 P^i(\frac{x-z}{\varepsilon}) w_0^i(z,t),$$
 (16)

where P^{j} is the pressure associated with the velocity U^{j} solution to (6). Setting the variation

$$z_{\varepsilon} = v_{\varepsilon} - W$$
 and $p_{z_{\varepsilon}} = p_{v_{\varepsilon}} - Q$. (17)

From (4) and (6), we can verify that $(z_{\varepsilon}, p_{z_{\varepsilon}})$ is solution to

$$\begin{cases} \frac{\partial z_{\varepsilon}}{\partial t} & -\nu \Delta z_{\varepsilon} + \nabla z_{\varepsilon} \cdot (w_{0} + W) + \nabla (w_{0} + W) \cdot z_{\varepsilon} + \nabla z_{\varepsilon} \cdot z_{\varepsilon} + \nabla p_{z_{\varepsilon}} \\ & = -\frac{\partial W}{\partial t} - \nabla w_{0} \cdot W - \nabla W \cdot w_{0} - \nabla W \cdot W & \text{in } \Omega_{z, \varepsilon} \times]0, T[, \\ \text{div } z_{\varepsilon} & = 0 & \text{in } \Omega_{z, \varepsilon} \times]0, T[, \\ z_{\varepsilon} & = -W & \text{on } \Gamma \times]0, T[, \\ z_{\varepsilon} & = -w_{0}(x, t) + w_{0}(z, t) & \text{on } \partial \mathcal{O}_{\varepsilon} \times]0, T[, \\ z_{\varepsilon}(., 0) & = 0 & \text{in } \Omega_{z, \varepsilon}. \end{cases}$$

$$(18)$$

The last boundary condition follows due to the fact that $U^j = -e_j$ on $\partial \mathbb{O}$. Moreover, since $|w_0|_{L^2(0,T;H^1(\Omega))} < \nu/k$, then ε sufficiently small,

$$|w_0 + W|_{L^2(0,T;H^1(\Omega_{z,s}))} \le \alpha < \nu/k.$$

Let R > 0 such that $\overline{O}_{z, \varepsilon} \subset B(z, R)$ and $\overline{B(z, R)} \subset \Omega$. Using the trace theorem, we obtain

$$||z_{\varepsilon}||_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))} \leq c \left(||\frac{\partial W}{\partial t}||_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} + ||W||_{L^{2}(0,T;H^{1}(\Omega_{R}))} + ||w_{0}(x,t) - w_{0}(z,t)||_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} + ||\nabla w_{0} \cdot W + \nabla W \cdot w_{0} + \nabla W \cdot W||_{L^{2}(0,T;H^{-1}(\Omega_{z,\varepsilon}))} \right),$$

$$(19)$$

where $\Omega_R = \Omega \setminus \overline{B(z, R)}$.

Using (5) and the variable change $x = z + \varepsilon y$, we obtain

$$\begin{split} \|\frac{\partial W}{\partial t}\|_{L^{2}(0,T;L^{2}(\Omega_{z,\,\varepsilon}))} &= \|\frac{\partial w_{0}}{\partial t}(z,.)\|_{L^{2}(0,T)}\|U(\frac{x-z}{\varepsilon})\|_{L^{2}(\Omega_{z,\,\varepsilon})} \\ &= \varepsilon^{3/2}\|\frac{\partial w_{0}}{\partial t}(z,.)\|_{L^{2}(0,T)}\|U\|_{L^{2}((\Omega_{z,\,\varepsilon})/\varepsilon)}. \end{split}$$

By the same way, we have

$$||W||_{L^{2}(0,T;H^{1}(\Omega_{R}))} \leq ||w_{0}(z,.)||_{L^{2}(0,T)} (||U(\frac{x-z}{\varepsilon})||_{L^{2}(\Omega_{R})} + ||\nabla_{x}U(\frac{x-z}{\varepsilon})||_{L^{2}(\Omega_{R})}),$$

$$\leq ||w_{0}(z,.)||_{L^{2}(0,T)} (\varepsilon^{3/2} ||U||_{L^{2}((\Omega_{R})/\varepsilon)} + \varepsilon^{1/2} ||\nabla_{y}U||_{L^{2}(\Omega_{R})/\varepsilon}).$$

Using [19] (see also [27]), the velocity field U^{j} , solution to the exterior Stokes problem, satisfies the estimate

$$||U^j||_{L^2((\Omega_p)/\varepsilon)} \le c\varepsilon^{-1/2}$$
 and $||\nabla_{\nu}U^j||_{L^2(\Omega_p)/\varepsilon} \le c\varepsilon^{1/2}$.

Then, using the smoothness of w_0 and the previous estimates, one can deduce

$$\left\|\frac{\partial W}{\partial t}\right\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \le c\varepsilon \text{ and } \|W\|_{L^{2}(0,T;H^{1}(\Omega_{R}))} \le c\varepsilon. \tag{20}$$

For the third term in (19). Expanding $w_0(x, t) = w_0(z, t) + \varepsilon \nabla w_0(\xi_y, t)y$ with $\xi_y \in \mathcal{O}_{z, \varepsilon}$ and using the fact that ∇w_0 is uniformly bounded, it follows that

$$||w_0(x,t)-w_0(z,t)||_{L^2(0,T;L^2(\Omega_{z,c}))} \le c\varepsilon.$$
 (21)

We now examine the last term in (19). Since $w_0 \in L^{\infty}(\Omega)$,

$$\begin{split} \|\nabla w_0 \cdot W + \nabla W \cdot w_0 &+ \nabla W \cdot W\|_{L^2(0,T;H^{-1}(\Omega_{z,\varepsilon}))} \le c \big(\|W\|_{L^2(0,T;H^{-1}(\Omega_{z,\varepsilon}))} \\ &+ \|\nabla W\|_{L^2(0,T;H^{-1}(\Omega_{z,\varepsilon}))} + \|\nabla W \cdot W\|_{L^2(0,T;H^{-1}(\Omega_{z,\varepsilon}))} \big), \\ &\le c \Big(\|W\|_{L^2(0,T;L^2(\Omega_{z,\varepsilon}))} + \|W\|_{L^2(0,T;H^1(\Omega_{z,\varepsilon}))} \|W\|_{L^2(0,T;H^1(\Omega_{z,\varepsilon}))} \Big), \end{split}$$

according to Lemma 4.2 in [17].

In addition, by Lemma 4.5 in [17], the variable change and the continuity of w_0 , we can deduce

$$\|W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \le c\varepsilon, \quad |W|_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))} \le c\varepsilon^{1/2}$$
 (22)

and then

$$\|\nabla w_0 \cdot W + \nabla W \cdot w_0 + \nabla W \cdot W\|_{L^2(0,T;H^{-1}(\Omega_{\tau,\epsilon}))} \le c\varepsilon. \tag{23}$$

Finally, combining (20), (21) and (23) we deduce that

$$||z_{\varepsilon}||_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))} \leq c\varepsilon.$$

5.2 Proof of Theorem 3.3

Let O be the pressure associated with the velocity W:

$$Q(x,t) = 4\pi\nu \Pi(x-z).w_0(z, t) = 4\pi\nu \sum_{j=1}^2 \Pi^j(x-z)w_0^j(z, t),$$

where Π^j is the pressure associated with the velocity E^j . Setting

$$z_{\varepsilon} = v_{\varepsilon} - \frac{1}{\log(\varepsilon)} W \text{ and } s_{\varepsilon} = p_{v_{\varepsilon}} - \frac{1}{\log(\varepsilon)} Q.$$
 (24)

From (1) and (3), we obtain that $(z_{\varepsilon}, s_{\varepsilon})$ is solution to

$$\begin{cases}
\frac{\partial z_{\varepsilon}}{\partial t} & -\nu \Delta z_{\varepsilon} + \nabla z_{\varepsilon} \cdot (w_{0} + \frac{1}{\log(\varepsilon)}W) + \nabla(w_{0} + \frac{1}{\log(\varepsilon)}W) \cdot z_{\varepsilon} + \nabla z_{\varepsilon} & \cdot z_{\varepsilon} + \nabla s_{\varepsilon} \\
& = -\frac{1}{\log(\varepsilon)} \left[\frac{\partial W}{\partial t} - \nabla w_{0} \cdot W - \nabla W \cdot w_{0} - \frac{1}{\log(\varepsilon)} \nabla W \cdot W \right] & \text{in } \Omega_{z,\varepsilon} \times]0, T[, \\
\text{div } z_{\varepsilon} & = 0 & \text{in } \Omega_{z,\varepsilon} \times]0, T[, \\
z_{\varepsilon} & = -\frac{1}{\log(\varepsilon)}W & \text{on } \Gamma \times]0, T[, \\
z_{\varepsilon} & = -w_{0}(x, t) - \frac{4\pi\nu}{\log(\varepsilon)}E(x - z)w_{0}(z, t) & \text{on } \partial \mathcal{O}_{z,\varepsilon} \times]0, T[.
\end{cases} \tag{25}$$

Using the relation $E((x-z)/\varepsilon) = E(x-z) + \frac{\log(\varepsilon)}{4\pi v}I$, the last boundary condition can be rewritten as

$$z_{\varepsilon} = -w_0(x, t) + w_0(z, t) - \frac{4\pi v}{\log(\varepsilon)} E((x - z)/\varepsilon) w_0(z, t) \text{ on } \partial \mathcal{O}_{z, \varepsilon} \times]0, T[.$$

Then, by an energy inequality [28], it follows

$$\|z_{\varepsilon}\|_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))}$$

$$\leq \frac{-c}{\log(\varepsilon)} \left[\|\frac{\partial W}{\partial t}\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} + \|W\|_{L^{2}(0,T;H^{1/2}(\Gamma))} + \log(\varepsilon) \|w_{0}(z+\varepsilon y,t) - w_{0}(z,t)\|_{L^{2}(0,T;H^{1/2}(\partial \mathcal{O}_{z,\varepsilon}))} + 4\pi \nu \|E((x-z)/\varepsilon)w_{0}(z,t)\|_{L^{2}(0,T;H^{1/2}(\partial \mathcal{O}_{z,\varepsilon}))} + \|\nabla w_{0} \cdot W + \nabla W \cdot w_{0} + \frac{1}{\log(\varepsilon)} \nabla W \cdot W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \right].$$

$$(26)$$

We estimate in the following each term in (26) separately.

We remark that:

- Since \emptyset is an open domain containing the origin, $\exists r > 0$ such that $B(0, r) \subset \emptyset$.
- $-\Omega$ is a bounded domain in such a way that $\exists R > 0$ such that $\overline{\Omega} \subset B(z,R), \ \forall z \in \Omega$.
- We have $\Omega_{z, \varepsilon}$ $z = \{x z, x \in \Omega_{z, \varepsilon}\} \subset C(0, r\varepsilon, R) = \{y \in \mathbb{R}^2; r\varepsilon < |y| < R\}$.

From the fact that $C(0, r\varepsilon, R) \subset \mathbb{R}^2 \setminus 0$, it follows that the function $\psi: y \mapsto \log(|y|)$ is smooth in $C(0, r\varepsilon, R)$ and we have $\|\psi\|_{0,C(0,r\varepsilon,R)} \le c$. Then, using the cylindrical coordinate system, one can prove that $\exists c > 0$, independent of ε , such that

$$||E(x-z)||_{L^{2}(0,T;L^{2}(\Omega))} \le ||E(y)||_{C(0,T\varepsilon,R)} \le c,$$
 (27)

$$||E(x-z)||_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \leq ||E(y)||_{C(0,r\varepsilon,R)} \leq c,$$

$$||\nabla E(x-z)||_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \leq c\sqrt{-\log(\varepsilon)}.$$
(28)

• Estimate of the first term in (26): Using that $w_0 \in H^1(0, T; H^1(\Omega))$, we obtain

$$\|\frac{\partial W}{\partial t}\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))}=4\pi\nu\|\frac{\partial w_{0}}{\partial t}(z,t)\|_{L^{2}(0,T)}\|E(x-z)\|_{L^{2}(\Omega_{z,\varepsilon})}=O(1).$$

• Estimate of the last term of (26):

Since w_0 and ∇w_0 belong to $L^{\infty}(\Omega)$, we have

$$\begin{split} &\|\nabla w_{0}\cdot W + \nabla W\cdot w_{0} + \frac{1}{\log(\varepsilon)}\nabla W\cdot W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \leq \\ &c\big(\|W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} + \|\nabla W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} + \frac{1}{\log(\varepsilon)}\|\nabla W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))}\|W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))}\big). \end{split}$$

Using the definition of *W*, we can deduce the following estimates

$$\|W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \leq c, \quad \|\nabla W\|_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))} \leq c\sqrt{-\log(\varepsilon)}.$$
 (29)

Yet, we have

$$\|\nabla w_0 \cdot W + \nabla W \cdot w_0 + \frac{1}{\log(\varepsilon)} \nabla W \cdot W\|_{L^2(0,T;L^2(\Omega_{z,\varepsilon}))} \le c\sqrt{-\log(\varepsilon)}. \tag{30}$$

• Estimate of boundary condition imposed on Γ :

Let $\widetilde{R} > 0$ such that $\mathcal{O}_{z,\varepsilon} \subset B(z,\widetilde{R})$ and $\overline{B(z,\widetilde{R})} \subset \Omega$. Since $z \notin \Omega_{\widetilde{R}} = \Omega \setminus \overline{B(z,\widetilde{R})}$, the function $x \mapsto E(x-z)$ belongs to $\mathcal{C}^1(\Omega_{\widetilde{R}})$. By the trace theorem, we have

$$\begin{split} \|W\|_{L^2(0,T;H^{1/2}(\Gamma)} &= 4\pi\nu \|w_0(z,t)\|_{L^2(0,T)} \|E(x-z)\|_{H^{1/2}(\Gamma)} \\ &\leq 4\pi\nu \|w_0(z,t)\|_{L^2(0,T)} \Big[\|E(x-z)\|_{L^2(\Omega_R)} + \|\nabla E(x-z)\|_{L^2(\Omega_R)} \Big]. \end{split}$$

Therefore, $\|W\|_{L^2(0,T;H^{1/2}(\varGamma)}$ is uniformly bounded with respect to ε .

• Estimate of boundary condition imposed on $\partial \mathcal{O}_{z, \varepsilon}$: Using the theorem of trace and the smoothness of w_0 in $\mathcal{O}_{z, \varepsilon} \times]0$, T[, one can obtain

$$||w_0(x,t)-w_0(z,t)||_{L^2(0,T;H^{1/2}(\partial\Omega_{z,c}))} \leq c\varepsilon.$$

Then, the first boundary term on $\partial \mathcal{O}_{z, \varepsilon}$ satisfies

$$\log(\varepsilon) \|w_0(x,t) - w_0(z,t)\|_{L^2(0,T;H^{1/2}(\partial\mathcal{O}_{z,\varepsilon}))} = o\left(\frac{-1}{\log(\varepsilon)}\right).$$

To estimate the last boundary term, we use that O contains the origin.

Setting $\mathcal{O}_r = \mathcal{O} \setminus \overline{B(0,r)}$ and $\mathcal{O}_{r,\varepsilon} = z + \varepsilon \mathcal{O}_r$. Using the theorem of trace and the variable change $x = z + \varepsilon y$, we obtain

$$\begin{split} \|E((x-z)/\varepsilon)w_{0}(z,t)\|_{L^{2}(0,T;H^{1/2}(\partial O_{z,\varepsilon}))} \\ &\leq \|w_{0}(z,t)\|_{L^{2}(0,T)} \Big(\|E((x-z)/\varepsilon)\|_{L^{2}(\mathcal{O}_{r,\varepsilon})} + \|\nabla_{x}E((x-z)/\varepsilon)\|_{L^{2}(\mathcal{O}_{r,\varepsilon})} \Big) \\ &\leq \|w_{0}(z,t)\|_{L^{2}(0,T)} \Big(\varepsilon \|E(y)\|_{L^{2}(\mathcal{O}_{r})} + \varepsilon^{1/2} \|\nabla_{y}E(y)\|_{L^{2}(\mathcal{O}_{r})} \Big). \end{split}$$

From the fact that $y \mapsto E(y)$ is sufficiently smooth in $\mathcal{O}_r \subset \mathbb{R}^2 \setminus \{0\}$, the last quantity is uniformly bounded and then

$$\frac{-4\pi\nu}{\log(\varepsilon)}\|E((x-z)/\varepsilon)w_0(z,t)\|_{L^2(0,T;H^{1/2}(\partial O_{z,\varepsilon}))}\leq \frac{-c\varepsilon^{1/2}}{\log(\varepsilon)}.$$

Finally, combining the above estimates, we obtain, $\exists c > 0$, independent of ε , such as

$$||z_{\varepsilon}||_{L^{2}(0,T;H^{1}(\Omega_{z,\varepsilon}))} \leq \frac{-c}{\log(\varepsilon)}$$

which ends the proof of Theorem 3.3.

5.3 Asymptotic analysis

This section deals with the proofs of the Theorems presented in paragraphs 3.2 and 3.3. Using the assumption (A),

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\,\varepsilon}}) - j(\Omega) = \int_{0}^{T} J_{\varepsilon}(w_{\varepsilon}(.,\,t)) dt - \int_{0}^{T} J_{0}(w_{0}(.,\,t)) dt$$

$$= \int_{0}^{T} DJ_{0}(w_{0}(.,\,t)) \Big(w_{\varepsilon}(.,\,t) - w_{0}(.,\,t)\Big) dt + \rho(\varepsilon) \delta \mathcal{J} + o(\rho(\varepsilon)),$$
(31)

where w_{ε} is extended by zero inside the domain $\mathcal{O}_{z,\varepsilon}$. Using Green formula and that $w_{\varepsilon} = 0$ in $\mathcal{O}_{\varepsilon}$, it follows

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\,\varepsilon}}) - j(\Omega) = -v \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} \nabla v_{\varepsilon} \nabla u_{0} \, dx \, dt - \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} \frac{\partial v_{\varepsilon}}{\partial t} \, u_{0} \, dx \, dt + \int_{0}^{T} \int_{0}^{T} \frac{\partial w_{0}}{\partial t} \, u_{0} \, dx \, dt + \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} \frac{\partial w_{0}}{\partial t} \, u_{0} \, dx \, dt + \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} (\nabla v_{\varepsilon} w_{0} + \nabla w_{0} v_{\varepsilon}) u_{0} \, dx \, dt + \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} (\nabla w_{0} w_{0}) u_{0} \, dx \, dt + \rho(\varepsilon) \, \delta \partial + o(\rho(\varepsilon)),$$

where u_0 is the solution to the associated adjoint problem. From (4) and the fact that $w_0 = 0$ on $\Gamma \times]0$, T[, we obtain

$$- v \int_{\mathbf{Q}_{z,\varepsilon}}^{T} \nabla v_{\varepsilon} \nabla u_{0} dx dt - \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} \frac{\partial v_{\varepsilon}}{\partial t} u_{0} dx dt - \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} w_{0} + \nabla w_{0} v_{\varepsilon}) u_{0} dx dt$$

$$= - \int_{0}^{T} \int_{\partial \Omega_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_{0} ds dt + \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} dx dt.$$
(32)

Therefore,

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\varepsilon}}) - j(\Omega)$$

$$= \int_{0}^{T} \int_{\mathbb{O}_{z,\varepsilon}} \frac{\partial w_{0}}{\partial t} u_{0} dx dt + v \int_{0}^{T} \int_{\mathbb{O}_{z,\varepsilon}} \nabla w_{0} \nabla u_{0} dx dt + 2 \int_{0}^{T} \int_{\mathbb{O}_{z,\varepsilon}} (\nabla w_{0} w_{0}) u_{0} dx dt$$

$$- \int_{0}^{T} \int_{\partial \mathbb{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_{0} ds dt + \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} dx dt + \rho(\varepsilon) \delta \mathcal{J}(z) + o(\rho(\varepsilon)).$$
(33)

We begin by giving the estimate of the first three terms in (33).

Lemma 5.1. The integral terms in (33) satisfy the estimate

$$\int_{0}^{T} \int_{\mathcal{O}_{x,\epsilon}} \frac{\partial w_0}{\partial t} u_0 dx dt + v \int_{0}^{T} \int_{\mathcal{O}_{x,\epsilon}} \nabla w_0 \nabla u_0 dx dt + 2 \int_{0}^{T} \int_{\mathcal{O}_{x,\epsilon}} (\nabla w_0 w_0) u_0 dx dt = O(\varepsilon^d).$$

Proof 5.2. Using the variable change $x = z + \varepsilon y$, the first integral term in (33) can be written

$$\int_{0}^{T} \int_{\mathcal{O}_{z,\varepsilon}} \frac{\partial w_0}{\partial t} u_0 dx dt = \varepsilon^d \int_{0}^{T} \int_{\mathcal{O}} \left(\frac{\partial w_0(z + \varepsilon y, t)}{\partial t} u_0(z + \varepsilon y, t) - \frac{\partial w_0(z, t)}{\partial t} u_0(z, t) dy dt \right) + \varepsilon^d |\mathcal{O}| \int_{0}^{T} \frac{\partial w_0(z, t)}{\partial t} u_0(z, t) dt,$$

where |O| denotes the Lebesgue measure of O.

Using that w_0 and u_0 are smooth near z, one can deduce that

$$\int_{0}^{T} \int_{0}^{\infty} \frac{\partial w_0}{\partial t} u_0 dx dt + \nu \int_{0}^{T} \int_{0}^{\infty} \nabla w_0 \nabla u_0 dx dt + 2 \int_{0}^{T} \int_{0}^{\infty} (\nabla w_0 w_0) u_0 dx dt = O(\varepsilon^d).$$

By the same arguments, we can estimate the two other terms in (33).

The shape function variation can be rewritten

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) - j(\Omega) = -\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) nu_{0} ds dt + \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} dx dt + \rho(\varepsilon) \delta \mathcal{J}(z) + o(\rho(\varepsilon)).$$

We are now ready to prove the established results in Theorems 3.5 and 3.7 and propositions 3.8 and 3.9.

5.3.1 Proof of Theorem 3.5

Using an integration by parts and the fact that $div(v_{\varepsilon}) = 0$ yield

$$\left| \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} \, dx \, dt \right| = \left| - \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla (u_{0}) \cdot (v_{\varepsilon})) \cdot v_{\varepsilon} dx \, dt \right|$$

$$\leq \| \nabla u_{0} \|_{L^{\infty}(\Omega_{z,\varepsilon})} \| v_{\varepsilon} \|_{L^{2}(\Omega_{z,\varepsilon})}^{2}$$

$$\leq 2 \| \nabla u_{0} \|_{L^{\infty}(\Omega_{z,\varepsilon})} (\| z_{\varepsilon} \|_{L^{2}(\Omega_{z,\varepsilon})}^{2} + \| W \|_{L^{2}(\Omega_{z,\varepsilon})}^{2})$$

$$\leq c\varepsilon^{2}.$$

$$(34)$$

Then, the shape function variation can be written

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\varepsilon}}) - j(\Omega) = -\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) nu_{0} ds dt + \varepsilon \delta \mathcal{J}(z) + o(\varepsilon).$$

From the definition of $(z_{\varepsilon}, s_{\varepsilon})$ and the variable change $x = z + \varepsilon y$, we have

$$\int_{0 \partial \mathcal{O}_{z,\varepsilon}}^{T} \int_{\sigma(v_{\varepsilon}, p_{v_{\varepsilon}})n u_{0} ds dt = \int_{0 \partial \mathcal{O}_{z,\varepsilon}}^{T} \int_{\sigma(z_{\varepsilon}, s_{\varepsilon})n u_{0} ds dt + \varepsilon \int_{0}^{T} w_{0}(z, t) \cdot \left(\int_{\partial \mathcal{O}} \sigma(U, P)(y) n(y) u_{0}(z + \varepsilon y, t) ds(y) \right) dt,$$

where $\sigma(U, P)n$ is the 3 × 3 matrix defined by

$$(\sigma(U, P)n)_{ij} = (\sigma(U^j, P^j)(y)n(y))_i, 1 \le i, j \le 3.$$

By the trace theorem, Theorem 3.1 and that u_0 is smooth in $\mathcal{O}_{z, \varepsilon}$,

$$\left|\int_{0}^{T}\int_{\partial \mathcal{O}_{z,\varepsilon}}\sigma(z_{\varepsilon}, s_{\varepsilon})n u_{0}dsdt\right| \leq \|\sigma(z_{\varepsilon}, s_{\varepsilon})n\|_{L^{2}(0,T;H^{-1/2}(\partial \mathcal{O}_{z,\varepsilon}))}\|u_{0}\|_{L^{2}(0,T;H^{1}(\mathcal{O}_{z,\varepsilon}))} = o(\varepsilon).$$

Making the variable change $x = z + \varepsilon y$, expanding $u_0(z + \varepsilon y, t) = u_0(z, t) + \varepsilon \nabla u_0(\xi_y, t)y$ with $\xi_y \in \mathcal{O}_{z, \varepsilon}$ and using that ∇u_0 is uniformly bounded, we obtain

$$\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_{0} ds dt$$

$$= \varepsilon \int_{0}^{T} w_{0}(z, t) \cdot \left(\int_{\partial \mathcal{O}} \sigma(U, P)(y) n ds(y) \right) u_{0}(z, t) dt$$

$$+ \varepsilon \int_{0}^{T} w_{0}(z, t) \left(\int_{\partial \mathcal{O}} \sigma(U, P)(y) n(y) [u_{0}(z + \varepsilon y, t) - u_{0}(z, t)] ds(y) \right) dt + o(\varepsilon).$$

Due to the jump condition of the single layer potential $\sigma(U^j, P^j)n = -\eta^j + \sigma(V^j, S^j)n$, where (V^j, S^j) is the solution to the interior problem

$$\begin{cases}
-\nu \Delta V^{j} + \nabla S^{j} &= 0 & \text{in} & \emptyset, \\
\operatorname{div} V^{j} &= 0 & \text{in} & \emptyset, \\
V^{j} &= U^{j} & \text{on} & \partial \emptyset.
\end{cases}$$

By the fact that div $\sigma(V^j, S^j) = \nu \Delta V^j - \nabla S^j = 0$ in \mathfrak{O} , we have $\int \sigma(V^j, S^j)(y) n \, ds = 0$.

Then, we obtain

$$\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_{0} ds dt = -\varepsilon \int_{0}^{T} w_{0}(z, t) \cdot \left(\int_{\partial \mathcal{O}} \eta(y) ds(y) u_{0}(z, t) \right) dt + o(\varepsilon).$$

Consequently, the shape function *j* admits the asymptotic expansion

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\,\varepsilon}}) = j(\Omega) + \varepsilon \left[\int\limits_0^T w_0(z,\,t). \mathcal{M}_{\mathcal{O}} u_0(z,\,\,t) dt + \delta \mathcal{J} \right] + o(\varepsilon),$$

where $\mathcal{M}_{\mathcal{O}}$ is the matrix given by

$$\mathcal{M}_{\mathcal{O}ij} = -\int\limits_{\partial\mathcal{O}} \eta_j^i(y) ds(y), \ 1 \le i, j \le 3.$$

5.3.2 Proof of Theorem 3.7

The shape function variation is given by

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\,\varepsilon}}) - j(\Omega) = -\int_{0\,\partial\mathbb{O}_{z,\,\varepsilon}}^T \int_{0\,\partial\mathbb{O}_{z,\,\varepsilon}} \sigma(\nu_\varepsilon, p_{\nu_\varepsilon}) nu_0 ds dt + \int_{0}^T \int_{\Omega_{z,\,\varepsilon}} (\nabla \nu_\varepsilon \nu_\varepsilon) u_0 dx dt + \frac{-1}{\log(\varepsilon)} \delta \mathcal{J}(z) + o(\frac{-1}{\log(\varepsilon)}).$$

Recall that the term (W, Q) describing the perturbation due to the presence of a small obstacle $\mathcal{O}_{Z, \mathcal{E}}$ is given by: $\forall (x, t) \in \Omega_{z, \varepsilon} \times]0, T[,$

$$W(x,t) = 4\pi\nu \sum_{j=1}^{2} [E^{j}(x-z)w_{0}(z,t)]e_{j}, \ Q(x,t) = 4\pi\nu \sum_{j=1}^{2} \Pi^{j}(x-z)w_{0}^{j}(z,t),$$

where $E^{j}(y) = E(y)e_{i}$ and $\Pi^{j}(y) = \Pi(y).e_{i}$, $1 \le j \le 2$.

Applying an integration by parts and using the fact that $div(v_{\varepsilon}) = 0$ provides

$$\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} dx dt = -\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla u_{0} \cdot v_{\varepsilon}) \cdot v_{\varepsilon} dx dt.$$

Then,

$$|\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} (\nabla v_{\varepsilon} v_{\varepsilon}) u_{0} \, dx \, dt| \leq \|\nabla u_{0}\|_{L^{\infty}(\Omega_{z,\varepsilon})} \|v_{\varepsilon}\|_{L^{2}(\Omega_{z,\varepsilon})}^{2}$$

$$\leq 2 \|\nabla u_{0}\|_{L^{\infty}(\Omega_{z,\varepsilon})} [\|z_{\varepsilon}\|_{L^{2}(\Omega_{\varepsilon})}^{2} + \|W\|_{L^{2}(\Omega_{\varepsilon})}^{2}]$$

$$\leq c \left(\frac{-1}{\log(\varepsilon)}\right)^{2} = o\left(\frac{-1}{\log(\varepsilon)}\right). \tag{35}$$

It follows that

$$j(\Omega \setminus \overline{\mathbb{O}_{z,\,\varepsilon}}) - j(\Omega) = -\int\limits_{\mathfrak{O} \mathbb{O}_{z,\,\varepsilon}}^T \sigma(v_\varepsilon, p_{v_\varepsilon}) n u_0 ds dt + \frac{-1}{\log(\varepsilon)} \delta \mathcal{J}(z) + o(\frac{-1}{\log(\varepsilon)}).$$

Then, from the decomposition (24), one can derive

$$\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_{0} ds dt = \int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(z_{\varepsilon}, s_{\varepsilon}) n u_{0} ds dt
+ \frac{4\pi v}{\log(\varepsilon)} \int_{0}^{T} w_{0}(z, t) \left(\int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E, \Pi)(x - z) n u_{0}(x, t) ds(x) \right) dt,$$
(36)

where $\sigma(E,\Pi)n$ is the 2 × 2 matrix defined by $(\sigma(E,\Pi)n)_{i,j} = (\sigma(E^j,\Pi^j)n)_i, \ 1 \le i,j \le 2$. Using Theorem 3.3 and the smoothness of u_0 in $\mathcal{O}_{z,\mathcal{E}}$, it follows

$$\Big|\int_{0}^{T}\int_{\partial \mathcal{O}_{z,\varepsilon}}\sigma(z_{\varepsilon}, s_{\varepsilon})n u_{0}dsdt\Big|=o\Big(\frac{-1}{\log(\varepsilon)}\Big).$$

The second term in (36) can be written

$$\int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E,\Pi)(x-z)n \, u_0(x,t) ds(x) = \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E,\Pi)(x-z)n [u_0(x,t)-u_0(z,t)] ds(x) + \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E,\Pi)(x-z)n \, u_0(z,t) ds(x).$$

Using the trace theorem and the variable change $x = z + \varepsilon y$, one can obtain

$$\begin{split} &|\int_{0}^{T} w_{0}(z,t) \cdot \Big(\int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E,\Pi)(x-z)n[u_{0}(x,t)-u_{0}(z,t)]ds(x)\Big)dt| \\ &\leq c \, \|w_{0}(z,t)\|_{L^{2}(0,T)} \|\sigma(E,\Pi)(x-z)n\|_{H^{-1/2}(\partial \mathcal{O}_{z,\varepsilon})} \|u_{0}(x,t)-u_{0}(z,t)\|_{L^{2}(0,T;H^{1/2}(\mathcal{O}_{z,\varepsilon}))}. \end{split}$$

By the fact that u_0 is smooth in \mathcal{O}_z , ε , it follows

$$\lim_{\varepsilon \longrightarrow 0} \|u_0(x,t) - u_0(z,t)\|_{L^2(0,T;H^{1/2}(\mathcal{O}_{z,\varepsilon}))} = 0.$$

Recall that $B(0, r) \subset \mathcal{O}$, $\mathcal{O}_r = \mathcal{O} \setminus \overline{B(0, r)}$ and $\mathcal{O}_{r, \varepsilon} = z + \varepsilon \mathcal{O}_r \subset \mathcal{O}_{z, \varepsilon}$. Here, one can check that the function $x \mapsto \sigma(E, \Pi)(x-z)$ is smooth in $\mathcal{O}_{r, \varepsilon}$. Using the trace theorem, we prove that the quantity $\|\sigma(E, \Pi)(x-z)n\|_{H^{-1/2}(\partial \mathcal{O}_{z, \varepsilon})}$

is bounded with respect to ε , which implies

$$\frac{4\pi\nu}{\log(\varepsilon)}\int_{0}^{T}w_{0}(z,t)\cdot\bigg(\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma(E,\Pi)(x-z)n[u_{0}(x,t)-u_{0}(z,t)]ds(x)\bigg)dt=o\bigg(\frac{-1}{\log(\varepsilon)}\bigg).$$

Combining the above estimates, one can deduce

$$\begin{split} &\int\limits_0^T \int\limits_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, \, p_{v_{\varepsilon}}) n u_0 ds dt \\ &= \frac{4\pi v}{\log(\varepsilon)} \int\limits_0^T w_0(z, \, t) \cdot \Big(\int\limits_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(E, \Pi)(x-z) n ds(x)\Big) u_0(z, \, t) dt + o\Big(\frac{-1}{\log(\varepsilon)}\Big). \end{split}$$

Since $\operatorname{div}(\sigma(E^j, \Pi^j)(x-z)) = \delta_z e_j$ in $\mathfrak{O}_z, \varepsilon$, it follows

$$\int_{\partial \mathcal{O}_{\varepsilon}} \sigma(E(x-z), \Pi(x-z)) n \, ds = I,$$

where *I* is the 2×2 identity matrix.

Then, the last estimate becomes

$$\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(v_{\varepsilon}, p_{v_{\varepsilon}}) n u_0 ds dt = \frac{4\pi v}{\log(\varepsilon)} \int_{0}^{T} w_0(z, t) u_0(z, t) dt + o\left(\frac{-1}{\log(\varepsilon)}\right).$$
 (37)

Consequently, all shape functions j satisfying the assumption (A) admit the asymptotic expansion

$$j(\Omega \setminus \overline{\mathcal{O}_{z,\,\varepsilon}}) = j(\Omega) + \frac{-1}{\log(\varepsilon)} \left[4\pi\nu \int_{0}^{T} w_{0}(z, t) u_{0}(z, t) dt + \delta \mathcal{J}(z) \right] + o\left(\frac{-1}{\log(\varepsilon)}\right).$$

5.3.3 Proof of Proposition 3.8

Since the desired fluid flow state $W_d \in L^2(0, T; H^1(\Omega))$, the function J_0 is differentiable at $w_0(., t)$ and we have

$$DJ_0(w_0(., t))(v) = 2 \int_{\Omega} (w_0(., t) - W_d(., t)) v \, dx, \quad \forall v \in H^1(\Omega).$$

The variation of the associated shape function j is given by

$$j(\Omega_{z,\,\varepsilon}) - j(\Omega) = \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |w_{\varepsilon} - W_{d}|^{2} dx dt - \int_{0}^{T} \int_{\Omega} |w_{0} - W_{d}|^{2} dx dt$$

$$= \int_{0}^{T} DJ_{0}(w_{0})(w_{\varepsilon} - w_{0}) dt + \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |w_{\varepsilon} - w_{0}|^{2} dx dt$$

$$+ \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |w_{0}|^{2} dx dt - \int_{0}^{T} \int_{\Omega_{z,\,\varepsilon}} |W_{d}|^{2} dx dt.$$

Using the smoothness of w_0 and W_d in Ω , one can conclude that

$$\int_{0}^{T} \int_{\mathcal{O}_{z,\varepsilon}} |w_0|^2 dx dt = o(\varepsilon) \text{ and } \int_{0}^{T} \int_{\mathcal{O}_{z,\varepsilon}} |\mathcal{W}_d|^2 dx dt = o(\varepsilon).$$

For the two-dimensional case: Using the decomposition (24), it follows

$$\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |w_{\varepsilon} - w_{0}|^{2} \leq 2 \left[\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |z_{\varepsilon}|^{2} dx dt + \frac{1}{(\log(\varepsilon))^{2}} \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |W|^{2} dx dt \right].$$

From Theorem 3.3, one can check

$$\int_{0}^{T} \int_{\Omega_{\varepsilon,\varepsilon}} |z_{\varepsilon}|^{2} dx dt = o\left(\frac{-1}{\log(\varepsilon)}\right).$$

Making use of (27), one can deduce

$$||W||_{L^{2}(0,T;L^{2}(\Omega_{z,\varepsilon}))}=4\pi\nu||w_{0}(z,t)||_{L^{2}(0,T)}||E(x-z)||_{L^{2}(\Omega_{z,\varepsilon})}=O(1).$$

Then, it follows

$$\frac{1}{(\log(\varepsilon))^2}\int_{0}^{T}\int_{\Omega_{z,\varepsilon}}|W|^2dxdt=o\left(\frac{-1}{\log(\varepsilon)}\right).$$

For the three-dimensional case: Using the decomposition (17), it follows

$$\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |w_{\varepsilon} - w_{0}|^{2} \leq 2 \left(\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |z_{\varepsilon}|^{2} dx dt + \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |W|^{2} dx dt \right).$$

Using Theorem 3.1 and the change of variable, one can check

$$\int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |z_{\varepsilon}|^{2} dx dt = o(\varepsilon) \text{ and } \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |W|^{2} dx dt = o(\varepsilon).$$

Therefore the function J_{ε} satisfies the assumption (A) with

$$DJ_0(w_0(., t))(v) = 2 \int_{\Omega} (w_0(., t) - W_d(., t)) v \, dx, \quad \forall v \in H^1(\Omega),$$
$$\delta J(x) = 0, \quad \forall x \in \Omega.$$

5.3.4 Proof of Proposition 3.9

The function J_0 is differentiable at $w_0(., t)$ and we have

$$DJ_0(w_0(., t))(v) = 2v \int_{\Omega} (\nabla w_0(., t) - \nabla W_d(., t)) \nabla v \, dx, \quad \forall v \in H^1(\Omega).$$

The variation of the associated shape function j is given by

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \int_{0}^{T} DJ_{0}(w_{0})(w_{\varepsilon} - w_{0}) dt - \nu \int_{0}^{T} \int_{0_{z,\varepsilon}} |\nabla W_{d}|^{2} dx dt$$

$$+ \nu \int_{0}^{T} \int_{0_{z,\varepsilon}} |\nabla w_{0}|^{2} dx dt + \nu \int_{0}^{T} \int_{\Omega_{z,\varepsilon}} |\nabla w_{\varepsilon} - \nabla w_{0}|^{2} dx dt.$$
(38)

Thanks to the regularity of w_0 and W_d in $O_{z, \varepsilon}$, one can derive

$$\int_{0}^{T} \int_{0_{Z,\varepsilon}} v |\nabla w_0|^2 dx dt = o(\varepsilon), \quad \int_{0}^{T} \int_{0_{Z,\varepsilon}} v |\nabla W_d|^2 dx dt = o(\varepsilon).$$

For the two-dimensional case: By an adaptation of the technique used in the proof of Theorem 3.7, one can derive

$$\int_{0}^{T} \int_{0}^{T} \sigma(z_{\varepsilon}, s_{\varepsilon}) n w_{0} ds dt = \frac{-4\pi \nu}{\log(\varepsilon)} \int_{0}^{T} |w_{0}(z, t)|^{2} dt + o(\frac{-1}{\log(\varepsilon)}).$$

Therefore, the function I_{ε} satisfies the assumption (A) with

$$DJ_0(w_0(., t))(v) = 2 v \int_{\Omega} (\nabla w_0(., t) - \nabla W_d(., t)) \nabla v \, dx, \quad \forall v \in H^1(\Omega),$$

and
$$\delta \mathcal{J}(x) = 4\pi v \int_{0}^{T} |w_0(z, t)|^2 dt$$
, $\forall x \in \Omega$.

For the three-dimensional case: By an adaptation of the technique used in the proof of Theorem 3.5, one can derive

$$\int_{0}^{T} \int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(z_{\varepsilon}, s_{\varepsilon}) n \, w_0 \, ds \, dt = \varepsilon \left[\int_{0}^{T} w_0(z, t) . \mathcal{M}_{\mathcal{O}} w_0(z, t) dt \right] + o(\varepsilon).$$

Therefore, the function I_{ε} satisfies the assumption (A) with

$$DJ_0(w_0(., t))(v) = 2 v \int_{\Omega} (\nabla w_0(., t) - \nabla W_d(., t)) \nabla v dx, \quad \forall v \in H^1(\Omega),$$

and
$$\delta \mathcal{J}(z) = \int_{0}^{T} w_0(z, t) . \mathcal{M}_{\mathcal{O}} w_0(z, t) dt, \ \forall z \in \Omega.$$

6 Conclusion

This paper deals with non-stationary Navier-Stokes topological optimization problem. In the theoretical part of this work, we have established a topological asymptotic formula describing the shape function variation related to a small Dirichlet geometric perturbation.

The obtained theoretical results are exploited for building a topological optimization algorithm for solving the Tesla micro-valve optimization problem. We illustrate the strengths of this approach namely the ability to find optimal design based only on boundary conditions and constraints information without the need of an initial design.

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