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A class of semipositone p-Laplacian problems with a critical growth reaction term

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Abstract: We prove the existence of ground state positive solutions for a class of semipositone p-Laplacian problems with a critical growth reaction term. The proofs are established by obtaining crucial uniform $C^{1,\alpha}$ a priori estimates and by concentration compactness arguments. Our results are new even in the semilinear case $p = 2$.

Keywords: critical semipositone p-Laplacian problems, ground state positive solutions, concentration compactness, uniform $C^{1,\alpha}$ a priori estimates

MSC: Primary 35B33, Secondary 35J92, 35B09, 35B45

1 Introduction

Consider the p-superlinear semipositone p-Laplacian problem

$$\begin{cases} -\Delta_p\, u = u^{q-1} - \mu & \text{in } \Omega \\ \qquad u > 0 & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, $p < q \le p^*$, $\mu > 0$ is a parameter, and $p^* = Np/(N-p)$ is the critical Sobolev exponent. The scaling $u \mapsto \mu^{1/(q-1)}\, u$ transforms the first equation in (1.1) into

$$-\Delta_p\, u = \mu^{(q-p)/(q-1)} \left(u^{q-1} - 1 \right),$$

so in the subcritical case $q < p^*$, it follows from the results in Castro et al.[1] and Chhetri et al.[2] that this problem has a weak positive solution for sufficiently small $\mu > 0$ when $p > 1$ (see also Unsurangie [3], Allegretto et al.[4], Ambrosetti et al.[5], and Caldwell et al.[6] for the case when $p = 2$). On the other hand, in the critical case $q = p^*$, it follows from a standard argument involving the Pohozaev identity for the p-Laplacian (see Guedda and Véron [7, Theorem 1.1]) that problem (1.1) has no solution for any $\mu > 0$ when Ω is star-shaped. The purpose of the present paper is to show that this situation can be reversed by the addition of lower-order terms, as was observed in the positone case by Brézis and Nirenberg in the celebrated paper [8]. However, this extension to the semipositone case is not straightforward as $u = 0$ is no longer a subsolution, making it much harder to find a positive solution as was pointed out in Lions [9]. The positive solutions that we obtain here are ground states, i.e., they minimize the energy among all positive solutions.

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We study the Brézis-Nirenberg type critical semipositone p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda, \mu > 0$ are parameters. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with the norm given by

$$\|u\|^p = \int_\Omega |\nabla u|^p \, dx.$$

For a given $\lambda > 0$, the energy of a weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.2) is given by

$$I_\mu(u) = \int_\Omega \left(\frac{|\nabla u|^p}{p} - \frac{\lambda u^p}{p} - \frac{u^{p^*}}{p^*} + \mu u \right) dx,$$

and clearly all weak solutions lie on the set

$$\mathcal{N}_\mu = \left\{ u \in W_0^{1,p}(\Omega) : u > 0 \text{ in } \Omega \text{ and } \int_\Omega |\nabla u|^p \, dx = \int_\Omega \left(\lambda u^p + u^{p^*} - \mu u \right) dx \right\}.$$

We will refer to a weak solution that minimizes I_μ on $\mathcal{N}_\mu$ as a ground state. Let

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} \tag{1.3}$$

be the first Dirichlet eigenvalue of the p-Laplacian, which is positive. We will prove the following existence theorem.

Theorem 1.1. *If $N \geq p^2$ and $\lambda \in (0, \lambda_1)$, then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem* (1.2) *has a ground state solution $u_\mu \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

The scaling $u \mapsto \mu^{-1/(p^*-p)} u$ transforms the first equation in the critical semipositone p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + \mu \left(u^{p^*-1} - 1 \right) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

into

$$-\Delta_p u = \lambda u^{p-1} + u^{p^*-1} - \mu^{(p^*-1)/(p^*-p)},$$

so as an immediate corollary we have the following existence theorem for problem (1.4).

Theorem 1.2. *If $N \geq p^2$ and $\lambda \in (0, \lambda_1)$, then there exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$, problem* (1.4) *has a ground state solution $u_\mu \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

We would like to emphasize that Theorems 1.1 and 1.2 are new even in the semilinear case $p = 2$.

The outline of the proof of Theorem 1.1 is as follows. We consider the modified problem

$$\begin{cases} -\Delta_p u = \lambda u_+^{p-1} + u_+^{p^*-1} - \mu f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $u_+(x) = \max\{u(x), 0\}$ and

$$f(t) = \begin{cases} 1, & t \geq 0 \\ 1 - |t|^{p-1}, & -1 < t < 0 \\ 0, & t \leq -1. \end{cases}$$

Weak solutions of this problem coincide with critical points of the C^1-functional

$$I_\mu(u) = \int_\Omega \left(\frac{|\nabla u|^p}{p} - \frac{\lambda u_+^p}{p} - \frac{u_+^{p^*}}{p^*} \right) dx + \mu \left[\int_{\{u \geq 0\}} u \, dx \right.$$
$$\left. + \int_{\{-1<u<0\}} \left(u - \frac{|u|^{p-1} u}{p} \right) dx - \left(1 - \frac{1}{p} \right) |\{u \leq -1\}| \right], \quad u \in W_0^{1,p}(\Omega),$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$. Recall that I_μ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or the $(\mathrm{PS})_c$ condition for short, if every sequence $(u_j) \subset W_0^{1,p}(\Omega)$ such that $I_\mu(u_j) \to c$ and $I_\mu'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence for I_μ, has a convergent subsequence. As we will see in Lemma 2.1 in the next section, it follows from concentration compactness arguments that I_μ satisfies the $(\mathrm{PS})_c$ condition for all

$$c < \frac{1}{N} S^{N/p} - \left(1 - \frac{1}{p} \right) \mu \, |\Omega| \,,$$

where S is the best Sobolev constant (see (2.1)). First we will construct a mountain pass level below this threshold for compactness for all sufficiently small $\mu > 0$. This part of the proof is more or less standard. The novelty of the paper lies in the fact that the solution u_μ of the modified problem (1.5) thus obtained is positive, and hence also a solution of our original problem (1.2), if μ is further restricted. Note that this does not follow from the strong maximum principle as usual since $-\mu f(0) < 0$. This is precisely the main difficulty in finding positive solutions of semipositone problems (see Lions [9]). We will prove that for every sequence $\mu_j \to 0$, a subsequence of u_{μ_j} is positive in Ω. The idea is to show that a subsequence of u_{μ_j} converges in $C_0^1(\overline{\Omega})$ to a solution of the limit problem

$$\begin{cases} -\Delta_p \, u = \lambda u^{p-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This requires a uniform $C^{1,\alpha}(\overline{\Omega})$ estimate of u_{μ_j} for some $\alpha \in (0, 1)$. We will obtain such an estimate by showing that u_{μ_j} is uniformly bounded in $W_0^{1,p}(\Omega)$ and uniformly equi-integrable in $L^{p^*}(\Omega)$, and applying a result of de Figueiredo et al. [10]. The proof of uniform equi-integrability in $L^{p^*}(\Omega)$ involves a second (nonstandard) application of the concentration compactness principle. Finally, we use the mountain pass characterization of our solution to show that it is indeed a ground state.

Remark 1.3. *Establishing the existence of solutions to the critical semipositone problem*

$$\begin{cases} -\Delta_p \, u = \mu \left(u^{p-1} + u^{p^*-1} - 1 \right) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for small μ remains open.

2 Preliminaries

Let

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p\,dx}{\left(\int_\Omega |u|^{p^*}\,dx\right)^{p/p^*}} \tag{2.1}$$

be the best constant in the Sobolev inequality, which is independent of Ω. The proof of Theorem 1.1 will make use of the following compactness result.

Lemma 2.1. *For any fixed $\lambda, \mu > 0$, I_μ satisfies the* $(\mathrm{PS})_c$ *condition for all*

$$c < \frac{1}{N}\, S^{N/p} - \left(1 - \frac{1}{p}\right) \mu\, |\Omega|\,. \tag{2.2}$$

Proof. Let (u_j) be a $(\mathrm{PS})_c$ sequence. First we show that (u_j) is bounded. We have

$$\begin{aligned} I_\mu(u_j) = \int_\Omega \left(\frac{|\nabla u_j|^p}{p} - \frac{\lambda u_{j+}^p}{p} - \frac{u_{j+}^{p^*}}{p^*}\right) dx + \mu \Bigg[\int_{\{u_j \ge 0\}} u_j\,dx \\ + \int_{\{-1<u_j<0\}} \left(u_j - \frac{|u_j|^{p-1}\,u_j}{p}\right) dx - \left(1 - \frac{1}{p}\right) \left|\{u_j \le -1\}\right| \Bigg] = c + \mathrm{o}(1) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_\mu'(u_j)\,v = \int_\Omega \left(|\nabla u_j|^{p-2}\,\nabla u_j \cdot \nabla v - \lambda u_{j+}^{p-1}\,v - u_{j+}^{p^*-1}\,v\right) dx + \mu \Bigg[\int_{\{u_j \ge 0\}} v\,dx \\ + \int_{\{-1<u_j<0\}} \left(1 - |u_j|^{p-1}\right) v\,dx \Bigg] = \mathrm{o}(1)\,\|v\| \quad \forall v \in W_0^{1,p}(\Omega). \end{aligned} \tag{2.4}$$

Taking $v = u_j$ in (2.4), dividing by p, and subtracting from (2.3) gives

$$\frac{1}{N} \int_\Omega u_{j+}^{p^*}\,dx \le c + \left(1 - \frac{1}{p}\right) \mu\,|\Omega| + \mathrm{o}(1)\left(\|u_j\| + 1\right), \tag{2.5}$$

and it follows from this, (2.3), and the Hölder inequality that (u_j) is bounded in $W_0^{1,p}(\Omega)$.

Since (u_j) is bounded, so is (u_{j+}), a renamed subsequence of which then converges to some $v \ge 0$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, p^*)$ and a.e. in Ω, and

$$|\nabla u_{j+}|^p\,dx \xrightarrow{w^*} \kappa, \qquad u_{j+}^{p^*}\,dx \xrightarrow{w^*} \nu \tag{2.6}$$

in the sense of measures, where κ and ν are bounded nonnegative measures on $\overline{\Omega}$ (see, e.g., Folland [11]). By the concentration compactness principle of Lions [12, 13], then there exist an at most countable index set I and points $x_i \in \overline{\Omega}$, $i \in I$ such that

$$\kappa \ge |\nabla v|^p\,dx + \sum_{i\in I} \kappa_i\,\delta_{x_i}, \qquad \nu = v^{p^*}\,dx + \sum_{i\in I} \nu_i\,\delta_{x_i}, \tag{2.7}$$

where $\kappa_i, \nu_i > 0$ and $\nu_i^{p/p^*} \le \kappa_i/S$. We claim that $I = \emptyset$. Suppose by contradiction that there exists $i \in I$. Let $\varphi : \mathbb{R}^N \to [0,1]$ be a smooth function such that $\varphi(x) = 1$ for $|x| \le 1$ and $\varphi(x) = 0$ for $|x| \ge 2$. Then set

$$\varphi_{i,\rho}(x) = \varphi\left(\frac{x - x_i}{\rho}\right), \quad x \in \mathbb{R}^N$$

for $i \in I$ and $\rho > 0$, and note that $\varphi_{i,\rho} : \mathbb{R}^N \to [0,1]$ is a smooth function such that $\varphi_{i,\rho}(x) = 1$ for $|x - x_i| \le \rho$ and $\varphi_{i,\rho}(x) = 0$ for $|x - x_i| \ge 2\rho$. The sequence $(\varphi_{i,\rho}\, u_{j+})$ is bounded in $W_0^{1,p}(\Omega)$ and hence taking $v = \varphi_{i,\rho}\, u_{j+}$ in (2.4) gives

$$\int_\Omega \left(\varphi_{i,\rho}\, |\nabla u_{j+}|^p + u_{j+}\, |\nabla u_{j+}|^{p-2}\, \nabla u_{j+} \cdot \nabla \varphi_{i,\rho} - \lambda\, \varphi_{i,\rho}\, u_{j+}^p - \varphi_{i,\rho}\, u_{j+}^{p^*} + \mu\, \varphi_{i,\rho}\, u_{j+}\right) dx = o(1). \tag{2.8}$$

By (2.6),

$$\int_\Omega \varphi_{i,\rho}\, |\nabla u_{j+}|^p\, dx \to \int_\Omega \varphi_{i,\rho}\, d\kappa, \qquad \int_\Omega \varphi_{i,\rho}\, u_{j+}^{p^*}\, dx \to \int_\Omega \varphi_{i,\rho}\, d\nu.$$

Denoting by C a generic positive constant independent of j and ρ,

$$\left| \int_\Omega \left(u_{j+}\, |\nabla u_{j+}|^{p-2}\, \nabla u_{j+} \cdot \nabla \varphi_{i,\rho} - \lambda\, \varphi_{i,\rho}\, u_{j+}^p + \mu\, \varphi_{i,\rho}\, u_{j+}\right) dx \right| \le C\left[\left(\frac{1}{\rho} + \mu\right) I_j^{1/p} + I_j\right],$$

where

$$I_j := \int_{\Omega \cap B_{2\rho}(x_i)} u_{j+}^p\, dx \to \int_{\Omega \cap B_{2\rho}(x_i)} v^p\, dx \le C\rho^p \left(\int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*}\, dx\right)^{p/p^*}.$$

So passing to the limit in (2.8) gives

$$\int_\Omega \varphi_{i,\rho}\, d\kappa - \int_\Omega \varphi_{i,\rho}\, d\nu \le C\left[(1 + \mu\rho)\left(\int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*}\, dx\right)^{1/p^*} + \int_{\Omega \cap B_{2\rho}(x_i)} v^p\, dx\right].$$

Letting $\rho \searrow 0$ and using (2.7) now gives $\kappa_i \le \nu_i$, which together with $\nu_i > 0$ and $\nu_i^{p/p^*} \le \kappa_i/S$ then gives $\nu_i \ge S^{N/p}$. On the other hand, passing to the limit in (2.5) and using (2.6) and (2.7) gives

$$\nu_i \le N\left[c + \left(1 - \frac{1}{p}\right)\mu\, |\Omega|\right] < S^{N/p}$$

by (2.2), a contradiction. Hence $I = \emptyset$ and

$$\int_\Omega u_{j+}^{p^*}\, dx \to \int_\Omega v^{p^*}\, dx. \tag{2.9}$$

Passing to a further subsequence, u_j converges to some u weakly in $W_0^{1,p}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, p^*)$, and a.e.in Ω. Since

$$|u_{j+}^{p^*-1}\, (u_j - u)| \le u_{j+}^{p^*} + u_{j+}^{p^*-1}\, |u| \le \left(2 - \frac{1}{p^*}\right) u_{j+}^{p^*} + \frac{1}{p^*}\, |u|^{p^*}$$

by Young's inequality,

$$\int_\Omega u_{j+}^{p^*-1}\, (u_j - u)\, dx \to 0$$

by (2.9) and the dominated convergence theorem. Then taking $v = u_j - u$ in (2.4) gives

$$\int_\Omega |\nabla u_j|^{p-2}\, \nabla u_j \cdot \nabla(u_j - u)\, dx \to 0,$$

so $u_j \to u$ in $W_0^{1,p}(\Omega)$ for a renamed subsequence (see, e.g., Perera et al.[14, Proposition 1.3]). □

The infimum in (2.1) is attained by the family of functions

$$u_\varepsilon(x) = \frac{C_{N,p}\, \varepsilon^{(N-p)/p^2}}{(\varepsilon + |x|^{p/(p-1)})^{(N-p)/p}}, \qquad \varepsilon > 0$$

when $\Omega = \mathbb{R}^N$, where the constant $C_{N,p} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^{p^*} \, dx = S^{N/p}.$$

Without loss of generality, we may assume that $0 \in \Omega$. Let $r > 0$ be so small that $B_{2r}(0) \subset \Omega$, take a function $\psi \in C_0^\infty(B_{2r}(0), [0,1])$ such that $\psi = 1$ on $B_r(0)$, and set

$$\tilde{u}_\varepsilon(x) = \psi(x)\, u_\varepsilon(x), \qquad v_\varepsilon(x) = \frac{\tilde{u}_\varepsilon(x)}{\left(\int_\Omega \tilde{u}_\varepsilon^{p^*} \, dx\right)^{1/p^*}},$$

so that $\int_\Omega v_\varepsilon^{p^*} \, dx = 1$. Then we have the well-known estimates

$$\int_\Omega |\nabla v_\varepsilon|^p \, dx \le S + C\varepsilon^{(N-p)/p}, \tag{2.10}$$

$$\int_\Omega v_\varepsilon^p \, dx \ge \begin{cases} \dfrac{1}{C}\, \varepsilon^{p-1}, & N > p^2 \\[2ex] \dfrac{1}{C}\, \varepsilon^{p-1}\, |\log \varepsilon|, & N = p^2, \end{cases} \tag{2.11}$$

where $C = C(N,p) > 0$ is a constant (see, e.g., Drábek and Huang [15]).

3 Proof of Theorem 1.1

First we show that I_μ has a uniformly positive mountain pass level below the threshold for compactness given in Lemma 2.1 for all sufficiently small $\mu > 0$. Let v_ε be as in the last section.

Lemma 3.1. *There exist $\mu_0, \rho, c_0 > 0$, $R > \rho$, and $\beta < \frac{1}{N} S^{N/p}$ such that the following hold for all $\mu \in (0, \mu_0)$:*

(i) $\|u\| = \rho \Rightarrow I_\mu(u) \ge c_0$,

(ii) $I_\mu(tv_\varepsilon) \le 0$ *for all* $t \ge R$ *and* $\varepsilon \in (0, 1]$,

(iii) *denoting by* $\Gamma = \left\{ y \in C([0,1], W_0^{1,p}(\Omega)) : y(0) = 0,\ y(1) = Rv_\varepsilon \right\}$ *the class of paths joining the origin to* Rv_ε,

$$c_0 \le c_\mu := \inf_{y \in \Gamma} \max_{u \in y([0,1])} I_\mu(u) \le \beta - \left(1 - \frac{1}{p}\right) \mu\, |\Omega| \tag{3.1}$$

for all sufficiently small $\varepsilon > 0$,

(iv) I_μ *has a critical point* u_μ *at the level* c_μ.

Proof. By (1.3) and (2.1),

$$I_\mu(u) \ge \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^p - \frac{S^{-p^*/p}}{p^*} \|u\|^{p^*} - \left(1 - \frac{1}{p}\right) \mu\, |\Omega|,$$

and (i) follows from this for sufficiently small $\rho, c_0, \mu > 0$ since $\lambda < \lambda_1$.

Since $v_\varepsilon \geq 0$,

$$I_\mu(tv_\varepsilon) = \frac{t^p}{p} \int\limits_\Omega \left(|\nabla v_\varepsilon|^p - \lambda v_\varepsilon^p\right) dx - \frac{t^{p^*}}{p^*} + \mu t \int\limits_\Omega v_\varepsilon \, dx$$

for $t \geq 0$. By the Hölder's and Young's inequalities,

$$\mu t \int\limits_\Omega v_\varepsilon \, dx \leq \mu t \, |\Omega|^{1-1/p} \left(\int\limits_\Omega v_\varepsilon^p \, dx \right)^{1/p} \leq C_\lambda \, \mu^{p/(p-1)} + \frac{\lambda t^p}{2p} \int\limits_\Omega v_\varepsilon^p \, dx,$$

where

$$C_\lambda = \left(1 - \frac{1}{p}\right) \left(\frac{2}{\lambda}\right)^{1/(p-1)} |\Omega| \, ,$$

so

$$I_\mu(tv_\varepsilon) \leq \frac{t^p}{p} \int\limits_\Omega \left(|\nabla v_\varepsilon|^p - \frac{\lambda}{2} v_\varepsilon^p\right) dx - \frac{t^{p^*}}{p^*} + C_\lambda \, \mu^{p/(p-1)}. \tag{3.2}$$

Then by (2.10) and for $\varepsilon, \mu \in (0, 1]$,

$$I_\mu(tv_\varepsilon) \leq (S + C) \, \frac{t^p}{p} - \frac{t^{p^*}}{p^*} + C_\lambda,$$

from which (ii) follows for sufficiently large $R > \rho$.

The first inequality in (3.1) is immediate from (i) since $R > \rho$. Maximizing the right-hand side of (3.2) over $t \geq 0$ gives

$$c_\mu \leq \frac{1}{N} \left[\int\limits_\Omega \left(|\nabla v_\varepsilon|^p - \frac{\lambda}{2} v_\varepsilon^p\right) dx \right]^{N/p} + C_\lambda \, \mu^{p/(p-1)},$$

and (2.10) and (2.11) imply that the integral on the right-hand side is strictly less than S for all sufficiently small $\varepsilon > 0$ since $N \geq p^2$ and $\lambda > 0$, so the second inequality in (3.1) holds for sufficiently small $\mu > 0$.

Finally, (iv) follows from (i)–(iii), Lemma 2.1, and the mountain pass lemma (see Ambrosetti and Rabinowitz [16]). □

Next we show that u_μ is uniformly bounded in $W_0^{1,p}(\Omega)$ and uniformly equi-integrable in $L^{p^*}(\Omega)$, and hence also uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ by de Figueiredo et al. [10, Proposition 3.7], for all sufficiently small $\mu \in (0, \mu_0)$.

Lemma 3.2. *There exists $\mu_\star \in (0, \mu_0]$ such that the following hold for all $\mu \in (0, \mu_\star)$:*

(i) *u_μ is uniformly bounded in $W_0^{1,p}(\Omega)$,*

(ii) $\displaystyle\int\limits_E |u_\mu|^{p^*} dx \to 0$ *as $|E| \to 0$, uniformly in μ,*

(iii) *u_μ is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

Proof. We have

$$I_\mu(u_\mu) = \int\limits_\Omega \left(\frac{|\nabla u_\mu|^p}{p} - \frac{\lambda u_{\mu+}^p}{p} - \frac{u_{\mu+}^{p^*}}{p^*} \right) dx + \mu \left[\int\limits_{\{u_\mu \geq 0\}} u_\mu \, dx \right.$$

$$\left. + \int\limits_{\{-1 < u_\mu < 0\}} \left(u_\mu - \frac{|u_\mu|^{p-1} u_\mu}{p} \right) dx - \left(1 - \frac{1}{p}\right) |\{u_\mu \leq -1\}| \right] = c_\mu \tag{3.3}$$

and

$$I'_\mu(u_\mu)\, v = \int_\Omega \left(|\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla v - \lambda u_{\mu+}^{p-1}\, v - u_{\mu+}^{p^*-1}\, v \right) dx + \mu \left[\int_{\{u_\mu \ge 0\}} v\, dx \right.$$

$$\left. + \int_{\{-1<u_\mu<0\}} \left(1 - |u_\mu|^{p-1} \right) v\, dx \right] = 0 \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.4}$$

Taking $v = u_\mu$ in (3.4), dividing by p, and subtracting from (3.3) gives

$$\frac{1}{N} \int_\Omega u_{\mu+}^{p^*}\, dx \le c_\mu + \left(1 - \frac{1}{p} \right) \mu\, |\Omega| \le \beta \tag{3.5}$$

by (3.1), and (i) follows from this, (3.4) with $v = u_\mu$, and the Hölder inequality.

If (ii) does not hold, then there exist sequences $\mu_j \to 0$ and (E_j) with $|E_j| \to 0$ such that

$$\varliminf \int_{E_j} |u_{\mu_j}|^{p^*}\, dx > 0. \tag{3.6}$$

Since (u_{μ_j}) is bounded by (i), so is (u_{μ_j+}), a renamed subsequence of which then converges to some $v \ge 0$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, p^*)$ and a.e. in Ω, and

$$|\nabla u_{\mu_j+}|^p\, dx \xrightarrow{w^*} \kappa, \qquad u_{\mu_j+}^{p^*}\, dx \xrightarrow{w^*} \nu \tag{3.7}$$

in the sense of measures, where κ and ν are bounded nonnegative measures on $\overline{\Omega}$. By Lions [12, 13], then there exist an at most countable index set I and points $x_i \in \overline{\Omega},\ i \in I$ such that

$$\kappa \ge |\nabla v|^p\, dx + \sum_{i \in I} \kappa_i\, \delta_{x_i}, \qquad \nu = v^{p^*}\, dx + \sum_{i \in I} \nu_i\, \delta_{x_i}, \tag{3.8}$$

where $\kappa_i, \nu_i > 0$ and $\nu_i^{p/p^*} \le \kappa_i / S$. Suppose I is nonempty, say, $i \in I$. An argument similar to that in the proof of Lemma 2.1 shows that $\kappa_i \le \nu_i$, so $\nu_i \ge S^{N/p}$. On the other hand, passing to the limit in (3.5) with $\mu = \mu_j$ and using (3.7) and (3.8) gives $\nu_i \le N\beta < S^{N/p}$, a contradiction. Hence $I = \emptyset$ and

$$\int_\Omega u_{\mu_j+}^{p^*}\, dx \to \int_\Omega v^{p^*}\, dx.$$

As in the proof of Lemma 2.1, a further subsequence of (u_{μ_j}) then converges to some u in $W_0^{1,p}(\Omega)$, and hence also in $L^{p^*}(\Omega)$, and a.e. in Ω. Then

$$\int_{E_j} |u_{\mu_j}|^{p^*}\, dx \le \int_\Omega \left| |u_{\mu_j}|^{p^*} - |u|^{p^*} \right| dx + \int_{E_j} |u|^{p^*}\, dx \to 0,$$

contradicting (3.6).

Finally, (iii) follows from (i), (ii), and de Figueiredo et al. [10, Proposition 3.7]. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We claim that u_μ is positive in Ω, and hence a weak solution of problem (1.2), for all sufficiently small $\mu \in (0, \mu_*)$. It suffices to show that for every sequence $\mu_j \to 0$, a subsequence of u_{μ_j} is positive in Ω. By Lemma 3.2 (iii), a renamed subsequence of u_{μ_j} converges to some u in $C^1_0(\overline{\Omega})$. We have

$$I_{\mu_j}(u_{\mu_j}) = \int_\Omega \left(\frac{|\nabla u_{\mu_j}|^p}{p} - \frac{\lambda u_{\mu_j+}^p}{p} - \frac{u_{\mu_j+}^{p^*}}{p^*} \right) dx + \mu_j \Bigg[\int_{\{u_{\mu_j} \ge 0\}} u_{\mu_j}\, dx$$

$$+ \int_{\{-1 < u_{\mu_j} < 0\}} \left(u_{\mu_j} - \frac{|u_{\mu_j}|^{p-1}\, u_{\mu_j}}{p} \right) dx - \left(1 - \frac{1}{p} \right) \left| \{ u_{\mu_j} \le -1 \} \right| \Bigg] = c_{\mu_j} \ge c_0$$

by (3.1) and

$$I'_{\mu_j}(u_{\mu_j})\, v = \int_\Omega \left(|\nabla u_{\mu_j}|^{p-2}\, \nabla u_{\mu_j} \cdot \nabla v - \lambda u_{\mu_j+}^{p-1}\, v - u_{\mu_j+}^{p^*-1}\, v \right) dx + \mu_j \Bigg[\int_{\{u_{\mu_j} \ge 0\}} v\, dx$$

$$+ \int_{\{-1 < u_{\mu_j} < 0\}} \left(1 - |u_{\mu_j}|^{p-1} \right) v\, dx \Bigg] = 0 \quad \forall v \in W^{1,p}_0(\Omega),$$

and passing to the limits gives

$$\int_\Omega \left(\frac{|\nabla u|^p}{p} - \frac{\lambda u_+^p}{p} - \frac{u_+^{p^*}}{p^*} \right) dx \ge c_0$$

and

$$\int_\Omega \left(|\nabla u|^{p-2}\, \nabla u \cdot \nabla v - \lambda u_+^{p-1}\, v - u_+^{p^*-1}\, v \right) dx = 0 \quad \forall v \in W^{1,p}_0(\Omega),$$

so u is a nontrivial weak solution of the problem

$$\begin{cases} -\Delta_p\, u = \lambda u_+^{p-1} + u_+^{p^*-1} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u > 0$ in Ω and its interior normal derivative $\partial u / \partial \nu > 0$ on $\partial\Omega$ by the strong maximum principle and the Hopf lemma for the p-Laplacian (see Vázquez [17]). Since $u_{\mu_j} \to u$ in $C^1_0(\overline{\Omega})$, then $u_{\mu_j} > 0$ in Ω for all sufficiently large j.

It remains to show that u_μ minimizes I_μ on $\mathcal{N}_\mu$ when it is positive. For each $w \in \mathcal{N}_\mu$, we will construct a path $y_w \in \Gamma$ such that

$$\max_{u \in y_w([0,1])} I_\mu(u) = I_\mu(w).$$

Since

$$I_\mu(u_\mu) = c_\mu \le \max_{u \in y_w([0,1])} I_\mu(u)$$

by the definition of c_μ, the desired conclusion will then follow. First we note that the function

$$g(t) = I_\mu(tw) = \frac{t^p}{p} \int_\Omega \left(|\nabla w|^p - \lambda w^p \right) dx - \frac{t^{p^*}}{p^*} \int_\Omega w^{p^*}\, dx + \mu t \int_\Omega w\, dx, \quad t \ge 0$$

has a unique maximum at $t = 1$. Indeed,

$$g'(t) = t^{p-1} \int_\Omega \left(|\nabla w|^p - \lambda w^p \right) dx - t^{p^*-1} \int_\Omega w^{p^*}\, dx + \mu \int_\Omega w\, dx$$

$$= \left(t^{p-1} - t^{p^*-1} \right) \int_\Omega \left(|\nabla w|^p - \lambda w^p \right) dx + \left(1 - t^{p^*-1} \right) \mu \int_\Omega w\, dx$$

since $w \in \mathcal{N}_\mu$, and the last two integrals are positive since $\lambda < \lambda_1$ and $w > 0$, so $g'(t) > 0$ for $0 \le t < 1$, $g'(1) = 0$, and $g'(t) < 0$ for $t > 1$. Hence

$$\max_{t \ge 0} I_\mu(tw) = I_\mu(w) > 0$$

since $g(0) = 0$. In view of Lemma 3.1 (ii), now it suffices to observe that there exists $\widetilde{R} > \max\{1, R\}$ such that

$$I_\mu(\widetilde{R}u) = \frac{\widetilde{R}^p}{p} \int_\Omega \left(|\nabla u|^p - \lambda u^p\right) dx - \frac{\widetilde{R}^{p^*}}{p^*} \int_\Omega u^{p^*}\, dx + \mu\widetilde{R} \int_\Omega u\, dx \le 0$$

for all u on the line segment joining w to v_ε since all norms on a finite dimensional space are equivalent. □

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References

[1] Alfonso Castro, Djairo G. de Figueredo, and Emer Lopera. Existence of positive solutions for a semipositone p-Laplacian problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 146(3):475–482, 2016.

[2] M. Chhetri, P. Drábek, and R. Shivaji. Existence of positive solutions for a class of p-Laplacian superlinear semipositone problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(5):925–936, 2015.

[3] S. Unsurangie. *Existence of a solution for a wave and elliptic Dirichlet problem*. PhD thesis, University of North Texas, Denton, 1988.

[4] W. Allegretto, P. Nistri, and P. Zecca. Positive solutions of elliptic nonpositone problems. *Differential Integral Equations*, 5(1):95–101, 1992.

[5] A. Ambrosetti, D. Arcoya, and B. Buffoni. Positive solutions for some semi-positone problems via bifurcation theory. *Differential Integral Equations*, 7(3-4):655–663, 1994.

[6] Scott Caldwell, Alfonso Castro, Ratnasingham Shivaji, and Sumalee Unsurangsie. Positive solutions for classes of multiparameter elliptic semipositone problems. *Electron. J. Differential Equations*, pages No. 96, 10 pp. (electronic), 2007.

[7] Mohammed Guedda and Laurent Véron. Quasilinear elliptic equations involving critical Sobolev exponents. *Nonlinear Anal.*, 13(8):879–902, 1989.

[8] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[9] P.-L. Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.*, 24(4):441–467, 1982.

[10] Djairo G. de Figueiredo, Jean-Pierre Gossez, and Pedro Ubilla. Local "superlinearity" and "sublinearity" for the p-Laplacian. *J. Funct. Anal.*, 257(3):721–752, 2009.

[11] Gerald B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[12] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[13] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985.

[14] Kanishka Perera, Ravi P. Agarwal, and Donal O'Regan. *Morse theoretic aspects of p-Laplacian type operators*, volume 161 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[15] Pavel Drábek and Yin Xi Huang. Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbf{R}^N$ with critical Sobolev exponent. *J. Differential Equations*, 140(1):106–132, 1997.

[16] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[17] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.