Research Article

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Regularity of solutions of the parabolic normalized *p*-Laplace equation

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Abstract: The parabolic normalized p-Laplace equation is studied. We prove that a viscosity solution has a time derivative in the sense of Sobolev belonging locally to L^2.

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1 Introduction

We consider viscosity solutions of the *normalized p-Laplace* equation

$$\frac{\partial u}{\partial t} = |\nabla u|^{2-p} \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty, \tag{1.1}$$

in $\Omega_T = \Omega \times (0, T)$, Ω being a domain in $\mathbb{R}^n$. Formally, the equation reads

$$\frac{\partial u}{\partial t} = \Delta u + (p-2)|\nabla u|^{-2} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

In the linear case $p = 2$, we have the heat equation $u_t = \Delta u$, and also for $n = 1$, the equation reduces to the heat equation $u_t = (p-1)u_{xx}$. At the limit $p = 1$, we obtain the equation for motion by mean curvature. We aim at showing that the time derivative $\frac{\partial u}{\partial t}$ exists in the Sobolev sense and belongs to $L^2_{\mathrm{loc}}(\Omega_T)$. We also study the second derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$.

There has been some recent interest in connection with stochastic game theory, where the equation appears, cf. [7]. From our point of view, the work [3] is of actual interest, because there it is shown that the time derivative u_t of the viscosity solutions exists and is locally bounded, provided that the lateral boundary values are smooth. Thus, the boundary values control the time regularity. If no such assumptions about the behaviour at the lateral boundary $\partial\Omega \times (0, T)$ are made, a conclusion like $u_t \in L^\infty_{\mathrm{loc}}(\Omega_T)$ is in doubt. Our main result is the following, where we unfortunately have to restrict p.

Theorem 1.1. *Suppose that $u = u(x, t)$ is a viscosity solution of the normalized p-Laplace equation in Ω_T. If $\frac{6}{5} < p < \frac{14}{5}$, then the Sobolev derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist and belong to $L^2_{\mathrm{loc}}(\Omega_T)$.*

We emphasize that no assumptions on the boundary values are made for this interior estimate. Our method of proof is based on a verification of the identity

$$\int_0^T \int_\Omega u\phi_t \, dx \, dt = -\int_0^T \int_\Omega U\phi \, dx \, dt, \quad \phi \in C_0^\infty(\Omega_T),$$

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where we have to prove that the function U, which is the right-hand side of equation (1.1), belongs to $L^2_{\mathrm{loc}}(\Omega_T)$. Thus, the second spatial derivatives D^2u are crucial (local boundedness of ∇u was proven in [2, 3] and interior Hölder estimates for the gradient in [6]). The elliptic case has been studied in [1].

In the range $1 < p < 2$, one can bypass the question of second derivatives.

Theorem 1.2. *Suppose that $u = u(x, t)$ is a viscosity solution of the normalized p-Laplace equation in Ω_T. If $1 < p < 2$, then the Sobolev derivative $\frac{\partial u}{\partial t}$ exists and belongs to $L^2_{\mathrm{loc}}(\Omega_T)$.*

To avoid the problem of vanishing gradient, we first study the regularized equation

$$\frac{\partial u^\epsilon}{\partial t} = (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \nabla u^\epsilon\big). \tag{1.2}$$

Here the classical parabolic regularity theory is applicable. The equation was studied by Does in [3], where an estimate of the gradient ∇u^ϵ was found with Bernstein's method. We shall prove a maximum principle for the gradient. Further, we differentiate equation (1.2) with respect to the space variables and derive estimates for u^ϵ, which are passed over to the solution u of (1.1).

Analogous results seem to be possible to reach through the *Cordes condition*. This also restricts the range of valid exponents p. We have refrained from this approach, mainly since the absence of zero (lateral) boundary values produces many undesired terms to estimate. Finally, we mention that the limits $\frac{6}{5}$ and $\frac{14}{5}$ in Theorem 1.1 are evidently an artifact of the method. It would be interesting to know whether the theorem is valid in the whole range $1 < p < \infty$. In any case, our method is not capable to reach all exponents.

2 Preliminaries

Notation. The gradient of a function $f\colon \Omega_T \to \mathbb{R}$ is

$$\nabla f = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\Big)$$

and its Hessian matrix is

$$(D^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad |D^2 f|^2 = \sum_{i,j=1}^{n} \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big)^2.$$

We shall, occasionally, use the abbreviation

$$u_j = \frac{\partial u}{\partial x_j}, \quad u_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k}$$

for partial derivatives. Young's inequality

$$|ab| \le \delta \frac{|a|^p}{p} + \Big(\frac{1}{\delta}\Big)^{q-1} \frac{|b|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

is often referred to. Finally, the summation convention is used when convenient.

Viscosity solutions. The normalized p-Laplace equation is not in divergence form. Thus, the concept of weak solutions with test functions under the integral sign is problematic. Fortunately, the modern concept of viscosity solutions works well. The existence and uniqueness of viscosity solutions of the normalized p-Laplace equation was established in [2]. We recall the definition.

Definition 2.1. We say that an upper semi-continuous function u is a *viscosity subsolution* of equation (1.1) if for all $\phi \in C^2(\Omega_T)$, we have

$$\phi_t \le \Big(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\Big)\phi_{x_i x_j}$$

at any interior point (x, t) where $u - \phi$ attains a local maximum, provided $\nabla\phi(x, t) \neq 0$. Further, at any interior

point (x, t) where $u - \phi$ attains a local maximum and $\nabla\phi(x, t) = 0$, we require

$$\phi_t \le (\delta_{ij} + (p-2)\eta_i\eta_j)\phi_{x_ix_j}$$

for some $\eta \in \mathbb{R}^n$, with $|\eta| \le 1$.

Definition 2.2. We say that a lower semi-continuous function u is a *viscosity supersolution* of equation (1.1) if for all $\phi \in C^2(\Omega_T)$, we have

$$\phi_t \ge \Big(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\Big)\phi_{x_ix_j}$$

at any interior point (x, t) where $u - \phi$ attains a local minimum, provided $\nabla\phi(x, t) \neq 0$. Further, at any interior point (x, t) where $u - \phi$ attains a local minimum and $\nabla\phi(x, t) = 0$, we require

$$\phi_t \ge (\delta_{ij} + (p-2)\eta_i\eta_j)\phi_{x_ix_j}$$

for some $\eta \in \mathbb{R}^n$, with $|\eta| \le 1$.

Definition 2.3. A continuous function u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

For a detailed discussion on the definition at critical points, we refer to [5]. The reason behind the choice of $\eta \in \mathbb{R}^n$ is given in [5, Section 2]. The viscosity solutions of equation (1.2) are defined in a similar manner, except that now $\nabla\phi(x, t) = 0$ is not a problem.

Maximum principle for the gradient. In order to estimate the time derivative, we need bounds on the second derivatives of u^ϵ (and also on its gradient). If we first assume that u^ϵ is C^1 on the parabolic boundary $\partial_{\text{par}}\Omega_T$, we get bounds on the gradient in all of Ω_T. This follows from the following maximum principle.

Proposition 2.4 (Maximum principle). *Let u^ϵ be a solution of equation* (1.2). *If $\nabla u^\epsilon \in C^1(\overline{\Omega}_T)$, then*

$$\max_{\overline{\Omega}_T}\{|\nabla u^\epsilon|\} = \max_{\partial_{\text{par}}\Omega_T}\{|\nabla u^\epsilon|\}.$$

Proof. With some modifications, a proof can be extracted from [3]. We give a direct proof. To this end, consider

$$V^\epsilon(x, t) = |\nabla u^\epsilon|^2 + \epsilon^2.$$

To find the partial differential equation satisfied by V^ϵ, we calculate[1]

$$V^\epsilon_i = 2u^\epsilon_v u^\epsilon_{iv}, \quad V^\epsilon_{ij} = 2u^\epsilon_{vj}u^\epsilon_{iv} + 2u^\epsilon_v u^\epsilon_{ijv}, \quad u^\epsilon_i u^\epsilon_j V^\epsilon_{ij} = \frac{1}{2}|\nabla V^\epsilon|^2 + 2u^\epsilon_i u^\epsilon_j u^\epsilon_v u^\epsilon_{ijv}.$$

Writing equation (1.1) in the form

$$u^\epsilon_t = \Big(\delta_{ij} + (p-2)\frac{u^\epsilon_i u^\epsilon_j}{|\nabla u^\epsilon|^2 + \epsilon^2}\Big)u^\epsilon_{ij},$$

we find

$$\frac{1}{2}V^\epsilon_t = u^\epsilon_v\frac{\partial}{\partial x_v}u^\epsilon_t = u^\epsilon_v\Delta u^\epsilon_v - \frac{p-2}{2(V^\epsilon)^2}|\langle\nabla u^\epsilon, \nabla V^\epsilon\rangle|^2 + \frac{p-2}{V^\epsilon}\Big(\frac{1}{4}|\nabla V^\epsilon|^2 + \frac{1}{2}u^\epsilon_v u^\epsilon_\mu V^\epsilon_{v\mu}\Big).$$

Rearranging and using

$$\Delta V^\epsilon = 2|D^2u^\epsilon|^2 + 2\langle\nabla u^\epsilon, \nabla\Delta u^\epsilon\rangle,$$

we arrive at the following differential equation for V^ϵ:

$$V^\epsilon_t = \Delta V^\epsilon - 2|D^2u^\epsilon|^2 - \frac{p-2}{(V^\epsilon)^2}|\langle\nabla u^\epsilon, \nabla V^\epsilon\rangle|^2 + \frac{p-2}{V^\epsilon}\Big\{\frac{1}{2}|\nabla V^\epsilon|^2 + u^\epsilon_v u^\epsilon_\mu V^\epsilon_{v\mu}\Big\}. \tag{2.1}$$

Let

$$w(x, t) = |\nabla u^\epsilon(x, t)|^2 + \epsilon^2 - \alpha t = V^\epsilon(x, t) - \alpha t \quad \text{for } \alpha > 0.$$

1 Sum over repeated indices.

Suppose that w^ϵ has an *interior* maximum point at (x_0, t_0). At this point, $V^\epsilon(x_0, t_0) > 0$, otherwise we would have $V^\epsilon(x, t) \equiv 0$ in Ω_T, in which case there is nothing to prove. By the infinitesimal calculus,

$$\nabla w(x_0, t_0) = 0 \quad \text{and} \quad w_t(x_0, t_0) \geq 0,$$

where we have included the case $t_0 = T$. Further, the matrix $D^2 w(x_0, t_0)$ is negative semidefinite. Using equation (2.1) and noting that $\nabla w = \nabla V^\epsilon$ and $D^2 w = D^2 V^\epsilon$, we get, at (x_0, t_0),

$$\begin{aligned} 0 \leq w_t &= V_t^\epsilon - \alpha \\ &= \Delta V^\epsilon - 2|D^2 u^\epsilon|^2 - \frac{p-2}{(V^\epsilon)^2}|\langle \nabla u^\epsilon, \nabla V^\epsilon \rangle|^2 + \frac{p-2}{V^\epsilon}\Big\{\frac{1}{2}|\nabla V^\epsilon|^2 + u_\nu^\epsilon u_\mu^\epsilon V_{\nu\mu}^\epsilon\Big\} - \alpha \\ &= \Big(\delta_{ij} + (p-2)\frac{u_i^\epsilon u_j^\epsilon}{V^\epsilon}\Big) w_{ij}^\epsilon - 2|D^2 u^\epsilon|^2 - \alpha \leq -\alpha, \end{aligned}$$

since the matrix A, with elements $A_{ij} = \delta_{ij} + (p-2)\frac{u_i^\epsilon u_j^\epsilon}{V^\epsilon}$, is positive semidefinite. To avoid the contradiction $\alpha \leq 0$, w must attain its maximum on the parabolic boundary.

Hence, for any $(x, t) \in \Omega_T$, we have

$$V^\epsilon(x, t) - \alpha t \leq \max_{\partial_{\mathrm{par}}\Omega_T} \{V^\epsilon(x, t) - \alpha t\} \leq \max_{\partial_{\mathrm{par}}\Omega_T} V^\epsilon(x, t).$$

We finish the proof by sending $\alpha \to 0^+$. □

With no assumptions for u^ϵ on the parabolic boundary, we need a stronger result, taken from [3, p. 381].

Theorem 2.5. *Let u^ϵ be a solution of equation* (1.2)*, with $u^\epsilon(x, 0) = u_0(x)$. Then*

$$|\nabla u^\epsilon(x, t)| \leq C_{n,p}\|u_0\|_{L^\infty(\Omega_T)}\Big\{1 + \Big(\frac{1}{\operatorname{dist}((x, t), \partial_{\mathrm{par}}\Omega_T)}\Big)^2\Big\}.$$

Note that no condition on the lateral boundary $\partial\Omega \times [0, T]$ was used. By continuity,

$$|\nabla u^\epsilon(x, t)| \leq C_{n,p}\|u^\epsilon(\cdot, t_0)\|_\infty\Big\{1 + \Big(\frac{1}{\operatorname{dist}((x, t), \partial_{\mathrm{par}}\Omega_T)}\Big)^2\Big\}$$

for $x \in D \subset\subset \Omega$ and $0 < t_0 \leq t \leq T - t_0$. The estimate

$$\|\nabla u^\epsilon\|_{L^\infty(D\times[t_0, T-t_0])} \leq C\|u^\epsilon\|_{L^\infty(\Omega_T)}\Big\{1 + \Big(\frac{1}{\operatorname{dist}(D, \partial_{\mathrm{par}}\Omega_T)}\Big)^2\Big\} \tag{2.2}$$

follows. (Here one can pass to the limit as $\epsilon \to 0$.)

The proof of the lemma below, a simple special case of the Miranda–Talenti lemma, can be found for smooth functions in [4, p. 308]. If f is not smooth, we perform a *strictly* interior approximation, so that no boundary integrals appear (which is possible since $\xi \in C_0^\infty$).

Lemma 2.6 (Miranda–Talenti). *Let $\xi \in C_0^\infty(\Omega_T)$ and $f \in L^2(0, T, W^{2,2}(\Omega))$. Then*

$$\int_0^T \int_\Omega |\Delta(\xi f)|^2 \, dx \, dt = \int_0^T \int_\Omega |D^2(\xi f)|^2 \, dx \, dt.$$

3 Regularization

The next lemma tells us that the solutions of (1.2) converge locally uniformly to the viscosity solution of (1.1).

Lemma 3.1. *Let u be a viscosity solution of equation* (1.1) *and let u^ϵ be the classical solution of the regularized equation* (1.2) *with boundary values*

$$u = u^\epsilon \quad \text{on } \partial_{\mathrm{par}}\Omega_T.$$

Then $u^\epsilon \to u$ uniformly on compact subsets of Ω_T.

Proof. By Theorem 2.5, we can use Ascoli's theorem to extract a convergent subsequence u^{ϵ_j} converging locally uniformly to some continuous function, namely, $u^{\epsilon_j} \to v$. We claim that v is a viscosity solution of equation (1.1). The lemma then follows by uniqueness.

We demonstrate that v is a viscosity subsolution. (A symmetric proof shows that v is a viscosity supersolution.) Assume that $v - \phi$ attains a strict local maximum at $z_0 = (x_0, t_0)$. Since $u^\epsilon \to v$ locally uniformly, there are points

$$z_\epsilon \to z_0$$

such that $u^\epsilon - \phi$ attains a local maximum at z_ϵ. If $\nabla\phi(z_0) \neq 0$, then $\nabla\phi(z_\epsilon) \neq 0$ for all $\epsilon > 0$ small enough, and at z_ϵ, we have

$$\phi_t \leq \left(\delta_{ij} + (p-2)\frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2 + \epsilon^2}\right)\phi_{x_i x_j}. \tag{3.1}$$

Letting $\epsilon \to 0$, we see that v satisfies Definition 2.3 when $\nabla\phi(z_0) \neq 0$. If $\nabla\phi(z_0) = 0$, let

$$\eta_\epsilon = \frac{\nabla\phi(z_\epsilon)}{\sqrt{|\nabla\phi(z_\epsilon)|^2 + \epsilon^2}}.$$

Since $|\eta_\epsilon| \leq 1$, there is a subsequence such that $\eta_{\epsilon_k} \to \eta$ when $k \to \infty$ for some $\eta \in \mathbb{R}^n$, with $|\eta| \leq 1$. Passing to the limit $\epsilon_k \to 0$ in equation (3.1), we see that v is a viscosity subsolution. □

Our proof of Theorem 1.1 consists in showing that the second derivatives D^2u^ϵ belong locally to L^2 with a bound independent of ϵ. Once this is established, we see that

$$(|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}}\nabla u^\epsilon\big) = \Delta u^\epsilon + \frac{p-2}{|\nabla u^\epsilon|^2 + \epsilon^2}\langle \nabla u^\epsilon, D^2u^\epsilon \nabla u^\epsilon\rangle \leq C_{p,n}|D^2u^\epsilon|.$$

Hence, for any bounded subdomain $D \subset\subset \Omega_T$,

$$\big\|(|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}}\nabla u^\epsilon\big)\big\|_{L^2(D)} \leq C,$$

with C independent of ϵ. By this uniform bound, there exists a subsequence such that, as $j \to \infty$,

$$(|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{p-2}{2}}\nabla u^{\epsilon_j}\big) \to U \quad \text{weakly in } L^2(D).$$

In particular, this means that $U \in L^2(D)$ and for any $\phi \in C_0^\infty(D)$, we have

$$\lim_{j\to\infty} \int_0^T \int_D \phi(|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{p-2}{2}}\nabla u^{\epsilon_j}\big)\,dx\,dt = \int_0^T \int_D \phi U\,dx\,dt.$$

If u is the unique viscosity solution of (1.1), we invoke Lemma 3.1 and the calculations above to find, for any test function $\phi \in C_0^\infty(D)$,

$$\begin{aligned}
\int_0^T \int_D u\frac{\partial\phi}{\partial t}\,dx\,dt &= \lim_{j\to\infty} \int_0^T \int_D u^{\epsilon_j}\frac{\partial\phi}{\partial t}\,dx\,dt \\
&= -\lim_{j\to\infty} \int_0^T \int_D \phi(|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{2-p}{2}} \operatorname{div}\big((|\nabla u^{\epsilon_j}|^2 + \epsilon_j^2)^{\frac{p-2}{2}}\nabla u^{\epsilon_j}\big)\,dx\,dt \\
&= -\int_0^T \int_D \phi U\,dx\,dt.
\end{aligned}$$

This shows that the Sobolev derivative u_t exists and, since the previous equation holds for any subdomain $D \subset\subset \Omega_T$, we conclude that $\frac{\partial u}{\partial t} = U \in L^2_{\text{loc}}(\Omega_T)$. To complete the proof of Theorem 1.1, it remains to establish the missing local bound of $\|D^2u^\epsilon\|_{L^2}$ uniformly in ϵ.

4 The differentiated equation

We shall derive a fundamental identity. Let

$$v^\epsilon = |\nabla u^\epsilon|^2, \quad V^\epsilon = |\nabla u^\epsilon|^2 + \epsilon^2.$$

Differentiating equation (1.2) with respect to the variable x_j, we obtain

$$\frac{\partial}{\partial t} u_j^\epsilon = \frac{2-p}{2}(V^\epsilon)^{-\frac{p}{2}} v_j^\epsilon \operatorname{div}((V^\epsilon)^{\frac{p-2}{2}} \nabla u^\epsilon) + (V^\epsilon)^{\frac{2-p}{2}} \operatorname{div}[((V^\epsilon)^{\frac{p-2}{2}} \nabla u^\epsilon)_j].$$

Take $\xi \in C_0^\infty(\Omega_T)$, with $\xi \geq 0$. Multiply both sides of the equation by $\xi^2 V^\epsilon u_j^\epsilon$ and sum j from 1 to n. Integrate over Ω_T, using integration by parts and keeping in mind that ξ is compactly supported in Ω_T, to obtain

$$\begin{aligned} -\frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon \, dx\, dt = \frac{2-p}{2}\int_0^T\int_\Omega \xi^2 (V^\epsilon)^{-\frac{p}{2}} \langle \nabla u^\epsilon, \nabla v^\epsilon\rangle \operatorname{div}((V^\epsilon)^{\frac{p-2}{2}} \nabla u^\epsilon)\, dx\, dt \\ - \int_0^T\int_\Omega \frac{\partial}{\partial x_j}\{(V^\epsilon)^{\frac{p-2}{2}} u_k^\epsilon\} \frac{\partial}{\partial x_k}\{\xi^2 (V^\epsilon)^{\frac{2-p}{2}} u_j^\epsilon\}\, dx\, dt. \end{aligned}$$

Writing out the derivatives gives the fundamental formula

$$\begin{aligned} \mathrm{I} + \mathrm{II} :=& \int_0^T\int_\Omega \xi^2 |D^2 u^\epsilon|^2 \, dx\, dt + \frac{p-2}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi^2 \langle \nabla u^\epsilon, \nabla v^\epsilon\rangle \Delta u^\epsilon \, dx\, dt \\ =& \frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon \, dx\, dt + (2-p)\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi \langle \nabla u^\epsilon, \nabla v^\epsilon\rangle \langle \nabla u^\epsilon, \nabla \xi\rangle \, dx\, dt - \int_0^T\int_\Omega \xi \langle \nabla v^\epsilon, \nabla \xi \, dx\, dt \\ =:& \ \mathrm{III} + \mathrm{IV} - \mathrm{V}. \end{aligned}$$

In the next section we shall bound the main term I uniformly with respect to ϵ.

5 Estimate of the second derivatives

We shall provide an estimate of the main term I. First, we record the elementary inequality

$$|\nabla v^\epsilon|^2 = |2D^2 u^\epsilon \nabla u^\epsilon|^2 \leq 4|D^2 u^\epsilon|^2 v^\epsilon. \tag{5.1}$$

One dimension. As an exercise, we show that in this case, the second derivatives are locally bounded in L^2 for any $1 < p < \infty$. In one dimension, equation (1.1) reads

$$u_t = |u_x|^{2-p}\frac{\partial}{\partial x}\{|u_x|^{p-2}u_x\} = (p-1)u_{xx}.$$

We absorb the terms IV and V, using Young's inequality and inequality (5.1). For any $\delta > 0$,

$$\begin{aligned} \int_0^T\int_\Omega \xi^2\Big(\frac{\partial^2 u^\epsilon}{\partial x^2}\Big)^2 \Bigg(1 + (p-2)\frac{(\frac{\partial u^\epsilon}{\partial x})^2}{(\frac{\partial u^\epsilon}{\partial x})^2 + \epsilon^2} - \delta(|p-2|+1)\Bigg) dx\, dt \\ \leq \frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon \, dx\, dt + \frac{|p-2|+1}{\delta}\int_0^T\int_\Omega V^\epsilon |\nabla \xi|^2 \, dx\, dt. \end{aligned}$$

Applying Theorem 2.5 we see that the right-hand side is bounded by a constant independent of $\epsilon > 0$. We have

$$1 + (p-2)\frac{(\frac{\partial u^\epsilon}{\partial x})^2}{(\frac{\partial u^\epsilon}{\partial x})^2 + \epsilon^2} \geq \min\{1, p-1\} > 0.$$

It follows that $\frac{\partial^2 u^\epsilon}{\partial x^2} \in L^2$ locally for any $p \in (1, \infty)$.

General n. We assume for the moment that $1 < p < 2$. We rewrite the term II involving the Laplacian as

$$\frac{2-p}{2}\frac{1}{V^\epsilon}\xi^2\langle\nabla u^\epsilon, \nabla v^\epsilon\rangle\Delta u^\epsilon = \frac{2-p}{2}\frac{1}{V^\epsilon}\xi\langle\nabla u^\epsilon, \nabla v^\epsilon\rangle\{\Delta(\xi u^\epsilon) - 2\langle\nabla u^\epsilon, \nabla\xi\rangle - u^\epsilon\Delta\xi\}.$$

Upon this rewriting, the term IV disappears from the equation. We focus our attention on the term involving $\Delta(\xi u^\epsilon)$. By Lemma 2.6,

$$\int_0^T\int_\Omega |D^2(\xi u^\epsilon)|^2\,dx\,dt = \int_0^T\int_\Omega |\Delta(\xi u^\epsilon)|^2\,dx\,dt.$$

Differentiating, we see that

$$(\xi u^\epsilon)_i = \xi_i u^\epsilon + \xi u_i^\epsilon, \quad (\xi u^\epsilon)_{ij} = \xi_{ij}u^\epsilon + u_i^\epsilon\xi_j + \xi_i u_j^\epsilon + \xi u_{ij}^\epsilon.$$

It follows that

$$|D^2(\xi u^\epsilon)|^2 = \xi^2|D^2u^\epsilon|^2 + f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon),$$

where $f(u^\epsilon, \nabla u_i^\epsilon, D^2u^\epsilon)$ depends only linearly on the second derivatives u_{ij}^ϵ:

$$\begin{aligned} f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon) = (u^\epsilon)^2|D^2\xi|^2 + 4u^\epsilon\langle\nabla\xi, D^2\xi\nabla u^\epsilon\rangle + 4\xi\langle\nabla\xi, D^2u^\epsilon\nabla u^\epsilon\rangle \\ + 2|\nabla\xi|^2|\nabla u^\epsilon|^2 + 2|\langle\nabla u^\epsilon, \nabla\xi\rangle|^2 + 2u^\epsilon\xi\,\mathrm{trace}\{(D^2\xi)(D^2u^\epsilon)\}. \end{aligned}$$

By Young's inequality, we obtain

$$\frac{2-p}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\langle\nabla u^\epsilon, \nabla v^\epsilon\rangle\Delta(\xi u^\epsilon)\,dx\,dt \le \frac{5}{4}(2-p)\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dx\,dt + \frac{2-p}{4}\int_0^T\int_\Omega f(u^\epsilon, \nabla u^\epsilon, D^2u^\epsilon)\,dx\,dt.$$

Inserting this into the main equation gives

$$\begin{aligned} \mathrm{I}^* := \Big(1 - \frac{5}{4}(2-p)\Big)\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dx\,dt &\le \frac{1}{2}\int_0^T\int_\Omega \xi\xi_t V^\epsilon\,dx\,dt - \int_0^T\int_\Omega \xi\langle\nabla v^\epsilon, \nabla\xi\rangle\,dx\,dt \\ &\quad + \frac{2-p}{2}\int_0^T\int_\Omega f(u^\epsilon, u_i^\epsilon, u_{ij}^\epsilon)\,dx\,dt \\ &\quad + \frac{2-p}{2}\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\langle\nabla u^\epsilon, \nabla v^\epsilon\rangle u^\epsilon\Delta\xi\,dx\,dt. \\ &=: \mathrm{III} - \mathrm{V} + \mathrm{VI} + \mathrm{VII}. \end{aligned}$$

All terms containing D^2u^ϵ can be absorbed by the new main term I^*. To this end, we use Young's inequality with a small parameter $\delta > 0$ to balance the terms.[2] For term V, we have

$$\int_0^T\int_\Omega \xi\langle\nabla v^\epsilon, \nabla\xi\rangle\,dx\,dt \le \delta\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dx\,dt + \frac{1}{\delta}\int_0^T\int_\Omega V^\epsilon|\nabla\xi|^2\,dx\,dt.$$

Similarly, for term VII,

$$\int_0^T\int_\Omega \frac{1}{V^\epsilon}\xi\langle\nabla u^\epsilon, \nabla v^\epsilon\rangle u^\epsilon\Delta\xi\,dx\,dt \le 2\delta_1\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2 + \frac{1}{\delta_1}\int_0^T\int_\Omega |u^\epsilon|^2|\Delta\xi|^2\,dx\,dt.$$

2 The parameter δ is to be made so small that terms like $\delta\int_0^T\int_\Omega \xi^2|D^2u^\epsilon|^2\,dx\,dt$ can be moved over to the left-hand side.

Using similar inequalities for the term involving $f(u^\epsilon, \nabla u^\epsilon, D^2 u^\epsilon)$ and choosing the parameters small enough in Young's inequality, we find,

$$\int_0^T \int_\Omega \xi^2 |D^2 u^\epsilon|^2 \, dx \, dt \le C \iint_{\{\xi \neq 0\}} ((u^\epsilon)^2 + |\nabla u^\epsilon|^2) \, dx \, dt, \tag{5.2}$$

where C is independent of ϵ but depends on $\|\xi\|_{C^2}$, provided that $1 - \frac{5}{4}(2-p) > 0$, i.e., $p > \frac{6}{5}$. This is now a decisive restriction. Invoking Lemma 3.1 and estimate (2.2), we deduce that the majorant in (5.2) is independent of ϵ.

A symmetric proof when $p > 2$ shows that equation (5.2) holds when $p < \frac{14}{5}$.

6 The case $1 < p < 2$

In this section, we give a proof of Theorem 1.2. To this end, let $\xi \in C_0^\infty(\Omega_T)$, with $0 \le \xi \le 1$. We claim that

$$\int_0^T \int_\Omega \xi^2 \Big(\frac{\partial u^\epsilon}{\partial t} \Big)^2 \, dx \, dt \le 4 \|V^\epsilon\|_\infty^2 \Big\{ \int_0^T \int_\Omega |\nabla \xi|^2 \, dx \, dt + \frac{1}{p} \int_0^T \int_\Omega \xi |\xi_t| \, dx \, dt \Big\}, \tag{6.1}$$

where the supremum norm of $V^\epsilon = |\nabla u^\epsilon|^2 + \epsilon^2$ is taken locally, over the support of ξ. Here, u^ϵ is the solution of the regularized equation (1.2). This is enough to complete the proof of Theorem 1.2, in virtue of Theorem 2.5.

Multiplying the regularized equation (1.2) by $(|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \xi^2 u_t^\epsilon$ yields

$$\begin{aligned} \xi^2 (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} (u_t^\epsilon)^2 &= \xi^2 u_t^\epsilon \operatorname{div}\big((|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \nabla u^\epsilon \big) \\ &= \operatorname{div}\big(\xi^2 u_t^\epsilon (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \nabla u^\epsilon \big) - (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla (\xi^2 u_t^\epsilon) \rangle. \end{aligned}$$

The integral of the divergence term vanishes by Gauss's theorem and, upon integration, we have

$$\begin{aligned} \int_0^T \int_\Omega \xi^2 (V^\epsilon)^{\frac{p-2}{2}} (u_t^\epsilon)^2 \, dx \, dt &= - \int_0^T \int_\Omega (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla (\xi^2 u_t^\epsilon) \rangle \, dx \, dt \\ &= -2 \int_0^T \int_\Omega \xi (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla \xi \rangle u_t^\epsilon \, dx \, dt - \int_0^T \int_\Omega \xi^2 (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla u_t^\epsilon \rangle \, dx \, dt. \end{aligned}$$

The first integral on the right-hand side can be absorbed by the left-hand side by choosing $\sigma = \frac{1}{2}$ in

$$\big| 2\xi (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla \xi \rangle u_t^\epsilon \big| \le \sigma \xi^2 (V^\epsilon)^{\frac{p-2}{2}} (u_t^\epsilon)^2 + \frac{1}{\sigma} (V^\epsilon)^{\frac{p-2}{2}} |\nabla u^\epsilon|^2 |\nabla \xi|^2,$$

and integrating.

For the last term, the decisive observation is that

$$\frac{1}{p} \frac{\partial}{\partial t} (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p}{2}} = (|\nabla u^\epsilon|^2 + \epsilon^2)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla u_t^\epsilon \rangle = (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla u_t^\epsilon \rangle.$$

We use this in the last integral on the right-hand side to obtain

$$\begin{aligned} - \int_0^T \int_\Omega \xi^2 (V^\epsilon)^{\frac{p-2}{2}} \langle \nabla u^\epsilon, \nabla u_t^\epsilon \rangle \, dx \, dt &= - \int_0^T \int_\Omega \frac{\partial}{\partial t} \Big\{ \frac{\xi^2}{p} (V^\epsilon)^{\frac{p}{2}} \Big\} \, dx \, dt + \frac{2}{p} \int_0^T \int_\Omega \xi \xi_t (V^\epsilon)^{\frac{p}{2}} \, dx \, dt \\ &= - \int_\Omega \Big[\frac{\xi^2}{p} (V^\epsilon)^{\frac{p}{2}} \Big]_{t=0}^{t=T} dx + \frac{2}{p} \int_0^T \int_\Omega \xi \xi_t (V^\epsilon)^{\frac{p}{2}} \, dx \, dt \\ &= \frac{2}{p} \int_0^T \int_\Omega \xi \xi_t (V^\epsilon)^{\frac{p}{2}} \, dx \, dt. \end{aligned}$$

To sum up, we have now the final estimate

$$\frac{1}{2}\int_0^T\int_\Omega \xi^2 (V^\epsilon)^{\frac{p-2}{2}} (u_t^\epsilon)^2 \, dx\, dt \le 2\int_0^T\int_\Omega (V^\epsilon)^{\frac{p-2}{2}} |\nabla u^\epsilon|^2 |\nabla \xi|^2 \, dx\, dt + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t (V^\epsilon)^{\frac{p}{2}} \, dx\, dt$$
$$\le 2\int_0^T\int_\Omega (V^\epsilon)^{\frac{p}{2}} |\nabla \xi|^2 \, dx\, dt + \frac{2}{p}\int_0^T\int_\Omega \xi\xi_t (V^\epsilon)^{\frac{p}{2}} \, dx\, dt.$$

So far, our calculations are valid in the full range $1 < p < \infty$. For $1 < p < 2$, we have

$$(V^\epsilon)^{\frac{p-2}{2}} \ge \|V^\epsilon\|_\infty^{\frac{p-2}{2}},$$

where the supremum norm is taken over the support of ξ. Hence, equation (6.1) holds for $1 < p < 2$ and the proof of Theorem 1.2 is complete.

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