Research Article

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A trace problem for associate Morrey potentials

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Abstract: As a solution to the restriction question for associate Morrey potentials (Question 1.1), this paper characterizes a Radon measure μ on $\mathbb{R}^n$ such that the Riesz operator I_α maps the associate Morrey spaces $H_1^{p,\kappa} \subset H^{p,\kappa}$ to the μ-induced Morrey spaces $L_\mu^{q,\lambda} \subset L_{\mu,\infty}^{q,\lambda}$ continuously. The discovered restriction/trace principle (Theorem 1.2) is brand-new, and its demonstration is non-trivial.

Keywords: Trace, associate Morrey potential, embedding

MSC 2010: 31B15, 42B35, 46E35

1 Introduction

1.1 Description of Question 1.1

Stemming from an extensive investigation of [2, 7, 9, 10, 45], this paper attempts to answer the coming-up-next restriction question, i.e., a trace problem for associate Morrey potentials. Namely, the Riesz images $I_\alpha H_1^{p,\kappa} \subset I_\alpha H^{p,\kappa}$ of the spaces $H_1^{p,\kappa} \subset H^{p,\kappa}$ associated with the Morrey spaces $L_\infty^{\frac{p}{p-1},\kappa} \supset L^{\frac{p}{p-1},\kappa}$ that have been proven to be of independent interest in geometric potential analysis (see, e.g., [5]) but also of great value in treating the local behaviour of an appropriate solution to certain elliptic partial differential equation or the incompressible Navier-Stokes system (see, e.g., [11, 14–16, 20–22, 28, 29, 31–35, 38, 39, 42, 43]):

Question 1.1. Let μ be a Radon measure on the Euclidean space $\mathbb{R}^n$. What ball condition must μ have so that the Riesz mapping $f \mapsto I_\alpha f$ is continuous (i.e., the potential $I_\alpha f$ is restricted to the support of μ) in the chain

$$H_1^{p,\kappa} \subset H^{p,\kappa} \xrightarrow{I_\alpha} L_\mu^{q,\lambda} \subset L_{\mu,\infty}^{q,\lambda}$$

for the associate Morrey spaces $H_1^{p,\kappa} \subset H^{p,\kappa}$ and the μ-based Morrey spaces $L_\mu^{q,\lambda} \subset L_{\mu,\infty}^{q,\lambda}$?

A resolution of the above question relies on the following fundamentals.

First of all, from a geometric measure theory perspective, a Radon measure on $\mathbb{R}^n$ is defined as a Borel regular measure assigning finite mass to any compact subset of $\mathbb{R}^n$ (cf. [18, p. 5]). For a given Radon measure μ on $\mathbb{R}^n$ and a nonnegative number β, denote by

$$|||\mu|||_\beta \equiv \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\beta}\mu(B(x,r))$$

the β-th order ball variation of μ. Trivially, $|||\mu|||_0$ is the total variation of μ. Obviously, this variation $|||\mu|||_\beta$ is finite whenever $\beta \in \{1, \dots, n\}$ and μ is the β-dimensional Hausdorff measure $\mathcal{H}^\beta$ restricted on a smooth manifold $\mathbb{M} \subset \mathbb{R}^n$ with $\mathcal{H}^\beta(\mathbb{M}) < \infty$.

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Next, given $\alpha \in (0, n)$ the Riesz operator I_α acting on a Lebesgue ν-measurable function f in $\mathbb{R}^n$ is determined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |y - x|^{\alpha - n} f(y)\, d\nu(y),$$

whose constant multiple

$$u = \Big(\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})} \Big) I_\alpha f$$

defines an inverse for a power of the Laplace operator on $\mathbb{R}^n$ (cf. [5, p. 9]), i.e., such an singular integral u solves the $\frac{\alpha}{2}$-th order Laplace equation $(-\Delta)^{\frac{\alpha}{2}} u = f$ using the Fourier transform.

After that, for $(q, \lambda) \in [1, \infty) \times (0, n]$, let $L^{q,\lambda}_{\mu,\infty}$ be the weak Morrey space of all μ-measurable functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^{q,\lambda}_{\mu,\infty}} = \sup_{(x,r,t) \in \mathbb{R}^n \times (0,\infty) \times (0,\infty)} \Big(r^{\lambda - n} t^q \mu(\{y \in B(x, r) : |f(y)| > t\}) \Big)^{\frac{1}{q}} < \infty,$$

where $B(x, r)$ is the Euclidean open ball with centre x and radius r. Naturally, as a proper subspace of $L^{q,\lambda}_{\mu,\infty}$, the symbol $L^{q,\lambda}_{\mu}$ is employed to represent the Morrey space of all μ-measurable functions f with

$$\|f\|_{L^{q,\lambda}_{\mu}} = \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} \bigg(r^{\lambda - n} \int_{B(x,r)} |f|^q\, d\mu \bigg)^{\frac{1}{q}} < \infty.$$

In particular, $L^q_{\mu,\infty} = L^{q,n}_{\mu,\infty}$ and $L^q_\mu = L^{q,n}_\mu$. Moreover, if $\mu = \nu$ (the n-dimensional Lebesgue measure on $\mathbb{R}^n$) and $(q, \lambda) = (p, \kappa) \in [1, \infty) \times (0, n]$, then $L^{q,\lambda}_{\mu,\infty}$ and $L^{q,\lambda}_\mu$ will be simply written as $L^{p,\kappa}_\infty$ and $L^{p,\kappa}$ respectively, and the endpoint spaces $L^{p,n}_\infty$ and $L^{p,n}$ coincide with the weak-L^p space L^p_∞ and L^p, respectively; but if $q = \infty$, then $L^{\infty,\kappa}_\infty = L^{\infty,\kappa} = L^\infty$. Furthermore, for $p \in [1, \infty]$ and $p' \equiv p/(p-1)$ with $\infty \equiv 1/(1-1)$ and $1 \equiv \infty/(\infty - 1)$, set

$$H^{p,\kappa}_1 = \bigg\{ \nu - \text{measurable functions } f : \|f\|_{H^{p,\kappa}_1} \equiv \sup_{\|g\|_{L^{p',\kappa}_\infty} \le 1} \bigg| \int_{\mathbb{R}^n} fg\, d\nu \bigg| < \infty \bigg\} \equiv [L^{p',\kappa}_\infty]^\star,$$

$$H^{p,\kappa} = \bigg\{ \nu - \text{measurable functions } f : \|f\|_{H^{p,\kappa}} \equiv \sup_{\|g\|_{L^{p',\kappa}} \le 1} \bigg| \int_{\mathbb{R}^n} fg\, d\nu \bigg| < \infty \bigg\} \equiv [L^{p',\kappa}]^\star.$$

Naturally, $H^{p,\kappa}_1$ and $H^{p,\kappa}$ are called the associate spaces $[L^{p',\kappa}_\infty]^\star$ and $[L^{p',\kappa}]^\star$ of the Morrey spaces $L^{p',\kappa}_\infty$ and $L^{p',\kappa}$, respectively (cf. [36]). Especially, one has $H^{1,\kappa}_1 = H^{1,\kappa} = L^1$ (for $0 < \kappa \le n$) and $H^{\infty,n}_1 \subsetneq H^{\infty,n} = L^\infty$ (due to $L^1 = L^{1,n} \subsetneq L^{1,n}_\infty$, cf. [17]).

Last but not least, note that $x \mapsto |x|^{-\frac{\kappa}{p}}$ is an element of $L^{p,\kappa}$ for $0 \le \kappa < n$ (cf. [46, p. 587]) and that $x \mapsto (1 + |x|^\eta)^{-1}$ belongs to $H^{p,\kappa}$ for $\eta > \frac{\kappa}{p}$ (cf. [24, Example 2.5]). So, the next implication seems to be very natural and motivative (cf. [26, p.85] and [6, Theorem 3.3]):

$$\|f\|_{H^{p,\kappa}_1} \gtrsim \|f\|_{H^{p,\kappa}} \gtrsim \|f\|_{L^q} \gtrsim \|f\|_{L^q_\infty} \quad \text{for } q = \frac{pn}{n + (n - \kappa)(p-1)} \le p.$$

In the above and below, $X \gtrsim Y$ stands for that there exists a constant $c > 0$ such that $X \ge cY$, $X \gtrsim Y \gtrsim X$ will be simply written as $X \approx Y$, and 1_E represents the characteristic function of a set $E \subseteq \mathbb{R}^n$.

1.2 Answering Question 1.1

Working on the above question plus the last implication one discovers the following trace and embedding principle for the associate Morrey potentials $I_\alpha H^{p,\kappa}$ (regarded as the space of tempered distributions having derivatives of order α in the associate Morrey space $H^{p,\kappa}$).

Theorem 1.2. *Assume* $(\alpha, \beta, \lambda) \in (0, n) \times (0, n] \times (0, n]$.

(i) *If*

$$1 < q = \frac{\beta + \lambda - n}{n - \alpha} < \infty, \quad 0 < \kappa \le n < \beta + \lambda \le 2n,$$

then $I_\alpha : H^{1,\kappa} \to L^{q,\lambda}_\mu$ *is continuous if and only if*

$$|||\mu|||_{\alpha,\beta,\lambda} \equiv \sup_{K \subset \mathbb{R}^n \text{ compact with } \nu(K)>0} \frac{\|I_\alpha 1_K\|_{L^{q,\lambda}_\mu}}{\nu(K)} < \infty,$$

while $I_\alpha : H^{1,\kappa} \to L^{q,\lambda}_{\mu,\infty}$ *is continuous if and only if* $|||\mu|||_\beta < \infty$.

(ii) *If*

$$1 < q = \frac{p(\beta + \lambda - n)}{\kappa + p(n - \alpha - \kappa)} \le p < \infty, \quad 0 < \kappa < n < \beta + \lambda \le 2n,$$

then $I_\alpha : H^{p,\kappa}_1 \subset H^{p,\kappa} \to L^{q,\lambda}_\mu \subset L^{q,\lambda}_{\mu,\infty}$ *is continuous if and only if* $|||\mu|||_\beta < \infty$.

(iii) *If*

$$1 < q = \frac{\beta + \lambda - n}{n - \alpha - \kappa} < \infty, \quad 0 < \kappa < n < \beta + \lambda \le 2n,$$

then $I_\alpha : H^{\infty,\kappa} \to L^{q,\lambda}_\mu$ *or* $I_\alpha : H^{\infty,\kappa} \to L^{q,\lambda}_{\mu,\infty}$ *is continuous if and only if* $|||\mu|||_\beta < \infty$.

Interestingly, as an immediate application of Theorem 1.2, the next corollary completely describes the embedding of the k-th order associate Morrey–Sobolev space into $L^{q,\lambda}_\mu$. More precisely, if

$$\nabla^k f = \begin{cases} (-\Delta)^{\frac{k}{2}} f & \text{for } k \text{ even}, \\ \nabla(-\Delta)^{\frac{k-1}{2}} f & \text{for } k \text{ odd}, \end{cases}$$

then

$$|f| \lesssim I_k(|\nabla^k f|)$$

for all C^∞-functions f with compact support in $\mathbb{R}^n$ (see, e.g., [4, p. 393] or [30, p. 19]), and hence applying Theorem 1.2, together with a small modification of the argument used in the proof of [30, p. 71, Theorem] as $p = 1$, produces the following corollary.

Corollary 1.3. *If* $(k, \beta, \lambda) \in \{1, \dots, n-1\} \times (0, n] \times (0, n]$, $1 \le p \le \infty$,

$$1 < q = \frac{\beta + \lambda - n}{p^{-1}\kappa + n - k - \kappa}, \quad 0 < \kappa < n < \beta + \lambda \le 2n,$$

then

$$\sup_{0<|\nabla^k f| \in H^{p,\kappa}} \|f\|_{L^{q,\lambda}_\mu} \big\| |\nabla^k f| \big\|^{-1}_{H^{p,\kappa}} < \infty \iff |||\mu|||_\beta < \infty.$$

Through exploiting the local nature of some indicator-like functions, introducing Lemmas 2.1, 3.1 and 4.1, and utilising the dual structure of a local L^q_μ-space, the non-trivial arguments for (i), (ii) and (iii) of Theorem 1.2 and their crucial Remarks 2.2, 3.2 and 4.2 will be arranged in Sections 2, 3 and 4, respectively. It is believed that the techniques developed in the forthcoming three sections can be applied to investigate a similar trace problem for the following Cordes–Nirenberg potentials (see, e.g., [16, 26, 27, 34]):

$$\left\{ I_\alpha f : \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(y)|^p |x - y|^{-\gamma} \, d\nu(y) < \infty \right\}, \quad 1 \le p < \infty, 0 < \gamma < n.$$

Nevertheless, this investigation will be the subject of a future article.

2 Demonstration of Theorem 1.2 (i)

2.1 A lemma for Theorem 1.2(i)

To prove Theorem 1.2 (i), one needs the following result which improves [2, Theorem 5.1].

Lemma 2.1. *If* $0 < \alpha < n$, $0 < \lambda \le \kappa \le n$,

$$1 \le p < \frac{\kappa}{\alpha}, \quad n - \alpha p < \beta \le n, \quad 1 \le q = \frac{p(\beta + \lambda - n)}{\kappa - \alpha p} \le \frac{pn}{\kappa - \alpha p},$$

then

$$|||\mu|||_\beta < \infty \Leftrightarrow \sup_{0 \not\equiv f \in L^{p,\kappa}} \|I_\alpha f\|_{L^{q,\lambda}_{\mu,\infty}} \|f\|^{-1}_{L^{p,\kappa}} < \infty.$$

Proof. Suppose

$$\sup_{0 \not\equiv f \in L^{p,\kappa}} \|I_\alpha f\|_{L^{q,\lambda}_{\mu,\infty}} \|f\|^{-1}_{L^{p,\kappa}} < \infty.$$

For a given ball $B(x, r)$, we compute (cf. [6])

$$\|1_{B(x,r)}\|_{L^{p,\kappa}} \approx r^{\frac{\kappa}{p}}, \quad r^\alpha \lesssim \inf_{y \in B(x,r)} I_\alpha 1_{B(x,r)}(y), \quad \mu(B(x,r)) \lesssim \mu(\{y \in B(x,r) : r^\alpha \lesssim I_\alpha 1_{B(x,r)}(y)\}).$$

Note that $1_{B(x,r)}$ belongs to $L^{p,\kappa}$, so

$$\mu(B(x,r)) r^{\alpha q + \lambda - n} \lesssim \mu(\{y \in B(x,r) : r^\alpha \lesssim I_\alpha 1_{B(x,r)}(y)\}) r^{\alpha q + \lambda - n} \lesssim \|1_{B(x,r)}\|^q_{L^{p,\kappa}} \lesssim r^{\frac{\kappa q}{p}}.$$

This inequality ensures that $|||\mu|||_\beta < \infty$ due to

$$0 < \beta = (\kappa p^{-1} - \alpha) q + n - \lambda \le n.$$

Next, suppose $|||\mu|||_\beta < \infty$. Without loss of generality, one may assume $0 \le f \in L^{p,\kappa}$. For a positive number t and a given ball $B(x, r)$ with $B(x, r)^c = \mathbb{R}^n \setminus B(x, r)$ let $0 \le f \in L^{p,\kappa}$, $f_1 = f 1_{B(x,r)}$, $f_2 = f 1_{B(x,r)^c}$,

$$F_t = \{y \in B(x,r) : I_\alpha f(y) > t\}, \quad F_{t,1} = \{y \in B(x,r) : I_\alpha f_1(y) > t\}, \quad F_{t,2} = \{y \in B(x,r) : I_\alpha f_2(y) > t\}.$$

Then

$$\mu(F_t) \le \mu(F_{t/2,1}) + \mu(F_{t/2,2}).$$

It remains to control the last two terms separately.

Step 1: In order to prove the desired estimate

$$\mu(F_{t/2,1}) \lesssim |||\mu|||_\beta t^{-\frac{p(\kappa+\beta-n)}{\kappa-\alpha p}} r^{n-\kappa} \quad \text{for all } (x, r, t) \in \mathbb{R}^n \times (0,\infty) \times (0,\infty),$$

one utilises the Fubini theorem to obtain

$$\begin{aligned}
t\mu(F_{t,1}) &\le \int_{F_{t,1}} I_\alpha f_1(z)\, d\mu(z) \\
&= \int_{F_{t,1}} \left(\int_{\mathbb{R}^n} |y - z|^{\alpha - n} f_1(y)\, d\nu(y) \right) d\mu(z) \\
&\lesssim \int_{F_{t,1}} \left(\int_0^\infty \left(\int_{B(z,s)} f_1(y)\, d\nu(y) \right) s^{\alpha - n - 1}\, ds \right) d\mu(z) \\
&\lesssim \int_0^\infty \left(\int_{F_{t,1}} \int_{B(z,s)} f_1(y)\, d\nu(y)\, d\mu(z) \right) s^{\alpha - n - 1}\, ds \\
&\lesssim \mathrm{J}_0(\delta) + \mathrm{J}_\infty(\delta),
\end{aligned}$$

where $\delta > 0$ will be determined later with

$$\mathrm{J}_j(\delta) = \begin{cases} \displaystyle\int_0^\delta \left(\int_{F_{t,1}} \int_{B(z,s)} f_1(y)\, d\nu(y)\, d\mu(z) \right) s^{\alpha - n - 1}\, ds & \text{for } j = 0, \\ \displaystyle\int_\delta^\infty \left(\int_{F_{t,1}} \int_{B(z,s)} f_1(y)\, d\nu(y)\, d\mu(z) \right) s^{\alpha - n - 1}\, ds & \text{for } j = \infty. \end{cases}$$

First, if $p > 1$, then an application of the Hölder inequality with $p' = p/(p-1)$ plus $f \in L^{p,\kappa}$ derives

$$\begin{aligned}
\int_{F_{t,1}}\int_{B(z,s)} f_1(y)\,d\nu(y)\,d\mu(z) &\le \int_{\mathbb{R}^n} \mu\big(B(y,s)\cap F_{t,1}\big)(1_{B(x,r)}f)(y)\,d\nu(y)\\
&\le \bigg(\int_{B(x,r)} (f(y))^p\,d\nu(y)\bigg)^{\frac{1}{p}}\bigg(\int_{B(x,r)}\Big(\mu\big(B(y,s)\cap F_{t,1}\big)\Big)^{p'}\,d\nu(y)\bigg)^{\frac{1}{p'}}\\
&\le r^{\frac{n-\kappa}{p}}\|f\|_{L^{p,\kappa}}\bigg(\int_{B(x,r)}\Big(\mu\big(B(y,s)\cap F_{t,1}\big)\Big)\Big(\mu\big(B(y,s)\cap F_{t,1}\big)\Big)^{p'-1}\,d\nu(y)\bigg)^{\frac{1}{p'}}\\
&\le r^{\frac{n-\kappa}{p}}\|f\|_{L^{p,\kappa}}s^{\frac{\beta}{p}}|||\mu|||_\beta^{\frac{1}{p}}\bigg(\int_{B(x,r)}\mu\big(B(y,s)\cap F_{t,1}\big)\,d\nu(y)\bigg)^{\frac{1}{p'}}\\
&\le r^{\frac{n-\kappa}{p}}\|f\|_{L^{p,\kappa}}s^{\frac{\beta}{p}}|||\mu|||_\beta^{\frac{1}{p}}\bigg(\int_{B(x,r)}1_{B(y,s)}(z)\,d\nu(z)\bigg)^{\frac{1}{p'}}\big(\mu(F_{t,1})\big)^{\frac{1}{p'}}\\
&\le r^{\frac{n-\kappa}{p}}\|f\|_{L^{p,\kappa}}s^{\frac{\beta}{p}+\frac{n}{p'}}|||\mu|||_\beta^{\frac{1}{p}}\big(\mu(F_{t,1})\big)^{\frac{1}{p'}}.
\end{aligned}$$

At the same time, if $p = 1$, then a slight modification of the above estimation gives

$$\int_{F_{t,1}}\int_{B(z,s)} f_1(y)\,d\nu(y)\,d\mu(z) \le r^{n-\kappa}\|f\|_{L^{1,\kappa}}s^\beta|||\mu|||_\beta.$$

Second, for $\mathrm{J}_0(\delta)$ one uses the foregoing inequalities to gain that if $p > 1$, then

$$\begin{aligned}
\mathrm{J}_0(\delta) &= \int_0^\delta\bigg(\int_{F_{t,1}}\int_{B(z,s)} f_1(y)\,d\nu(y)\,d\mu(z)\bigg)s^{\alpha-n-1}\,ds\\
&\lesssim r^{\frac{n-\kappa}{p}}\|f\|_{L^{p,\kappa}}|||\mu|||_\beta^{\frac{1}{p}}\big(\mu(F_{t,1})\big)^{\frac{1}{p'}}\int_0^\delta s^{\frac{\beta}{p}+\frac{n}{p'}+\alpha-n-1}\,ds\\
&\approx r^{\frac{n-\kappa}{p}}\delta^{\alpha+\frac{\beta-n}{p}}\|f\|_{L^{p,\kappa}}|||\mu|||_\beta^{\frac{1}{p}}\big(\mu(F_{t,1})\big)^{\frac{1}{p'}},
\end{aligned}$$

and if $p = 1$, then

$$\mathrm{J}_0(\delta) \lesssim r^{n-\kappa}\delta^{\alpha+\beta-n}\|f\|_{L^{1,\kappa}}|||\mu|||_\beta.$$

Of course, the condition $\beta > n - \alpha p$ is needed for the treatment of $\mathrm{J}_0(\delta)$. Meanwhile, for $\mathrm{J}_\infty(\delta)$ one utilises $f \in L^{p,\kappa}$ to deduce

$$\mathrm{J}_\infty(\delta) \lesssim \|f\|_{L^{p,\kappa}}\int_\delta^\infty s^{\alpha-\frac{\kappa}{p}-1}\mu(F_{t,1})\,ds \approx \delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,1})\|f\|_{L^{p,\kappa}}.$$

Here the condition $\alpha p < \kappa$ has been used.

Third, putting the estimates for $\mathrm{J}_0(\delta)$ and $\mathrm{J}_\infty(\delta)$ together gives

$$t\mu(F_{t,1}) \lesssim \begin{cases} r^{\frac{n-\kappa}{p}}\delta^{\alpha+\frac{\beta-n}{p}}\|f\|_{L^{p,\kappa}}|||\mu|||_\beta^{\frac{1}{p}}\big(\mu(F_{t,1})\big)^{\frac{1}{p'}} + \delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,1})\|f\|_{L^{p,\kappa}} & \text{for } p > 1,\\ r^{n-\kappa}\delta^{\alpha+\beta-n}\|f\|_{L^{1,\kappa}}|||\mu|||_\beta + \delta^{\alpha-\kappa}\mu(F_{t,1})\|f\|_{L^{1,\kappa}} & \text{for } p = 1.\end{cases}$$

Now, choosing

$$\delta = \bigg(\frac{\mu(F_{t,1})}{r^{n-\kappa}|||\mu|||_\beta}\bigg)^{\frac{1}{\beta+\kappa-n}}$$

gives

$$\mu(F_{t,1}) \lesssim |||\mu|||_\beta r^{n-\kappa}\big(t^{-1}\|f\|_{L^{p,\kappa}}\big)^{\frac{p(n-\beta-\kappa)}{\alpha p-\kappa}}.$$

Step 2: In order to prove the estimate

$$\mu(F_{t/2,2}) \lesssim \|\!|\mu|\!\|_\beta r^{n-\kappa}(t^{-1}\|f\|_{L^{p,\kappa}})^{\frac{p(\kappa+\beta-n)}{\kappa-\alpha p}} \quad \text{for all } (x,r,t)\in\mathbb{R}^n\times(0,\infty)\times(0,\infty),$$

one can borrow the idea used in Step 1 to get

$$t\mu(F_{t,2}) \lesssim \mathsf{K}_0(\delta)+\mathsf{K}_\infty(\delta),$$

where

$$\mathsf{K}_j(\delta)=\begin{cases}\displaystyle\int_0^\delta\Big(\int_{F_{t,2}}\int_{B(z,s)} f_2(y)\,d\nu(y)\,d\mu(z)\Big)s^{\alpha-n-1}\,ds & \text{for } j=0,\\ \displaystyle\int_\delta^\infty\Big(\int_{F_{t,2}}\int_{B(z,s)} f_2(y)\,d\nu(y)\,d\mu(z)\Big)s^{\alpha-n-1}\,ds & \text{for } j=\infty.\end{cases}$$

Notice that

$$y\in B(x,r)^c \text{ and } z\in B(y,s)\cap F_{t,2} \;\Rightarrow\; y\in B(x,r+s).$$

So, if $p>1$, then

$$\begin{aligned}\int_{F_{t,2}}\int_{B(z,s)} f_2(y)\,d\nu(y)\,d\mu(z) &\lesssim \int_{B(x,r+s)} f(y)\Big(\int_{\mathbb{R}^n}1_{B(y,s)}(z)1_{F_{t,2}}(z)\,d\mu(z)\Big)\,d\nu(y)\\ &\lesssim \Big(\int_{B(x,r+s)}(f(y))^p\,d\nu(y)\Big)^{\frac1p}\Big(\int_{B(x,r+s)}\Big(\int_{\mathbb{R}^n}1_{B(y,s)}(z)1_{F_{t,2}}(z)\,d\mu(z)\Big)^{p'}\,d\nu(y)\Big)^{\frac{1}{p'}}\\ &\lesssim \|f\|_{L^{p,\kappa}}(r+s)^{\frac{n-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}s^{n+\frac{\beta-n}{p}}\|\!|\mu|\!\|_\beta^{\frac1p},\end{aligned}$$

and if $p=1$, then

$$\int_{F_{t,2}}\int_{B(z,s)} f_2(y)\,d\nu(y)\,d\mu(z)\lesssim \|f\|_{L^{1,\kappa}}(r+s)^{n-\kappa}s^\beta\|\!|\mu|\!\|_\beta,$$

whence

$$\mathsf{K}_0(\delta)\lesssim\begin{cases}\|f\|_{L^{p,\kappa}}\delta^{\alpha+\frac{\beta-n}{p}}(\delta+r)^{\frac{n-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}\|\!|\mu|\!\|_\beta^{\frac1p} & \text{for } p>1,\\ \|f\|_{L^{1,\kappa}}\delta^{\alpha+\beta-n}(\delta+r)^{n-\kappa}\|\!|\mu|\!\|_\beta & \text{for } p=1.\end{cases}$$

Similarly, one has

$$\mathsf{K}_\infty(\delta)\lesssim\|f\|_{L^{p,\kappa}}\int_\delta^\infty s^{\alpha-\frac{\kappa}{p}-1}\mu(F_{t,2})\,ds\approx\delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,2})\|f\|_{L^{p,\kappa}}.$$

Putting together the estimates for $\mathsf{K}_0(\delta)$ and $\mathsf{K}_\infty(\delta)$ yields

$$t\mu(F_{t,2})\lesssim\begin{cases}\|f\|_{L^{p,\kappa}}\big(\delta^{\alpha+\frac{\beta-n}{p}}(\delta+r)^{\frac{n-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}\|\!|\mu|\!\|_\beta^{\frac1p}+\delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,2})\big) & \text{for } p>1,\\ \|f\|_{L^{1,\kappa}}\big(\delta^{\alpha+\beta-n}(\delta+r)^{n-\kappa}\|\!|\mu|\!\|_\beta+\delta^{\alpha-\kappa}\mu(F_{t,2})\big) & \text{for } p=1.\end{cases}$$

Below is a treatment of two cases: $\delta\ge r$ and $\delta\le r$.

Case $\delta\ge r$. Under this situation, one has

$$\begin{cases}\delta^{\alpha+\frac{\beta-n}{p}}(\delta+r)^{\frac{n-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}\|\!|\mu|\!\|_\beta^{\frac1p}+\delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,2})\lesssim\delta^{\alpha+\frac{\beta-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}\|\!|\mu|\!\|_\beta^{\frac1p}+\delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,2}) & \text{for } p>1,\\ \delta^{\alpha+\beta-n}(\delta+r)^{n-\kappa}\|\!|\mu|\!\|_\beta+\delta^{\alpha-\kappa}\mu(F_{t,2})\lesssim\delta^{\alpha+\beta-\kappa}\|\!|\mu|\!\|_\beta+\delta^{\alpha-\kappa}\mu(F_{t,2}) & \text{for } p=1,\end{cases}$$

and hence selecting

$$\delta\approx\Big(\frac{\mu(F_{t,2})}{\|\!|\mu|\!\|_\beta}\Big)^{\frac1\beta}$$

yields

$$\left(\frac{\mu(F_{t,2})}{|||\mu|||_\beta}\right)^{\frac{\kappa-\alpha p}{\beta p}} \lesssim t^{-1}\|f\|_{L^{p,\kappa}}.$$

Taking the $p(\beta+\kappa-n)/(\kappa-\alpha p)$-th power on this last estimate gives

$$\mu(F_{t,2}) \lesssim |||\mu|||_\beta (t^{-1}\|f\|_{L^{p,\kappa}})^{\frac{p(\beta+\kappa-n)}{\kappa-\alpha p}} \left(\frac{\mu(F_{t,2})}{|||\mu|||_\beta}\right)^{\frac{n-\kappa}{\beta p}} \lesssim (t^{-1}\|f\|_{L^{p,\kappa}})^{\frac{p(\beta+\kappa-n)}{\kappa-\alpha p}} |||\mu|||_\beta r^{n-\kappa}.$$

Case $\delta \le r$. Under this situation, one has

$$\begin{cases} \delta^{\alpha+\frac{\beta-n}{p}}(\delta+r)^{\frac{n-\kappa}{p}}(\mu(F_{t,2}))^{\frac{1}{p'}}|||\mu|||_\beta^{\frac{1}{p}} + \delta^{\alpha-\frac{\kappa}{p}}\mu(F_{t,2}) \le \left(\dfrac{r^{\frac{n-\kappa}{p}}}{\delta^{\frac{n-\beta}{p}-\alpha}}\right)(\mu(F_{t,2}))^{\frac{1}{p'}}|||\mu|||_\beta^{\frac{1}{p}} + \dfrac{\mu(F_{t,2})}{\delta^{\frac{\kappa}{p}-\alpha}} & \text{for } p>1, \\ \delta^{\alpha+\beta-n}(\delta+r)^{n-\kappa}|||\mu|||_\beta + \delta^{\alpha-\kappa}\mu(F_{t,2}) \le \delta^{\alpha+\beta-n}r^{n-\kappa}|||\mu|||_\beta + \delta^{\alpha-\kappa}\mu(F_{t,2}) & \text{for } p=1. \end{cases}$$

By selecting

$$\delta \approx \left(\frac{r^{n-\kappa}|||\mu|||_\beta}{\mu(F_{t,2})}\right)^{\frac{1}{n-\kappa-\beta}},$$

one achieves

$$t \lesssim \|f\|_{L^{p,\kappa}}\left(\frac{r^{n-\kappa}|||\mu|||_\beta}{\mu(F_{t,2})}\right)^{\frac{\alpha-\kappa}{n-\kappa-\beta}},$$

thereby arriving at

$$\mu(F_{t,2}) \lesssim |||\mu|||_\beta (t^{-1}\|f\|_{L^{p,\kappa}})^{\frac{p(\kappa+\beta-n)}{\kappa-\alpha p}} r^{n-\kappa}.$$

Now, a combination of the preceding two steps gives

$$\mu(F_t) \lesssim |||\mu|||_\beta (t^{-1}\|f\|_{L^{p,\kappa}})^{\frac{p(\kappa+\beta-n)}{\kappa-\alpha p}} r^{n-\kappa} \quad \text{for all } (x,r,t)\in\mathbb{R}^n\times(0,\infty)\times(0,\infty).$$

Since $\lambda \le \kappa$, one gets

$$q = \frac{p(\beta+\lambda-n)}{\kappa-\alpha p} \le \frac{p(\beta+\kappa-n)}{\kappa-\alpha p} = \tilde{q},$$

thereby implying

$$(\mu(F_t))^{\frac{1}{\tilde{q}}} = (\mu(F_t))^{\frac{1}{q}}(\mu(F_t))^{\frac{1}{\tilde{q}}-\frac{1}{q}} \ge (\mu(F_t))^{\frac{1}{q}} r^{\beta(\frac{1}{\tilde{q}}-\frac{1}{q})}|||\mu|||_\beta^{\frac{1}{\tilde{q}}-\frac{1}{q}}.$$

As a result, one establishes

$$\mu(F_t) \lesssim |||\mu|||_\beta t^{-q} r^{n-\lambda}\|f\|_{L^{p,\kappa}}^q \quad \text{for all } (x,r,t)\in\mathbb{R}^n\times(0,\infty)\times(0,\infty),$$

as desired. □

2.2 Proof of Theorem 1.2 (i)

On the one hand, since

$$\|1_K\|_{H^{1,\kappa}} = \|1_K\|_{L^1} = v(K) \quad \text{for all compact } K\subset\mathbb{R}^n,$$

one has

$$\sup_{0\ne f\in H^{1,\kappa}} \|I_\alpha f\|_{L^{q,\lambda}_\mu}\|f\|_{H^{1,\kappa}}^{-1} < \infty \Rightarrow |||\mu|||_{\alpha,\beta,\lambda} < \infty.$$

Conversely, assume $|||\mu|||_{\alpha,\beta,\lambda} < \infty$. Without loss of generality, one may assume $0 \le f \in H^{1,\kappa} = L^1$. Then such a function f can be written as $f = \lim_{j\to\infty} f_j$ (cf. [44, p. 88]), where

$$f_j(y) = \sum_{i=1}^{j2^j}(j-1)2^{-j}1_{K_{i,j}}(y) + j1_{K_j}(y), \quad K_{i,j} = \{y\in\mathbb{R}^n : (i-1)2^{-j} \le f(y) < i2^{-j}\}, \quad K_j = \{y\in\mathbb{R}^n : f(y)\ge j\}.$$

Thus, applying the Minkowski inequality for $L^q_\mu(B(x,r))$ (all L^q_μ-integrable functions on a given ball $B(x,r)$) derives

$$\begin{aligned}\|I_\alpha f_j\|_{L^q_\mu(B(x,r))} &\le \sum_{i=1}^{j2^j}(j-1)2^{-j}\|I_\alpha 1_{K_{i,j}}\|_{L^q_\mu(B(x,r))} + j\|I_\alpha 1_{K_j}\|_{L^q_\mu(B(x,r))}\\ &\le r^{\frac{n-\lambda}{q}}\left(\sum_{i=1}^{j2^j}(j-1)2^{-j}\|I_\alpha 1_{K_{i,j}}\|_{L^{q,\lambda}_\mu} + j\|I_\alpha 1_{K_j}\|_{L^{q,\lambda}_\mu}\right)\\ &\le r^{\frac{n-\lambda}{q}}|||\mu|||_{\alpha,\beta,\lambda}\left(\sum_{i=1}^{j2^j}(j-1)2^{-j}v(K_{i,j}) + jv(K_j)\right)\\ &\le r^{\frac{n-\lambda}{q}}|||\mu|||_{\alpha,\beta,\lambda}\|f_j\|_{L^1}.\end{aligned}$$

Now, sending j to ∞ leads to

$$\|I_\alpha f\|_{L^q_\mu(B(x,r))} \lesssim r^{\frac{n-\lambda}{q}}|||\mu|||_{\alpha,\beta,\lambda}\|f\|_{L^1},$$

whence

$$\|I_\alpha f\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||_{\alpha,\beta,\lambda}\|f\|_{H^{1,\kappa}} \approx |||\mu|||_{\alpha,\beta,\lambda}\|f\|_{H^{1,\kappa}_1}.$$

On the other hand, an application of Lemma 2.1 with $0 \le \lambda \le \kappa = n$, $p = 1$ and $q = (\beta+\lambda-n)/(n-\alpha) \ge 1$ gives

$$|||\mu|||_\beta < \infty \Leftrightarrow \sup_{0\ne f\in L^1}\|I_\alpha f\|_{L^{q,\lambda}_{\mu,\infty}}\|f\|^{-1}_{L^1} < \infty.$$

So, if $|||\mu|||_\beta < \infty$, then $I_\alpha\colon H^{1,\kappa} \to L^{q,\lambda}_{\mu,\infty}$ is continuous, and vice versa.

Remark 2.2. In the first equivalence of Theorem 1.2 (i), $|||\mu|||_{\alpha,\beta,\lambda} < \infty$ implies easily $|||\mu|||_\beta < \infty$ (cf. the beginning of the argument for the proof of Lemma 2.1). However, if $\lambda = n$, $\mu = v$ (ensuring $\beta = n$) and $q = n/(n-\alpha)$, then

$$|||\mu|||_{\alpha,\beta,\lambda} \ge \|I_\alpha 1_{B(0,1)}\|_{L^{n/(n-\alpha)}}\big(v(B(0,1))\big)^{-1} = \infty,$$

and hence $I_\alpha L^1$ is not subset of $L^{\frac{n}{n-\alpha}}$ according to the first equivalence of Theorem 1.2 (i).

Moreover, as a key tool of verifying the second equivalence of Theorem 1.2 (i), Lemma 2.1 can be used, along with the well-known layer-cake formula, to imply the following. If $0 < \alpha < n$, $0 < \lambda \le \kappa \le n$,

$$1 \le p < \frac{\kappa}{\alpha}, \quad n-\alpha p < \beta \le n, \quad 1 \le \hat q < q = \frac{p(\beta+\lambda-n)}{\kappa-\alpha p}, \quad |||\mu|||_\beta < \infty, \quad f \in L^{p,\kappa},$$

then

$$\int_{B(x,r)}|I_\alpha f(y)|^{\hat q}\,d\mu(y) = \left(\int_0^\delta + \int_\delta^\infty\right)\mu\big(\{y\in B(x,r): |I_\alpha f(y)| > t\}\big)\,dt^{\hat q} \lesssim |||\mu|||_\beta\big(r^\beta\delta^{\hat q} + \|f\|^q_{L^{p,\kappa}}r^{n-\lambda}\delta^{\hat q-q}\big),$$

and hence

$$\int_{B(x,r)}|I_\alpha f(y)|^{\hat q}\,d\mu(y) \lesssim |||\mu|||_\beta\|f\|^{\hat q}_{L^{p,\kappa}}r^{\beta+\frac{\hat q(n-\lambda-\beta)}{q}},$$

through choosing $\delta = (r^{n-\lambda-\beta}\|f\|^q_{L^{p,\kappa}})^{\frac{1}{q}}$. Consequently,

$$I_\alpha f \in L^{\hat q,\hat\lambda}_\mu \quad \text{with } \hat\lambda = n-\beta+\frac{\hat q(\kappa-\alpha p)}{p} < n.$$

In other words,

$$|||\mu|||_\beta < \infty \Rightarrow \sup_{0\ne f\in L^{p,\kappa}}\|I_\alpha f\|_{L^{\hat q,\hat\lambda}_\mu}\|f\|^{-1}_{L^{p,\kappa}} < \infty.$$

Naturally, if $\hat q = p$ in the last implication, then

$$|||\mu|||_\beta < \infty \Leftrightarrow \sup_{0\ne f\in L^{p,\kappa}}\|I_\alpha f\|_{L^{p,\tilde\lambda=n-\beta+\kappa-\alpha p}_\mu}\|f\|^{-1}_{L^{p,\kappa}} < \infty,$$

whose speciality $\mu = v$ recovers the second imbedding of [8, Theorem 16 (i)] which in turn yields the following new k-th order Morrey–Poincaŕe inequality:

$$\|f\|_{L^{p,\kappa-kp}} \lesssim \big\||\nabla^k f|\big\|_{L^{p,\kappa}} \quad \text{for all } \big\||\nabla^k f|\big\|_{L^{p,\kappa}} > 0 \text{ with } kp < \kappa \le n.$$

3 Demonstration of Theorem 1.2 (ii)

3.1 A lemma for Theorem 1.2 (ii)

To demonstrate Theorem 1.2(ii), one needs an equivalent characterisation of $|||\mu|||_\beta$.

Lemma 3.1. *For $(\beta,\gamma)\in[0,n]\times[0,n]$, let*

$$|||\mu|||_{\beta,\gamma}\equiv\sup_{(x,y,r)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,\infty)} r^{-\beta}\int_{B(x,r)}|z-y|^{-\gamma}\,d\mu(z)$$

be the (β,γ)-order ball variation of μ. If $\gamma\le\beta$, then

$$|||\mu|||_\beta\le|||\mu|||_{\beta-\gamma,\gamma}.$$

Moreover, if $\gamma<\beta$, then

$$|||\mu|||_\beta\gtrsim|||\mu|||_{\beta-\gamma,\gamma}.$$

But, if $0<\gamma=\beta$, then there exists a Radon measure μ_0 such that

$$|||\mu_0|||_\beta<\infty,\quad |||\mu_0|||_{\beta-\gamma,\gamma}=\infty.$$

Proof. On the one hand, one utilises the above definition to discover that if $\gamma\le\beta$, then

$$|||\mu|||_{\beta-\gamma,\gamma}\ge\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-(\beta-\gamma)}\int_{B(x,r)}|z-x|^{-\gamma}\,d\mu(z)\ge\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\beta}\mu(B(x,r))=|||\mu|||_\beta.$$

On the other hand, if $\gamma<\beta$, then [29, Lemma 1.27] is employed to estimate

$$\begin{aligned}|||\mu|||_{\beta-\gamma,\gamma}&\approx\sup_{(x,y,r)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,\infty)} r^{-(\beta-\gamma)}\bigg(\int_0^r+\int_r^\infty\bigg)\mu(B(x,r)\cap B(y,s))s^{-\gamma-1}\,ds\\&\lesssim|||\mu|||_\beta\sup_{(x,y,r)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,\infty)} r^{-(\beta-\gamma)}\bigg(\int_0^r s^{\beta-\gamma-1}\,ds+\int_r^\infty r^\beta s^{-\gamma-1}\,ds\bigg)\approx|||\mu|||_\beta.\end{aligned}$$

Unfortunately, this last estimation is not valid for $\gamma=\beta=j$ (a natural number less than n). In fact, for the origin $\{0\}^{n-j}$ of $\mathbb{R}^{n-j}$, the Radon measure $d\mu_j=1_{\mathbb{R}^j\times\{0\}^{n-j}}dv$ satisfies

$$|||\mu_j|||_j<\infty,\quad |||\mu_j|||_{0,j}=\infty.$$ □

3.2 Proof of Theorem 1.2 (ii)

The argument comprises two parts.

Part 1: Verify that I_α maps $H^{p,\kappa}$ to either $L^{q,\lambda}_\mu$ or $L^{q,\lambda}_{\mu,\infty}$ continuously if and only if $|||\mu|||_\beta<\infty$.
Since $L^{q,\lambda}_\mu\subset L^{q,\lambda}_{\mu,\infty}$, it is enough to check

$$\sup_{0\not\equiv f\in H^{p,\kappa}}\|I_\alpha f\|_{L^{q,\lambda}_{\mu,\infty}}\|f\|^{-1}_{H^{p,\kappa}}<\infty\ \Rightarrow\ |||\mu|||_\beta<\infty\ \Rightarrow\ \sup_{0\not\equiv f\in H^{p,\kappa}}\|I_\alpha f\|_{L^{q,\lambda}_\mu}\|f\|^{-1}_{H^{p,\kappa}}<\infty.$$

To this end, recall that (cf. [6, Theorem 3.3]) if $f\in H^{p,\kappa}$, then

$$\|f\|_{H^{p,\kappa}}\approx\inf\bigg\{\sum_j|c_j|:\ f=\sum_j c_j a_j\bigg\},$$

where a_j is a $(p,n-\kappa)$-atom, that is, a_j is supported on a ball $B_j\subset\mathbb{R}^n$ and satisfies

$$\|a_j\|_{L^p}\le\big(v(B_j)\big)^{\frac{\kappa-n}{np'}},\quad\text{where } p'=\frac{p}{p-1}.$$

So, for a given ball $B(x,r)$, the function

$$f_{x,r}=\big(\nu(B(x,r))\big)^{\frac{\kappa-n}{np'}-\frac{1}{p}}1_{B(x,r)}$$

belongs to $H^{p,\kappa}$. Consequently,

$$\|f_{x,r}\|_{H^{p,\kappa}}\approx 1,\quad r^{\alpha-n+\frac{\kappa}{p'}}\le\inf_{y\in B(x,r)}I_\alpha f_{x,r}(y),\quad \mu\big(B(x,r)\big)\le\mu\big(\{y\in B(x,r):r^{\alpha-n+\frac{\kappa}{p'}}\lesssim I_\alpha f_{x,r}(y)\}\big).$$

Now, if

$$\sup_{0\not\equiv f\in H^{p,\kappa}}\|I_\alpha f\|_{L^{q,\lambda}_{\mu,\infty}}\|f\|^{-1}_{H^{p,\kappa}}<\infty,$$

then $\|I_\alpha f_{x,r}\|_{L^{q,\lambda}_{\mu,\infty}}\lesssim 1$, and hence

$$r^{\lambda-n+q(\alpha-n+\frac{\kappa}{p'})}\mu\big(B(x,r)\big)\lesssim r^{\lambda-n+q(\alpha-n+\frac{\kappa}{p'})}\mu\big(\{y\in B(x,r):r^{\alpha-n+\frac{\kappa}{p'}}\lesssim I_\alpha f_{x,r}(y)\}\big)\lesssim\|I_\alpha f_{x,r}\|^q_{L^{q,\lambda}_\mu}\lesssim 1.$$

This in turn validates $|||\mu|||_\beta<\infty$.

Next, suppose $|||\mu|||_\beta<\infty$. The desired continuity follows from a consideration of the forthcoming two cases.

Case $\lambda<n$. Since $0<n-\lambda<\beta\le 2n-\lambda$, Lemma 3.1 is utilised to ensure $|||\mu|||_{n-\lambda,\beta-(n-\lambda)}<\infty$. Let $f=\sum_j c_j a_j$ belong to $H^{p,\kappa}$. Given a ball $B(x,r)$ and the conjugate index $q'=q/(q-1)$, let $L^{q'}_\mu(B(x,r))$ be the μ-measurable functions g with

$$\|g\|_{L^{q'}_\mu(B(x,r))}=\bigg(\int_{B(x,r)}|g|^{q'}\,d\mu\bigg)^{\frac{1}{q'}}<\infty.$$

Then, from duality and the Fubini theorem it follows that

$$\begin{aligned}\bigg(\int_{B(x,r)}|I_\alpha f|^q\,d\mu\bigg)^{\frac1q}&=\sup_{\|g\|_{L^{q'}_\mu(B(x,r))}\le1}\bigg|\int_{B(x,r)}gI_\alpha f\,d\mu\bigg|\\&=\sup_{\|g\|_{L^{q'}_\mu(B(x,r))}\le1}\bigg|\int_{B(x,r)}g(z)\int_{\mathbb{R}^n}f(y)|z-y|^{\alpha-n}\,d\nu(y)\,d\mu(z)\bigg|\\&=\sup_{\|g\|_{L^{q'}_\mu(B(x,r))}\le1}\bigg|\int_{\mathbb{R}^n}\bigg(\int_{B(x,r)}g(z)|z-y|^{\alpha-n}\,d\mu(z)\bigg)f(y)\,d\nu(y)\bigg|\\&\le\sup_{\|g\|_{L^{q'}_\mu(B(x,r))}\le1}\sum_j|c_j|\int_{\mathbb{R}^n}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)|a_j(y)|\,d\nu(y).\end{aligned}$$

Next, choose the parameter pair $\{\alpha_1,\alpha_2\}$ such that

$$n-\beta<\alpha_1=2n-\beta-\lambda<n,\quad 0<\alpha_2<n,\quad \alpha q=\alpha_1+(q-1)\alpha_2.$$

Then the Hölder inequality on the supporting ball B_j of a_j, with $q\le p$, yields

$$\begin{aligned}&\int_{\mathbb{R}^n}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)|a_j(y)|\,d\nu(y)\\&\quad\le\int_{B_j}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)|a_j(y)|\,d\nu(y)\\&\quad\le\bigg(\int_{B_j}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)^{q'}d\nu(y)\bigg)^{\frac{1}{q'}}\bigg(\int_{B_j}|a_j|^q\,d\nu\bigg)^{\frac1q}\\&\quad\le\big(\nu(B_j)\big)^{\frac1q-\frac1p}\bigg(\int_{B_j}|a_j|^p\,d\nu\bigg)^{\frac1p}\bigg(\int_{B_j}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)^{q'}d\nu(y)\bigg)^{\frac{1}{q'}}\\&\quad\le\big(\nu(B_j)\big)^{\frac1q-1+\frac{\kappa}{p'n}}\bigg(\int_{B_j}\bigg(\int_{B(x,r)}|g(z)||z-y|^{\alpha-n}\,d\mu(z)\bigg)^{q'}d\nu(y)\bigg)^{\frac{1}{q'}}.\end{aligned}$$

Another application of the Hölder inequality, along with $\|\|\mu\|\|_{n-\lambda,\beta-(n-\lambda)} < \infty$, gives

$$\begin{aligned}\left(\int_{B(x,r)} |g(z)||z-y|^{\alpha-n}\,d\mu(z)\right)^{q'} &\le \left(\int_{B(x,r)} \frac{|g(z)|^{q'}}{|z-y|^{n-\alpha_2}}\,d\mu(z)\right)\left(\int_{B(x,r)} \frac{d\mu(z)}{|z-y|^{n-\alpha_1}}\right)^{q'-1}\\ &\lesssim \left(r^{n-\lambda}\|\|\mu\|\|_{n-\lambda,\beta-(n-\lambda)}\right)^{q'-1}\int_{B(x,r)} \frac{|g(z)|^{q'}}{|z-y|^{n-\alpha_2}}\,d\mu(z).\end{aligned}$$

Now, the foregoing two groups of estimates are put together, and using the Fubini theorem and [29, Lemma 1.30], as well as the equalities

$$q = \frac{p(\beta+\lambda-n)}{\kappa - p(\alpha+\kappa-n)}, \quad \alpha_2 = q'\Big(\alpha - \frac{2n-\beta-\lambda}{q}\Big), \quad \frac{1}{q} + \frac{\alpha_2}{nq'} = 1 - \frac{\kappa}{np'},$$

gives

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\int_{B(x,r)} |g(z)||z-y|^{\alpha-n}\,d\mu(z)\right)|a_j(y)|\,d\nu(y)\\ &\quad\lesssim \big(\nu(B_j)\big)^{\frac{1}{q}-1+\frac{\kappa}{p'n}} r^{\frac{n-\lambda}{q}} \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)}\left(\int_{B_j}\int_{B(x,r)} \frac{|g(z)|^{q'}}{|z-y|^{n-\alpha_2}}\,d\mu(z)\,d\nu(y)\right)^{\frac{1}{q'}}\\ &\quad\approx \big(\nu(B_j)\big)^{\frac{1}{q}-1+\frac{\kappa}{p'n}} r^{\frac{n-\lambda}{q}} \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)}\left(\int_{B(x,r)}\left(\int_{B_j} \frac{d\nu(y)}{|z-y|^{n-\alpha_2}}\right)|g(z)|^{q'}\,d\mu(z)\right)^{\frac{1}{q'}}\\ &\quad\lesssim \big(\nu(B_j)\big)^{\frac{1}{q}-1+\frac{\kappa}{p'n}+\frac{\alpha_2}{nq'}} r^{\frac{n-\lambda}{q}} \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)}\left(\int_{B(x,r)} |g(z)|^{q'}\,d\mu(z)\right)^{\frac{1}{q'}}\\ &\quad\approx r^{\frac{n-\lambda}{q}} \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)}\left(\int_{B(x,r)} |g(z)|^{q'}\,d\mu(z)\right)^{\frac{1}{q'}}.\end{aligned}$$

This in turn yields

$$\left(\int_{B(x,r)} |I_\alpha f|^q\,d\mu\right)^{\frac{1}{q}} \lesssim r^{\frac{n-\lambda}{q}} \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)} \sum_j |c_j|,$$

and so by Lemma 3.1,

$$\|I_\alpha f\|_{L^{q,\lambda}_\mu} \lesssim \|\|\mu\|\|^{\frac{1}{q}}_{n-\lambda,\beta-(n-\lambda)} \|f\|_{H^{p,\kappa}} \lesssim \|\|\mu\|\|^{\frac{1}{q}}_{\beta} \|f\|_{H^{p,\kappa}}.$$

Case $\lambda = n$. Under this situation, one has $L^{q,\lambda}_\mu = L^q_\mu$. So, a particular care is needed. However, the initial issue is to take such a parameter pair (α_1, α_2) such that

$$n - \beta < \alpha_1 < n, \quad 0 < \alpha_2 < n, \quad \alpha q = \alpha_1 + (q-1)\alpha_2.$$

Fix a point x_0 in the supporting ball B of a given $(p, n-\kappa)$-atom a and select $B(x_0, r_0)$ so that $\nu(B(x_0, r_0)) = \nu(B)$ and $B \subseteq B(x_0, 2r_0)$. One can use [29, Lemma 1.30] to get

$$\int_B |y-z|^{\alpha_2-n}\,d\nu(z) \lesssim \big(\nu(B)\big)^{\frac{\alpha_2}{n}},$$

and then use the Fubini theorem, [29, Lemma 1.27] and the Hölder inequality with $q \le p$ to compute

$$\begin{aligned}\int_{B(x_0,2r_0)}\left(\int_B |a(z)|^q |y-z|^{\alpha_1-n}\,d\nu(z)\right)d\mu(y) &= \int_B\left(\int_{B(x_0,2r_0)} |y-z|^{\alpha_1-n}\,d\mu(y)\right)|a(z)|^q\,d\nu(z)\\ &\lesssim \int_B\left(\int_0^{2r_0} + \int_{2r_0}^{\infty} \mu(B(x_0,2r_0)\cap B(z,s))s^{\alpha_1-n-1}\,ds\right)|a(z)|^q\,d\nu(z)\end{aligned}$$

$$\lesssim |||\mu|||_\beta \Big(\int_B |a|^q \, d\nu \Big) r_0^{\beta+\alpha_1-n}$$
$$\lesssim |||\mu|||_\beta \Big(\int_B |a|^p \, d\nu \Big)^{\frac{q}{p}} (\nu(B))^{1-\frac{q}{p}} r_0^{\beta+\alpha_1-n}$$
$$\approx |||\mu|||_\beta (\nu(B))^{\frac{\beta+\alpha_1}{n}+\frac{q\kappa}{n}(1-\frac{1}{p})-q}.$$

Now, one utilises $\alpha = \alpha_1/q + (q-1)\alpha_2/q$, the Hölder inequality and $q = \beta p/(p(n-\alpha)-(p-1)\kappa)$ to find the following estimation:

$$\int_{B(x_0,2r_0)} (I_\alpha|a|)^q \, d\mu \le \int_{B(x_0,2r_0)} \Big(\int_B \frac{|a(y)|^q d\nu(y)}{|y-z|^{n-\alpha_1}} \Big) \Big(\int_B \frac{d\nu(y)}{|y-z|^{n-\alpha_2}} \Big)^{q-1} d\mu(z)$$
$$\lesssim |||\mu|||_\beta (\nu(B))^{\frac{\beta+\alpha_1}{n}+\frac{q\kappa}{n}(1-\frac{1}{p})-q+\frac{\alpha_2(q-1)}{n}}$$
$$\approx |||\mu|||_\beta.$$

Meanwhile, a combination of the Minkowski inequality, [29, Lemma 1.27], the Hölder inequality and the fact that $q = \beta p/(p(n-\alpha)-(p-1)\kappa) > \beta/(n-\alpha)$ yields

$$\Big(\int_{B(x_0,2r_0)^c} (I_\alpha|a|)^q \, d\mu \Big)^{\frac{1}{q}} \le \int_B \Big(\int_{B(x_0,2r_0)^c} |y-z|^{(\alpha-n)q} \, d\mu(y) \Big)^{\frac{1}{q}} |a(z)| \, d\nu(z)$$
$$\lesssim \int_B \Big(\int_{2r_0}^\infty s^{(\alpha-n)q-1} \mu(B(x_0,2r_0)^c \cap B(z,s)) \, ds \Big)^{\frac{1}{q}} |a(z)| \, d\nu(z)$$
$$\lesssim \Big(\int_B |a|^p \, d\nu \Big)^{\frac{1}{p}} (\nu(B))^{1-\frac{1}{p}} (\nu(B))^{\frac{\alpha q+\beta}{nq}-1} |||\mu|||_\beta^{\frac{1}{q}}$$
$$\lesssim (\nu(B))^{\frac{\alpha q+\beta}{nq}-\frac{1}{p}+\frac{\kappa}{np'}-\frac{1}{p'}} |||\mu|||_\beta^{\frac{1}{q}}$$
$$\approx |||\mu|||_\beta^{\frac{1}{q}}.$$

Putting the integrals $\int_{B(x_0,2r_0)}(\cdots)\,d\mu$ and $\int_{B(x_0,2r_0)^c}(\cdots)\,d\mu$ together implies

$$\big\| I_\alpha|a| \big\|_{L^q_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}}.$$

Upon expressing $f \in H^{p,\kappa}$ as $f = \sum_j c_j a_j$, one uses the foregoing estimate and the Minkowski inequality to establish the following inequality:

$$\|I_\alpha f\|_{L^q_\mu} \le \sum_j |c_j| \big\| I_\alpha|a_j| \big\|_{L^q_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}} \sum_j |c_j|.$$

Consequently, one finds

$$\|I_\alpha f\|_{L^q_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}} \|f\|_{H^{p,\kappa}}.$$

Part 2: Verify that I_α maps $H^{p,\kappa}_1$ to either $L^{q,\lambda}_\mu$ or $L^{q,\lambda}_{\mu,\infty}$ continuously if and only if $|||\mu|||_\beta < \infty$.

Thanks to both $H^{p,\kappa}_1 \subset H^{p,\kappa}$ and $L^{q,\lambda}_\mu \subset L^{q,\lambda}_{\mu,\infty}$, using Part 1, one has that $|||\mu|||_\beta < \infty$ implies that I_α maps $H^{p,\kappa}_1$ to either $L^{q,\lambda}_\mu$ or $L^{q,\lambda}_{\mu,\infty}$ continuously. So, it remains to check that if $I_\alpha \colon H^{p,\kappa}_1 \to L^{q,\lambda}_{\mu,\infty}$ is continuous, then $|||\mu|||_\beta < \infty$. To this end, utilising [20, Lemma 3.1] one gets that $f \in H^{p,\kappa}_1$ if and only if there exist a sequence of numbers $\{c_j\}$ and a sequence of functions $\{a_j\}$ such that

$$f = \sum_j c_j a_j \quad \text{with } \|f\|_{H^{p,\kappa}_1} \approx \sum_j |c_j| < \infty,$$

for which there exists a ball B_j containing the support of a_j such that

$$(\nu(B_j))^{\frac{n-\kappa}{p'n}} \int_0^\infty \Big(\nu\big(\{x\in\mathbb{R}^n : 1_{B_j}(x)|a_j(x)|>t\}\big)\Big)^{\frac{1}{p}}\,dt \le 1.$$

Upon choosing

$$f_{x,r} = \big(\nu(B(x,r))\big)^{\frac{\kappa-n}{np'}-\frac{1}{p}} 1_{B(x,r)}$$

for a given ball $B(x,r)$, one gets

$$\|f_{x,r}\|_{H^{p,\kappa}_1} \lesssim 1.$$

Now, using the induced condition

$$\|I_\alpha f_{x,r}\|_{L^{q,\lambda}_{\mu,\infty}} \lesssim \|f_{x,r}\|_{H^{p,\kappa}_1}$$

and the calculation (with $f_{x,r}$) done at the beginning of Part 1, one can readily reach $|||\mu|||_\beta < \infty$.

Remark 3.2. If $H^{p,0}$ or $H^{p,n}$ is defined as the space of all ν-measurable functions f with

$$\|f\|_{H^{p,\kappa}} \approx \inf\left\{\sum_j |c_j| : f=\sum_j c_j a_j\right\} < \infty,$$

and a_j is a (p,n)-atom or a $(p,0)$-atom, i.e., a_j is supported on a ball $B_j\subset\mathbb{R}^n$ and satisfies $\|a_j\|_{L^p}\le(\nu(B_j))^{-\frac{1}{p'}}$ or $\|a_j\|_{L^p}\le 1$, then the above argument is still valid for $H^{p,0}$ and $H^{p,n}$, and hence Theorem 1.2 can be extended to $H^{p,0\le\kappa\le n}$. Therefore, not only this extension gives a new approach to [1, Theorem 2], but also the limiting case $\alpha\to 0$ of Theorem 1.2 (ii) for $\lambda=n$ and $\mu=\nu$ actually gives (cf. [26, p.85])

$$H^{p,\kappa}\subset L^q \quad \text{with } q=\frac{pn}{\kappa+p(n-\kappa)}.$$

Furthermore, one has the following cases:

(a) If $\lambda=n$ and $0<\kappa<n$, then Theorem 1.2 (ii) recovers [10, Theorem 2.4 (ii)].

(b) If $\mu=\nu$ (forcing $\beta=n$) and $\lambda=\kappa$ in Theorem 1.2 (ii), then

$$\|I_\alpha f\|_{L^{\tilde q=p\kappa/(\kappa+p(n-\alpha-\kappa)),\kappa}} \lesssim \|f\|_{H^{p,\kappa}} \quad \text{for all } f\in H^{p,\kappa},$$

which may be regarded as a sort of the associate representation of the first embedding of [8, Theorem 16 (ii)] (cf. [2, Theorem 3.1])

$$\|I_\alpha f\|_{L^{\bar q=p\kappa/(\kappa-\alpha p),\kappa}} \lesssim \|f\|_{L^{p,\kappa}} \quad \text{for all } f\in L^{p,\kappa},$$

due to $1/\tilde q+1/\bar q+\kappa/n=1$.

(c) If $p=q=(\beta+\lambda-n-\kappa)/(n-\alpha-\kappa)$, then Theorem 1.2 (ii) is utilised to deduce

$$|||\mu|||_\beta<\infty \;\Leftrightarrow\; \|I_\alpha f\|_{L^{p,\tilde\lambda=n+\kappa-\beta+p(n-\alpha-\kappa)}_\mu} \lesssim \|f\|_{H^{p,\kappa}} \quad \text{for all } f\in H^{p,\kappa},$$

which may be treated as a kind of the associate form of the last equivalence in Remark 2.2 thanks to $(\bar\lambda-\tilde\lambda)/(pn)+\kappa/n=1$.

4 Demonstration of Theorem 1.2 (iii)

4.1 A lemma for Theorem 1.2 (iii)

In accordance with [6, Theorem 2.3], [36, Theorem 4.1] and [12, Theorem 2.7], if $p\in(1,\infty)$ and $\gamma\in[0,n)$, then

$$H^{p,\gamma}=\left\{f\in L^p_{\rm loc} : \|f\|_{H^{p,\gamma}} \approx \inf_{w\in B_1^{(n-\gamma)}}\left(\int_{\mathbb{R}^n}|f|^p w^{1-p}\,d\nu\right)^{\frac{1}{p}}<\infty\right\},$$

where $B_1^{(n-\gamma)}$ comprises all nonnegative functions w with

$$\int_{\mathbb{R}^n} w\, d\Lambda_{n-\gamma}^{(\infty)} \equiv \int_0^\infty \Lambda_{n-\beta}^{(\infty)}(\{y \in \mathbb{R}^n : w(y) > t\})\, dt \le 1,$$

and, for $E \subset \mathbb{R}^n$, the symbol $\Lambda_{n-\gamma}^{(\infty)}(E)$ expresses the Hausdorff capacity of E with order $n-\beta$, i.e.,

$$\Lambda_{n-\gamma}^{(\infty)}(E) = \inf \sum_j r_j^{n-\gamma},$$

where the infimum ranges over all countable coverings of E by open balls of radius r_j.

To validate Theorem 1.2 (iii), it is required to push the above description of $H^{p,\gamma}$ to the limiting case $p = \infty$, that is, to characterise $H^{\infty,\gamma}$ by means of $\Lambda_{n-\gamma}^{(\infty)}$ as seen below.

Lemma 4.1. *For $\gamma \in (0, n)$, let $\mathsf{L}^{1,\gamma}$ be the Morrey space of all signed Radon (locally finite regular signed Borel) measures μ whose total variation measures $|\mu| \equiv \tilde{\mu}$ obey $\|\tilde{\mu}\|_\beta < \infty$, and set $L^1_{\Lambda_\gamma^{(\infty)}}$ be the class of all $\Lambda_\gamma^{(\infty)}$-quasi continuous functions f on $\mathbb{R}^n$ for which*

$$\|f\|_{L^1_{\Lambda_\gamma^{(\infty)}}} \equiv \int_{\mathbb{R}^n} |f|\, d\Lambda_\gamma^{(\infty)} < \infty.$$

Then

$$\mathsf{L}^{1,\gamma} = \left\{\mu \text{ signed Radon measure} : \|\mu\|_{[L^1_{\Lambda_\gamma^{(\infty)}}]^\star} \equiv \sup_{\|g\|_{L^1_{\Lambda_\gamma^{(\infty)}}} \le 1} \left| \int_{\mathbb{R}^n} g\, d\mu \right| < \infty \right\} \equiv [L^1_{\Lambda_\gamma^{(\infty)}}]^\star.$$

Moreover, if L_0^∞ is the class of L^∞-functions with compact support in $\mathbb{R}^n$, then

$$H^{\infty,\gamma} = \left\{f \in L_0^\infty : \|f\|_{\mathsf{H}^{\infty,\gamma}} \equiv \inf_{w \in B_1^{(n-\gamma)}} \|fw^{-1}\|_{L^\infty} < \infty\right\} \equiv \mathsf{H}^{\infty,\gamma}.$$

Consequently, $L^1_{\Lambda_{n-\gamma}^{(\infty)}}$ exists as a subspace of $H^{\infty,\gamma}$.

Proof. The first identity immediately follows from [3, Proposition 1]. The second needs a demonstration.

Suppose $f \in \mathsf{H}^{\infty,\gamma}$. Then $f \in L_0^\infty$ and $\|f\|_{\mathsf{H}^{\infty,\gamma}} < \infty$. Now for any $g \in L^{1,\kappa}$ with $\|g\|_{L^{1,\kappa}} \le 1$, one has (cf. [6, Theorem 2.2])

$$\|g\|_{L^{1,\gamma}} = \sup_{w \in B_1^{(n-\gamma)}} \int_{\mathbb{R}^n} |f| w\, d\nu \le 1.$$

This in turn implies

$$\left| \int_{\mathbb{R}^n} fg\, d\nu \right| \le \int_{\mathbb{R}^n} |g| w |f| w^{-1}\, d\nu \le \|g\|_{L^{1,\gamma}} \|fw^{-1}\|_{L^\infty} \quad \text{for all } w \in B_1^{(n-\gamma)},$$

and so $f \in H^{\infty,\gamma}$. As a consequence, $\mathsf{H}^{\infty,\gamma} \subset H^{\infty,\gamma}$.

Since $L^{1,\gamma}$ is just the associate space of $H^{\infty,\gamma}$, in short, $L^{1,\gamma} = [H^{\infty,\gamma}]^\star$, in order to prove $H^{\infty,\gamma} \subset \mathsf{H}^{\infty,\gamma}$ it is enough to check

$$[\mathsf{H}^{\infty,\gamma}]^\star \equiv \left\{f\ \nu\text{-measurable function} : \|f\|_{[\mathsf{H}^{\infty,\gamma}]^\star} \equiv \sup_{\|g\|_{\mathsf{H}^{\infty,\gamma}} \le 1} \left| \int_{\mathbb{R}^n} fg\, d\nu \right| < \infty \right\} \subset L^{1,\gamma}.$$

Now, if f lies in $[\mathsf{H}^{\infty,\gamma}]^\star$, then for a given ball $B(x_0, r_0)$ and any function g supported in $B(x_0, r_0)$ one has

$$\|g\|_{\mathsf{H}^{\infty,\gamma}} = \inf_{w \in B_1^{(n-\gamma)}} \|gw^{-1}\|_{L^\infty(B(x_0,r_0))}.$$

Since g vanishes in $\mathbb{R}^n \setminus B(x_0, r_0)$, one can choose

$$w_0 = r_0^{\beta-n} 1_{B(x_0,r_0)} \in B_1^{(n-\beta)},$$

and then get

$$\|g\|_{\mathsf{H}^{\infty,\gamma}} \lesssim \|g\|_{L^\infty(B(x_0,r_0))} r_0^{n-\gamma}.$$

Upon taking

$$g_0(x) = \begin{cases} 1_{B(x_0,r_0)}|f(x)|f(x)^{-1} & \text{for } f(x) \neq 0, \\ 0 & \text{for } f(x) = 0, \end{cases}$$

one achieves

$$\int_{B(x_0,r_0)} |f|\,d\nu = \left| \int_{\mathbb{R}^n} fg_0\,d\nu \right| \leq \|f\|_{[\mathsf{H}^{\infty,\gamma}]^\star} \|g_0\|_{\mathsf{H}^{\infty,\gamma}} \lesssim \|f\|_{[\mathsf{H}^{\infty,\gamma}]^\star} r_0^{n-\gamma},$$

thereby reaching $f \in L^{1,\gamma}$, and thus $[\mathsf{H}^{\infty,\gamma}]^\star \subset L^{1,\gamma}$, as desired.

To see that $L^1_{\Lambda^{(\infty)}_{n-\gamma}}$ is contained in $H^{\infty,\gamma}$, suppose $f \in L^1_{\Lambda^{(\infty)}_{n-\gamma}}$. Then $w = |f|\|f\|^{-1}_{L^1_{\Lambda^{(\infty)}_{n-\gamma}}}$ satisfies

$$\int_{\mathbb{R}^n} w\,d\Lambda^{(\infty)}_{n-\gamma} = \int_0^\infty \Lambda^{(\infty)}_{n-\gamma}(\{y \in \mathbb{R}^n : |f(y)|\|f\|^{-1}_{L^1_{\Lambda^{(\infty)}_{n-\gamma}}} > t\})\,dt \leq 1,$$

i.e., $w \in B_1^{(n-\gamma)}$, and hence

$$\|f\|_{H^{\infty,\gamma}} \lesssim \|fw^{-1}\|_{L^\infty} \lesssim \|f\|_{L^1_{\Lambda^{(\infty)}_{n-\gamma}}},$$

that is, $f \in H^{\infty,\gamma}$, as desired. □

4.2 Proof of Theorem 1.2 (iii)

For a given nonnegative Radon measure μ, according to Lemma 4.1 it is better to begin with proving that

$$I_\alpha \colon L^1_{\Lambda^{(\infty)}_{n-\kappa}} \to L^{q,\lambda}_\mu$$

is continuous if and only if

$$I_\alpha \colon L^1_{\Lambda^{(\infty)}_{n-\kappa}} \to L^{q,\lambda}_{\mu,\infty}$$

is continuous if and only if $\mu \in \mathsf{L}^{1,\beta}$. As a matter of fact, it is enough to check

$$|||\mu|||_\beta < \infty \Rightarrow \|I_\alpha f\|_{L^{q,\lambda}_\mu} \lesssim \|f\|_{L^1_{\Lambda^{(\infty)}_{n-\kappa}}} \quad \text{for all } f \in L^1_{\Lambda^{(\infty)}_{n-\kappa}}.$$

To this end, assume $|||\mu|||_\beta < \infty$. In accordance with [6, Remark 3.4], one has

$$\|f\|_{L^1_{\Lambda^{(\infty)}_{n-\kappa}}} \approx \inf\left\{ \sum_j |c_j| : f = \sum_j c_j a_j \right\} < \infty,$$

where a_j is a $(\infty, n-\kappa)$-atom, i.e., a_j is supported on a ball $B_j \subset \mathbb{R}^n$ and satisfies $\|a_j\|_{L^\infty} \leq (\nu(B_j))^{\frac{\kappa-n}{n}}$.

Next, for the given parameter $q = (\beta + \lambda - n)/(n - \alpha - \kappa) > 1$ and a function $f = \sum_j c_j a_j \in L^1_{\Lambda^{(\infty)}_{n-\kappa}}$, one can slightly modify the second part of the argument used for the proof of Theorem 1.2 (ii) (with not only putting $p' = 1$ when $p = \infty$ but also noticing that $\|a_j\|_{L^\infty} \leq (\nu(B_j))^{\frac{\kappa-n}{n}}$) to obtain

$$\|I_\alpha a_j\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}},$$

which, by using the Minkowski inequality, implies

$$\|I_\alpha f\|_{L^{q,\lambda}_\mu} \leq \sum_j |c_j| \|I_\alpha a_j\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}} \sum_j |c_j|.$$

Consequently, one finds

$$\|I_\alpha f\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||_\beta^{\frac{1}{q}} \|f\|_{L^1_{\Lambda^{(\infty)}_{n-\kappa}}}.$$

Finally, if $I_\alpha \colon H^{\infty,\kappa} \to L^{q,\lambda}_\mu$ is continuous, then $I_\alpha \colon H^{\infty,\kappa} \to L^{q,\lambda}_{\mu,\infty}$ is continuous, and hence $|||\mu|||_\beta < \infty$ follows from choosing $1_{B(x,r)}$ as the test function in $H^{\infty,\kappa}$. Now, if $|||\mu|||_\beta < \infty$, then the previous analysis ensures

that $I_\alpha\colon L^{1}_{\Lambda^{(\infty)}_{n-\kappa}} \to L^{q,\lambda}_{\mu}$ is continuous. Hence, for a given ball $B(x,r)$, any $w \in B_1^{(n-\kappa)}$ and any $g \in L^{q'}_{\mu}(B(x,r))$, one applies the Fubini theorem to compute

$$\begin{aligned}
\biggl(\int_{B(x,r)} |I_\alpha f|^q\,d\mu\biggr)^{\frac{1}{q}} &= \sup_{\|g\|_{L^{q'}_{\mu}(B(x,r))}\le 1}\biggl|\int_{B(x,r)} (I_\alpha f)g\,d\mu\biggr| \\
&\le \sup_{\|g\|_{L^{q'}_{\mu}(B(x,r))}\le 1}\int_{B(x,r)}\biggl(\int_{\mathbb{R}^n}|f(y)||y-z|^{\alpha-n}\,d\nu(y)\biggr)|g(z)|\,d\mu(z) \\
&= \sup_{\|g\|_{L^{q'}_{\mu}(B(x,r))}\le 1}\int_{\mathbb{R}^n}\biggl(\int_{B(x,r)}|g(z)||y-z|^{\alpha-n}\,d\mu(z)\biggr)|f(y)|\,d\nu(y) \\
&\le \|fw^{-1}\|_{L^\infty}\sup_{\|g\|_{L^{q'}_{\mu}(B(x,r))}\le 1}\int_{\mathbb{R}^n}\biggl(\int_{B(x,r)}|g(z)||y-z|^{\alpha-n}\,d\mu(z)\biggr)w(y)\,d\nu(y) \\
&= \|fw^{-1}\|_{L^\infty}\sup_{\|g\|_{L^{q'}_{\mu}(B(x,r))}\le 1}\int_{B(x,r)}(I_\alpha w)|g|\,d\mu \\
&= \|fw^{-1}\|_{L^\infty}\|I_\alpha w\|_{L^q_\mu(B(x,r))} \\
&\le \|fw^{-1}\|_{L^\infty} r^{\frac{n-\lambda}{q}}\|I_\alpha w\|_{L^{q,\lambda}_{\mu}} \\
&\lesssim \|fw^{-1}\|_{L^\infty} r^{\frac{n-\lambda}{q}}|||\mu|||_\beta^{\frac{1}{q}}\|w\|_{L^{1}_{\Lambda^{(\infty)}_{n-\kappa}}} \\
&\lesssim r^{\frac{n-\lambda}{q}}\|fw^{-1}\|_{L^\infty}|||\mu|||_\beta^{\frac{1}{q}}.
\end{aligned}$$

Therefore, one utilises Lemma 4.1 to achieve

$$\|I_\alpha f\|_{L^{q,\lambda}_{\mu}} \lesssim |||\mu|||_\beta^{\frac{1}{q}}\|f\|_{\mathrm{H}^{\infty,\kappa}} \lesssim |||\mu|||_\beta^{\frac{1}{q}}\|f\|_{H^{\infty,\kappa}}.$$

That is, $I_\alpha\colon H^{\infty,\kappa} \to L^{q,\lambda}_{\mu}$ is continuous.

Remark 4.2. Perhaps, it is worth pointing out that $\kappa = 0$ is suitably allowable in Theorem 1.2 (iii). To see this nature more clearly, see [3, Proposition 5] which says, as Fefferman–Stein's predual of BMO (John Nirenberg's space of functions of bounded mean oscillation on $\mathbb{R}^n$; see, e.g., [19, 25, 40]), that if H^1 stands for the real 1-Hardy space comprising all L^1-functions f for which the $\{1,\dots,n\} \ni j$-Riesz transform

$$y \mapsto R_jf(y) = \lim_{\epsilon\to 0}\Gamma\Big(\frac{n+1}{2}\Big)\pi^{-\frac{n+1}{2}}\int_{B(0,\epsilon)^c} z_j|z|^{-n-1}f(y-z)\,d\nu(z)$$

belongs to L^1, then

$$\|I_\alpha f\|_{L^{1}_{\Lambda^{(\infty)}_{n-\alpha}}} \lesssim \|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^{n}\|R_jf\|_{L^1} \quad \text{for all } f \in H^1,$$

and hence, by [3, Corollary],

$$\biggl|\int_{\mathbb{R}^n} f\,d\mu\biggr| \lesssim |||\mu|||_{n-\alpha}\|f\|_{H^1} \quad \text{for all } (\mu,f) \in \mathsf{L}^{1,n-\alpha}_{+}\times H^1,$$

where $\mathsf{L}^{1,n-\alpha}_{+}$ consists of all nonnegative elements in $\mathsf{L}^{1,n-\alpha}$. Also, recall that (cf., e.g., [23, Theorem 6.6.10]) $f \in H^1$ if and only if there exist a sequence of (∞,n)-atoms $\{a_j\}$ with $\int_{\mathbb{R}^n} a\,d\nu = 0$ and a sequence of numbers $\{c_j\}$ such that

$$f = \sum_j c_j a_j, \quad \|f\|_{H^1} \approx \sum_j |c_j| < \infty.$$

So, this leads to an investigation of the endpoint $\kappa = 0$ of Theorem 1.2 (iii) for $q = (\beta+\lambda-n)/(n-\alpha) > 1$.

On the one hand, if $I_\alpha\colon H^1 \to L^{q,\lambda}_{\mu}$ is continuous, then

$$\|I_\alpha a\|_{L^{q,\lambda}_{\mu}} \lesssim \|a\|_{H^1} \quad \text{for all } a \in H^1.$$

For $B(x_-, r) = B(x, r)$, $B(x_+, r)$ (whose distance to $B(x, r)$ is $3r$, i.e., $|x - x_+| = 5r$) and

$$a = \frac{1_{B(x_-,r)} - 1_{B(x_+,r)}}{\nu(B(x, 6r))} \in H^1 \quad \text{with } \|a\|_{H^1} \lesssim 1,$$

via a simple geometric computation, one gets

$$I_\alpha a(y) = \big(\nu(B(x, 6r))\big)^{-1}\bigg(\int\limits_{B(x_-,r)} \frac{d\nu(z)}{|z-y|^{n-\alpha}} - \int\limits_{B(x_+,r)} \frac{d\nu(z)}{|z-y|^{n-\alpha}}\bigg) \gtrsim r^{\alpha-n} \quad \text{for all } y \in B(x, r),$$

whence finding

$$1 \gtrsim \|I_\alpha a\|^q_{L^{q,\lambda}_\mu} \gtrsim r^{q(\alpha-n)+\lambda-n}\mu(B(x, r)) \gtrsim r^{-\beta}\mu(B(x, r)) \;\Rightarrow\; |||\mu|||_\beta < \infty.$$

On the other hand, assume $|||\mu|||_\beta < \infty$. Upon writing $f = \sum_{j=1}^\infty c_j a_j \in H^1$ and observing that the argument for the proof of Theorem 1.2 (iii) actually reveals

$$\|I_\alpha a_j\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||^{\frac{1}{q}}_\beta,$$

one utilises the Minkowski inequality to achieve

$$\|I_\alpha f\|_{L^{q,\lambda}_\mu} \le \sum_j |c_j| \|I_\alpha a_j\|_{L^{q,\lambda}_\mu} \lesssim |||\mu|||^{\frac{1}{q}}_\beta \sum_j |c_j| \approx |||\mu|||^{\frac{1}{q}}_\beta \|f\|_{H^1}.$$

Surprisingly, this last estimation for $\mu = \nu$, i.e., $I_\alpha H^1 \subset L^{\frac{\lambda}{n-\alpha},\lambda}$, is an associate form of [26, Theorem 2], which implies $I_\alpha H^1 \subset H^{p,p'\alpha}$, since $\lambda = p'\alpha$ amounts to $p = \lambda/(\lambda - \alpha)$. When $\lambda = n$, both go back to the well-known Stein–Weiss embedding $I_\alpha H^1 \subset L^{\frac{n}{n-\alpha}}$, see, e.g., [13, 37, 40, 41].

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