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Prescribed mean curvature equation on torus

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Abstract: Prescribed mean curvature problems on the torus have been considered in one dimension. In this paper, we prove the existence of a graph on the n-dimensional torus $\mathbb{T}^n$, the mean curvature vector of which equals the normal component of a given vector field satisfying suitable conditions for a Sobolev norm, the integrated value, and monotonicity.

Keywords: Prescribed mean curvature, fixed-point theorem

MSC 2010: 35J93

1 Introduction

In this paper, we consider the following prescribed mean curvature problem on the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$:

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \nu(\nabla u)\cdot g(x,u(x)) \quad \text{on } \mathbb{T}^n, \tag{1.1}$$

where ν is the unit normal vector of u, that is,

$$\nu(z) = \frac{1}{\sqrt{1+|z|^2}}(-z,1).$$

The vector field $g(x,x^{n+1}) : \mathbb{T}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$ is given, and we seek a solution u satisfying (1.1). The left-hand side of (1.1) represents the mean curvature of the graph of u, and the right-hand side is the normal component of the vector field g on the graph.

In the case of Dirichlet conditions of a bounded domain $\Omega \subset \mathbb{R}^n$, prescribed mean curvature problems have been studied by numerous researchers. Bergner [3] solved the Dirichlet problem in the case where the right-hand side of (1.1) is $H = H(x,u,\nu(\nabla u))$ under the assumptions of boundedness ($|H| < \infty$), monotonicity ($\partial_{n+1}H \geq 0$), and convexity of Ω. Under the same conditions for the function H, Marquardt [9] imposed a condition on $\partial\Omega$ depending on H that guarantees the existence of solutions even for a domain Ω that is not necessarily convex. In [13], we proved the existence of a solution only under the condition that the Sobolev norm of H is sufficiently small. In the case of a compact Riemannian manifold, Aubin [2] solved the linear elliptic problem $-\partial_i[a_{ij}(x)\partial_j u] = H(x)$ if the integrated value of H is zero. The assumption of the integrated value plays an important role in the existence of solutions to elliptic equations on a compact Riemannian manifold. Denny [4] solved the quasilinear elliptic problem $-\operatorname{div}(a(u(x))\nabla u) = H(x)$ on the torus $\mathbb{T}^n$ with $n = 2, 3$. Prescribed mean curvature problems on the one-dimensional torus

$$\Big(\frac{u'}{\sqrt{1+(u')^2}}\Big)' = H(x,u,u')$$

have been investigated for a wide variety of conditions H (we refer to, for example, [5, 7, 8, 10, 11, 14]).

As we noted in [13], the motivation for the present study comes from a singular perturbation problem, and we proved the following in [12]. Suppose a constant $\varepsilon > 0$ and functions $\phi_\varepsilon \in W^{1,2}$ and $g_\varepsilon \in W^{1,p}$,

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with $p > \frac{n+1}{2}$, satisfy

$$-\varepsilon\Delta\phi_\varepsilon + \frac{W'(\phi_\varepsilon)}{\varepsilon} = \varepsilon\nabla\phi_\varepsilon \cdot g_\varepsilon,$$
$$\int\Big(\frac{\varepsilon|\nabla\phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon}\Big)\,dx + \|g_\varepsilon\|_{W^{1,p}(\tilde{\Omega})} \le C,$$

where W is a double-well potential such as $W(\phi) = (1-\phi^2)^2$. Then the interface $\{\phi_\varepsilon = 0\}$ converges locally in the Hausdorff distance to a surface having a mean curvature given by $\nu \cdot g$ as $\varepsilon \to 0$. Here, ν is the unit normal vector of the surface, and g is the weak $W^{1,p}$ limit of g_ε. If the surface is represented locally as a graph of a function u on $\mathbb{T}^n$, we can observe that u satisfies (1.1). In this paper, we prove the existence of solutions to (1.1) assuming that the Sobolev norm of g is sufficiently small, g^{n+1} for the $(n+1)$-st component is monotonous, and the integrated value of g^{n+1} is zero. The following theorem is the main result.

Theorem 1.1. *Fix $\frac{n+1}{2} < p < n+1$ and $q = \frac{np}{n+1-p}$. Then there exists a constant $\varepsilon_1 = \varepsilon_1(n,p) > 0$ with the following property: If $\varepsilon < \varepsilon_1$, and*

$$g = (g^1, \dots, g^n, g^{n+1}) = (g', g^{n+1}) \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$$

satisfies the relations

$$\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} < \varepsilon^{\frac{2}{3}}, \tag{1.2}$$
$$\partial_{n+1}g^{n+1}(x,x^{n+1}) > \varepsilon + \varepsilon^{\frac{1}{2}}|\partial_{n+1}g'(x,x^{n+1})|, \tag{1.3}$$
$$\int_{\mathbb{T}^n} g^{n+1}(x,0) = 0, \tag{1.4}$$

then there exists a function $u \in W^{2,q}(\mathbb{T}^n)$ such that

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \nu(\nabla u)\cdot g(x,u(x)) \quad \text{on } \mathbb{T}^n. \tag{1.5}$$

Moreover, the following inequality holds:

$$\Big\|u - \int_{\mathbb{T}^n} u(y)\,dy\Big\|_{W^{2,q}(\mathbb{T}^n)} \le \varepsilon^{\frac{1}{2}}.$$

Assumptions (1.2) and (1.3) guarantee the existence and uniqueness of solutions to the linearized problem of (1.1) where a given function depends on ∇u. Equation (1.4) is necessary for the existence of solutions to elliptic equations on the torus. To the best of our knowledge, prescribed mean curvature problems on the torus in the general dimension have been insufficiently studied. However, we have proved the existence of the solution under natural assumptions.

The following is the method of proof. First, we find the conditions of H for the linearized problem of (1.1), i.e.

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla v|^2}}\Big) = H,$$

to have a unique solution. If we add a suitable constant term for any v, the function $\nu(\nabla v)\cdot g(x,v(x))$ satisfies the conditions. By estimating the norm of this solution with g, the mapping $T(v) = u$ has a fixed point using a fixed-point theorem, and Theorem 1.1 follows.

2 Proof of Theorem 1.1

A theorem that holds in the Euclidean space also holds on a torus, as we consider a function on a torus to be a periodic function in the Euclidean space.

Let $X(\mathbb{T}^n)$ be a function space on $\mathbb{T}^n$. We define a subspace $X_{\mathrm{ave}}(\mathbb{T}^n) \subset X(\mathbb{T}^n)$ as

$$X_{\mathrm{ave}} := \Big\{w \in X : \int_{\mathbb{T}^n} w = 0\Big\}.$$

Theorem 2.1. *Suppose $v \in C^1(\mathbb{T}^n)$ and $H \in L^2_{\mathrm{ave}}(\mathbb{T}^n)$. Then there exists a unique function $u \in W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n)$ such that*

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} H\phi$$

for all $\phi \in W^{1,2}(\mathbb{T}^n)$.

Proof. We define a function $B : W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n) \times W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n) \to \mathbb{R}$ by

$$B[w_1, w_2, v] := \int_{\mathbb{T}^n} \frac{\nabla w_1 \cdot \nabla w_2}{\sqrt{1+|\nabla v|^2}}.$$

By the Hölder inequality, we obtain

$$\begin{aligned} |B[w_1, w_2, v]| &\le \int_{\mathbb{T}^n} |\nabla w_1||\nabla w_2| \\ &\le \|\nabla w_1\|_{L^2(\mathbb{T}^n)} \|\nabla w_2\|_{L^2(\mathbb{T}^n)} \\ &\le \|w_1\|_{W^{1,2}(\mathbb{T}^n)} \|w_2\|_{W^{1,2}(\mathbb{T}^n)}. \end{aligned} \tag{2.1}$$

Using the Poincaré inequality, we have

$$\begin{aligned} |B[w, w, v]| &\ge \frac{1}{\sqrt{1+\|v\|^2_{C^1(\mathbb{T}^n)}}} \|\nabla w\|^2_{L^2(\mathbb{T}^n)} \\ &\ge \frac{1}{\sqrt{1+\|v\|^2_{C^1(\mathbb{T}^n)}}} \|\nabla w\|^2_{W^{1,2}(\mathbb{T}^n)}. \end{aligned} \tag{2.2}$$

By (2.1), (2.2), and the Lax–Milgram theorem, for any $H \in L^2_{\mathrm{ave}}(\mathbb{T}^n)$, there exists a unique function

$$u \in W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n)$$

such that

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \psi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} H\psi \tag{2.3}$$

for all $\psi \in W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n)$. For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we define $c_\phi := \int_{\mathbb{T}^n} \phi$ and $\tilde{\phi} := \phi - c_\phi \in W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n)$. By (2.3) and $H \in L^2_{\mathrm{ave}}(\mathbb{T}^n)$, we obtain

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \tilde{\phi}}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} H\tilde{\phi} = \int_{\mathbb{T}^n} H\phi.$$

Thus, Theorem 2.1 follows. □

We define a mollifier as follows:

$$\eta(x) := \begin{cases} C \exp\Big(\dfrac{1}{|x|^2-1}\Big) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \ge 1, \end{cases}$$

where the constant $C > 0$ is selected such that $\int_{\mathbb{R}^{n+1}} \eta = 1$. We define

$$\eta_\lambda(x) := \frac{1}{\lambda^n} \eta\Big(\frac{x}{\lambda}\Big).$$

For any $f \in L^2(\mathbb{T}^n \times (-1,1))$ and $x^{n+1} \in (-1+\lambda, 1-\lambda)$,

$$\begin{aligned} f_\lambda(x, x^{n+1}) &:= \int_{\mathbb{T}^n \times (-1,1)} \eta_\lambda(x-y, x^{n+1}-y^{n+1}) f(y, y^{n+1})\, dy \\ &= \int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) f(x-y, x^{n+1}-y^{n+1})\, dy, \end{aligned}$$

where $B^{n+1}(x, \lambda)$ is an open ball with center x and radius λ in $\mathbb{T}^n \times \mathbb{R}$. Moreover, for any

$$g \in W^{1,p}(\mathbb{T}^n \times (-1, 1); \mathbb{R}^{n+1}),$$

we define $g_\lambda := (g_\lambda^1, \dots, g_\lambda^n, g_\lambda^{n+1}) = (g'_\lambda, g_\lambda^{n+1})$.

Lemma 2.2. *Fix $\beta_1 > 0$ and $0 < \lambda < 1$. Suppose $v \in C^1(\mathbb{T}^n)$ satisfies $\|v\|_{C^1(\mathbb{T}^n)} < \beta_1$, and*

$$g \in W^{1,p}(\mathbb{T}^n \times (-1, 1); \mathbb{R}^{n+1})$$

satisfies

$$\partial_{n+1} g^{n+1}(x, x^{n+1}) > \beta_1 |\partial_{n+1} g'(x, x^{n+1})|.$$

For any positive constant $c_0 > 0$, if $v(\mathbb{T}^n) + c_0 \subset (-1 + \lambda, 1 - \lambda)$, then

$$\int_{\mathbb{T}^n} v(\nabla v) \cdot g_\lambda(x, v) < \int_{\mathbb{T}^n} v(\nabla v) \cdot g_\lambda(x, v + c_0).$$

Proof. From the assumptions, we compute

$$\begin{aligned}
&\int_{\mathbb{T}^n} v(\nabla v) \cdot (g_\lambda(x, v + c_0) - g_\lambda(x, v)) \\
&\quad = \int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_0} -\nabla v \cdot \partial_{n+1} g'_\lambda(x, t) + \partial_{n+1} g_\lambda^{n+1}(x, t)\, dt \\
&\quad \geq \int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_0} -\beta_1 |\partial_{n+1} g'_\lambda(x, t)| + \partial_{n+1} g_\lambda^{n+1}(x, t)\, dt \\
&\quad \geq \int_{\mathbb{T}^n} \frac{1}{\sqrt{1 + |\nabla v|^2}} \int_v^{v+c_0} \int_{\mathbb{T}^n \times (-1,1)} \eta_\lambda(x - y, t - y_{n+1}) \{-\beta_1 |\partial_{n+1} g'(y, y_{n+1})| + \partial_{n+1} g^{n+1}(y, y_{n+1})\}\, dt \\
&\quad > 0.
\end{aligned}$$

Lemma 2.2 follows. □

Lemma 2.3. *Suppose $g \in W^{1,p}(\mathbb{T}^n \times (-1, 1)$ and $v \in C^1(\mathbb{T}^n)$ with $\|v\|_{C^1(\mathbb{T}^n)} \leq \frac{7}{16}$. Let $q = \frac{np}{n+1-p}$. Then there exists a constant $c_1 = c_1(n, p) > 0$ such that, if $\lambda < \frac{1}{8}$,*

$$\|g_\lambda(\cdot, v(\cdot))\|_{L^q(\mathbb{T}^n)} \leq c_1 \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}.$$

Proof. By the same proof as in [13, Lemma 2.3], we obtain

$$\|g_\lambda(\cdot, v(\cdot))\|_{L^q(\mathbb{T}^n)} \leq c_2 \|g_\lambda\|_{W^{1,p}(\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8}))}, \tag{2.4}$$

where $c_2 = c_2(n, p) > 0$. Using the Hölder inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8})} |g_\lambda|^p\, dx &\leq \int_{\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8})} \Big(\int_{B^{n+1}(x,\lambda)} \eta_\lambda^{1-\frac{1}{p}+\frac{1}{p}}(x - y, x^{n+1} - y^{n+1}) |g(y, y^{n+1})|\, dy \Big)^p\, dx \\
&\leq \int_{\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8})} \Big(\int_{B^{n+1}(x,\lambda)} \eta_\lambda(x - y, x^{n+1} - y^{n+1}) |g(y, y^{n+1})|^p\, dy \Big)\, dx \\
&\leq \int_{\mathbb{T}^n \times (-1,1)} |g(y, y^{n+1})|^p \Big(\int_{B^{n+1}(y,\lambda)} \eta_\lambda(x - y, x^{n+1} - y^{n+1})\, dx \Big)\, dy \\
&= \int_{\mathbb{T}^n \times (-1,1)} |g(y, y^{n+1})|^p\, dy.
\end{aligned} \tag{2.5}$$

We can show that

$$\|\nabla g_\lambda\|_{L^p(\mathbb{T}^n \times (-\frac{7}{8}, \frac{7}{8}))} \leq \|\nabla g\|_{L^p(\mathbb{T}^n \times (-1,1))}$$

in the exact same manner, and Lemma 2.3 follows by (2.4) and (2.5). □

Theorem 2.4. *Suppose* $v \in C^1(\mathbb{T}^n)$ *and*

$$g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1}).$$

Then there exist constants $\varepsilon_2 = \varepsilon_2(n,p) > 0$ *such that, if* $\lambda < \frac{1}{8}$, $\varepsilon < \varepsilon_2$, *and* $\|v\|_{C^1(\mathbb{T}^n)} \le \varepsilon^{1/2}$, *then* g *satisfies* (1.2)–(1.4). *Then there exist a unique function* $u \in W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n)$ *and a unique constant* $-\frac{1}{4} < c_v < \frac{1}{4}$ *such that*

$$\int_{\mathbb{T}^n} \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla v|^2}} = \int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v + c_v)\phi \tag{2.6}$$

for all $\phi \in W^{1,2}(\mathbb{T}^n)$.

Proof. We define

$$F(t) := \int_{\mathbb{T}^n} \nu(\nabla v) \cdot g_\lambda(x, v + t).$$

The function F is continuous. Suppose that $\varepsilon < \frac{1}{16^2}$. We will consider that the domain of F is $[-\frac{1}{4}, \frac{1}{4}]$. By the mean value theorem, there exists a constant $c_3 = c_3(n,p) > 0$ such that

$$\begin{aligned} F\Big(\frac{1}{4}\Big) &= \int_{\mathbb{T}^n} (\nu(\nabla v) - \nu(0) + \nu(0)) \cdot g_\lambda\Big(x, v + \frac{1}{4}\Big) \\ &\ge -c_3 \|v\|_{C^1(\mathbb{T}^n)} \Big\| g_\lambda\Big(\cdot, v(\cdot) + \frac{1}{4}\Big)\Big\|_{L^q(\mathbb{T}^n)} + \int_{\mathbb{T}^n} g_\lambda^{n+1}\Big(x, v + \frac{1}{4}\Big). \end{aligned} \tag{2.7}$$

By Lemma 2.3 and $\|v + \frac{1}{4}\|_{C^1(\mathbb{T}^n)} \le \frac{5}{16}$, we obtain

$$\Big\| g_\lambda\Big(\cdot, v(\cdot) + \frac{1}{4}\Big)\Big\|_{L^q(\mathbb{T}^n)} \le c_1 \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}.$$

By (1.3) and (1.4), there exists a constant $c_4 = c_4(n) > 0$ such that

$$\begin{aligned} \int_{\mathbb{T}^n} g_\lambda^{n+1}\Big(x, v + \frac{1}{4}\Big) &= \int_{\mathbb{T}^n} \int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) g^{n+1}\Big(x - y, v + \frac{1}{4} - y^{n+1}\Big)\, dy\, dx \\ &> \int_{\mathbb{T}^n} \int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) g^{n+1}\Big(x - y, \frac{1}{16}\Big)\, dy\, dx \\ &> \int_{\mathbb{T}^n} \int_{B^{n+1}(0,\lambda)} \eta_\lambda(y, y^{n+1}) \Big(g^{n+1}(x - y, 0) + \frac{\varepsilon}{16}\Big)\, dy\, dx \\ &> \frac{c_4}{16}\varepsilon. \end{aligned} \tag{2.8}$$

By (1.2), (2.7)–(2.8), and $\|v\|_{C^1(\mathbb{T}^n)} < \varepsilon^{1/2}$, if

$$\varepsilon < \Big(\frac{c_4}{16 c_1 c_3}\Big)^6 =: \varepsilon_2(n,p),$$

then

$$\begin{aligned} F\Big(\frac{1}{4}\Big) &> -c_1 c_3 \|v\|_{C^1(\mathbb{T}^n)} \|g_\lambda\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))} + \frac{c_4}{16}\varepsilon \\ &> -c_1 c_3 \varepsilon^{\frac{7}{6}} + \frac{c_4}{16}\varepsilon \\ &> \varepsilon\Big(-c_1 c_3 \varepsilon^{\frac{1}{6}} + \frac{c_4}{16}\Big) \\ &> 0. \end{aligned}$$

Similarly, we can show that $F(-\frac{1}{4}) < 0$. By Lemma 2.2 and the mean value theorem, there exists a unique constant $-\frac{1}{4} < c_v < \frac{1}{4}$ that satisfies $F(c_v) = 0$. By using Theorem 2.1, Theorem 2.4 follows. □

Let us define an operator $T : \mathcal{A}(s) \to W^{1,2}_{\mathrm{ave}}(\mathbb{T}^n) \times [-\frac{1}{4}, \frac{1}{4}]$ by $T(v) = (T_1(v), T_2(v)) := (u, c_v)$ that satisfies (2.6), where

$$\mathcal{A}(s) := \{w \in W^{2,q}_{\mathrm{ave}}(\mathbb{T}^n) : \|w\|_{W^{2,q}(\mathbb{T}^n)} \le s\}.$$

Theorem 2.5. *There exist constants $\varepsilon_3 = \varepsilon_3(n,p) > 0$ and $c_5 = c_5(n,p) > 0$ such that, if $\lambda < \frac{1}{8}$, $\varepsilon < \min\{\varepsilon_2, \varepsilon_3\}$, $v \in \mathcal{A}(\varepsilon^{1/2})$, and $g \in W^{1,p}(\mathbb{T}^n \times (-1,1); \mathbb{R}^{n+1})$ satisfies (1.2)–(1.4), then*

$$\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq c_5 \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}.$$

Proof. We first assume that $v \in C^\infty(\mathbb{T}^n) \cap \mathcal{A}(\varepsilon^{1/2})$. Using [6, Corollary 8.11], we obtain $T_1(v) \in C^\infty(\mathbb{T}^n)$. Thus, we can rewrite (2.6) as

$$\frac{\Delta T_1(v)}{\sqrt{1+|\nabla v|^2}} + \nabla T_1(v) \cdot \nabla\Big(\frac{1}{\sqrt{1+|\nabla v|^2}}\Big) = -v(\nabla v) \cdot g_\lambda(x, v + T_2(v)).$$

Using [6, Theorem 9.11], we find that there exists a constant $c_6 = c_6(n,p) > 0$ such that

$$\begin{aligned}\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq c_6\Big(&\|T_1(v)\|_{L^q(\mathbb{T}^n)} + \|v(\nabla v) \cdot g_\lambda(x, v + T_2(v))\|_{L^q(\mathbb{T}^n)} \\ &+ \Big\|\nabla T_1(v) \cdot \nabla\Big(\frac{1}{\sqrt{1+|\nabla v|^2}}\Big)\Big\|_{L^q(\mathbb{T}^n)}\Big).\end{aligned} \tag{2.9}$$

Using Lemma 2.3, we obtain

$$\|v(\nabla v) \cdot g_\lambda(x, v + T_2(v))\|_{L^q(\mathbb{T}^n)} \leq c_1 \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}. \tag{2.10}$$

Using the Sobolev inequality, we find that there exists a constant $c_7 = c_7(n,p) > 0$ such that

$$\begin{aligned}\Big\|\nabla T_1(v) \cdot \nabla\Big(\frac{1}{\sqrt{1+|\nabla v|^2}}\Big)\Big\|_{L^q(\mathbb{T}^n)} &\leq \|T_1(v)\|_{C^1(\mathbb{T}^n)} \Big\|\nabla\Big(\frac{1}{\sqrt{1+|\nabla v|^2}}\Big)\Big\|_{L^q(\mathbb{T}^n)} \\ &\leq c_7 \|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \|v\|_{W^{2,q}(\mathbb{T}^n)}.\end{aligned} \tag{2.11}$$

Next, we estimate the term $\|T_1(v)\|_{L^q(\mathbb{T}^n)}$. If $q \leq 2$, then, by (2.2) and Lemma 2.3, we obtain

$$\begin{aligned}\|T_1(v)\|_{L^q(\mathbb{T}^n)} &\leq c_8(n,p) \|T_1(v)\|_{L^2(\mathbb{T}^n)} \\ &\leq c_9(n,p) B[T_1(v), T_1(v), v]^{\frac{1}{2}}. \\ &= c_9\Big(\int_{\mathbb{T}^n} \frac{\nabla T_1(v) \cdot \nabla T_1(v)}{\sqrt{1+|\nabla v|^2}}\Big)^{\frac{1}{2}} \\ &= c_9\Big(\int_{\mathbb{T}^n} v(\nabla v) \cdot g_\lambda(x, v + T_2(v)) T_1(v)\Big)^{\frac{1}{2}} \\ &\leq c_{10}(n,p) \|g\|^{\frac{1}{2}}_{W^{1,p}(\mathbb{T}^n)} \|T_1(v)\|^{\frac{1}{2}}_{L^\infty(\mathbb{T}^n)} \\ &\leq c_{11}(n,p) \|g\|_{W^{1,p}(\mathbb{T}^n)} + \frac{1}{4c_6} \|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}.\end{aligned} \tag{2.12}$$

If $q > 2$, by (2.12) and the Riesz–Thorin theorem, we obtain

$$\begin{aligned}\|T_1(v)\|_{L^q(\mathbb{T}^n)} &\leq \|T_1(v)\|^{\frac{1}{q}}_{L^2(\mathbb{T}^n)} \|T_1(v)\|^{1-\frac{1}{q}}_{L^2(\mathbb{T}^n)} \\ &\leq c_{12}(n,p) \|g\|^{\frac{1}{2q}}_{W^{1,p}(\mathbb{T}^n)} \|T_1(v)\|^{\frac{1}{2q}+1-\frac{1}{q}}_{L^\infty(\mathbb{T}^n)} \\ &\leq c_{13}(n,p) \|g\|_{W^{1,p}(\mathbb{T}^n)} + \frac{1}{4c_6} \|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}.\end{aligned} \tag{2.13}$$

By (2.9)–(2.13), there exists a constant $c_{14} = c_{14}(n,p) > 0$ such that

$$\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq c_{14}(\|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))} + \|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \|v\|_{W^{2,q}(\mathbb{T}^n)}) + \frac{1}{4} \|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)}.$$

If $\varepsilon < \frac{1}{16c_{14}^2}$, we obtain

$$\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} \leq 2c_{14} \|g\|_{W^{1,p}(\mathbb{T}^n \times (-1,1))}. \tag{2.14}$$

For the general case of $v \in W^{2,q}(\mathbb{T}^n)$, suppose that $\{v_m\}_{m\in\mathbb{N}} \in C^\infty(\mathbb{T}^n)$ converges to v in the sense of $C^1(\mathbb{T}^n)$. By (2.14), there exists a subsequence

$$\{v_{m_k}\}_{k\in\mathbb{N}} \subset \{v_m\}_{m\in\mathbb{N}}$$

such that $T_1(v_{m_k})$ converges to a function $w_\infty \in W^{2,q}(\mathbb{T}^n)$ in the sense of $C^1(\mathbb{T}^n)$, and $T_2(v_{m_k})$ converges to a constant $d_\infty \in [-\frac{1}{4}, \frac{1}{4}]$. For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} \nu(\nabla v)\cdot g_\lambda(x, v + d_\infty)\phi - \nu(\nabla v_{m_k})\cdot g_\lambda(x, v_{m_k} + T_2(v_{m_k}))\phi \\
&\qquad \le \int_{\mathbb{T}^n} |\phi||\nu(\nabla v) - \nu(\nabla v_{m_k})||g_\lambda(x, v_{m_k} + T_2(v_{m_k}))| + \int_{\mathbb{T}^n} |\phi|\Big|\int_{v_{m_k}+T_2(v_{m_k})}^{v+d_\infty} \partial_{n+1} g_\lambda(x, s)\Big| \\
&\qquad \to 0 \quad (k \to \infty)
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\int_{\mathbb{T}^n} \frac{\nabla w_\infty\cdot\nabla\phi}{\sqrt{1+|\nabla v|^2}} - \frac{\nabla T_1(v_{m_k})\cdot\nabla\phi}{\sqrt{1+|\nabla v_{m_k}|^2}} \\
&\qquad \le \int_{\mathbb{T}^n} \frac{(\nabla w_\infty - \nabla T_1(v_{m_k}))\cdot\nabla\phi}{\sqrt{1+|\nabla v|^2}} + \int_{\mathbb{T}^n} (\nabla T_1(v_{m_k})\cdot\nabla\phi)\Big(\frac{1}{\sqrt{1+|\nabla v|^2}} - \frac{1}{\sqrt{1+|\nabla v_{m_k}|^2}}\Big) \\
&\qquad \to 0 \quad (k \to \infty).
\end{aligned} \tag{2.16}$$

By (2.15) and (2.16), we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n} \frac{\nabla w_\infty\cdot\nabla\phi}{\sqrt{1+|\nabla v|^2}} - \nu(\nabla v)\cdot g_\lambda(x, v + d_\infty)\phi \\
&\qquad = \lim_{k\to\infty} \int_{\mathbb{T}^n} \frac{\nabla T_1(v_{m_k})\cdot\nabla\phi}{\sqrt{1+|\nabla v_{m_k}|^2}} - \nu(\nabla v_{m_k})\cdot g_\lambda(x, v_{m_k} + T_2(v_{m_k}))\phi \\
&\qquad = 0,
\end{aligned} \tag{2.17}$$

that is, $T(v) = (w_\infty, d_\infty)$. By (2.14) and (2.17), Theorem 2.5 follows. □

Next, we provide the fixed-point theorem, which is needed later ([1, Theorem 1]). An operator $T : X \to A$ is considered weakly sequentially continuous if, for every sequence $\{x_m\}_{m\in\mathbb{N}} \subset X$ and $x_\infty \in X$ such that x_m weakly converges to x_∞, $T(x_m)$ weakly converges to $T(x_\infty)$.

Theorem 2.6. *Let X be a metrizable, locally convex topological vector space and let Ω be a weakly compact convex subset of X. Then any weakly sequentially continuous map $T : \Omega \to \Omega$ has a fixed point.*

We first prove Theorem 1.1 in the case of g_λ.

Theorem 2.7. *There exists a constant $\varepsilon_4 = \varepsilon_4(n, p) > 0$ such that, if $\lambda < \frac{1}{8}$ and $\varepsilon < \varepsilon_4$, then*

$$g \in W^{1,p}(\mathbb{T}^n \times (-1, 1); \mathbb{R}^{n+1})$$

satisfies (1.2)–(1.4). *Then there exists a function $u_\lambda \in W^{2,q}(\mathbb{T}^n)$ such that*

$$-\operatorname{div}\Big(\frac{\nabla u_\lambda}{\sqrt{1+|\nabla u_\lambda|^2}}\Big) = \nu(\nabla u_\lambda)\cdot g_\lambda(x, u_\lambda(x)) \quad \text{on } \mathbb{T}^n. \tag{2.18}$$

Proof. The set $W^{2,q}(\mathbb{T}^n)$ is a metrizable, locally convex topological vector space, and the set $\mathcal{A}(\varepsilon^{1/2})$ is a weakly compact convex subset of $W^{2,q}(\mathbb{T}^n)$. By (1.2) and Theorem 2.5, if $\varepsilon < \min\{\varepsilon_2, \varepsilon_3, c_5^{-6}\} =: \varepsilon_4$, we have

$$\begin{aligned}
\|T_1(v)\|_{W^{2,q}(\mathbb{T}^n)} &\le c_5\|g\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} \\
&\le c_5\varepsilon^{\frac{1}{6}}\varepsilon^{\frac{1}{2}} \\
&\le \varepsilon^{\frac{1}{2}} \quad \text{for any } v \in \mathcal{A}(\varepsilon^{\frac{1}{2}}),
\end{aligned} \tag{2.19}$$

that is, $T_1(\mathcal{A}(\varepsilon^{1/2})) \subset \mathcal{A}(\varepsilon^{1/2})$. Suppose that $\{v_m\}_{m\in\mathbb{N}}$ weakly converges to v_∞ in the sense of $W^{2,q}(\mathbb{T}^n)$. According to Theorem 2.5, there exists a subsequence $\{v_{m_k}\}_{k\in\mathbb{N}} \subset \{v_m\}_{m\in\mathbb{N}}$ such that $T_1(v_{m_k})$ weakly converges to a function $w_\infty \in W^{2,q}(\mathbb{T}^n)$ in the sense of $W^{2,q}(\mathbb{T}^n)$, and $T_2(v_{m_k})$ converges to a constant $d_\infty \in [-\frac{1}{4}, \frac{1}{4}]$. By the same argument (2.15)–(2.17), for any $\phi \in W^{2,q}(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n} \frac{\nabla w_\infty \cdot \nabla \phi}{\sqrt{1+|\nabla v_\infty|^2}} - \nu(\nabla v_\infty)\cdot g_\lambda(x, v_\infty + d_\infty)\phi = 0,$$

that is, we obtain $\lim_{k\to\infty} T_1(v_{m_k}) = T_1(v_\infty)$ by the uniqueness of solution of Theorem 2.4. Therefore, every convergent subsequence of $\{T_1(v_m)\}$ converges to $T_1(v_\infty)$, and T_1 is a weakly sequentially continuous map. Using Theorem 2.6, we obtain a function $v_\lambda \in W^{2,q}_{\mathrm{ave}}(\mathbb{T}^n)$ satisfying

$$-\operatorname{div}\Big(\frac{\nabla v_\lambda}{\sqrt{1+|\nabla v_\lambda|^2}}\Big) = \nu(\nabla v_\lambda)\cdot g_\lambda(x, v_\lambda(x) + T_2(v_\lambda)) \quad \text{on } \mathbb{T}^n,$$

that is, $u_\lambda := v_\lambda + T_2(v_\lambda) \in W^{2,q}(\mathbb{T}^n)$ satisfying (2.18). □

Proof of Theorem 1.1. Suppose $u_\lambda \in W^{2,q}(\mathbb{T}^n)$ satisfies (2.18). By Theorem 2.5, there exists a convergent subsequence

$$\{u_{\lambda_k}\}_{k\in\mathbb{N}} \subset \{u_\lambda\}_{0<\lambda<\frac{1}{8}}$$

with a limit $u_\infty \in W^{2,q}(\mathbb{T}^n)$ in the sense of $C^1(\mathbb{T}^n)$ and $\lambda_k \to 0$. We show that u_∞ satisfies (1.5). For any $\phi \in W^{1,2}(\mathbb{T}^n)$, we obtain

$$\begin{aligned}\int_{\mathbb{T}^n} -\operatorname{div}\Big(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}} - \frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\Big)\phi &= \int_{\mathbb{T}^n}\Big(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}} - \frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\Big)\cdot\nabla\phi \\ &\to 0. \end{aligned} \tag{2.20}$$

Using Lemma 2.3, we have

$$\begin{aligned}&\int_{\mathbb{T}^n} \nu(\nabla u_{\lambda_k})\cdot g_{\lambda_k}(x, u_{\lambda_k}) - \nu(\nabla u_\infty)\cdot g(x, u_\infty) \\ &\quad= \int_{\mathbb{T}^n} (\nu(\nabla u_{\lambda_k}) - \nu(\nabla u_\infty))\cdot g_{\lambda_k}(x, u_{\lambda_k}) + \int_{\mathbb{T}^n} \nu(\nabla u_\infty)\cdot(g_{\lambda_k}(x, u_{\lambda_k}) - g(x, u_{\lambda_k})) \\ &\qquad+ \int_{\mathbb{T}^n} \nu(\nabla u_\infty)\cdot(g(x, u_{\lambda_k}) - 7g(x, u_\infty)) \\ &\quad= c_1\|\nu(\nabla u_{\lambda_k}) - \nu(\nabla u_\infty)\|_{C^0(\mathbb{T}^n)}\|g_{\lambda_k}\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} + c_1\|g_{\lambda_k} - g_\infty\|_{W^{1,p}(\mathbb{T}^n\times(-1,1))} + \int_{\mathbb{T}^n}\Big|\int_{u_\infty}^{u_{\lambda_k}} \partial_{n+1} g(x,s)\Big| \\ &\quad\to 0. \end{aligned} \tag{2.21}$$

By (2.20) and (2.21), we obtain

$$\begin{aligned}&\int_{\mathbb{T}^n} -\operatorname{div}\Big(\frac{\nabla u_\infty}{\sqrt{1+|\nabla u_\infty|^2}}\Big)\phi - \nu(\nabla u_\infty)\cdot g(x, u_\infty)\phi \\ &\qquad= \lim_{k\to\infty}\int_{\mathbb{T}^n} -\operatorname{div}\Big(\frac{\nabla u_{\lambda_k}}{\sqrt{1+|\nabla u_{\lambda_k}|^2}}\Big)\phi - \nu(\nabla u_{\lambda_k})\cdot g_{\lambda_k}(x, u_{\lambda_k})\phi \\ &\qquad\to 0. \end{aligned}$$

Thus, u_∞ satisfies (1.5) using the fundamental lemma of the calculus of variations. By (2.19), we obtain

$$\Big\|u_\infty - \int_{\mathbb{T}^n} u_\infty(y)\,dy\Big\|_{W^{2,q}(\mathbb{T}^n)} \le \varepsilon^{\frac{1}{2}},$$

and Theorem 1.1 follows. □

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