Research Article

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Bloom-type two-weight inequalities for commutators of maximal functions

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Abstract: We study Bloom-type two-weight inequalities for commutators of the Hardy-Littlewood maximal function and sharp maximal function. Some necessary and sufficient conditions are given to characterize the two-weight inequalities for such commutators.

Keywords: Hardy-Littlewood maximal function; sharp maximal function; commutator; weights; weighted BMO space

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1 Introduction and main results

Let T be a classical singular integral operator. In 1976, Coifman et al. [6] studied the commutator generated by T and a locally integrable function b as follows:

$$[b,T](f)(x) = T((b(x) - b(\cdot))f(\cdot))(x) = b(x)T(f)(x) - T(bf)(x). \tag{1.1}$$

A well-known result states that $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$ (bounded mean oscillation) (see [6] and [17] for details). In 1985, Bloom [4] considered two-weight behavior for the commutators of the Hilbert transform H. He proved that for two weights $\mu, \lambda \in A_p$, the commutator $[b,H] = bH(f) - H(bf)$ is bounded from $L^p(\mu)$ to $L^p(\lambda)$ if and only if the symbol b belongs to BMO_ν with $\nu = (\mu\lambda^{-1})^{1/p}$, where BMO_ν is a kind of weighted BMO space introduced by Muckenhoupt and Wheeden [23].

Segovia and Torrea [24] first extended the sufficient part of Bloom's result to general Calderón-Zygmund operators. They also characterized Bloom-type two-weight inequalities for maximal commutator of the Hardy-Littlewood maximal function. Soon afterward, García-Cuerva et al. [8] extended the sufficient part to a class of strongly singular integrals.

In recent years, Bloom-type inequalities have been at the focus of harmonic analysis. In 2017, Holmes et al. [13] extended Bloom's result to commutators of Calderón-Zygmund operators. For some more up-to-date results related to Bloom's result, we refer to [1,5,10,12–14,16,19–22] and references therein.

In this article, we will extend Bloom's results to commutators of the Hardy-Littlewood maximal function and sharp maximal function. Some necessary and sufficient conditions are given to characterize the two-weight inequalities for such commutators. We first recall some notations.

Let Q be a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure and by χ_Q the characteristic function of Q. For a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal function of f is defined by

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$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

and the sharp maximal function $M^\sharp$, introduced by Fefferman and Stein [7], is given by

$$M^\sharp(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x and $f_Q = |Q|^{-1}\int_Q f(x)\mathrm{d}x$.

Let b be a locally integrable function. Similar to (1.1), we can define two different kinds of commutators of the Hardy-Littlewood maximal function as follows.

The maximal commutator of the Hardy-Littlewood maximal function is given by

$$M_b(f)(x) = M((b(x) - b)f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The commutator generated by M and b is defined by

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Similarly, we can also define the commutator generated by $M^\sharp$ and b by

$$[b, M^\sharp](f)(x) = b(x)M^\sharp(f)(x) - M^\sharp(bf)(x).$$

Obviously, commutators M_b and $[b, M]$ essentially differ from each other. For example, M_b is positive and sublinear, but $[b, M]$ and $[b, M^\sharp]$ are neither positive nor sublinear.

In 2000, Bastero et al. [3] gave some characterizations for the boundedness of $[b, M]$ and $[b, M^\sharp]$ on L^p spaces. Certain BMO classes are characterized by the boundedness of the commutators. In 2014, Zhang and Wu [27] extended the results to the setting of variable exponent Lebesgue spaces.

A weight will always mean a nonnegative locally integrable function. As usual, we denote by A_p $(1 \le p \le \infty)$ the Muckenhoupt weights classes (see [9] and [25] for details). Let ω be a weight. For a function f and a measurable set E, we write

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) \mathrm{d}x \right)^{1/p} \quad \text{and} \quad \omega(E) = \int_E \omega(x) \mathrm{d}x.$$

For $\omega \in A_p (1 < p < \infty)$, we denote its conjugate weight by ω', i.e., $\omega' = \omega^{1-p'}$, where $1/p + 1/p' = 1$. It follows obviously from the definition of A_p that $\omega \in A_p$ if and only if $\omega' \in A_{p'}$.

Definition 1.1. [23] Let $\omega \in A_\infty$. We say that a locally integrable function b belongs to weighted BMO class BMO_ω if

$$\|b\|_{\mathrm{BMO}_\omega} := \sup_Q \frac{1}{\omega(Q)} \int_Q |b(x) - b_Q| \mathrm{d}x < \infty.$$

When $\omega = (\mu\lambda^{-1})^{1/p}$ with $\mu, \lambda \in A_p$, BMO_ω is also referred to as Bloom's two-weight BMO space in the literature (see, e.g., [5,12,13,19,22]).

For a function b, denote by $b^+(x) = \max\{b(x), 0\}$ and $b^-(x) = -\min\{b(x), 0\}$.

Let Q_0 be a cube. The Hardy-Littlewood maximal function relative to Q_0 is given by

$$M_{Q_0}(b)(x) = \sup_{\substack{Q \ni x \\ Q \subset Q_0}} \frac{1}{|Q|} \int_Q |b(y)| \mathrm{d}y,$$

where the supremum is taken over all the cubes Q with $Q \subset Q_0$ and $Q \ni x$.

Our results can be stated as follows:

Theorem 1.1. *Let* $1 < p < \infty$, $\mu, \lambda \in A_p$, *and* $\nu = (\mu\lambda^{-1})^{1/p}$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:

(1) $b \in \mathrm{BMO}_\nu$ *and* $b^-/\nu \in L^\infty$.

(2) $[b, M]$ *is bounded from* $L^p(\mu)$ to $L^p(\lambda)$.

(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\mu(Q)} \int_Q |b(x) - M_Q(b)(x)|^p \lambda(x) \mathrm{d}x \right)^{1/p} \leq C.$$

(4) $[b, M]$ *is bounded from* $L^{p'}(\lambda')$ *to* $L^{p'}(\mu')$, *where* $\mu' = \mu^{1-p'}$ *and* $\lambda' = \lambda^{1-p'}$.

(5) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\lambda'(Q)} \int_Q |b(x) - M_Q(b)(x)|^{p'} \mu'(x) \mathrm{d}x \right)^{1/p'} \leq C.$$

(6) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{\nu(Q)} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x \leq C.$$

Theorem 1.1 gives new characterizations for certain subclass of Bloom's two-weight BMO. The unweighted case of Theorem 1.1 was obtained in [3, Proposition 4]. Moreover, for $\omega \in A_p \cap A_{p'}$, let $\mu = \omega$ and $\lambda = \omega^{1-p}$ in Theorem 1.1, observe that $\omega \in A_{p'}$ implies $\lambda = \omega^{1-p} \in A_p$, we obtain the following characterization for one-weight case, which seems to be new.

Corollary 1.1. *Let* $1 < p < \infty$ *and* $\omega \in A_p \cap A_{p'}$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:

(1) $b \in \mathrm{BMO}_\omega$ *and* $b^-/\omega \in L^\infty$.

(2) $[b, M]$ *is bounded from* $L^p(\omega)$ *to* $L^p(\omega^{1-p})$.

(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - M_Q(b)(x)|^p \omega(x)^{1-p} \mathrm{d}x \right)^{1/p} \leq C.$$

(4) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{\omega(Q)} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x \leq C.$$

Remark 1.1. We would like to note that [26, Theorem 2] provides a result similar to the equivalence of (1) and (2) for bilinear case under the assumption $\omega \in A_1$. Our result essentially improves the corresponding linear case of it. The equivalence of (1), (3), and (4) is new.

Taking $\mu = \lambda = \omega \in A_p$ in Theorem 1.1, we deduce the following result.

Corollary 1.2. *Let* $1 < p < \infty$ *and* $\omega \in A_p$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:

(1) $b \in \mathrm{BMO}$ *and* $b^- \in L^\infty$.

(2) $[b, M]$ *is bounded from* $L^p(\omega)$ *to* $L^p(\omega)$.

(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - M_Q(b)(x)|^p \omega(x) \mathrm{d}x \right)^{1/p} \leq C.$$

(4) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x \leq C.$$

Remark 1.2. Corollary 1.2 was obtained by Ağcayazi and Zhang in [2, Corollary 4.2]. The equivalence of (1) and (2) was also proved by Hu and Wang in [15, Theorem 1.4] independently.

On the other hand, Ho [11, Theorem 3.1] proved that for $1 \leq p < \infty$ and $\omega \in A_p$, then $b \in \mathrm{BMO}$ if and only if

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x) \mathrm{d}x \right)^{1/p} < \infty.$$

Parallel to Ho's result, statement (3) can characterize a function $b \in \mathrm{BMO}$ with $b^- \in L^\infty$.

For the commutator of the sharp maximal function, we have similar results.

Theorem 1.2. *Let* $1 < p < \infty$, $\mu, \lambda \in A_p$, *and* $\nu = (\mu\lambda^{-1})^{1/p}$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:
(1) $b \in \mathrm{BMO}_\nu$ *and* $b^-/\nu \in L^\infty$.
(2) $[b, M^\sharp]$ *is bounded from* $L^p(\mu)$ *to* $L^p(\lambda)$.
(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\mu(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)|^p \lambda(x) \mathrm{d}x \right)^{1/p} \leq C.$$

(4) $[b, M^\sharp]$ *is bounded from* $L^{p'}(\lambda')$ *to* $L^{p'}(\mu')$.
(5) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\lambda'(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)|^{p'} \mu'(x) \mathrm{d}x \right)^{1/p'} \leq C.$$

(6) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{\nu(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)| \mathrm{d}x \leq C.$$

Theorem 1.2 gives new characterizations for certain subclass of Bloom's two-weight BMO class. When $\mu = \nu = 1$, the result was obtained in [3, Proposition 6]. Similar to Corollary 1.1, we have new characterizations for one-weight results for commutator $[b, M^\sharp]$ as follows.

Corollary 1.3. *Let* $1 < p < \infty$ *and* $\omega \in A_p \cap A_{p'}$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:
(1) $b \in \mathrm{BMO}_\omega$ *and* $b^-/\omega \in L^\infty$.
(2) $[b, M^\sharp]$ *is bounded from* $L^p(\omega)$ *to* $L^p(\omega^{1-p})$.
(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)|^p \omega(x)^{1-p} \mathrm{d}x \right)^{1/p} \leq C.$$

(4) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{\omega(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)| dx \le C.$$

By taking $\mu = \lambda = \omega \in A_p$ in Theorem 1.2, we have the following result, which was also obtained by Ağcayazi and Zhang [2, Corollary 4.3].

Corollary 1.4. *Let* $1 < p < \infty$, $\omega \in A_p$ *and* b *be a locally integrable function in* $\mathbb{R}^n$. *Then, the following statements are equivalent*:
(1) $b \in \mathrm{BMO}$ *and* $b^- \in L^\infty$.
(2) $[b, M]$ *is bounded from* $L^p(\omega)$ *to* $L^p(\omega)$.
(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)|^p \omega(x) dx \right)^{1/p} \le C.$$

(4) *There is a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)| dx \le C.$$

To compare our results with the ones for maximal commutator, we summarize some characterizations for Bloom's two-weight BMO and the two-weight boundedness of maximal commutator of the Hardy-Littlewood maximal function.

Theorem 1.3. *Let* $1 < p < \infty$, $\mu, \lambda \in A_p$, *and* $\nu = (\mu\lambda^{-1})^{1/p}$. *Suppose that* b *is a locally integrable function on* $\mathbb{R}^n$, *then the following statements are equivalent*:
(1) $b \in \mathrm{BMO}_\nu$.
(2) M_b *is bounded from* $L^p(\mu)$ *to* $L^p(\lambda)$.
(3) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\mu(Q)} \int_Q |b(x) - b_Q|^p \lambda(x) dx \right)^{1/p} \le C.$$

(4) M_b *is bounded from* $L^{p'}(\lambda')$ *to* $L^{p'}(\mu')$.
(5) *There is a constant* $C > 0$ *such that*

$$\sup_Q \left(\frac{1}{\lambda'(Q)} \int_Q |b(x) - b_Q|^{p'} \mu'(x) dx \right)^{1/p'} \le C.$$

The equivalence of (1) and (2) was proved in [24, Theorem 3] (see also [8, Theorem 2.4]). The equivalence of (1), (3), and (5) was given in [13, Theorem 4.1] for the dyadic version.

It is easy to see that the statements in Theorems 1.1 and 1.2, which can characterize a function $b \in \mathrm{BMO}_\nu$ with $b^-/\nu \in L^\infty$ and two-weight boundedness of $[b, M]$ and $[b, M^\sharp]$, correspond to the ones in Theorem 1.3 that characterize Bloom's two-weight BMO functions.

2 Preliminaries and lemmas

In this section, we present some lemmas that will be used in the proof of our results. The following weighted norm inequality for the Hardy-Littlewood maximal function is well known (see, for instance, [18] and [25] for details).

Lemma 2.1. *Let $1 < p < \infty$. Then, M is bounded from $L^p(\omega)$ to itself if and only if $\omega \in A_p$.*

For Bloom's weight, we have the following estimates (see [13, Lemma 2.7 and (2.15)]).

Lemma 2.2. [13] *Let $1 < p < \infty$, $\mu, \lambda \in A_p$, and $\nu = (\mu\lambda^{-1})^{1/p}$. Then, $\nu \in A_2$ and for any cube Q,*

$$\left(\int_Q \mu(x)\mathrm{d}x\right)^{1/p}\left(\int_Q \lambda(x)^{-p'/p}\mathrm{d}x\right)^{1/p'} \le C\nu(Q).$$

The following characterization for Bloom-type two-weight inequalities for maximal commutator M_b can be deduced from [24, Theorem 3] (see also [8, Theorem 2.4]).

Lemma 2.3. *Let $1 < p < \infty$, $\mu, \lambda \in A_p$, and $\nu = (\mu\lambda^{-1})^{1/p}$. For any locally integrable function b, the following statements are equivalent*:

(1) $b \in \mathrm{BMO}_\nu$.

(2) *M_b is bounded from $L^p(\mu)$ to $L^p(\lambda)$.*

The following lemma shows that the maximal commutator M_b pointwise controls the commutators $[b, M]$ and $[b, M^\sharp]$ when $b \ge 0$ (see [27, (3.1) and (3.2)] for details).

Lemma 2.4. *Let b and f be locally integrable functions and $b \ge 0$ a.e. in $\mathbb{R}^n$. Then, for any fixed $x \in \mathbb{R}^n$ such that $M(f)(x) < \infty$ and $0 \le b(x) < \infty$, we have*

$$|[b, M](f)(x)| \le M_b(f)(x). \tag{2.1}$$

and

$$|[b, M^\sharp](f)(x)| \le 2M_b(f)(x). \tag{2.2}$$

Proof. Let b and f be locally integrable and $b \ge 0$ a.e. From the proof of [27, (3.1)], we see that each step in (3.1) can be applied to such x that satisfies $0 \le b(x) < \infty$ and $M(f)(x) < \infty$. So does (3.2). This concludes Lemma 2.4. We omit the details. □

Remark 2.1. Observe that a locally integrable function is finite a.e. and $M^\sharp(f)(x) \le 2M(f)(x)$. It follows from Lemma 2.4 that if b is locally integrable and $b \ge 0$ a.e., then (2.1) and (2.2) hold for a.e. $x \in \mathbb{R}^n$, provided $M(f)$ is finite a.e. in $\mathbb{R}^n$.

To prove our theorems, we also need the following result.

Theorem 2.1. *Let $1 < p < \infty$, $\mu, \lambda \in A_p$, and $\nu = (\mu\lambda^{-1})^{1/p}$. If $b \in \mathrm{BMO}_\nu$ and $b \ge 0$, then $[b, M]$ and $[b, M^\sharp]$ are bounded from $L^p(\mu)$ to $L^p(\lambda)$.*

Proof. For $\mu \in A_p$ and $f \in L^p(\mu)$, by Lemma 2.1, we deduce that $M(f)(x)$ is finite a.e. in $\mathbb{R}^n$. Since $b \in \mathrm{BMO}_\nu$ implies that b is locally integrable, it follows from Remark 2.1 that

$$|[b, M](f)(x)| \le M_b(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

and

$$|[b, M^\sharp](f)(x)| \le 2M_b(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

By Lemma 2.3, we deduce that $[b, M]$ and $[b, M^\sharp]$ are bounded from $L^p(\mu)$ to $L^p(\lambda)$. □

3 Proof of the theorems

Proof of Theorem 1.1. We first prove the implication (1) ⇒ (2). Since $|b(x)| - b(x) = 2b^-(x)$ and $M(bf)(x) = M(|b|f)(x)$, we have

$$\begin{aligned}|[b,M](f)(x)| &\le |[b,M](f)(x) - [|b|,M](f)(x)| + |[|b|,M](f)(x)| \\ &\le |(b(x) - |b(x)|)M(f)(x)| + |M(bf)(x) - M(|b|f)(x)| + |[|b|,M](f)(x)| \\ &\le 2b^-(x)M(f)(x) + |[|b|,M](f)(x)|.\end{aligned} \tag{3.1}$$

Observe that $|b| \in \mathrm{BMO}_\nu$ when $b \in \mathrm{BMO}_\nu$ and M is bounded on $L^p(\mu)$ since $\mu \in A_p$. Since $b^-/\nu \in L^\infty$ and $\nu = (\mu\lambda^{-1})^{1/p}$, it follows from (3.1) and Theorem 2.1 that

$$\begin{aligned}\|[b,M](f)\|_{L^p(\lambda)} &\le 2\|b^-M(f)\|_{L^p(\lambda)} + \|[|b|,M](f)\|_{L^p(\lambda)} \\ &\le 2\|b^-/\nu\|_{L^\infty}\|M(f)\|_{L^p(\mu)} + \|[|b|,M](f)(x)\|_{L^p(\lambda)} \\ &\le C\|f\|_{L^p(\mu)}.\end{aligned}$$

Next, we will show the implication (2) ⇒ (3). For any cube $Q \subset \mathbb{R}^n$ and any $x \in Q$, we have (see [3, p. 3331])

$$M(\chi_Q)(x) = \chi_Q(x) \quad \text{and} \quad M(b\chi_Q)(x) = M_Q(b)(x).$$

Applying statement (2), we obtain that

$$\begin{aligned}\left(\frac{1}{\mu(Q)}\int_Q |b(x) - M_Q(b)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p} &= \left(\frac{1}{\mu(Q)}\int_Q |b(x)M(\chi_Q)(x) - M(b\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p} \\ &= \left(\frac{1}{\mu(Q)}\int_Q |[b,M](\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p} \\ &= \frac{1}{\mu(Q)^{1/p}}\|[b,M](\chi_Q)\|_{L^p(\lambda)} \\ &\le \frac{C}{\mu(Q)^{1/p}}\|\chi_Q\|_{L^p(\mu)} \\ &\le C.\end{aligned}$$

Thus, we obtain statement (3) since the constant C is independent of Q.

The proof of the implication (3) ⇒ (6) is easy. Indeed, by Hölder's inequality, Lemma 2.2, and statement (3), we have

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q |b(x) - M_Q(b)(x)|\mathrm{d}x &= \frac{1}{\nu(Q)}\int_Q |b(x) - M_Q(b)(x)|\lambda(x)^{1/p}\lambda(x)^{-1/p}\mathrm{d}x \\ &\le \frac{1}{\nu(Q)}\left(\int_Q |b(x) - M_Q(b)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p}\left(\int_Q \lambda(x)^{-p'/p}\mathrm{d}x\right)^{1/p'} \\ &\le C\left(\frac{1}{\mu(Q)}\int_Q |b(x) - M_Q(b)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p} \\ &\le C.\end{aligned}$$

This achieves statement (6) since Q is arbitrary and C is independent of Q.

Now, we prove the implication (6) ⇒ (1). First, we want to prove $b \in \mathrm{BMO}_\nu$. We use similar procedure to the proof of Proposition 4 in [3]. For any fixed cube Q, let

$$E = \{x \in Q : b(x) \le b_Q\} \quad \text{and} \quad F = \{x \in Q : b(x) > b_Q\}.$$

Then, (see [3, p. 3331])

$$\int_E |b(x) - b_Q| \mathrm{d}x = \int_F |b(x) - b_Q| \mathrm{d}x.$$

Since for any $x \in E$ we have $b(x) \le b_Q \le M_Q(b)(x)$,

$$|b(x) - b_Q| \le |b(x) - M_Q(b)(x)|, \quad \text{for any } x \in E.$$

This yields

$$\begin{aligned}\frac{1}{\nu(Q)} \int_Q |b(x) - b_Q| \mathrm{d}x &= \frac{1}{\nu(Q)} \int_{E \cup F} |b(x) - b_Q| \mathrm{d}x \\ &= \frac{2}{\nu(Q)} \int_E |b(x) - b_Q| \mathrm{d}x \\ &\le \frac{2}{\nu(Q)} \int_E |b(x) - M_Q(b)(x)| \mathrm{d}x \\ &\le \frac{2}{\nu(Q)} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x.\end{aligned} \tag{3.2}$$

Combining (3.2) and statement (6), we obtain that

$$\frac{1}{\nu(Q)} \int_Q |b(x) - b_Q| \mathrm{d}x \le C,$$

which implies $b \in \mathrm{BMO}_\nu$ by Definition 1.1.

Now, we will show $b^-/\nu \in L^\infty(\mathbb{R}^n)$. For any fixed cube Q, observe that

$$0 \le b^+(x) \le |b(x)| \le M_Q(b)(x), \quad \text{for a.e. } x \in Q.$$

We have for almost all $x \in Q$ that

$$0 \le b^-(x) \le M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x). \tag{3.3}$$

By Lemma 2.2, we have $\nu \in A_2$ since $\mu, \lambda \in A_p$. Then, it follows from the A_2 condition that $1/\nu$ is locally integrable and

$$\frac{1}{|Q|} \int_Q \frac{1}{\nu(x)} \mathrm{d}x \le C \frac{|Q|}{\nu(Q)}. \tag{3.4}$$

By (3.4), (3.3), and statement (6), we deduce

$$\begin{aligned}\left(\frac{1}{|Q|} \int_Q b^-(x) \mathrm{d}x\right)\left(\frac{1}{|Q|} \int_Q \frac{1}{\nu(x)} \mathrm{d}x\right) &\le \frac{C}{\nu(Q)} \int_Q b^-(x) \mathrm{d}x \\ &\le \frac{C}{\nu(Q)} \int_Q |b(x) - M_Q(b)(x)| \mathrm{d}x \\ &\le C.\end{aligned} \tag{3.5}$$

Applying the Lebesgue differentiation theorem, we can achieve that $b^-/\nu \in L^\infty$. Indeed, denote by $\mathcal{L}(b)$ and $\mathcal{L}(1/\nu)$ the sets of all Lebesgue points of b and $1/\nu$, and then, $\mathbb{R}^n \backslash (\mathcal{L}(b) \cap \mathcal{L}(1/\nu))$ is a set of measure zero.

For any $x_0 \in \mathcal{L}(b) \cap \mathcal{L}(1/\nu)$, it follows from the Lebesgue differentiation theorem that

$$\lim_{\substack{|Q| \to 0 \\ Q \ni x_0}} \left(\frac{1}{|Q|} \int_Q b^-(x) \mathrm{d}x\right)\left(\frac{1}{|Q|} \int_Q \frac{1}{\nu(x)} \mathrm{d}x\right) = \frac{b^-(x_0)}{\nu(x_0)}.$$

This together with (3.5) concludes that $b^-(x_0)/\nu(x_0) \le C$. Thus, we have $b^-/\nu \in L^\infty$.

Finally, to complete the proof of Theorem 1.1, we need to verify the implications (1) ⇒ (4), (4) ⇒ (5), and (5) ⇒ (6). Observe that for $\mu, \lambda \in A_p$, we have $\mu', \lambda' \in A_{p'}$ and $\nu = (\mu\lambda^{-1})^{1/p} = (\lambda'\mu'^{-1})^{1/p'}$. So, the procedure we just used on the ordered-group (μ, λ, ν, p) also applies to the ordered-group $(\lambda', \mu', \nu, p')$. Thus, we can use the same procedure to conclude the implications (1) ⇒ (4), (4) ⇒ (5), and (5) ⇒ (6). We omit the details.

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Reasoning as the proof of Theorem 1.1, we only prove the implications (1) ⇒ (2) ⇒ (3) ⇒ (6) and (6) ⇒ (1).

We first prove the implication (1) ⇒ (2). By the definition of $[b, M^\sharp]$, we have

$$\begin{aligned}|[b, M^\sharp](f)(x) - [|b|, M^\sharp](f)(x)| &\le |b(x)M^\sharp(f)(x) - |b(x)|M^\sharp(f)(x)| + |M^\sharp(bf)(x) - M^\sharp(|b|f)(x)|\\ &\le 2b^-(x)M^\sharp(f)(x) + |M^\sharp((|b| - b)f)(x)|\\ &\le 2b^-(x)M^\sharp(f)(x) + M^\sharp(2b^-f)(x).\end{aligned}$$

This gives

$$|[b, M^\sharp](f)(x)| \le 2b^-(x)M^\sharp(f)(x) + M^\sharp(2b^-f)(x) + |[|b|, M^\sharp](f)(x)|. \tag{3.6}$$

Since $M^\sharp(f)(x) \le 2M(f)(x)$, by Lemma 2.1, we have that $M^\sharp$ is bounded on both $L^p(\mu)$ and $L^p(\lambda)$. Observe that $|b| \in \mathrm{BMO}_\nu$ when $b \in \mathrm{BMO}_\nu$ and note that $b^-/\nu \in L^\infty$ and $\nu = (\mu\lambda^{-1})^{1/p}$. It follows from (3.6) and Theorem 2.1 that

$$\begin{aligned}\|[b, M^\sharp](f)\|_{L^p(\lambda)} &\le 2\|b^-M^\sharp(f)\|_{L^p(\lambda)} + \|M^\sharp(2b^-f)\|_{L^p(\lambda)} + \|[|b|, M^\sharp](f)\|_{L^p(\lambda)}\\ &\le 2\|b^-/\nu\|_{L^\infty}\|M^\sharp(f)\|_{L^p(\mu)} + 2\|b^-f\|_{L^p(\lambda)} + C\|f\|_{L^p(\mu)}\\ &\le 2\|b^-/\nu\|_{L^\infty}\|f\|_{L^p(\mu)} + C\|f\|_{L^p(\mu)}\\ &\le C\|f\|_{L^p(\mu)}.\end{aligned}$$

This concludes that $[b, M^\sharp]$ is bounded from $L^p(\mu)$ to $L^p(\lambda)$.

Next, we prove the implication (2) ⇒ (3). For any fixed a cube Q, we have $M^\sharp(\chi_Q)(x) = \frac{1}{2}$ for $x \in Q$ (see [3, p. 3333] for details). Note statement (2) that $[b, M^\sharp]$ is bounded from $L^p(\mu)$ to $L^p(\lambda)$, and then, we have

$$\begin{aligned}&\left(\frac{1}{\mu(Q)}\int_Q |b(x) - 2M^\sharp(b\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p}\\ &= \frac{2}{\mu(Q)^{1/p}}\left(\int_Q |b(x)M^\sharp(\chi_Q)(x) - M^\sharp(b\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p}\\ &= \frac{2}{\mu(Q)^{1/p}}\left(\int_Q |[b, M^\sharp](\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\right)^{1/p}\\ &\le \frac{2}{\mu(Q)^{1/p}}\|[b, M^\sharp](\chi_Q)\|_{L^p(\lambda)}\\ &\le \frac{C}{\mu(Q)^{1/p}}\|\chi_Q\|_{L^p(\mu)}\\ &\le C.\end{aligned}$$

Since the constant C is independent of Q, this concludes the proof of (2) ⇒ (3).

It is easy to prove the implication (3) ⇒ (6). Indeed, by Hölder's inequality, Lemma 2.2, and statement (3), we have

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q|b(x)-2M^\sharp(b\chi_Q)(x)|\mathrm{d}x &= \frac{1}{\nu(Q)}\int_Q|b(x)-2M^\sharp(b\chi_Q)(x)|\lambda(x)^{1/p}\lambda(x)^{-1/p}\mathrm{d}x\\
&\le \frac{1}{\nu(Q)}\bigg(\int_Q|b(x)-2M^\sharp(b\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\bigg)^{1/p}\bigg(\int_Q\lambda(x)^{-p'/p}\mathrm{d}x\bigg)^{1/p'}\\
&\le C\bigg(\frac{1}{\mu(Q)}\int_Q|b(x)-2M^\sharp(b\chi_Q)(x)|^p\lambda(x)\mathrm{d}x\bigg)^{1/p}\\
&\le C.\end{aligned}$$

Finally, to prove the implication (6) ⇒ (1), we can argue as mentioned earlier. For any fixed cube Q, it was proved by Bastro et al. in [3, (2)] that

$$|b_Q| \le 2M^\sharp(b\chi_Q)(x), \quad \text{for } x \in Q. \tag{3.7}$$

Let $E = \{x \in Q : b(x) \le b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$ as mentioned earlier. Then (see [3, p. 3331]),

$$\int_E|b(x)-b_Q|\mathrm{d}x = \int_F|b(x)-b_Q|\mathrm{d}x.$$

Since $b(x) \le b_Q \le |b_Q| \le 2M^\sharp(b\chi_Q)(x)$ for $x \in E$, we have

$$|b(x)-b_Q| \le |b(x)-2M^\sharp(b\chi_Q)(x)|, \quad x \in E.$$

Similar to (3.2), we have

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q|b(x)-b_Q|\mathrm{d}x &= \frac{2}{\nu(Q)}\int_E|b(x)-b_Q|\mathrm{d}x\\
&\le \frac{2}{\nu(Q)}\int_Q|b(x)-2M^\sharp(b\chi_Q)(x)|\mathrm{d}x.\end{aligned} \tag{3.8}$$

Combining (3.8) and statement (6), we conclude that $b \in \mathrm{BMO}_\nu$.

To complete the proof, we need to show $b^-/\nu \in L^\infty$. By (3.7) again, we obtain for $x \in Q$,

$$2M^\sharp(b\chi_Q)(x) - b(x) \ge |b_Q| - b(x) = |b_Q| - b^+(x) + b^-(x).$$

Since $\mu, \lambda \in A_p$, by Lemma 2.2, one has $\nu \in A_2$. Using (3.4) again, we have

$$\begin{aligned}&\frac{1}{\nu(Q)}\int_Q|2M^\sharp(b\chi_Q)(x)-b(x)|\mathrm{d}x\\
&= \frac{1}{|Q|}\cdot\frac{|Q|}{\nu(Q)}\int_Q|2M^\sharp(b\chi_Q)(x)-b(x)|\mathrm{d}x\\
&\ge \bigg(\frac{1}{|Q|}\int_Q|2M^\sharp(b\chi_Q)(x)-b(x)|\mathrm{d}x\bigg)\bigg(\frac{1}{|Q|}\int_Q\frac{1}{\nu(x)}\mathrm{d}x\bigg)\\
&\ge \bigg(\frac{1}{|Q|}\int_Q(2M^\sharp(b\chi_Q)(x)-b(x))\mathrm{d}x\bigg)\bigg(\frac{1}{|Q|}\int_Q\frac{1}{\nu(x)}\mathrm{d}x\bigg)\\
&\ge \bigg(\frac{1}{|Q|}\int_Q(|b_Q|-b^+(x)+b^-(x))\mathrm{d}x\bigg)\bigg(\frac{1}{|Q|}\int_Q\frac{1}{\nu(x)}\mathrm{d}x\bigg)\\
&= \bigg(|b_Q| - \frac{1}{|Q|}\int_Q b^+(x)\mathrm{d}x + \frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x\bigg)\bigg(\frac{1}{|Q|}\int_Q\frac{1}{\nu(x)}\mathrm{d}x\bigg).\end{aligned}$$

This together with statement (6) gives

$$\left(|b_Q| - \frac{1}{|Q|}\int_Q b^+(x)\mathrm{d}x + \frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{\nu(x)}\mathrm{d}x\right) \le C. \tag{3.9}$$

By the same reason as in the prove of Theorem 1.1, letting $|Q| \to 0$ with $Q \ni x$ in (3.9), Lebesgue differentiation theorem assures that the limit of the left-hand side of (3.9) equals almost everywhere to

$$\frac{|b(x)| - b^+(x) + b^-(x)}{\nu(x)} = \frac{2b^-(x)}{\nu(x)}.$$

Putting this fact together with (3.9) gives that $b^-/\nu \in L^\infty$.

This completes the proof of Theorem 1.2. □

Before ending this article, we give another proof of the equivalence of (1), (3), and (5) in Theorem 1.3, which is different from the ones provided in [13, Theorem 4.1].

Proof of Theorem 1.3. Since the equivalence of (1) and (2) is given in [24, Theorem 3], it is sufficient to prove the implications (2) ⇒ (3) ⇒ (1) and (1) ⇒ (4) ⇒ (5) ⇒ (1).

For any cube Q, noting statement (2) that M_b is bounded from $L^p(\mu)$ to $L^p(\lambda)$, we have

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q |b(x) - b_Q|^p\lambda(x)\mathrm{d}x &\le \frac{1}{\mu(Q)}\int_Q\left(\frac{1}{|Q|}\int_Q |b(x) - b(y)|dy\right)^p\lambda(x)\mathrm{d}x\\ &\le \frac{1}{\mu(Q)}\int_Q (M_b(\chi_Q)(x))^p\lambda(x)\mathrm{d}x\\ &\le \frac{1}{\mu(Q)}\|M_b(\chi_Q)\|^p_{L^p(\lambda)}\\ &\le \frac{C}{\mu(Q)}\|\chi_Q\|^p_{L^p(\mu)}\\ &\le C,\end{aligned}$$

which concludes the implication (2) ⇒ (3).

Now, we prove the implication (3) ⇒ (1). Given a cube Q, by Hölder's inequality, Lemma 2.2, and statement (3), we have

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q |b(x) - b_Q|\mathrm{d}x &= \frac{1}{\nu(Q)}\int_Q |b(x) - b_Q|\lambda(x)^{1/p}\lambda(x)^{-1/p}\mathrm{d}x\\ &\le \frac{1}{\nu(Q)}\left(\int_Q |b(x) - b_Q|^p\lambda(x)\mathrm{d}x\right)^{1/p}\left(\int_Q \lambda(x)^{-p'/p}\mathrm{d}x\right)^{1/p'}\\ &\le C\left(\frac{1}{\mu(Q)}\int_Q |b(x) - b_Q|^p\lambda(x)\mathrm{d}x\right)^{1/p}\\ &\le C.\end{aligned}$$

This achieves $b \in \mathrm{BMO}_\nu$ by Definition 1.1.

Observe that $1 < p' < \infty$, $\mu', \lambda' \in A_{p'}$, and $\nu = (\mu\lambda^{-1})^{1/p} = (\lambda'\mu'^{-1})^{1/p'}$. Applying the equivalence of (1) and (2) to the ordered-group $(\lambda', \mu', \nu, p')$ gives the equivalence of (1) and (4). Using the procedure in proving the implications (2) ⇒ (3) ⇒ (1) to the ordered-group $(\lambda', \mu', \nu, p')$, we conclude the implications (4) ⇒ (5) and (5) ⇒ (1). □

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