Research Article

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Metric quasiconformality and Sobolev regularity in non-Ahlfors regular spaces

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Abstract: Given a homeomorphism $f : X \to Y$ between Q-dimensional spaces X, Y, we show that f satisfying the metric definition of quasiconformality outside suitable exceptional sets implies that f belongs to the Sobolev class $N^{1,p}_{\mathrm{loc}}(X;\, Y)$, where $1 \le p \le Q$, and also implies one direction of the geometric definition of quasiconformality. Unlike previous results, we only assume a pointwise version of Ahlfors Q-regularity, which in particular enables various weighted spaces to be included in the theory. Notably, even in the classical Euclidean setting, we are able to obtain new results using this approach. In particular, in spaces including the Carnot groups, we are able to prove the Sobolev regularity $f \in N^{1,Q}_{\mathrm{loc}}(X;\, Y)$ without the strong assumption of the infinitesimal distortion h_f belonging to $L^\infty(X)$.

Keywords: quasiconformal mapping; Newton-Sobolev mapping; modulus of a curve family; absolute continuity; Ahlfors regularity; weighted space

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1 Introduction

Consider two metric measure spaces (X, d, μ) and (Y, d_Y, ν), and a mapping $f : X \to Y$. For every $x \in X$ and $r > 0$, one defines

$$L_f(x, r) \coloneqq \sup\{d_Y(f(y), f(x)) : d(y, x) \le r\}$$

and

$$l_f(x, r) \coloneqq \inf\{d_Y(f(y), f(x)) : d(y, x) \ge r\},$$

and then

$$H_f(x, r) \coloneqq \frac{L_f(x, r)}{l_f(x, r)};$$

we interpret this to be ∞ if the denominator is zero. A homeomorphism $f : X \to Y$ is (metric) quasiconformal if there is a number $1 \le H < \infty$ such that

$$H_f(x) \coloneqq \limsup_{r\to 0} H_f(x, r) \le H \tag{1.1}$$

for all $x \in X$.

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In the case where X and Y are Ahlfors Q-regular spaces, with $Q > 1$, the homeomorphism f is said to satisfy the *analytic definition* of quasiconformality if $f \in N^{1,Q}_{\mathrm{loc}}(X; Y)$ and the minimal Q-weak upper gradient g_f to the Qth power is bounded pointwise by a constant C times the Jacobian J_f; see Section 2 for definitions. Moreover, f is said to satisfy the *geometric definition* of quasiconformality if for every family of curves Γ in X, we have

$$\frac{1}{C}\mathrm{Mod}_Q(\Gamma) \le \mathrm{Mod}_Q(f(\Gamma)) \le C\mathrm{Mod}_Q(\Gamma)$$

for some constant $C \ge 1$. It has been a problem of wide interest to study the equivalence between these definitions and in particular to examine whether the metric definition implies the other definitions. It has turned out possible to significantly relax the metric definition and still obtain at least that f is a Sobolev mapping. Results in this direction have been proven in Euclidean and Carnot-Carathéodory spaces by Gehring [7,8], Marguilis and Mostow [19], Balogh and Koskela [1], Kallunki and Koskela [15], Kallunki and Martio [16], and Koskela and Rogovin [17]. These articles show that one does not need the condition $H_f(x) \le H$ at *every* point x, and that instead of H_f, one can consider

$$h_f(x) := \liminf_{r\to 0} H_f(x, r).$$

Heinonen and Koskela [11,12] and Heinonen et al. [13, Theorem 9.8] have studied the equivalence between the definitions in the more general setting of an Ahlfors Q-regular metric space supporting a Poincaré inequality.

Balogh et al. [2] and Williams [21] show that to go from the metric definition of quasiconformality to the fact that $f \in N^{1,1}_{\mathrm{loc}}(X; Y)$, the assumption of a Poincaré inequality is in fact not necessary. In Lahti and Zhou [18] the authors of the present article show that it is possible to largely remove the assumption of Ahlfors regularity as well. Now we give a similar result when $1 < p \le Q$. In fact we also cover the case $p = 1$, but in this case, the assumptions needed by Lahti and Zhou [18] are mostly even weaker than here, so this is not our main focus; the case $p = Q$ is of greatest interest in the theory of quasiconformal mappings.

Our main theorem is the following.

Theorem 1.1. *Let $\Omega \subset X$ be open and bounded, let $f : \Omega \to f(\Omega) \subset Y$ be a homeomorphism such that $f(\Omega)$ is open and $\nu(f(\Omega)) < \infty$, and*

(1) *Suppose there exists a μ-measurable set $E \subset \Omega$ such that in $\Omega\setminus E$ there exist μ-measurable functions $1 < Q(x) < \infty$ and $0 < R(x) < \infty$ with*

$$\limsup_{r\to 0}\frac{\mu(B(x,r))}{r^{Q(x)}} < R(x)\liminf_{r\to 0}\frac{\nu(B(f(x),r))}{r^{Q(x)}} \quad \text{for } \mu\text{-a.e. } x \in \Omega\setminus E.$$

Suppose also that there is a Borel regular outer measure $\tilde{\mu} \ge \mu$ on X, which is doubling within a ball $2B_0$ with $\Omega \subset B_0$, and

$$\limsup_{r\to 0}\frac{\tilde{\mu}(B(x,r))}{r^{Q(x)}} < \infty, \quad \liminf_{r\to 0}\frac{\nu(B(f(x),r))}{r^{Q(x)}} > 0 \quad \text{for all } x \in \Omega\setminus E.$$

(2) *Suppose $Q := \inf_{x\in\Omega\setminus E} Q(x) > 1$ and let $1 \le p \le Q$. Assume that*

$$\mathrm{Mod}_p(\{\gamma \subset \Omega : \mathcal{H}^1(f(\gamma\cap E)) > 0\}) = 0; \tag{1.2}$$

in particular, it is enough to assume that E is the union of a countable set and a set with σ-finite codimension p Hausdorff measure.

(3) *Finally, assume that $p \le q \le Q$ and that*

$$\begin{cases} \dfrac{Q(\cdot)-q}{Q(\cdot)}(R(\cdot)h_f(\cdot)^{Q(\cdot)})^{q/(Q(\cdot)-q)} \in L^1(\Omega\setminus E) & \text{if } 1 \le q < Q \\ R(\cdot)^{1/Q(\cdot)}h_f(\cdot) \in L^\infty(\Omega\setminus E) & \text{if } q = Q. \end{cases}$$

Then it follows that $f \in D^p(\Omega; Y)$.

In the case $p = Q = Q(x)$ *for* μ-a.e. $x \in \Omega \setminus E$, *for every curve family* Γ *in* Ω, *we also obtain*

$$\mathrm{Mod}_Q(\Gamma) \le C \mathrm{Mod}_Q(f(\Gamma)) \tag{1.3}$$

with $C = \|R(\cdot)h_f(\cdot)^Q\|_{L^\infty(\Omega)}$.

If X is proper and supports a $(1, p)$*-Poincaré inequality, and* μ *is doubling, then* $f \in D^q(\Omega; Y)$.

Note that by $D^q(\Omega; Y)$, we mean $D^q(\Omega, \mu; Y)$; here, $D^q(\Omega; Y)$ is the Dirichlet space, that is, f is not required to be in $L^q(\Omega; Y)$.

While rather technical in its full generality, the theorem will be seen to have several corollaries that improve on known results; in Corollary 6.1, we make a comparison with Williams [21, Corollary 1.3]. The main difference with the previous work of Balogh et al. [2] and Williams [21] is that instead of local Ahlfors Q-regularity, we merely assume the pointwise conditions of (1), as well as the fact that μ is controlled by a doubling measure $\widetilde{\mu}$. Note that the "dimension" $Q(x)$ as well as the "density" $R(x)$ are allowed to vary from point to point, but typically the space X at the point x and the space Y at the point $f(x)$ need to have the same dimension for the first condition in (1) to be satisfied. Mostly, we are interested in the case where $Q(x)$ is constant, but the flexibility provided by the function $R(x)$ makes it easy to include weighted spaces in the theory. Indeed, $R(x)$ then represents the ratio of the weights in the two spaces at the points x and $f(x)$, as we will see in Corollary 1.2.

Since we work in spaces without a specific dimension, we consider a codimension p Hausdorff measure, which however reduces, up to a constant, to the $(Q - p)$-dimensional Hausdorff measure in the Ahlfors Q-regular case. The set E can also contain any countable set; in our generality, even a single point could have infinite codimension p Hausdorff measure.

Similarly to the study by Williams [21] and several other previous works, in the study by Lahti and Zhou [18], we relied on constructing a sequence of "almost upper gradients" $\{g_i\}_{i=1}^\infty$. Since we worked in the case $p = 1$, we then had to prove the sequence to be *equi-integrable* to find a weakly converging subsequence. In the present article we mostly deal with the case $1 < p \le Q$, and so our methods are quite different from [18], but partially in the same vein as those of Williams [21]. The key step is to prove boundedness of the sequence $\{g_i\}_{i=1}^\infty$ in L^p and then use reflexivity to find a weakly converging subsequence, and to obtain a p-weak upper gradient of f at the limit.

The inequality (1.3) is one direction of the geometric definition of quasiconformality. There is no hope in general to obtain the opposite estimate

$$\mathrm{Mod}_Q(f(\Gamma)) \le C \mathrm{Mod}_Q(\Gamma),$$

essentially because all of the assumptions of Theorem 1.1 still hold if we make ν bigger, but this tends to increase $\mathrm{Mod}_Q(f(\Gamma))$. For a counterexample, see Example 6.3.

From Theorem 1.1, we obtain the following corollary for weighted spaces. By a weight, we simply mean a nonnegative locally integrable function.

Corollary 1.2. *Let* $1 \le p \le q \le Q$. *Let* (X_0, d, μ_0) *and* (Y_0, d_Y, ν_0) *be Ahlfors Q-regular spaces, with* $Q > 1$. *Let X and Y be the same metric spaces but equipped with the weighted measures* $d\mu = w\, d\mu_0$ *and* $d\nu = w_Y\, d\nu_0$, *where* $w_Y > 0$ *is pointwise represented by*

$$w_Y(y) = \liminf_{r \to 0} \frac{1}{\nu_0(B(y, r))} \int_{B(y,r)} w_Y \, d\nu_0, \quad y \in Y. \tag{1.4}$$

Let $\Omega \subset X$ *be open and bounded and let* $f : \Omega \to f(\Omega) \subset Y$ *be a homeomorphism with* $f(\Omega)$ *open and* $\nu(f(\Omega)) < \infty$. *Moreover*:

(1) *Suppose* $w \le \widetilde{w}$ *for some weight* $\widetilde{w}$ *for which* $d\widetilde{\mu} := \widetilde{w}\, d\mu_0$ *is doubling, and suppose there is a set* $E \subset \Omega$ *that is the union of a countable set and a set with* σ*-finite* $\widetilde{\mathcal{H}}^p$*-measure, and*

$$\limsup_{r\to 0}\frac{\widetilde{\mu}(B(x,r))}{r^Q}<\infty \quad \textit{for all } x\in\Omega\setminus E;$$

(2) *Assume that* $h_f<\infty$ *in* $\Omega\setminus E$ *and that*

$$\begin{cases}\left(\dfrac{w(\cdot)}{w_Y(f(\cdot))}h_f(\cdot)^Q\right)^{q/(Q-q)}\in L^1(\Omega) & \text{if} \quad 1\le q<Q;\\ \dfrac{w(\cdot)}{w_Y(f(\cdot))}h_f(\cdot)^Q\in L^\infty(\Omega) \quad \text{if } q=Q.\end{cases}$$

Then $f\in D^p(\Omega;\,Y)$. *In the case* $p=Q$, *for every curve family* Γ *in* Ω, *we also obtain*

$$\mathrm{Mod}_Q(\Gamma)\le C\,\mathrm{Mod}_Q(f(\Gamma))$$

with $C=\|w(\cdot)w_Y(f(\cdot))^{-1}h_f(\cdot)^Q\|_{L^\infty(\Omega)}$.

Furthermore, if X *is proper and supports a* $(1,p)$*-Poincaré inequality, and* μ *is doubling, then* $f\in D^q(\Omega;\,Y)$.

Again by $D^q(\Omega;\,Y)$, we mean $D^q(\Omega,\mu;\,Y)$.

We observe that quite general weights w are allowed in the space X, and thus, we can include many spaces in the theory where the measure μ is not Ahlfors regular or even doubling. Moreover, contrary to all previous works, to the best of our knowledge, we do not require h_f to be essentially bounded to obtain $f\in D^Q(\Omega;\,Y)$ or $f\in N^{1,Q}(\Omega;\,Y)$. Instead, h_f can be large in regions where the weight w is small, see Example 6.2.

In the space Y, even more general weights are allowed. Note that the choice of weight w_Y does not change the classes $D^p(\Omega;\,Y)$ and $N^{1,p}(\Omega;\,Y)$, whose definition depends only on Y as a metric space and is independent of the choice of the measure ν on Y. Thus, we can choose any weight w_Y that is convenient for us. Already in the classical setting of the unweighted plane, there are simple examples of $N^{1,2}$-mappings $f\colon\mathbb{R}^2\to\mathbb{R}^2$ for which h_f is not in $L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, and so the previous results, such as those of [2,21], do not tell us that $f\in N^{1,2}_{\mathrm{loc}}(\mathbb{R}^2;\,\mathbb{R}^2)$. However, we can detect the Sobolev property of many such mappings from the next corollary, simply by equipping Y with a suitable weight w_Y; see Example 6.3.

Corollary 1.3. *Let* $1\le q\le n\in\mathbb{N}\setminus\{1\}$. *Let* $w_Y\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *be represented by* (1.4), *with* $w_Y>0$, *and* $d\nu:=w_Y\,d\mathcal{L}^n$. *Let* $\Omega\subset\mathbb{R}^n$ *be open and bounded and let* $f\colon\Omega\to f(\Omega)\subset\mathbb{R}^n$ *be a homeomorphism with* $f(\Omega)$ *open and* $\nu(f(\Omega))<\infty$. *Suppose there is a set* $E\subset\Omega$ *that has* σ*-finite* $\mathcal{H}^{n-1}$*-measure, and* $h_f<\infty$ *in* $\Omega\setminus E$. *Finally, assume that*

$$\begin{cases}\left(\dfrac{h_f(\cdot)^n}{w_Y(f(\cdot))}\right)^{q/(n-q)}\in L^1(\Omega) & \text{if } 1\le q<n;\\ \dfrac{h_f(\cdot)^n}{w_Y(f(\cdot))}\in L^\infty(\Omega) \quad \text{if } q=n.\end{cases}$$

Then $f\in D^q(\Omega;\,\mathbb{R}^n)$. *In the case* $q=n$, *for every curve family* Γ *in* Ω, *we have*

$$\mathrm{Mod}_n(\Gamma)\le C\,\mathrm{Mod}_n(f(\Gamma))$$

with $C=\|w_Y(f(\cdot))^{-1}h_f(\cdot)^n\|_{L^\infty(\Omega)}$.

In Carnot groups, we obtain an analogous result.

Corollary 1.4. *Let* G *be a Carnot group of homogeneous dimension* $Q>1$. *Let* $w_Y\in L^1_{\mathrm{loc}}(G)$ *be represented by* (1.4), *with* $w_Y>0$, *and* $d\nu:=w_Y\,d\mathcal{L}^n$, *where* $\mathcal{L}^n$ *is the Haar measure on* G. *Let* $\Omega\subset G$ *be open and bounded and let* $f\colon\Omega\to f(\Omega)\subset G$ *be a homeomorphism with* $f(\Omega)$ *open and* $\nu(f(\Omega))<\infty$. *Suppose there is a set* $E\subset\Omega$ *that has* σ*-finite* $\mathcal{H}^{Q-1}$*-measure, and* $h_f<\infty$ *in* $\Omega\setminus E$. *Finally, assume that*

$$\begin{cases} \left(\dfrac{h_f(\cdot)^Q}{w_Y(f(\cdot))}\right)^{q/(Q-q)} \in L^1(\Omega) & \textit{if } 1 \le q < Q; \\ \dfrac{h_f(\cdot)^Q}{w_Y(f(\cdot))} \in L^\infty(\Omega) & \textit{if } q = Q. \end{cases}$$

Then $f \in D^q(\Omega; G)$. *In the case* $q = Q$, *for every curve family* Γ *in* Ω, *we have*

$$\mathrm{Mod}_Q(\Gamma) \le C\mathrm{Mod}_Q(f(\Gamma))$$

with $C = \|w_Y(f(\cdot))^{-1} h_f(\cdot)^Q\|_{L^\infty(\Omega)}$.

This article is organized in the following way: we give definitions and notation in Section 2 and study the exceptional set E in Section 3. In Section 4, we study absolute continuity on curves and further preliminary results, and then in Section 5, we prove the results given here in Section 1. Finally, in Section 6, we give examples and applications of our main results.

2 Definitions and notation

Throughout the article, we consider two metric measure spaces (X, d, μ) and (Y, d_Y, ν), where μ and ν are Borel regular outer measures, such that the measure of every ball is finite in both spaces. We understand balls $B(x, r)$, with $x \in X$ and $0 < r < \infty$, to be open. We also assume X to be connected and Y to be separable. To avoid certain pathologies, we assume that X consists of at least two points.

We say that X is metric doubling with constant $M \in \mathbb{N}$ if every ball $B(x, r)$ can be covered by M balls of radius $r/2$. This definition works also for subsets $A \subset X$, by considering (A, d) as a metric space.

We will often work also with another Borel regular outer measure $\tilde{\mu}$ on X, which we assume to satisfy a doubling condition. We say that $\tilde{\mu}$ is doubling with constant $C_d \ge 1$ within an open set $W \subset X$ if

$$0 < \tilde{\mu}(B(x, 2r)) \le C_d \tilde{\mu}(B(x, r)) < \infty$$

for every ball $B(x, r) \subset W$. For a ball $B = B(x, r)$, we sometimes use the abbreviation $2B = B(x, 2r)$; note that in metric spaces, a ball (as a set) does not necessarily have a unique center point and radius, but when using this abbreviation, we will understand that these have been prescribed.

Given $Q \ge 1$, we say that μ is locally Ahlfors Q-regular if for every $z \in X$, there is $R > 0$ and a constant $C_A \ge 1$ such that

$$C_A^{-1} r^Q \le \mu(B(x, r)) \le C_A r^Q$$

for all $x \in B(z, R)$ and $0 < r \le R$.

If these conditions hold for every $x \in X$ and $0 < r < 2\mathrm{diam} X$ with C_d (resp. C_A) replaced by a universal constant, we say that μ is doubling, or that μ (or X) is Ahlfors Q-regular.

Let $\Omega \subset X$ always be an open set. If $f : \Omega \to f(\Omega) \subset Y$ is a homeomorphism with $f(\Omega)$ open, the pull-back of the measure ν is the measure on Ω given by

$$f_\# \nu(D) := \nu(f(D))$$

for every Borel set $D \subset \Omega$. Note that since f is a homeomorphism, $f_\# \nu$ defines a Borel measure. The Jacobian of f is then defined by

$$J_f(x) := \lim_{r \to 0} \frac{\nu(f(B(x, r)))}{\mu(B(x, r))} = \lim_{r \to 0} \frac{f_\# \nu(B(x, r))}{\mu(B(x, r))} \tag{2.1}$$

at every $x \in \Omega$, where the limit exists.

The n-dimensional Lebesgue measure is denoted by $\mathcal{L}^n$.

The s-dimensional Hausdorff content is denoted by $\mathcal{H}^s_R$, with $R > 0$ and $s \geq 0$, and the s-dimensional Hausdorff measure is denoted by $\mathcal{H}^s$; these definitions extend automatically to metric spaces.

For a mapping $f : X \to Y$, we define

$$\operatorname{lip}_f(x) := \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{d_Y(f(y), f(x))}{r}, \quad x \in X. \tag{2.2}$$

This is easily seen to be a Borel function.

A continuous mapping from a compact interval into X is said to be a rectifiable curve if it has finite length. A rectifiable curve γ always admits an arc-length parametrization, so that we obtain a curve $\gamma : [0, \ell_\gamma] \to X$ (for a proof, see, e.g., [9, Theorem 3.2]). We will only consider curves that are rectifiable and arc-length parametrized. If $\gamma : [0, \ell_\gamma] \to X$ is a curve and $g : X \to [0, \infty]$ is a Borel function, we define

$$\int_\gamma g\,ds := \int_0^{\ell_\gamma} g(\gamma(s))\,ds.$$

We will always assume that $1 \leq p < \infty$, though often we will specify a more restricted range for p. The p-modulus of a family of curves Γ is defined by

$$\operatorname{Mod}_p(\Gamma) := \inf \int_X \rho^p\,d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0, \infty]$ such that $\int_\gamma \rho\,ds \geq 1$ for every curve $\gamma \in \Gamma$. If a property holds apart from a curve family with zero p-modulus, we say that it holds for p-a.e. curve.

Recall that $\Omega \subset X$ is always an open set. Next we give (special cases of) Mazur's lemma and Fuglede's lemma, see, e.g., [20, Theorem 3.12] and [4, Lemma 2.1], respectively.

Lemma 2.1. *Let $\{g_i\}_{i\in\mathbb{N}}$ be a sequence with $g_i \to g$ weakly in $L^p(\Omega)$. Then there exist convex combinations $\widehat{g}_i := \sum_{j=i}^{N_i} a_{i,j} g_j$, for some $N_i \in \mathbb{N}$, such that $\widehat{g}_i \to g$ in $L^p(\Omega)$.*

Lemma 2.2. *Let $\{g_i\}_{i=1}^\infty$ be a sequence of functions with $g_i \to g$ in $L^p(\Omega)$. Then for p-a.e. curve γ in Ω, we have*

$$\int_\gamma g_i\,ds \to \int_\gamma g\,ds \quad \text{as } i \to \infty.$$

Definition 2.3. Let $f : \Omega \to Y$. We say that a Borel function $g : \Omega \to [0, \infty]$ is an upper gradient of f in Ω if

$$d_Y(f(\gamma(0)), f(\gamma(\ell_\gamma))) \leq \int_\gamma g\,ds \tag{2.3}$$

for every curve $\gamma : [0, \ell_\gamma] \to \Omega$. If $g : \Omega \to [0, \infty]$ is a μ-measurable function and (2.3) holds for p-a.e. curve in Ω, we say that g is a p-weak upper gradient of f in Ω.

We fix some $y_0 \in Y$, and we say that $f \in L^p(\Omega; Y)$ if f is μ-measurable and $d_Y(f(\cdot), y_0) \in L^p(\Omega)$.

Definition 2.4. The Newton-Sobolev space $N^{1,p}(\Omega; Y)$ consists of those mappings $f \in L^p(\Omega; Y)$ for which there exists an upper gradient $g \in L^p(\Omega)$.

The Dirichlet space $D^p(\Omega; Y)$ consists of those mappings $f : \Omega \to Y$ that have an upper gradient $g \in L^p(\Omega)$.

In the classical setting of $X = Y = \mathbb{R}^n$, both spaces equipped with the Lebesgue measure, and assuming f is continuous, we have $f \in N^{1,p}(X; Y)$ if and only if $f \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n)$, see, e.g., [4, Theorem A.2].

We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega; Y)$ if for every $x \in \Omega$ there is $r > 0$ such that $f \in N^{1,p}(B(x, r); Y)$; other local spaces of mappings are defined analogously.

It is known that for every $f \in D^p_{\mathrm{loc}}(\Omega;\, Y)$, there exists a minimal p-weak upper gradient of f in Ω, denoted by g_f, satisfying $g_f \le g\,\mu$-a.e. in Ω for every p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of f in Ω, see [14, Theorem 6.3.20].

We say that X supports a $(1,p)$-Poincaré inequality if every ball in X has nonzero μ-measure, and there exist constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, every $u : X \to \mathbb{R}$ that is integrable on balls, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \mathrm{d}\mu \le C_P r \fint_{B(x,\lambda r)} g \mathrm{d}\mu, \tag{2.4}$$

where

$$u_{B(x,r)} \coloneqq \fint_{B(x,r)} u \mathrm{d}\mu \coloneqq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \mathrm{d}\mu.$$

3 The exceptional set E

In this section, we consider some preliminary results and use them to study the exceptional set E. Recall that the standing assumptions on the spaces X, Y are listed in the first paragraph of Section 2.

In Balogh e al. [2] and Williams [21], it is assumed that the exceptional set E is σ-finite with respect to the $(Q-p)$-dimensional Hausdorff measure. Since we do not assume the space to be Ahlfors regular, we instead consider a codimension p Hausdorff measure. Specifically, we consider it with respect to the measure $\widetilde{\mu}$, which will always be assumed to satisfy a doubling condition.

Definition 3.1. Let $1 \le p < \infty$. For any set $A \subset X$ and $0 < R < \infty$, the restricted Hausdorff content of codimension p is defined by

$$\widetilde{\mathcal{H}}^p_R(A) \coloneqq \inf\left\{\sum_j \frac{\widetilde{\mu}(B(x_j, r_j))}{r_j^p} : A \subset \bigcup_j B(x_j, r_j),\, r_j \le R\right\},$$

where we consider finite and countable coverings. The codimension p Hausdorff measure of $A \subset X$ is then defined by

$$\widetilde{\mathcal{H}}^p(A) \coloneqq \lim_{R \to 0} \widetilde{\mathcal{H}}^p_R(A).$$

If $\widetilde{\mu}$ is Ahlfors Q-regular, then $\widetilde{\mathcal{H}}^p$ is easily seen to be comparable to $\mathcal{H}^{Q-p}$ when $1 \le p \le Q$.

The following result is often used and a proof can be found in the study by Bojarski [5, Lemma 4.2].

Lemma 3.2. *Let $1 < p < \infty$. Let $\Omega \subset X$ be open and suppose $\widetilde{\mu}$ is doubling with constant C_d within a ball $2B_0$, with $\Omega \subset B_0$. Then for any finite or countable collection of balls $\{B_j = B(x_j, r_j)\}_j$ with $6B_j \subset \Omega$, and for numbers $a_j \ge 0$, we have*

$$\int_X \left(\sum_j a_j \chi_{6B_j}\right)^p \mathrm{d}\widetilde{\mu} \le C_0 \int_X \left(\sum_j a_j \chi_{B_j}\right)^p \mathrm{d}\widetilde{\mu}$$

for a constant C_0 that only depends on C_d and p.

We have the following "continuity from below" for the modulus of families of curves; for a proof, see the study by Ziemer [23, Lemma 2.3] or by Heinonen et al. [14, Proposition 5.2.11].

Lemma 3.3. *Let $1 < p < \infty$. If $\{\Gamma_j\}_{j=1}^{\infty}$ is a sequence of families of curves such that $\Gamma_j \subset \Gamma_{j+1}$ for all j, then*

$$\operatorname{Mod}_p\left(\bigcup_{j=1}^{\infty}\Gamma_j\right) = \lim_{j\to\infty}\operatorname{Mod}_p(\Gamma_j).$$

In the case $p = 1$, from [18, Lemma 5.2], we have the following result for the exceptional set. By a slight abuse of notation, we denote the image of a curve γ also by the same symbol.

Lemma 3.4. *Let $\Omega \subset X$ be open and $f : \Omega \to Y$ continuous. Suppose $\widetilde{\mu} \geq \mu$ is doubling within Ω, and that $E \subset \Omega$ has σ-finite $\widetilde{\mathcal{H}}^1$-measure. Then $\operatorname{Mod}_1(\Gamma) = 0$ for the curve family*

$$\Gamma := \{\gamma \subset \Omega : \mathcal{H}^1(f(\gamma \cap E)) > 0\}.$$

In the case $p > 1$, we obtain the following result, which is similar to [2, Lemma 3.5].

Lemma 3.5. *Let $1 < p < \infty$. Let $\Omega \subset X$ be open and suppose $\widetilde{\mu} \geq \mu$ is doubling within a ball $2B_0$, with $\Omega \subset B_0$. Suppose $E \subset \Omega$ is a set with $\widetilde{\mathcal{H}}^p(E) < \infty$. Then $\operatorname{Mod}_p(\Gamma) = 0$ for the curve family*

$$\Gamma := \{\gamma \subset \Omega : \#(\gamma \cap E) = \infty\}.$$

Here, $\#(\gamma \cap E)$ is the cardinality of $\gamma \cap E$.

Proof. For $j, k \in \mathbb{N}$, define

$$\Gamma_{j,k} := \{\gamma \in \Gamma : \exists\{x_1, \dots, x_j\} \in \gamma \cap E \quad \text{s.t. } d(x_l, x_m) > 1/k \quad \text{for all } l \neq m\}.$$

Fix $j, k \in \mathbb{N}$. There is a (finite or countable) cover of E by balls $\{B_n = B(y_n, r_n)\}_n$ such that $r_n \leq (10k)^{-1}$, $B(y_n, 6r_n) \subset \Omega$, and

$$\sum_n \frac{\widetilde{\mu}(B(y_n, r_n))}{r_n^p} < \widetilde{\mathcal{H}}^p(E) + 1.$$

By the 5-covering lemma [14, p. 60], we find an index set I such that the balls $\{B(y_n, r_n)\}_{n\in I}$ are disjoint, and the balls $\{B(y_n, 5r_n)\}_{n\in I}$ cover E. Define the function

$$\rho := \frac{1}{j}\sum_{n\in I}\frac{\chi_{B(y_n,6r_n)}}{r_n}.$$

Let $\gamma \in \Gamma_{j,k}$ and consider the points $\{x_1, \dots, x_j\}$. Each of these points is contained in a different ball $B(y_n, 5r_n)$, $n \in I$, and necessarily travels at least the distance r_n in the ball $B(y_n, 6r_n)$. Thus,

$$\int_\gamma \rho \, ds = \frac{1}{j}\sum_{n\in I}\int_\gamma \frac{\chi_{B(y_n,6r_n)}}{r_n} ds \geq \frac{1}{j} \times j \times \frac{r_n}{r_n} = 1,$$

and so ρ is admissible for $\Gamma_{j,k}$. Hence,

$$\begin{aligned}
\operatorname{Mod}_p(\Gamma_{j,k}) &\leq \int_X \rho^p \, d\mu \\
&\leq \int_X \rho^p \, d\widetilde{\mu} \\
&\leq \frac{1}{j^p}\int_X\left(\sum_{n\in I}\frac{\chi_{B(y_n,6r_n)}}{r_n}\right)^p d\widetilde{\mu} \\
&\leq \frac{C_0}{j^p}\int_X\left(\sum_{n\in I}\frac{\chi_{B(y_n,r_n)}}{r_n}\right)^p d\widetilde{\mu} \quad \text{by Lemma 3.2} \\
&= \frac{C_0}{j^p}\sum_{n\in I}\frac{\widetilde{\mu}(B(y_n, r_n))}{r_n^p} \\
&\leq \frac{C_0}{j^p}(\widetilde{\mathcal{H}}^p(E) + 1).
\end{aligned}$$

Then by Lemma 3.3, we obtain

$$\mathrm{Mod}_p\left(\bigcup_{k=1}^{\infty}\Gamma_{j,k}\right) = \lim_{k\to\infty}\mathrm{Mod}_p(\Gamma_{j,k}) \le \frac{1}{j^p}(\widetilde{\mathcal{H}}^p(E)+1).$$

Finally, note that $\Gamma = \bigcap_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\Gamma_{j,k}$. Thus, $\mathrm{Mod}_p(\Gamma) = 0$. □

Lemma 3.6. *Let $1 < p < \infty$. Let $\Omega \subset X$ be open and suppose $\tilde{\mu} \ge \mu$ is doubling within a ball $2B_0$, with $\Omega \subset B_0$. Suppose $E \subset \Omega$ has σ-finite $\widetilde{\mathcal{H}}^p$-measure. Then $\mathrm{Mod}_p(\Gamma) = 0$ for the curve family*

$$\Gamma \coloneqq \{\gamma \subset \Omega : \gamma \cap E \text{ is uncountable}\}.$$

Proof. We have $E = \bigcup_{k=1}^{\infty}E_k$ with $\widetilde{\mathcal{H}}^p(E_k) < \infty$ for every $k \in \mathbb{N}$. Define the families

$$\Gamma_k \coloneqq \{\gamma \subset \Omega : \#(\gamma \cap E_k) = \infty\}.$$

By Lemma 3.5, we have $\mathrm{Mod}_p(\Gamma_k) = 0$ for every $k \in \mathbb{N}$. Clearly $\Gamma \subset \bigcup_{k=1}^{\infty}\Gamma_k$, and so also $\mathrm{Mod}_p(\Gamma) = 0$. □

Now we obtain the following result for the exceptional set E.

Proposition 3.7. *Let $\Omega \subset X$ be open and suppose $\tilde{\mu} \ge \mu$ is doubling within a ball $2B_0$, with $\Omega \subset B_0$. Let $1 \le p < \infty$. Suppose $E = E_1 \cup E_2 \subset \Omega$ such that E_1 is an at most countable set, and E_2 is σ-finite with respect to $\widetilde{\mathcal{H}}^p$. Let $f : \Omega \to Y$ be continuous. Then*

$$\mathrm{Mod}_p(\{\gamma \subset \Omega : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0.$$

Proof. For every curve $\gamma \subset \Omega$, the set $f(\gamma \cap E_1)$ is at most countable, and for p-a.e. curve $\gamma \subset \Omega$, we have $\mathcal{H}^1(f(\gamma \cap E_2)) = 0$ by Lemmas 3.4 and 3.6. □

Remark 3.8. Let $Q > 1$. A countable set is always σ-finite with respect to the $Q - p$-dimensional Hausdorff measure, when $1 \le p \le Q$. Thus, when considering this measure, the set E_1 could always be included in the set E_2 above. However, in our setting, it can happen that even a single point has infinite $\widetilde{\mathcal{H}}^p$-measure, as can be seen from equipping $\mathbb{R}^n$ with the weighted measure $\mathrm{d}\mu = d\tilde{\mu} = |x|^{-\alpha}\,\mathrm{d}\mathcal{L}^n$, with $n - p < \alpha < n$.

In an Ahlfors Q-regular space, $\widetilde{\mathcal{H}}^p$ is clearly comparable to the $(Q - p)$-dimensional Hausdorff measure, when $1 \le p \le Q$. Thus, in such a space our condition on E reduces to that required in the study by Balogh et al. [2,21].

4 Preliminary results

In this section, we record and prove further preliminary results, such as covering theorems and basic results on absolute continuity on curves.

An obvious question concerning the Jacobian (2.1) is the existence of the limit. For this, we consider the following definitions and facts. Let Z be a separable metric space. For $x \in Z$, a closed ball is defined by

$$\overline{B}(x, r) \coloneqq \{y \in Z : d(y, x) \le r\}.$$

Definition 4.1. A covering $\mathcal{F}$ of a set $A \subset Z$, consisting of closed balls $\overline{B}(x, r)$, is called a fine covering if

$$\inf\{r > 0 : \overline{B}(x, r) \in \mathcal{F}\} = 0$$

for every $x \in A$.

Definition 4.2. Let Z be equipped with a locally finite Borel regular outer measure μ_0. We say that μ_0 is a Vitali measure in Z if for every set $A \subset Z$ and every fine covering $\mathcal{F}$ of A, consisting of closed balls $\overline{B}$, there is a subcollection $\mathcal{G} \subset \mathcal{F}$ consisting of pairwise disjoint balls such that

$$\mu_0\left(A \setminus \bigcup_{\overline{B} \in \mathcal{G}} \overline{B}\right) = 0.$$

By locally finite, we mean that for every $x \in Z$, there is $r > 0$ such that $\mu_0(B(x, r)) < \infty$. For a proof of the following fact, see the study by Heinonen et al. [14, Theorem 3.4.3].

Proposition 4.3. *Suppose $\widetilde{\mu}$ is a Borel regular, locally finite outer measure on Z and*

$$\limsup_{r \to 0} \frac{\widetilde{\mu}(B(x, 2r))}{\widetilde{\mu}(B(x, r))} < \infty$$

for $\widetilde{\mu}$-a.e. $x \in Z$. Then $\widetilde{\mu}$ is a Vitali measure in Z.

In our setting, we obtain the following.

Lemma 4.4. *Let $\Omega \subset X$ be open and suppose there exists a Borel regular outer measure $\widetilde{\mu} \geq \mu$ on X that is doubling within Ω. Then μ is a Vitali measure in the metric space (Ω, d).*

Note that in Theorem 1.1, we assume that $\widetilde{\mu}$ is doubling within $2B_0$ with $\Omega \subset B_0$, so then in particular $\widetilde{\mu}$ is doubling within Ω.

Proof. By Proposition 4.3, we know that $\widetilde{\mu}$ is a Vitali measure in the metric space (Ω, d). Since $\mu \leq \widetilde{\mu}$, from Definition 4.2, it clearly follows that μ is then also a Vitali measure in (Ω, d). □

We have the following Lebesgue-Radon-Nikodym differentiation theorem, see the study by Heinonen et al. [14, p. 82].

Theorem 4.5. *Suppose Z is equipped with a Vitali measure μ_0, and let κ be a Borel regular, locally finite measure on Z. Then there exists a decomposition of κ into the absolutely continuous and singular parts*

$$d\kappa = d\kappa^a + d\kappa^s = a\,\mathrm{d}\mu_0 + d\kappa^s,$$

where

$$a(x) := \lim_{r \to 0} \frac{\kappa(\overline{B}(x, r))}{\mu_0(\overline{B}(x, r))} \quad \text{for } \mu_0\text{-a.e. } x \in Z. \tag{4.1}$$

Returning to our setting, let $\Omega \subset X$ be open and suppose there exists a Borel regular outer measure $\widetilde{\mu} \geq \mu$ on X, which is doubling within Ω. Let $f : \Omega \to f(\Omega) \subset Y$ be a homeomorphism, with $f(\Omega)$ open. We can now decompose

$$df_\# \nu = d(f_\# \nu)^a + d(f_\# \nu)^s = a\,\mathrm{d}\mu + d(f_\# \nu)^s \quad \text{in} \quad \Omega.$$

By (2.1) and (4.1), we know that in fact $J_f = a$. Thus,

$$\nu(f(\Omega)) = (f_\# \nu)(\Omega) \geq (f_\# \nu)^a(\Omega) = \int_\Omega J_f \,\mathrm{d}\mu.$$

We record:

$$J_f \text{ exists } \mu\text{-a.e. in } \Omega \quad \text{and} \quad \int_\Omega J_f \,\mathrm{d}\mu \leq \nu(f(\Omega)). \tag{4.2}$$

Later we will use this together with the assumption $\nu(f(\Omega)) < \infty$.

The following covering result is from the study by Lahti and Zhou [18, Lemma 3.1]; it is similar to Lemma 2.2 of Balogh et al. [2].

Lemma 4.6. *Let $A \subset X$ be bounded and metric doubling with constant M, and let $\mathcal{F}$ be a collection of balls $\{B(x, r_x)\}_{x\in A}$ with radius at most $0 < R < \infty$. Then there exist finite or countable subcollections $\mathcal{G}_1, \dots, \mathcal{G}_N$, with $N = M^4$ and $\mathcal{G}_j = \{B_{j,l} = B(x_{j,l}, r_{j,l})\}_l$, such that*

(1) $A \subset \bigcup_{j=1}^N \bigcup_l B_{j,l}$;

(2) *If $j \in \{1, \dots, N\}$ and $l \neq m$, then $x_{j,l} \in X \backslash B_{j,m}$ and $x_{j,m} \in X \backslash B_{j,l}$;*

(3) *If $j \in \{1, \dots, N\}$ and $l \neq m$, then $\frac{1}{2}B_{j,l} \cap \frac{1}{2}B_{j,m} = \varnothing$.*

The following two lemmas concerning absolute continuity on curves are essentially well known, see the study by Zürcher [24, Lemma 3.6], but we do not know a source for the precise formulations that we need, so we provide full proofs. Recall the definition of lip_h from (2.2).

Lemma 4.7. *Let $-\infty < a < b < \infty$. Let $h : [a, b] \to Y$ be a continuous mapping such that $\mathrm{lip}_h(t) < \infty$ for all $t \in A$ for some $\mathcal{L}^1$-measurable set $A \subset [a, b]$. Then*

$$\mathcal{H}^1(h(A)) \leq \int_A \mathrm{lip}_h(t)\,\mathrm{d}t.$$

Proof. We can assume that $A \subset (a, b)$. Define $\mathrm{lip}_h^\vee := \max\{\mathrm{lip}_h, 1\}$. We let

$$A_k := \{t \in A : 2^{k-1} \leq \mathrm{lip}_h^\vee < 2^k\}, \quad k \in \mathbb{N},$$

so that $\bigcup_{k=1}^\infty A_k = A$. Fix $\delta > 0$. For each $k \in \mathbb{N}$, choose an open set U_k with $A_k \subset U_k \subset (a, b)$ and

$$\mathcal{L}^1(U_k) \leq \mathcal{L}^1(A_k) + \frac{\delta}{2^{2k}}.$$

For every $t \in A_k$, we note that for all $r > 0$ the set $h(B(t, r))$ is contained in the ball $B(h(t), \sup_{s\in B(t,r)} d_Y(h(s), h(t)))$, and so we can estimate

$$\liminf_{r\to 0} \frac{\mathcal{H}^1_\delta(h(B(t, r)))}{r} \leq 2 \liminf_{r\to 0} \frac{\sup_{s\in B(t,r)} d_Y(h(s), h(t))}{r} = 2\mathrm{lip}_h(t) < 2^{k+1}.$$

Thus, for each $k \in \mathbb{N}$ and every $t \in A_k$, we find $r_t > 0$ such that

$$B(t, r_t) \subset U_k \quad \text{and} \quad \mathcal{H}^1_\delta(h(B(t, r_t))) \leq 2^{k+1} r_t.$$

We can choose an at most countable subcollection $\{B_j = B(t_j, r_j)\}_j$ that covers A_k such that every $x \in (a, b)$ is contained in at most 2 balls B_j, see the study by Engelking [6, Proposition 3.2.2]. Thus,

$$\begin{aligned}
\mathcal{H}^1_\delta(h(A)) &\leq \sum_{k=1}^\infty \mathcal{H}^1_\delta(h(A_k)) \\
&\leq \sum_{k=1}^\infty \sum_{j\in\mathbb{N}: x_j\in A_k} \mathcal{H}^1_\delta(h(B_j)) \\
&\leq \sum_{k=1}^\infty 2^{k+1} \sum_{j\in\mathbb{N}: x_j\in A_k} r_j \\
&\leq \sum_{k=1}^\infty 2^{k+1} \mathcal{L}^1(U_k) \\
&\leq \sum_{k=1}^\infty 2^{k+1} \left(\mathcal{L}^1(A_k) + \frac{\delta}{2^{2k}}\right) \\
&\leq 4 \int_A \mathrm{lip}_h^\vee \mathrm{d}t + 2\delta.
\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\mathcal{H}^1(h(A)) \le 4\int_A \operatorname{lip}_h^\vee \mathrm{d}t.$$

This is rather close to the desired result, but we only use this to conclude the following absolute continuity:

$$\text{if } N \subset A \text{ with } \mathcal{L}^1(N) = 0, \text{ then } \mathcal{H}^1(h(N)) = 0. \tag{4.3}$$

Now fix $\varepsilon > 0$. We can assume that $\operatorname{lip}_h \chi_A \in L^1([a, b])$. For every $t \in A$, we have

$$\liminf_{r\to 0} \frac{\mathcal{H}^1_\varepsilon(h(\overline{B(t,r)}))}{r} \le 2\liminf_{r\to 0} \frac{\sup_{s\in B(t,r)} d_Y(h(s), h(t))}{r} = 2\operatorname{lip}_h(t).$$

Thus, for every Lebesgue point $t \in A$ of the function $\operatorname{lip}_h \chi_A$, we find an arbitrarily small radius r_t such that

$$\mathcal{H}^1_\varepsilon(h(\overline{B(t,r_t)})) \le 2(\operatorname{lip}_h(t) + \varepsilon) r_t \le (1+\varepsilon) \int_{B(t,r_t)\cap A} (\operatorname{lip}_h + \varepsilon)\mathrm{d}s.$$

By the Vitali covering theorem (recall Proposition 4.3), we find a collection $\{\overline{B_k} = \overline{B(x_k, r_k)}\}_{k=1}^\infty$ of disjoint balls covering $A\setminus N$ for some $N \subset A$ with $\mathcal{L}^1(N) = 0$. Then

$$\begin{aligned}
\mathcal{H}^1_\varepsilon(h(A)) &\le \mathcal{H}^1_\varepsilon(h(A\setminus N)) + \mathcal{H}^1_\varepsilon(h(N)) \\
&\le \sum_{k=1}^\infty \mathcal{H}^1_\varepsilon(h(\overline{B_k})) + 0 \quad \text{by (4.3)} \\
&\le \sum_{k=1}^\infty (1+\varepsilon) \int_{B_k\cap A} (\operatorname{lip}_h + \varepsilon)\mathrm{d}s \\
&= (1+\varepsilon)\int_A (\operatorname{lip}_h + \varepsilon)\mathrm{d}s.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain the result. □

Lemma 4.8. *Let $\Omega \subset X$ be open, let $f : \Omega \to Y$ be continuous, and let $\gamma : [0, \ell_\gamma] \to \Omega$ be a curve such that $\operatorname{lip}_f(\gamma(t)) < \infty$ for all $t \in A \subset [0, \ell_\gamma]$, where A is $\mathcal{L}^1$-measurable. Then*

$$\mathcal{H}^1(f(\gamma(A))) \le \int_A \operatorname{lip}_f(\gamma(t))\mathrm{d}t.$$

Proof. For the mapping $h := f\circ\gamma : [0, \ell_\gamma] \to Y$, by the fact that γ is a 1-Lipschitz mapping, for all $t \in A$ we have

$$\begin{aligned}
\operatorname{lip}_h(t) &= \liminf_{r\to 0} \sup_{s\in B(t,r)} \frac{d_Y(f\circ\gamma(s), f\circ\gamma(t))}{r} \\
&\le \liminf_{r\to 0} \sup_{y\in B(\gamma(t),r)} \frac{d_Y(f(y), f(\gamma(t)))}{r} \\
&= \operatorname{lip}_f(\gamma(t)).
\end{aligned}$$

Now the result follows from Lemma 4.7. □

We will use the following theorem of Williams [22, Theorem 1.1]. The symbol g_f denotes the minimal Q-weak upper gradient of f in Z.

Theorem 4.9. *Let $1 < Q < \infty$; let Z and W be separable, locally finite metric measure spaces; and let $f : Z \to W$ be a homeomorphism. Then the following condition* (1) *implies* (2):

(1) $f \in N^{1,Q}_{\mathrm{loc}}(Z;W)$ *and for* μ*-a.e.* $x \in Z$,

$$g_f(x)^Q \leq KJ_f(x);$$

(2) *For every family* Γ *of curves in* Z,

$$\mathrm{Mod}_Q(\Gamma) \leq K\mathrm{Mod}_Q(f(\Gamma)).$$

Note that $f(\Gamma)$ means the curves $f \circ \gamma$, $\gamma \in \Gamma$, reparametrized by arc-length. We will apply the implication (1) ⇒ (2) with the choices $Z = \Omega \subset X$ and $W = f(\Omega) \subset Y$. In the study by Williams [22] it is in fact shown that (2) also implies (1), but it is additionally assumed (in the first paragraph of Section 2 in [22]) that the supports of the measures that Z and W are equipped with are the entire spaces; this condition is not needed in the proof of (1) ⇒ (2).

We define the Hardy-Littlewood maximal function of a locally integrable nonnegative function $g \in L^1_{\mathrm{loc}}(X)$ by

$$Mg(x) := \sup_{0<r<\infty} \fint_{B(x,r)} g\,\mathrm{d}\mu, \quad x \in X.$$

The following fact is well known, but we present it in a slightly different form than what is usual, so we also sketch a proof. Recall the Poincaré inequality from (2.4).

Proposition 4.10. *Let* $1 \leq p < q < \infty$. *Suppose* X *is proper and supports a* $(1,p)$*-Poincaré inequality, and* μ *is doubling. Let* $\Omega \subset X$ *be open and suppose* $f : \Omega \to Y$ *is continuous and that* $g \in L^q(\Omega)$ *is a p-weak upper gradient of f in* Ω. *Then* $f \in D^q(\Omega;\, Y)$.

Proof. We can interpret g to be zero extended to the whole space, so that $g \in L^q(X)$. Consider a ball $B(z,r) \subset B(z, 3\lambda r) \subset \Omega$, where $\lambda \geq 1$ is the dilation constant from the Poincaré inequality. By a telescoping argument, see the study by [14, Theorem 8.1.7], for all $x, y \in B(z,r)$, we obtain

$$d_Y(f(x), f(y)) \leq Cd(x,y)([Mg^p(x)]^{1/p} + [Mg^p(y)]^{1/p})$$

for some $C > 0$ that only depends on the constants of the doubling and Poincaré conditions. Here, $[Mg^p]^{1/p} \in L^q(X)$ by the Hardy-Littlewood maximal theorem, see the study by Heinonen et al. [14, Theorem 3.5.6]. Moreover, $f \in L^q(B(x,r))$ since X is proper and so f is bounded in $B(x,r)$. It follows that f is in the *Hajłasz-Sobolev space* $M^{1,q}(B(z,r);\, Y)$. Then by the proof of [14, Lemma 10.2.5], we know that $3C[Mg^p]^{1/p}$ is an upper gradient of f in $B(z,r)$. Since we can cover Ω by countably many such balls, it follows that $3C[Mg^p]^{1/p}$ is an upper gradient of f in Ω, and so, $f \in D^q(\Omega;\, Y)$. □

5 Proofs of the main results

In this section, we prove our main result, Theorem 1.1, and also Corollaries 1.2–1.4.

Throughout this section, we assume that $\Omega \subset X$ is nonempty, open, and bounded, and that $f : \Omega \to f(\Omega) \subset Y$ is a homeomorphism with $f(\Omega)$ open and $\nu(f(\Omega)) < \infty$.

We will consider the following Lusin property on curves.

Definition 5.1. Let $\gamma : [0, \ell_\gamma] \to X$ be a curve and let $A \subset X$, and $f : X \to Y$. We say that f satisfies the N_A-property on γ if for every $N \subset [0, \ell_\gamma]$ with $\mathcal{L}^1(N) = 0$, we have

$$\mathcal{H}^1(f(\gamma(N) \cap A)) = 0.$$

We call the N_X-property simply the N-property.

We give the proof of Theorem 1.1 in the following two propositions. The idea of separating the argument into two steps, first considering absolute continuity on curves and then the Dirichlet seminorm, comes from Williams [21].

Proposition 5.2. *Suppose there exists a Borel regular outer measure $\tilde{\mu} \geq \mu$ on X, which is doubling within a ball $2B_0$ with $\Omega \subset B_0$, and that there exist a set $E \subset \Omega$ and a function $1 < Q(x) < \infty$ on $\Omega \backslash E$, with*

$$\limsup_{r\to 0}\frac{\tilde{\mu}(B(x,r))}{r^{Q(x)}} < \infty \quad \textit{and} \quad \liminf_{r\to 0}\frac{\nu(B(f(x),r))}{r^{Q(x)}} > 0 \quad \textit{for all } x \in \Omega\backslash E.$$

Suppose $Q \coloneqq \inf_{x\in\Omega\backslash E} Q(x) > 1$ and let $1 \leq p \leq Q$. Suppose also that

$$\mathrm{Mod}_p(\{\gamma \subset \Omega : \mathcal{H}^1(f(\gamma \cap E)) > 0\}) = 0 \tag{5.1}$$

and $h_f < \infty$ in $\Omega\backslash E$. Then f satisfies the N-property on p-a.e. curve γ in Ω.

Proof. Define the sets A_k, $k \in \mathbb{N}$, as subsets of $\Omega\backslash E$ such that $\mathrm{lip}_f(x) > 1$, $Q(x) \leq k$, $h_f(x) \leq k$, as well as

$$\limsup_{r\to 0}\frac{\tilde{\mu}(B(x,r))}{r^{Q(x)}} < k \quad \text{and} \quad \liminf_{r\to 0}\frac{\nu(B(f(x),r))}{r^{Q(x)}} > 1/k \quad \text{for all } x \in A_k. \tag{5.2}$$

Then $\Omega \cap \{\mathrm{lip}_f > 1\}\backslash E = \bigcup_{k=1}^{\infty} A_k$.

Fix $k \in \mathbb{N}$ and let $A \coloneqq A_k$. Fix $\delta > 0$. For every $x \in A$, we can choose a radius $0 < r_x < \delta$ sufficiently small so that

- $B(x, 2r_x) \subset \Omega$, $B(f(x), L_f(x, r_x)) \subset f(\Omega)$, and $L_f(x, r_x) \leq \delta/2$ (since f is continuous);
- by the fact that $\mathrm{lip}_f(x) > 1$, we can obtain

$$\frac{L_f(x, r_x)}{r_x} > 1; \tag{5.3}$$

- by (5.2),

$$\sup_{0<r\leq r_x}\frac{\tilde{\mu}(B(x,r))}{r^{Q(x)}} \leq k \quad \text{and} \quad \inf_{0<r\leq L_f(x,r_x)}\frac{\nu(B(f(x),r))}{r^{Q(x)}} \geq 1/k. \tag{5.4}$$

Finally, since $h_f(x) = \liminf_{r\to 0} H_f(x, r)$, and noting that $h_f(x) \geq 1$ for every $x \in \Omega$ by the fact that X is connected, we can also choose r_x to have

$$H_f(x, r_x) \leq 2h_f(x) \leq 2k. \tag{5.5}$$

From the fact that $\tilde{\mu}$ is doubling within some ball $2B_0$ with $A \subset \Omega \subset B_0$, we obtain that (A, d) is metric doubling, see [3, Proposition 3.4]. Thus, we can apply Lemma 4.6 to the covering $\mathcal{G} \coloneqq \{B(x, r_x)\}_{x\in A}$, to extract subcoverings $\mathcal{G}_1, \ldots, \mathcal{G}_N$, with

$$\mathcal{G}_j = \{B_{j,l} = B(x_{j,l}, r_{j,l})\}_l$$

and having the good properties given in the Lemma. Define

$$g \coloneqq 2\sum_{j=1}^{N}\sum_{l}\frac{L_f(x_{j,l}, r_{j,l})}{r_{j,l}}\chi_{2B_{j,l}}.$$

Consider a curve γ in Ω with $\mathrm{diam}\gamma > \delta$. If γ intersects $B_{j,l}$, then $\mathcal{H}^1(\gamma \cap 2B_{j,l}) > r_{j,l}$. Thus we have

$$\int_\gamma g\,ds \geq 2\sum_{\gamma\cap B_{j,l}\neq\varnothing} L_f(x_{j,l}, r_{j,l}) \geq \sum_{\gamma\cap B_{j,l}\neq\varnothing}\mathrm{diam}(f(B_{j,l})) \geq \mathcal{H}^1_\delta(f(\gamma \cap A)), \tag{5.6}$$

where the last inequality holds since the balls $B_{j,l}$ satisfying $\gamma \cap B_{j,l} \neq \varnothing$ cover $\gamma \cap A$ and so the sets $f(B_{j,l})$ with $\gamma \cap B_{j,l} \neq \varnothing$ cover $f(\gamma \cap A)$.

In general, note that if balls $B(z_1, r_1)$ and $B(z_2, r_2)$ are disjoint and contained in Ω, then from the definition of $l_f(\cdot,\cdot)$, it follows that

$$f(z_2) \notin B(f(z_1), l_f(z_1, r_1)) \quad \text{and} \quad f(z_1) \notin B(f(z_2), l_f(z_2, r_2))$$

and thus,

$$B(f(z_1), \frac{1}{2}l_f(z_1, r_1)) \cap B(f(z_2), \frac{1}{2}l_f(z_2, r_2)) = \varnothing.$$

Thus, for every $j \in \{1, \dots, N\}$, we have

$$B\left(f(x_{j,l}), \frac{L_f(x_{j,l}, r_{j,l})}{2H_f(x_{j,l}, r_{j,l})}\right) \cap B\left(f(x_{j,m}), \frac{L_f(x_{j,m}, r_{j,m})}{2H_f(x_{j,m}, r_{j,m})}\right) = \varnothing \quad \text{for all } l \neq m,$$

and so by (5.5),

$$B\left(f(x_{j,l}), \frac{L_f(x_{j,l}, r_{j,l})}{4k}\right) \cap B\left(f(x_{j,m}), \frac{L_f(x_{j,m}, r_{j,m})}{4k}\right) = \varnothing \quad \text{for all } l \neq m. \tag{5.7}$$

Denote $Q_{j,l} \coloneqq Q(x_{j,l})$ and $L_{j,l} \coloneqq L(x_{j,l}, r_{j,l})$. From the fact that $\tilde{\mu} \geq \mu$, we obtain

$$\begin{aligned}
\int_\Omega g^Q \mathrm{d}\mu &\leq \int_\Omega g^Q \mathrm{d}\tilde{\mu} \\
&= 2^Q \int_\Omega \left(\sum_{j=1}^N \sum_l \frac{L_{j,l}}{r_{j,l}} \chi_{2B_{j,l}}\right)^Q \mathrm{d}\tilde{\mu} \\
&= (2N)^Q \sum_{j=1}^N \int_\Omega \left(\sum_l \frac{L_{j,l}}{r_{j,l}} \chi_{2B_{j,l}}\right)^Q \mathrm{d}\tilde{\mu} \\
&\leq (2N)^Q C_0 \sum_{j=1}^N \int_\Omega \left(\sum_l \frac{L_{j,l}}{r_{j,l}} \chi_{\frac{1}{3}B_{j,l}}\right)^Q \mathrm{d}\tilde{\mu} \quad \text{by Lemma 3.2} \\
&= (2N)^Q C_0 \sum_{j=1}^N \sum_l \left(\frac{L_{j,l}}{r_{j,l}}\right)^Q \tilde{\mu}\left(\frac{1}{3}B_{j,l}\right) \\
&\leq (2N)^Q C_0 \sum_{j=1}^N \sum_l \left(\frac{L_{j,l}}{r_{j,l}}\right)^{Q_{j,l}} \tilde{\mu}\left(\frac{1}{3}B_{j,l}\right)
\end{aligned}$$

by (5.3) and the fact that $Q \leq Q_{j,l}$. By using (5.4) and abbreviating $\sum_{j=1}^N \sum_l$ by $\sum_{j,l}$, we continue the estimate

$$\begin{aligned}
\int_\Omega g^Q \mathrm{d}\mu &\leq (2N)^Q k C_0 \sum_{j,l} \left(\frac{L_{j,l}}{r_{j,l}}\right)^{Q_{j,l}} r_{j,l}^{Q_{j,l}} \\
&= (2N)^Q k C_0 \sum_{j,l} L_{j,l}^{Q_{j,l}} \\
&\leq (2N)^Q k C_0 (4k)^k \sum_{j,l} \left(\frac{L_{j,l}}{4k}\right)^{Q_{j,l}} \quad \text{since } Q_{j,l} \leq k \\
&\leq (2N)^Q k^2 (4k)^k C_0 \sum_{j,l} \nu(B(f(x_{j,l}), L_{j,l}/4k)) \quad \text{by (5.4)} \\
&\leq (2N)^Q k^2 (4k)^k C_0 \nu(f(\Omega)) \quad \text{by (5.7)} \\
&< \infty.
\end{aligned}$$

Recall that $k \in \mathbb{N}$ is kept fixed, but g depends on $\delta > 0$. Now we can choose functions g with the choices $\delta = 1/i$, to obtain a sequence $\{g_i\}_{i=1}^\infty$ that is bounded in $L^Q(\Omega)$. By the reflexivity of the space $L^Q(\Omega)$, we find a subsequence (not relabeled) and $g \in L^Q(\Omega)$ such that $g_i \to g$ weakly in $L^Q(\Omega)$. By Mazur's and Fuglede's lemmas (Lemmas 2.1 and 2.2), we find convex combinations $\hat{g}_i \coloneqq \sum_{j=i}^{N_i} a_{i,j} g_j$ such that for Q-a.e. curve γ' in Ω, we have

$$\int_{\gamma'} g \mathrm{d}s = \lim_{i\to\infty} \int_{\gamma'} \hat{g}_i \mathrm{d}s \geq \lim_{i\to\infty} \mathcal{H}^1_{1/i}(f(\gamma' \cap A)) = \mathcal{H}^1(f(\gamma' \cap A)), \tag{5.8}$$

the last equality follows from (5.6).

By the properties of modulus, see the study by Björn and Björn [4, Lemma 1.34], for Q-a.e. curve γ in Ω we have that the aforementioned holds for every subcurve γ' of γ. For Q-a.e. curve γ in Ω, we also have that $\int_\gamma g\,ds < \infty$, which follows from the definition of the Q-modulus. Fix a curve γ satisfying the aforementioned two conditions. We can write any open $U \subset (0, \ell_\gamma)$ as a union of pairwise disjoint intervals $U = \bigcup_{j=1}^\infty (a_j, b_j)$, and then

$$\begin{aligned}\int_U g(\gamma(s))\mathrm{d}s &= \sum_{j=1}^\infty \int_{(a_j,b_j)} g(\gamma(s))\mathrm{d}s \\ &\geq \sum_{j=1}^\infty \mathcal{H}^1(f(\gamma((a_j,b_j)) \cap A)) \quad \text{by (5.8)} \\ &\geq \mathcal{H}^1(f(\gamma(U) \cap A))\end{aligned} \tag{5.9}$$

by the subadditivity of the $\mathcal{H}^1$-measure. Then for any Borel set $S \subset (0, \ell_\gamma)$, we can let $\varepsilon > 0$ and find an open set U such that $S \subset U \subset (0, \ell_\gamma)$ and

$$\begin{aligned}\int_S g(\gamma(s))\mathrm{d}s &\geq \int_U g(\gamma(s))\mathrm{d}s - \varepsilon \\ &\geq \mathcal{H}^1(f(\gamma(U) \cap A)) - \varepsilon \quad \text{by (5.9)} \\ &\geq \mathcal{H}^1(f(\gamma(S) \cap A)) - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\int_S g(\gamma(s))\mathrm{d}s \geq \mathcal{H}^1(f(\gamma(S) \cap A)),$$

which in particular proves the N_A-property for Q-a.e. curve γ in Ω. Then the property also holds for p-a.e. curve γ in Ω, since $p \leq Q$, see e.g. [4, Proposition 2.45]. Recall that so far $k \in \mathbb{N}$ was fixed and $A = A_k$.

In total, since $\Omega = \bigcup_{k=1}^\infty A_k \cup E \cup \{\mathrm{lip}_f \leq 1\}$, for p-a.e. curve γ in Ω, we have that if $N \subset [0, \ell_\gamma]$ with $\mathcal{L}^1(N) = 0$, then

$$\begin{aligned}\mathcal{H}^1(f(\gamma(N))) &\leq \sum_{k=1}^\infty \mathcal{H}^1(f(\gamma(N) \cap A_k)) + \mathcal{H}^1(f(\gamma(N) \cap \{\mathrm{lip}_f \leq 1\})) + \mathcal{H}^1(f(\gamma \cap E)) \\ &= 0 + 0 + 0\end{aligned}$$

by the N_{A_k}-property proved just above, Lemma 4.8, and the assumption (5.1). Thus, f satisfies the N-property on p-a.e. curve γ in Ω. □

Proposition 5.3. *Suppose there exists a Borel regular outer measure $\tilde{\mu} \geq \mu$ on X, which is doubling within Ω. Suppose also that there exists a μ-measurable set $E \subset \Omega$ and μ-measurable functions $1 < Q(x) < \infty$ and $0 < R(x) < \infty$ in $\Omega \setminus E$ such that*

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^{Q(x)}} < R(x) \liminf_{r \to 0} \frac{\nu(B(f(x),r))}{r^{Q(x)}} \quad \textit{for } \mu\text{-a.e. } x \in \Omega\setminus E. \tag{5.10}$$

Suppose $Q := \inf_{x\in\Omega\setminus E} Q(x) > 1$ and let $1 \leq q \leq Q$. Assume also that

$$\begin{cases} \dfrac{Q(\cdot) - q}{Q(\cdot)}(R(\cdot)h_f(\cdot)^{Q(\cdot)})^{q/(Q(\cdot)-q)} \in L^1(\Omega\setminus E) & \textit{if } 1 \leq q < Q; \\ R(\cdot)^{1/Q(\cdot)}h_f(\cdot) \in L^\infty(\Omega\setminus E) & \textit{if } q = Q. \end{cases}$$

Then $\mathrm{lip}_f \in L^q(\Omega\setminus E, \mu)$. In the case $q = Q = Q(x)$ for μ-a.e. $x \in \Omega\setminus E$, we also obtain

$$\mathrm{lip}_f(x)^Q \leq \|R(\cdot)h_f(\cdot)^Q\|_{L^\infty(\Omega)} J_f(x) \quad \textit{for } \mu\text{–a.e. } x \in \Omega\setminus E.$$

Proof. For μ-a.e. $x \in \Omega\backslash E$, by (5.10), we have for some $0 < C(x) < \infty$ and some sufficiently small $r_x > 0$ that

$$\frac{\mu(B(x, r_x))}{r_x^{Q(x)}} < C(x)R(x) \quad \text{and} \quad \frac{\nu(B(f(x), l_f(x, r_x)))}{l_f(x, r_x)^{Q(x)}} > C(x). \tag{5.11}$$

Choosing r_x smaller, if necessary, we also have $l_f(x, r_x) < \infty$ and

$$B(f(x), L_f(x, r_x)) \subset f(\Omega).$$

Fix $\varepsilon > 0$. Since $h_f(x) = \liminf_{r\to 0} H_f(x, r)$, and noting that $h_f(x) \geq 1$ for every $x \in \Omega$, we can choose r_x, so that we also have

$$H_f(x, r_x) \leq (1+\varepsilon)h_f(x). \tag{5.12}$$

Note that $B(f(x), l_f(x, r_x)) \subset f(B(x, r_x))$. Thus, we estimate

$$\begin{aligned} \frac{\nu(f(B(x, r_x)))}{\mu(B(x, r_x))} &\geq \frac{\nu(B(f(x), l_f(x, r_x)))}{\mu(B(x, r_x))} \\ &\geq \frac{C(x) l_f(x, r_x)^{Q(x)}}{C(x)R(x) r_x^{Q(x)}} \quad \text{by (5.11)} \\ &= \frac{1}{R(x)} \left(\frac{l_f(x, r_x)}{r_x} \right)^{Q(x)} \\ &= \frac{1}{R(x)} \left(\frac{L_f(x, r_x)}{H_f(x, r_x) \cdot r_x} \right)^{Q(x)} \\ &\geq \frac{1}{R(x)(1+\varepsilon)^{Q(x)}} \left(\frac{L_f(x, r_x)}{h_f(x) \cdot r_x} \right)^{Q(x)} \end{aligned} \tag{5.13}$$

by (5.12). Recall from (4.2) that the Jacobian J_f exists μ-a.e. in Ω. Since (5.13) holds for arbitrarily small r_x, we can take the limit $\liminf_{r_x \to 0}$ to obtain at μ-a.e. $x \in \Omega\backslash E$ that

$$J_f(x) \geq \frac{1}{R(x)(1+\varepsilon)^{Q(x)}} \frac{\operatorname{lip}_f(x)^{Q(x)}}{h_f(x)^{Q(x)}}.$$

Thus,

$$\operatorname{lip}_f(x) \leq (1+\varepsilon) R(x)^{1/Q(x)} h_f(x) J_f(x)^{1/Q(x)} \tag{5.14}$$

and so for any $1 \leq q < Q$, we obtain by Young's inequality

$$\begin{aligned} \operatorname{lip}_f(x)^q &\leq (1+\varepsilon)^q R(x)^{q/Q(x)} h_f(x)^q J_f(x)^{q/Q(x)} \\ &\leq \frac{Q(x)-q}{Q(x)} (R(x)(1+\varepsilon)^{Q(x)} h_f(x)^{Q(x)})^{q/(Q(x)-q)} + J_f(x), \end{aligned}$$

where we estimated simply $q/Q(x) \leq 1$ for the second term. By using also (4.2), we conclude

$$\int_{\Omega\backslash E} \operatorname{lip}_f(x)^q \,\mathrm{d}\mu(x) \leq (1+\varepsilon)^{qQ/(Q-q)} \int_{\Omega\backslash E} \frac{Q(x)-q}{Q(x)} (R(x) h_f(x)^{Q(x)})^{q/(Q(x)-q)} \mathrm{d}\mu(x) + \nu(f(\Omega)) < \infty$$

by assumption.

In the case $q = Q$, from (5.14), we estimate simply

$$\operatorname{lip}_f(x)^Q \leq \|(1+\varepsilon)R(\cdot)^{1/Q(\cdot)} h_f(\cdot)\|^Q_{L^\infty(\Omega\backslash E)} J_f(x)^{Q/Q(x)} \quad \text{for } \mu\text{-a.e. } x \in \Omega\backslash E. \tag{5.15}$$

Since $Q/Q(x) \leq 1$ and $J_f \in L^1(\Omega)$, and $\mu(\Omega) < \infty$, also $J_f(\cdot)^{Q/Q(\cdot)} \in L^1(\Omega)$. Thus, $\operatorname{lip}_f \in L^Q(\Omega\backslash E)$.

In the case $Q(x) = Q$ for μ-a.e. $x \in \Omega$, (5.15) gives

$$\operatorname{lip}_f(x)^Q \le \|(1+\varepsilon)R(\cdot)h_f(\cdot)^Q\|_{L^\infty(\Omega)}J_f(x) \quad \text{for } \mu\text{-a.e. } x \in \Omega\setminus E.$$

Letting $\varepsilon \to 0$, we obtain the last conclusion. □

Proof of Theorem 1.1. By Proposition 5.3, we know that $\operatorname{lip}_f(x) < \infty$ for μ-a.e. $x \in \Omega\setminus E$, and so we know that for p-a.e. curve γ in Ω, we have $\mathcal{L}^1(N_\gamma) = 0$ for

$$N_\gamma := \{t \in [0, \ell_\gamma] : \gamma(t) \in \Omega\setminus E \quad \text{and} \quad \operatorname{lip}_f(\gamma(t)) = \infty\}.$$

By Proposition 3.7 combined with Proposition 5.2, for p-a.e. curve γ in Ω, we thus have $\mathcal{H}^1(f(\gamma(N_\gamma))) = 0$. Denoting the end points of γ by x, y, we obtain

$$\begin{aligned} d_Y(f(x), f(y)) &\le \mathcal{H}^1(f(\gamma([0,\ell_\gamma]\setminus N_\gamma))) \\ &= \mathcal{H}^1(f(\gamma([0,\ell_\gamma]\setminus N_\gamma)\setminus E)) \quad \text{by Proposition 3.7} \\ &\le \int_\gamma \operatorname{lip}_f \chi_{\Omega\setminus E}\, ds \end{aligned}$$

by Lemma 4.8 with $A = [0, \ell_\gamma] \cap \gamma^{-1}(\Omega\setminus E)\setminus N_\gamma$. Thus, $\operatorname{lip}_f\chi_{\Omega\setminus E}$ is a p-weak upper gradient of f in Ω. We also have $\operatorname{lip}_f\chi_{\Omega\setminus E} \in L^q(\Omega) \subset L^p(\Omega)$ by Proposition 5.3, so we conclude that $f \in D^p(\Omega; Y)$.

Since $\operatorname{lip}_f\chi_{\Omega\setminus E}$ is a p-weak upper gradient of f in Ω, for the minimal p-weak upper gradient, we obtain $g_f \le \operatorname{lip}_f\chi_{\Omega\setminus E}\,\mu$-a.e. in Ω. In the case $p = Q = Q(x)$ for μ-a.e. $x \in \Omega\setminus E$, Proposition 5.3 now gives for μ-a.e. $x \in \Omega$ that

$$g_f(x)^Q \le \|R(\cdot)h_f(\cdot)^Q\|_{L^\infty(\Omega)}J_f(x).$$

Then Theorem 4.9 gives for every curve family Γ in Ω that

$$\operatorname{Mod}_Q(\Gamma) \le \|R(\cdot)h_f(\cdot)^Q\|_{L^\infty(\Omega)}\operatorname{Mod}_Q(f(\Gamma)).$$

Finally, if X is proper and supports a $(1, p)$-Poincaré inequality, and μ is doubling, then by Proposition 4.10, we obtain $f \in D^q(\Omega; Y)$. □

Next we consider weighted spaces. By a weight, we simply mean a nonnegative locally integrable function. Suppose Y is equipped with an Ahlfors regular measure ν_0, and then we add a weight w_Y. Note that w_Y has Lebesgue points ν_0-a.e., see the study by Heinonen [10, Theorem 1.8]. For our purposes, it is natural to consider the pointwise representative

$$w_Y(y) = \liminf_{r\to 0} \frac{1}{\nu_0(B(y,r))} \int_{B(y,r)} w_Y\, d\nu_0, \quad y \in Y. \tag{5.16}$$

Proof of Corollary 1.2. We can choose $R(x) := (1+\varepsilon)w(x)/w_Y(f(x))$ for arbitrarily small $\varepsilon > 0$, and then, the result follows from Theorem 1.1. □

Proof of Corollaries 1.3 and 1.4. The Euclidean space (resp. Carnot group), equipped with the Lebesgue measure (resp. Haar measure $\mathcal{L}^n$), supports a $(1, 1)$-Poincaré inequality and is equipped with an Ahlfors n-regular (resp. Q-regular), and hence doubling, measure. Thus, these corollaries follow from Corollary 1.2 with $w = \tilde{w} = 1$. □

Remark 5.4. Concerning the assumption $\nu(f(\Omega)) < \infty$ that we make throughout, note that as a continuous mapping f is bounded in every $\Omega' \Subset \Omega$ (i.e. $\overline{\Omega'}$ is a compact subset of Ω). Thus, we have $\nu(f(\Omega')) < \infty$ and also $f \in L^p(\Omega', Y)$, so if $f \in D^p(\Omega'; Y)$, then in fact $f \in N^{1,p}(\Omega'; Y)$. But since we do not assume X to be proper, there may not be many such sets Ω', and so we prefer to simply assume $\nu(f(\Omega)) < \infty$.

6 Examples

In this section, we discuss various examples and applications of our main results.

Corollary 1.2 applies to a wide range of weighted spaces. For example, in $\mathbb{R}^n$, we can consider any weight $w \le \widetilde{w}$, where $\widetilde{w}$ is p-admissible or a p-Muckenhoupt weight for which

$$\limsup_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} \widetilde{w}\,\mathrm{d}\mathcal{L}^n < \infty$$

apart from at most countably many $x \in \mathbb{R}^n$. See the study by Björn and Björn [4, Appendix A.2] for a discussion on these concepts.

We recall that part of the analytic definition of quasiconformality is that $f \in N^{1,Q}_{\mathrm{loc}}(X;\,Y)$, and so the case $p = Q$ is of particular interest in the theory. To the best of our knowledge, in all previous results, the assumption $h_f \in L^\infty(\Omega)$ has been made to obtain $f \in N^{1,Q}_{\mathrm{loc}}(X;\,Y)$ or $f \in D^Q_{\mathrm{loc}}(X;\,Y)$. As an elementary application of our results, we note that this strong assumption can be relaxed if the weight is small, where h_f is large. The following corollary is essentially the case $p = Q$ of [21, Corollary 1.3], but there the weight was simply $w = 1$.

Corollary 6.1. *Let $Q > 1$. Let (X_0, d, μ_0) and (Y, d_Y, ν) be Ahlfors Q-regular spaces. Let $X = X_0$ as a metric space but equipped with the weighted measure $\mathrm{d}\mu = w\,\mathrm{d}\mu_0$, with $0 \le w \le 1$. Let $\Omega \subset X$ be open and bounded and let $f : \Omega \to f(\Omega) \subset Y$ be a homeomorphism with $f(\Omega)$ open and $\nu(f(\Omega)) < \infty$. Suppose there is an at most countable set $E \subset \Omega$ such that $h_f < \infty$ in $\Omega \backslash E$, and $w(\cdot)h_f(\cdot)^Q \in L^\infty(\Omega)$. Then $f \in D^Q(\Omega, \mu;\, Y)$.*

Proof. This follows from Corollary 1.2; note that we can choose $\widetilde{w} \equiv 1$. □

Next we give a more concrete example.

Example 6.2. Consider the square $\Omega := (-1,1) \times (-1,1)$ on the unweighted plane $X = Y = \mathbb{R}^2$. Let $0 < b \le 1/2$ and consider the homeomorphism $f : \Omega \to \Omega$

$$f(x_1, x_2) := \begin{cases} (x_1, x_2^b), & x_2 \ge 0, \\ (x_1, -|x_2|^b), & x_2 \le 0. \end{cases}$$

Essentially, f maps squares centered at the origin to rectangles that become thinner and thinner near the origin.

By symmetry, it will be enough to study the behavior of f in the unit square $S := (0,1) \times (0,1)$. There we have

$$Df(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & bx_2^{b-1} \end{bmatrix},$$

and so

$$|Df| = \sqrt{1 + (bx_2^{b-1})^2} \ge bx_2^{b-1}.$$

Thus, $|Df| \notin L^2(S)$ so that f is not in the classical Dirichlet space $D^2_{\mathrm{euc}}(S;\,S)$, and then $f \notin D^2(S;\,S)$ by [4, Corollary A.4]. In particular, $f \notin D^2(\Omega;\,\Omega)$.

Clearly, f maps a small square centered at $(x_1, x_2) \in S$ with side length ε to a rectangle centered at $f(x_1, x_2)$ and with side lengths ε and $bx_2^{b-1}\varepsilon + o(\varepsilon)$. This means that

$$h_f(x_1, x_2) = bx_2^{b-1} \quad \text{in } S_1 := \{(x_1, x_2) \in S : x_2 \le b^{1/(1-b)}\}$$

and

$$h_f(x_1, x_2) = b^{-1}x_2^{1-b} \quad \text{in}\, S_2 := \{(x_1, x_2) \in S : x_2 \geq b^{1/(1-b)}\}.$$

Of these two sets, S_1 is the relevant one for us, since it contains a neighborhood of the x_1-axis in S. Indeed, h_f blows up on the x_1-axis, and so it is not in $L^\infty(\Omega)$. Thus, the conditions of Corollary 1.2 with $p = q = 2$ are not fulfilled, as of course they should not be.

On the other hand, let $X = \mathbb{R}^2$ equipped with a weighted Lebesgue measure, with weight $w(x_1, x_2) = \min\{1, |x_2|^{2-2b}\}$. Now for $(x_1, x_2) \in S_1$, we have

$$w(x_1, x_2)h_f(x_1, x_2)^2 = b^2,$$

and then, it is clear that $wh_f^2 \in L^\infty(S)$, and then by symmetry, $wh_f^2 \in L^\infty(\Omega)$. Moreover, $h_f < \infty$ in $\Omega \setminus E$, with E consisting of the x_1-axis intersected with Ω. We let $\widetilde{w} := w$ and note that the corresponding weighted Lebesgue measure is doubling. Then is straightforward to check that E has finite $\widetilde{\mathcal{H}}^2$-measure (the codimension 2 Hausdorff measure with respect to $\widetilde{w}\mathrm{d}\mathcal{L}^2$). Now Corollary 1.2 gives $f \in D^2(\Omega, w\mathrm{d}\mathcal{L}^2; \Omega)$ and then in fact $f \in N^{1,2}(\Omega, w\mathrm{d}\mathcal{L}^2; \Omega)$.

We conclude that in a suitable weighted space, we can show f to be a Newton-Sobolev mapping despite the fact that h_f is not essentially bounded.

In the setting of Theorem 1.1, we can also consider spaces whose dimension varies between different parts of the space. In the study by Lahti and Zhou [18, Example 6.2], we consider some such spaces in the case $p = 1$; in the case $1 < p \leq Q$, the situation is quite similar, so we do not go into more detail here.

Although Corollary 6.1 discusses equipping (X, μ_0) with a (small) weight, it is perhaps even more interesting to equip the space Y with a weight. This can be done in a very flexible way, since the space $D^p(\Omega; Y)$ does not depend on the measure ν that Y is equipped with. In this way, we can often avoid the strong assumption $h_f \in L^\infty(\Omega)$ even in unweighted Euclidean spaces. We use almost the same construction as in Example 6.2; this is also similar to [18, Example 6.8], where we considered the case $p = 1$.

Example 6.3. Consider the square $\Omega := (-1, 1) \times (-1, 1)$ on the unweighted plane $X = Y = \mathbb{R}^2$. Now choose $1 < b < \infty$ and consider the homeomorphism $f : \Omega \to \Omega$

$$f(x_1, x_2) := \begin{cases} (x_1, x_2^b), & x_2 \geq 0, \\ (x_1, -|x_2|^b), & x_2 \leq 0. \end{cases}$$

In the unit square $S := (0, 1) \times (0, 1)$, we have

$$Df(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & bx_2^{b-1} \end{bmatrix},$$

and so,

$$|Df| = \sqrt{1 + (bx_2^{b-1})^2} \leq 1 + bx_2^{b-1}.$$

Thus, $|Df| \in L^2(S)$ and then in fact $|Df| \in L^2(\Omega)$ and $f \in N^{1,2}(\Omega; \Omega)$.

Moreover,

$$h_f(x_1, x_2) = b^{-1}x_2^{1-b} \quad \text{in } S_1 := \{(x_1, x_2) \in S : x_2 \leq b^{1/(1-b)}\}$$

and

$$h_f(x_1, x_2) = bx_2^{b-1} \quad \text{in } S_2 := \{(x_1, x_2) \in S : x_2 \geq b^{1/(1-b)}\}.$$

Again, S_1 is the relevant set for us. Obviously $h_f \notin L^\infty(S_1)$ and then $h_f \notin L^\infty(\Omega)$, since h_f blows up on the x_1-axis. Thus, the previous results in the literature do not detect that $f \in N^{1,2}(\Omega; \Omega)$.

On the other hand, we can equip Y with the weight $w_Y(y_1, y_2) = |y_2|^u$, $-1 < u < 0$. Now for $(x_1, x_2) \in \Omega$,

$$w_Y(f(x)) = ||x_2|^b|^u = |x_2|^{bu},$$

and so for $(x_1, x_2) \in S_1$,

$$\frac{h_f(x_1,x_2)^2}{w_Y(f(x_1,x_2))} = b^{-2}x_2^{2-2b}x_2^{-bu} = b^{-2}x_2^{2-2b-bu}, \tag{6.1}$$

which is constant and thus in $L^\infty(S_1)$ if $b = 2/(2+u)$. Going over the values $-1 < u < 0$, we conclude that all values $1 < b < 2$ are allowed. Since S_1 contains a neighborhood of the x_1-axis in S, clearly the quantity $h_f(x_1,x_2)^2/w_Y(f(x_1,x_2))$ is then in $L^\infty(S)$, and then by symmetry in $L^\infty(\Omega)$. Moreover, $h_f < \infty$ in $\Omega\backslash E$, with E consisting of the x_1-axis intersected with Ω, so that E has finite $\mathcal{H}^1$-measure. Thus, Corollary 1.3 implies that $f \in N^{1,2}(\Omega;\,\Omega)$, and that for every curve family Γ in Ω, we have

$$\operatorname{Mod}_2(\Gamma) \le C\operatorname{Mod}_2(f(\Gamma)) \tag{6.2}$$

with $C = \|w_Y(f(\cdot))^{-1}h_f(\cdot)^2\|_{L^\infty(\Omega)}$.

Consider the curve families

$$\Gamma_i := \{\gamma_t(s) = (t,s), \quad 0 < s < 1/i\}_{t\in(-1,1)}, \quad i \in \mathbb{N}.$$

The families $f(\Gamma_i)$, when reparametrized by arc-length, are

$$f(\Gamma_i) = \{\gamma_t(s) = (t,s), \quad 0 < s < 1/i^b\}_{t\in(-1,1)}.$$

The function $\rho_i = i\chi_{(-1,1)\times(0,1/i)}$ is admissible for Γ_i, and so

$$\operatorname{Mod}_2(\Gamma_i) \le \int_\Omega \rho_i^2 \, d\mathcal{L}^2 = 2i.$$

Then suppose g_i is an admissible function for $\operatorname{Mod}_2(f(\Gamma_i))$. Here, we again consider $Y = \mathbb{R}^2$ equipped with the Lebesgue measure $\mathcal{L}^2$. By using Hölder's inequality and then Fubini's theorem, we estimate

$$\begin{aligned}\frac{2}{i^b}\int_{(-1,1)\times(0,1/i^b)} g_i^2 \, d\mathcal{L}^2 &\ge \left(\int_{(-1,1)\times(0,1/i^b)} g_i \, d\mathcal{L}^2\right)^2 \\ &= \left(\int_{-1}^{1}\int_0^{1/i^b} g_i(x_1,x_2)\,dx_2\,dx_1\right)^2 \\ &\ge \left(\int_{-1}^{1} dx_1\right)^2 = 4.\end{aligned}$$

In total, we obtain

$$\operatorname{Mod}_2(\Gamma_i) \le 2i \quad \text{and} \quad \operatorname{Mod}_2(f(\Gamma_i)) \ge 2i^b,$$

where the latter is much larger when i is large. Thus, there can be no constant C for which we would have the reverse inequality to (6.2), namely:

$$\operatorname{Mod}_2(f(\Gamma)) \le C\operatorname{Mod}_2(\Gamma)$$

for every curve family Γ in Ω. Of course, the same will happen if we equip Y with the larger weighted measure $w_Y d\mathcal{L}^2$ as we did above. This "shortcoming" is natural: h_f not being essentially bounded means that f is certainly not a (metric) quasiconformal mapping. But we were still able to detect that $f \in N^{1,2}(\Omega;\,\Omega)$, as well as prove the one-sided inequality (6.2), solely by using information about the size of h_f.

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References

[1] Z. Balogh and P. Koskela, *Quasiconformality, quasisymmetry, and removability in Loewner spaces*, Duke Math. J. **101** (2000), no. 3, 554–577.

[2] Z. Balogh, P. Koskela, and S. Rogovin, *Absolute continuity of quasiconformal mappings on curves*, Geom. Funct. Anal. **17** (2007), no. 3, 645–664.

[3] A. Björn and J. Björn, *Local and semilocal Poincaré inequalities on metric spaces*, J. Math. Pures Appl. **119** (2018), no. 9, 158–192.

[4] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17, European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

[5] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Complex Analysis, Joensuu, 1987, pp. 52–68, Lecture Notes in Math., 1351, Springer, Berlin, 1988.

[6] R. Engelking, *Dimension theory*, Translated from the Polish and revised by the author North-Holland Math. Library, 19 North-Holland Publishing Co., Amsterdam-Oxford-New York – PWN—Polish Scientific Publishers, Warsaw, 1978, x+314 pp.

[7] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[8] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[9] P. Hajłasz, *Sobolev spaces on metric-measure spaces. Heat kernels and analysis on manifolds, graphs, and metric spaces*, Paris, 2002, p. 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[10] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext. Springer-Verlag, New York, 2001. x+140 pp.

[11] J. Heinonen and P. Koskela, *Definitions of quasiconformality*, Invent. Math. **120** (1995), no. 1, 61–79.

[12] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), no. 1, 1–61.

[13] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. **85** (2001), 87–139.

[14] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev spaces on metric measure spaces, An approach based on upper gradients*, New Mathematical Monographs, vol. 27, Cambridge University Press, Cambridge, 2015. p. xii+434 pp.

[15] S. Kallunki and P. Koskela, *Exceptional sets for the definition of quasiconformality*, Amer. J. Math. **122** (2000), no. 4, 735–743.

[16] S. Kallunki and O. Martio, *ACL homeomorphisms and linear dilatation*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1073–1078.

[17] P. Koskela and S. Rogovin, *Linear dilatation and absolute continuity*, Ann. Acad. Sci. Fenn. Math. **30** (2005), no. 2, 385–392.

[18] P. Lahti and X. Zhou, *Quasiconformal and Sobolev mappings in non-Ahlfors regular metric spaces*, Tohoku Math. J., (2023). https://arxiv.org/abs/2106.03602.

[19] G. A. Margulis and G. D. Mostow, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. **5** (1995), no. 2, 402–433.

[20] W. Rudin, *Functional analysis*, 2nd edn, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. xviii+424 pp.

[21] M. Williams, *Dilatation, pointwise Lipschitz constants, and condition N on curves*, Michigan Math. J. **63** (2014), no. 4, 687–700.

[22] M. Williams, *Geometric and analytic quasiconformality in metric measure spaces*, Proc. Amer. Math. Soc. **140** (2012), no. 4, 1251–1266.

[23] W. Ziemer, *Extremal length and p-capacity*, Michigan Math. J. **16** (1969), 43–51.

[24] T. Zürcher, *Local Lipschitz numbers and Sobolev spaces*, Michigan Math. J. **55** (2007), no. 3, 561–574.