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Growth Competitions on Spherically Symmetric Riemannian Manifolds

Abstract: We propose a model for a growth competition between two subsets of a Riemannian manifold. The sets grow at two different rates, avoiding each other. It is shown that if the competition takes place on a surface which is rotationally symmetric about the starting point of the slower set, then if the surface is conformally equivalent to the Euclidean plane, the slower set remains in a bounded region, while if the surface is nonpositively curved and conformally equivalent to the hyperbolic plane, both sets may keep growing indefinitely.

1 Introduction

Consider two subsets A, B of, say, the Euclidean plane, which evolve over time, $A = A_t$, $B = B_t$, according to the following simple rules: both sets begin as singletons, $A_0 = \{q\}$, $B_0 = \{p\}$ for some points $p \neq q$, and expand at rates $\lambda > 1$ and 1 respectively, without intersecting each other. If a point belongs to one of the sets at a certain time, then it remains in the set forever. What will the sets $A_\infty := \bigcup_t A_t$ and $B_\infty := \bigcup_t B_t$ look like?

This *growth competition* can take place on essentially any metric space. Its precise formulation is given in the following section, and its existence and uniquness on complete Riemannian manifolds is established. We then study growth competitions on Riemannian manifolds which are spherically symmetric about the point p. The cases of the Euclidean plane and the hyperbolic plane exhibit contrasting behaviors: on the Euclidean plane, the faster set A will eventually trap the set B in a bounded region, while on the hyperbolic plane, if the two sets begin sufficiently far apart, then there is *coexistence*, i.e., both sets keep expanding forever. In fact, we show:

Theorem 1.1. *Let M be a two-dimensional, complete, non-compact Riemannian manifold, which is rotationally symmetric about $p \in M$. Let $q \in M$ and $\lambda > 1$, and let $\{A_t, B_t\}$ be the corresponding growth competition. Then*

1. *If M is parabolic, then B_∞ is bounded.*
2. *If M is hyperbolic and nonpositively curved, then B_∞ is unbounded if $d(p, q)$ is sufficiently large.*

Here *parabolic* (resp. *hyperbolic*) stands for "conformally equivalent to the Euclidean (resp. hyperbolic) plane".

The problem of determining the shapes of the competing sets was suggested by Itai Benjamini [1], and is loosely inspired by probabilistic competitions on $\mathbb{Z}^d$ and other graphs, see [5], [7]. A related problem, concerning a strategy to control a forest fire on the Euclidean plane, was introduced by Bressan, see [2], [3], [4]. In [6], Bressan's fire confinement problem is treated using the apparatus of viscosity solutions to the Hamilton-Jacobi equation. This could be the appropriate framework for dealing with problems such as the

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(a)   (b)
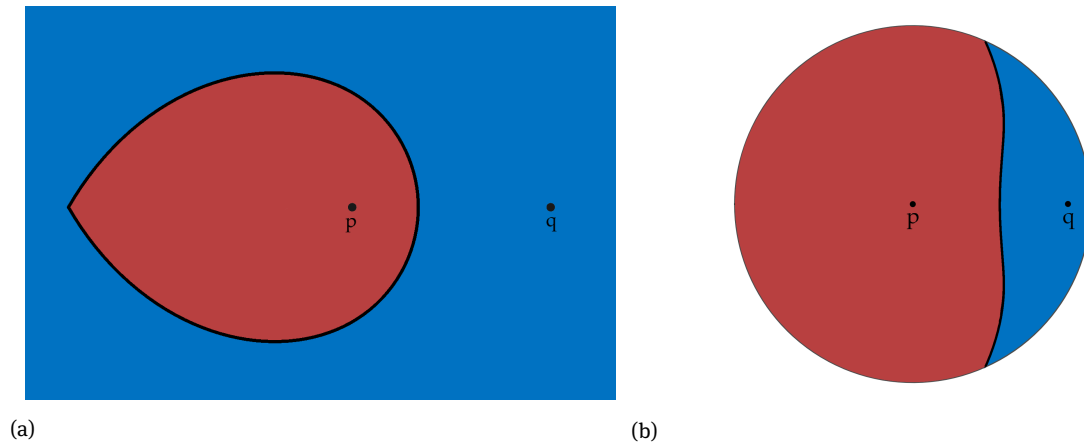
Figure 1: The sets A_∞ (blue) and B_∞ (red) for growth competitions on (a) the Euclidean plane (b) the hyperbolic plane.

one introduced here. However, as we are interested mostly in the relationship between coexistence and the underlying geometry, we have managed to define the competition in a way which enables an elementary proof of existence and uniqueness of the solution, while still capturing the essence of the problem.

Given the connection presented here between the conformal type of a simply-connected surface, and the outcome of growth competitions on it, it is now natural to ask:

Problem. *Prove or disprove each of the following statements.*

1. *On a hyperbolic surface, for every $\lambda > 1$ there is some choice of p, q such that B_∞ is unbounded.*
2. *On a parabolic surface, for every $\lambda > 1$ and every $L > 0$, there is some choice of p, q such that $d(p, q) > L$ and B_∞ is bounded.*

2 Growth competitions

Let X be a metric space, let $p \neq q \in X$, and let $\lambda > 1$. Let $\{A_t, B_t\}_{t \geq 0}$ be two families of subsets of X. Say that a path $\gamma : [0, T] \to X$ *avoids A* (resp. *avoids B*) if

- $\gamma(0) = p$ (resp. $\gamma(0) = q$),
- γ is 1-Lipschitz (resp. λ-Lipschitz)
- $\gamma(s) \notin A_s$ for all $s \in [0, T]$ (resp. $\gamma(s) \notin B_s$ for all $s \in [0, T]$).

The pair A_t, B_t will be called a *growth competition between p and q*, if

- A_t, B_t are increasing in t, i.e., $A_{t_1} \subseteq A_{t_2}$ and $B_{t_1} \subseteq B_{t_2}$ for all $0 \leq t_1 \leq t_2$,
- $A_0 = \{q\}$, $B_0 = \{p\}$, and for all $t > 0$:

$$A_t = \bigcup_\gamma \gamma([0, t]) \qquad \text{and} \qquad B_t = \bigcup_\eta \eta([0, t)) \tag{2.1}$$

where the union on the left is over paths avoiding B, and the union on the right is over paths avoiding A.

Denote also

$$A_\infty = \bigcup_{t \geq 0} A_t \qquad \text{and} \qquad B_\infty = \bigcup_{t \geq 0} B_t. \tag{2.2}$$

In the rest of this section, we prove existence and uniqueness of growth competitions on complete Riemannian manifolds. Fix a manifold (M, g), two points $p \neq q \in M$, and $\lambda > 1$.

The following lemma is quite evident from the definitions.

Lemma 2.1. *Let $\{A_t, B_t\}$ be a growth competition between p and q. Then $A_t \cap B_{t'} = \varnothing$ for all $t, t' \in [0, \infty]$.*

Proof. Let $0 < t < \infty$ and let $x \in A_t$. There exists a path $\gamma : [0, t] \to M$ avoiding B such that $x = \gamma(t_0)$ for some $0 \le t_0 \le t$, and since $\gamma|_{[0, t_0]}$ is also a curve avoiding B, $x \in A_{t_0}$. Let $\eta : [0, t') \to M$ be a curve avoiding A. We claim that $\eta(s) \ne x$ for all $s \in [0, t')$. Indeed,

- If $s < t_0$ then $\eta(s) \in B_{t_0}$, so $\eta(s) \ne x = \gamma(t_0)$ because γ avoids B.
- If $s \ge t_0$ then $x = \gamma(t_0) \in A_{t_0} \subseteq A_s$, so $\eta(s) \ne x$ since η avoids A.

Thus $\eta(s) \ne x$ for all $s \in [0, t')$, and since η is an arbitrary curve avoiding A, $x \notin B_{t'}$. This proves that $A_t, B_{t'}$ are disjoint for all $t, t' \ge 0$, and it follows that A_∞ and B_∞ are disjoint. $\qquad\square$

Denote by d the distance function of (M, g). For $x \in M$ and $R > 0$, denote by $\mathcal{B}(x, R) = \{y \in M \mid d(x, y) < R\}$ the open ball of radius R centered at x. Another fact which follows trivially from the definitions is the following.

Lemma 2.2. $A_t \subseteq \overline{\mathcal{B}(q, \lambda t)}$ *and* $B_t \subseteq \mathcal{B}(p, t)$ *for all $t \ge 0$.*

We shall now construct a certain subset of M and prove that it must coincide with B_∞. This will leave us with a unique candidate for a growth competition.

For a subset $S \subseteq M$, denote by d_S the *intrinsic metric of $M \setminus S$*, which is the metric on $M \setminus S$ defined by

$$d_S(x, y) = \inf\{\text{Length}(\gamma) \mid \gamma : [0, T] \to M \setminus S \text{ is a Lipschitz path}, \ \gamma(0) = x, \gamma(T) = y\}.$$

If the set S is open, then by the Arzelà–Ascoli theorem, the metric space $(M \setminus S, d_S)$ is a geodesic metric space, i.e., for each $x, y \in M \setminus S$ there exists a path γ realizing their intrinsic distance. If $M \setminus S$ is not path-connected, then d_S might attain the value ∞.

Define a sequence $\Omega_n \subseteq M$ for $n \ge 0$ by recursion as follows. Set $\Omega_0 := \varnothing$. Having defined $\Omega_0, \dots, \Omega_n$, set

$$\Omega_{n+1} = \Omega_n \cup \{x \in M \setminus \Omega_n \mid d_{\Omega_n}(x, q) > \lambda d(x, p)\}. \tag{2.3}$$

In particular,

$$\Omega_1 = \{x \in M \mid d(x, q) > \lambda d(x, p)\}. \tag{2.4}$$

Finally, set

$$\Omega := \bigcup_{n=1}^{\infty} \Omega_n.$$

Lemma 2.3. *The sets Ω_n are open.*

Proof. Ω_0 is trivially open. Assume that Ω_n is open. Then the distance function $d_{\Omega_n}(\cdot, q)$ is lower semicontinuous on $M \setminus \Omega_n$ with respect to the metric d, so the set $M \setminus \Omega_{n+1}$ is closed in $M \setminus \Omega_n$, and therefore in M. Thus Ω_{n+1} is open. $\qquad\square$

Lemma 2.4. *Let $n \ge 0$. Then*

$$d_{\Omega_n}(x, q) \ge \lambda d(x, p) \qquad \text{for all } x \in \partial\Omega_n, \text{ and} \tag{2.5}$$

$$d_{\Omega_n}(x, q) = \lambda d(x, p) \qquad \text{for all } x \in \partial\Omega_{n+1}. \tag{2.6}$$

Proof. For $n = 0$, (2.5) is vacuous and (2.6) is obvious from (2.4). Now let $n \ge 1$ and assume that the claim holds for $n - 1$. Let $x \in \partial\Omega_n$. Then since $\Omega_{n-1} \subseteq \Omega_n$, we have

$$d_{\Omega_n}(x, q) \ge d_{\Omega_{n-1}}(x, q) = \lambda d(x, p).$$

Now let $x \in \partial\Omega_{n+1}$. Then in particular $x \in M \setminus \Omega_{n+1}$, which by definition means $d_{\Omega_n}(x, q) \le \lambda d(x, p)$. Now, either $x \in \partial\Omega_n$, in which case reverse inequality holds by (2.5), or $x \notin \overline{\Omega_n}$, and then the reverse inequality holds since d_{Ω_n} is continuous on $M \setminus \overline{\Omega_n}$ and $x \in \partial\Omega_{n+1}$. $\qquad\square$

A subset $S \subseteq M$ is said to be *star-shaped* about a point $x_0 \in S$ if for every $x \in S$, any minimizing geodesic joining x_0 and x lies inside S.

Lemma 2.5. *The set Ω_n is star-shaped about p for all $n \geq 0$. Thus Ω is star-shaped about p.*

Proof. For $n = 0$ there is nothing to prove. Let $n \geq 1$ and let $x \in \Omega_n$. We must show that a minimizing geodesic joining p and x lies in Ω_n. By induction we may assume that $x \notin \Omega_{n-1}$, so that $d_{\Omega_{n-1}}(x, q) > \lambda d(x, p)$. Let γ be a unit-speed minimizing geodesic from p to x. By induction, there exists some $t_0 > 0$ such that $\gamma(t) \in \Omega_{n-1}$ exactly when $t < t_0$, and $x = \gamma(t_1)$ for some $t_1 \geq t_0$. Since $\gamma|_{[t_0,t_1]}$ lies outside Ω_{n-1}, we have that $d_{\Omega_{n-1}}(\gamma(t), x) = t_1 - t$ for all $t_0 \leq t \leq t_1$, and so

$$d_{\Omega_{n-1}}(\gamma(t), q) \geq d_{\Omega_{n-1}}(x, q) - d_{\Omega_{n-1}}(x, \gamma(t)) > \lambda d(x, p) - (t_1 - t) = (\lambda - 1)t_1 + t \geq \lambda t$$

for all $t_0 \leq t \leq t_1$, which implies that $\gamma(t) \in \Omega_n$ as desired. □

The following lemma states the key property of Ω.

Lemma 2.6. *For all $x \in M \setminus \Omega$,*

$$d_\Omega(x, q) \leq \lambda d(x, p), \tag{2.7}$$

with equality if and only if $x \in \partial\Omega$.

Proof. Let $x \in \partial\Omega$. For each $n \geq 1$, let $x_n \in \overline{\Omega_n}$ be a point of minimal distance to x. Clearly $x_n \to x$ as $n \to \infty$. The point x_n must lie in $\partial\Omega_n$, for otherwise we would have $x \in \Omega_n \subseteq \Omega$.
Now, on one hand, by (2.6),

$$\lambda d(x_n, p) = d_{\Omega_{n-1}}(x_n, q) \leq d_{\Omega_{n-1}}(x_n, x) + d_{\Omega_{n-1}}(x, q) \leq d_{\Omega_n}(x_n, x) + d_\Omega(x, q), \tag{2.8}$$

and taking $n \to \infty$ we get $\lambda d(x, p) \leq d_\Omega(x, q)$. On the other hand, again by (2.6), there are curves γ_n joining q and x_n of length $\lambda d(x_n, p)$ and lying outside Ω_{n-1}, and by the Arzelá-Ascoli theorem, a subsequence of them converges to a curve γ lying outside Ω_{n-1} for all n, and thus outside Ω, and of length $\lambda d(x, p)$. This implies that $d_\Omega(x, q) \leq \lambda d(x, p)$, and together with (2.8) we have equality.
Now let $x \in M \setminus \overline{\Omega}$, and let γ by a unit-speed minimizing geodesic from p to x. Since Ω is star-shaped, we there exist $0 < t_0 < t_1$ such that $\gamma(t_1) = x$ and $\gamma(t) \in \Omega$ exactly when $t < t_0$. Let $x' = \gamma(t_0)$. Then $x' \in \partial\Omega$, and since $\gamma|_{[t_0,t_1]}$ lies outside Ω, we get

$$d(x, p) = d(x, x') + d(x', p) = d_\Omega(x, x') + \lambda^{-1} d_\Omega(x', q) > \lambda^{-1}(d_\Omega(x, x') + d_\Omega(x', q))$$
$$\geq \lambda^{-1} d_\Omega(x, q). \qquad\square$$

We now have what we need in order to argue that Ω coincides with B_∞, and write down the solution to the competition.

Proposition 2.7. *Let $\lambda > 1$. Let $\{A_t, B_t\}$ be a growth competition between p, q. Then $B_\infty = \Omega$ and $A_\infty = M \setminus \Omega$.*

Proof. For the inclusion $\Omega \subseteq B_\infty$, we must show that $\Omega_n \subseteq B_\infty$ for all $n \in \mathbb{N}$.
For $n = 0$ there is nothing to prove. Let $n \geq 1$. Let γ be a unit speed geodesic emanating from p, and suppose that γ does not avoid A. Let $t_0 = \inf\{t \mid \gamma(t) \in A_t\}$. There is a sequence of curves η_k avoiding B such that $\eta_k(s_k) = \gamma(s_k)$ for some $s_k \searrow t_0$. By induction, $\eta_k|_{[0,s_k]}$ lie outside Ω_{n-1}, so $d_{\Omega_{n-1}}(\gamma(s_k), q) \leq \text{Length}(\eta_k|_{[0,s_k]}) \leq \lambda s_k$ since η_k are λ-Lipschitz. Taking $k \to \infty$ we see that $\gamma(t_0) \notin \Omega_{n-1}$ and $d_{\Omega_{n-1}}(\gamma(t_0), q) \leq \lambda d(\gamma(t_0), p)$, whence $\gamma(t_0) \notin \Omega_n$. Thus any unit speed geodesic emanating from p and staying inside Ω_n is a curve avoiding A. Since Ω_n is star-shaped about p, this proves that $\Omega_n \subseteq B_\infty$ for all $n \in \mathbb{N}$ whence $\Omega \subseteq B_\infty$.
In the other direction, let γ be a 1-Lipschitz path with $\gamma(0) = p$, and suppose that $\gamma(t_1) \notin \Omega$ for some t_1. Then by Lemma 2.6, there is a path η of length at most $\lambda d(\gamma(t_1), p) \leq \lambda t_1$ joining q and $\gamma(t_1)$ and lying entirely outside Ω, and if we take η to be minimizing and parametrized by speed λ then $\eta(t_2) = \gamma(t_1)$ for some $t_2 \leq t_1$, and $d(\eta(t), p) \geq \lambda^{-1} d_\Omega(\eta(t), q) = t$ for all $t \in [0, t_1]$ by Lemma 2.6, whence η is a curve avoiding B and

therefore $\gamma(t_1) = \eta(t_2) \in A_{t_2} \subseteq A_{t_1}$. It follows that γ does not avoid A. Thus curves avoiding A cannot leave Ω and therefore $B_\infty \subseteq \Omega$.

We have shown that $B_\infty = \Omega$. Lemma 2.6 implies that $M \setminus \Omega$ is path-connected, so for every $x \in M \setminus \Omega$, there is a curve η lying outside Ω and joining q to x. Since $B_\infty \subseteq \Omega$, the curve η avoids B and therefore $x \in A_\infty$. On the other hand, by Lemma 2.1, $A_\infty \subseteq M \setminus \Omega$. □

Corollary 2.8. *For each $p, q \in M$ and $\lambda > 1$, there is a unique growth competition, given by*

$$
\begin{aligned}
A_t &= \overline{\mathcal{B}}_\Omega(q, \lambda t) \\
B_t &= \Omega \cap \mathcal{B}(p, t),
\end{aligned}
\tag{2.9}
$$

where $\overline{\mathcal{B}}_\Omega(x, R)$ denotes the closed ball of radius R in the metric space $(M \setminus \Omega, d_\Omega)$.

Proof. Proposition 2.7, Lemma 2.7 and the fact that Ω is star-shaped about p, imply that any growth competition must take the form (2.9). So it remains to show that this is indeed a growth competition. Clearly any path starting at p and staying inside Ω avoids A, and any path starting at q and not intersecting Ω avoids B. So we have the inclusions $\subseteq$ in (2.1).

For the inclusions $\supseteq$, first let γ be a curve avoiding A. The same argument as in the proof of Proposition 2.7 implies that γ remains inside Ω, and since it is 1-Lipschitz, it is contained in $\mathcal{B}(p, t)$. Now let $\gamma : [0, t] \to M$ be a curve avoiding B; we argue by induction that it does not enter Ω_n. Again $n = 0$ is trivial. Let $n \geq 0$ and assume that $\gamma(t_0) \in \Omega_{n+1}$. By the definition of Ω_{n+1}, and since γ does not intersect Ω_n by induction, $\mathrm{Length}(\gamma|_{[0,t_0]}) > \lambda d(x, p)$. Since γ is λ-Lipschitz, it follows that $t_0 > d(x, p)$, whence $\gamma(t_0) \in B_{t_0}$, which is a contradiction to the assumption that γ avoids B. Thus $\gamma(s) \notin \Omega_n$ for all $s \in [0, t]$ and all $n \in \mathbb{N}$, whence $\gamma(s) \notin \Omega$ for all $s \in [0, t]$. Since γ is λ-Lispchitz, $\gamma(s) \in \overline{\mathcal{B}}_\Omega(q, \lambda t)$ for all $s \in [0, t]$. This finishes the proof of the inclusions $\supseteq$ in (2.1). □

3 Coexistence on spherically symmetric manifolds

We now restrict our attention to growth competitions taking place on complete, non-compact manifolds which are spherically symmetric about the point p. By spherically symmetric we mean that the metric takes the form

$$
g = dr^2 + G(r)^2 d\theta^2
\tag{3.1}
$$

where $(r, \theta) \in (0, R) \times S^{n-1}$ are polar normal coordinates centered at p, $d\theta^2$ is the standard metric on $S_p M \cong S^{n-1}$, where $S_x M$ denotes the unit sphere of the tangent space to M at x, and G is a smooth positive function satisfying $G \to 0$ and $G/r \to 1$ as $r \to 0$. Since M is noncompact, the coordinates (r, θ) are global, i.e. $R = \infty$, and M is diffeomorphic to $\mathbb{R}^n$. Set

$$
\ell := d(p, q).
$$

Let $u_0 \in S_p M$ denote the initial velocity vector of the unit-speed geodesic from p to q, so that in polar coordinates, $q = (\ell, u_0)$.

Coexistence in this setting is related to the convergence of the integral

$$
I := \int_1^\infty \frac{1}{G(r)} dr.
\tag{3.2}
$$

Proposition 3.1. *Suppose that M is spherically symmetric about p, and that $I = \infty$. Then B_∞ is bounded.*

Remark. *In particular, if $M = \mathbb{R}^n$ then B_∞ is bounded.*

Proof. We prove that on every geodesic ray emanating from p, there is a point which can be reached by a path avoiding B within some time T independent of θ. Since B_∞ is star-shaped, the proposition will follow because then $B_\infty \subseteq \mathcal{B}(p, T)$.

Let $\theta \in S_pM$ and let η be a unit-speed great circle in S_pM with $\eta(0) = u_0$ and $\eta(\tau) = \theta$ for some $\tau \le \pi$. Define a path $\gamma : [0, T] \to M$ in polar coordinates by

$$\gamma(t) = (\ell + t, \eta(\alpha(t)))$$

where

$$\alpha(t) := \sqrt{\lambda^2 - 1} \int_0^t G(\ell + s)^{-1} ds.$$

Since α tends to ∞ with t, we can choose T such that $\alpha(T) = \tau$, and therefore $\gamma(T) = (\ell + T, \eta(\tau)) = (\ell + t, \theta)$ lies on the ray from p with direction θ. The path γ avoids B; indeed, $\gamma(0) = (\ell, u_0) = q$, γ is λ-Lipschitz because

$$|\dot{\gamma}(t)|^2 = |(1, \alpha'(t)\dot{\eta}(\alpha(t)))|^2 = 1 + G(\ell + t)^2(\lambda^2 - 1)G(\ell + t)^{-2}|\dot{\eta}(\alpha(t))|^2 = \lambda^2,$$

and $d(\gamma(t), p) = \ell + t > t$ for all $t \in [0, T]$, so $\gamma(t) \in M \setminus \mathcal{B}(p, t) \subseteq M \setminus B_t$. □

The condition $I = \infty$ is not necessary. First, observe that if ℓ is small, then the competition resembles a Euclidean competition and we cannot expect coexistence, no matter what G is. Second, having $I < \infty$ does not prevent M from containing spheres centered around p with arbitrarily large radius yet arbitrarily small surface area, and if q lies on such a sphere then the set A will conquer the entire sphere quickly, trapping B within the ball it bounds. Thus, at least as long as p remains at the origin, some extra assumption is needed in order to enable coexistence. Motivated by Itai Benjamini's observation [1] that coexistence is possible on Gromov-hyperbolic spaces, we add the assumption that M is nonpositively curved, i.e. that all its sectional curvatures are nonpositive.

Proposition 3.2. *Suppose that M is spherically symmetric about p and nonpositively curved, and that $I < \infty$. Then there exists $L > 0$ such that if $\ell > L$ then B_∞ is unbounded.*

Remark. Note that if the sectional curvature of M is bounded from above by a negative constant then automatically $I < \infty$. In this case one can use the argument from [1] to prove that coexistence is possible.

Recall that $B_\infty = \Omega$ by Proposition 2.7. Since M is nonpositively curved, any two points are joined by a unique minimizing geodesic. Say that a point $x \in M$ is *visible* if the unique minimizing geodesic joining q and x does not intersect Ω, and that x is *visible$_n$* if this geodesic does not intersect Ω_n.

Since Ω is open and star-shaped, there is a function $f : S_pM \to (0, \infty]$ such that Ω is given in polar normal coordinates by the relation

$$\Omega = \{(r, \theta) \mid r < f(\theta)\}$$

(note that f may attain the value ∞). Similarly, let f_1 denote the function corresponding to the star-shaped set Ω_1.

Lemma 3.3. *Suppose that M is spherically symmetric about p and nonpositively curved. A geodesic from q meets $\partial\Omega_1$ at most twice. If it meets $\partial\Omega_1$ at $x_1 = (r_1, \theta_1)$ and $x_2 = (r_2, \theta_2)$, and $\angle(\theta_1, u_0) < \angle(\theta_2, u_0)$, then x_1 is visible$_1$, while x_2 is not.*

Here $\angle(v, w) = \arccos \langle v, w \rangle$ for $v, w \in S_xM$, $x \in M$.

Proof. Let γ be a unit-speed geodesic emanating from q. The function

$$h(t) := \lambda d(\gamma(t), p) - d(\gamma(t), q) = \lambda d(\gamma(t), p) - 1$$

is strictly convex since M is nonpositively curved, unless γ passes through p, in which case $h(t) = \lambda|\ell - t| - 1$. In both cases, h vanishes at most twice, and is negative between its zeros; hence $\gamma^{-1}(\Omega_1)$ is an interval (possibly empty) and $\gamma^{-1}(\partial\Omega_1)$ consists of at most two points; if it consists of exactly two points $t_1 < t_2$, then only $\gamma(t_1)$ is visible$_1$, since $\gamma|_{[0,t_2]}$ intersects Ω_1 and $\gamma|_{[0,t_1]}$ does not. Let

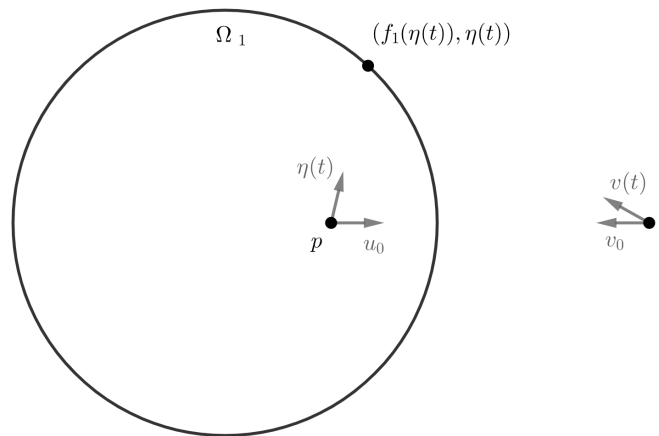$$\gamma(t) = (\rho(t), \eta(t))$$

Figure 2: Proof of Lemma 3.4.

be the expression of γ in polar coordinates. Since M is spherically symmetric, the curve η lies on a great circle in S_pM, and $\angle(\eta(t), \eta(0)) = \angle(\eta(t), u_0)$ is strictly increasing. Therefore, if x_1, x_2 are as in the statement of the lemma, and t_i satisfy $\gamma(t_i) = x_i$, then $t_1 < t_2$, hence x_1 is visible$_1$ while x_2 is not. □

Lemma 3.4. *Suppose that M is spherically symmetric about p and nonpositively curved. Then there exists $0 < a < \pi$ such that $(f_1(\theta), \theta) \in \partial\Omega_1$ is visible$_1$ exactly when $\angle(\theta, u_0) \leq a$. Moreover, a is bounded away from π in terms of λ alone.*

Proof. Identifying M with the tangent space to M at p, the set Ω_1 is invariant under orthogonal transformations fixing the line through p and q. By this symmetry, it suffices to consider a unit-speed great circle $\eta : [0, T] \to S_pM$ joining $-u_0$ to u_0 and to show that there exists some $t_0 \in [0, T]$ such that $(f_1(\eta(t)), \eta(t))$ is visible$_1$ exactly when $t \geq t_0$, and that $\angle(\eta(t_0), u_0) \leq c(\lambda) < \pi$.

For every $t \in [0, T]$, let $v(t) \in S_qM$ be the initial velocity of the unit-speed geodesic joining q to $(f_1(\eta(t)), \eta(t))$. As t varies, v traces an arc in S_qM, which, by symmetry, is contained in a great circle in S_qM. If v_0 is the initial velocity of the unit-speed geodesic from q to p, then $v(0) = v(T) = v_0$ (see Figure 2).

By Lemma 3.3, the path v may visit each point at most twice, so since v lies on a great circle and its endpoints coincide, the function $\angle(v(t), v_0)$ attains a unique maximum, say at $t = t_0$, and every value except $v(t_0)$ must therefore be attained by v exactly twice, once in $[0, t_0)$ and again in $(t_0, T]$. Since $\angle(\eta(t), u_0)$ is decreasing in t, it follows from Lemma 3.3 that $(f(\eta(t), \eta(t))$ is visible$_1$ exactly when $t \geq t_0$.
Set $\theta_0 := \eta(t_0)$ and $\alpha_0 := \angle(\eta(t_0), u_0)$. Let $\gamma : [0, S] \to M$ be a unit speed geodesic joining q to $x_0 := (f_1(\theta_0), \theta_0)$. Since γ intersects Ω_1 only at x_0, it is tangent to $\partial\Omega_1$ at x_0, so by the definition of Ω_1,

$$0 = \frac{d}{dt}\Big|_{t=S} \Big(\lambda d(\gamma(t), p) - d(\gamma(t), q)\Big) = \lambda \langle \dot\gamma(S), \partial/\partial_r \rangle - 1,$$

whence $\alpha_1 := \angle(\dot\gamma(S), \partial/\partial_r) = \arccos(\lambda^{-1})$. Now, both α_0 and α_1 are angles in the triangle $\triangle pqx_0$, so since M is nonpositively curved, $\alpha_0 \leq \pi - \arccos(\lambda^{-1})$. This finishes the proof. □

Corollary 3.5. *Under the assumptions and notations of the previous lemma, a point $x = (f(\theta), \theta) \in \partial\Omega$ is visible if and only if $\angle(\theta, u_0) \leq a$, if and only if $x \in \partial\Omega_1$.*

Proof. Suppose that x is visible. Then $d_\Omega(x, q) = d(x, q)$, so by Lemma 2.6, $d(x, q) = \lambda d(x, p)$ whence $x \in \partial\Omega_1$. Since x is visible and $\Omega_1 \subseteq \Omega$, x is visible$_1$, and therefore by Lemma 3.4, $\angle(\theta, u_0) \leq a$.

Suppose that $\angle(\theta, u_0) \leq a$, and let $x_1 := (f_1(\theta), \theta) \in \partial\Omega_1$. Then by Lemma 3.4, x_1 is visible$_1$. We argue by induction that $x_1 \in \partial\Omega_n$ and x_1 is visible$_n$ for all $n \in \mathbb{N}$.

Indeed, suppose that $x_1 \in \partial \Omega_n$ and x_1 is visible$_n$. Then $d_{\Omega_n}(x_1, q) = d(x_1, q) = \lambda d(x_1, p)$ so $x \in \partial \Omega_{n+1}$. If γ is the unit-speed geodesic from q to x_1, then since γ does not intersect Ω_n nor Ω_1,

$$d_{\Omega_n}(\gamma(t), q) = d(\gamma(t), q) \le \lambda d(\gamma(t), p)$$

so $\gamma(t) \notin \Omega_{n+1}$ for all t. Hence x_1 is visible$_{n+1}$. This completes the induction step. Thus $x_1 \in \partial \Omega_n$ and x_1 is visible$_n$ for all $n \in \mathbb{N}$, whence $x_1 \in \partial \Omega$. It follows that $x = x_1$, because x and x_1 share the same θ-coordinate and both lie on $\partial \Omega$. In particular, $x \in \partial \Omega_1$.

Finally, if $x \in \partial \Omega_1$ then $d(x, q) = \lambda d(x, p) = d_{\Omega}(x, q)$ by Lemma 2.6, so x is visible. □

Proof of Proposition 3.2. Let $\theta \in S_p M$ satisfy $f(\theta) < \infty$, i.e.

$$x := (f(\theta), \theta) \in \partial \Omega.$$

We will show that

$$\angle(\theta, u_0) < \pi, \tag{3.3}$$

provided that ℓ is larger that some $L > 0$ independent of θ. This proves that $f(-u_0) = \infty$, whence $\Omega(= B_\infty)$ contains the ray $\{\theta = -u_0\}$ and is therefore unbounded.

Let $\gamma : [0, T] \to M \setminus \Omega$ be a minimal path joining q and x in $M \setminus \Omega$, parametrized by speed λ, so that

$$d_{\Omega}(\gamma(t), q) = \lambda t \qquad \text{for all } t \in [0, T].$$

We now prove that there is some $t_0 > 0$ such that $\gamma|_{[0,t_0]}$ is a geodesic, while $\gamma(t) \in \partial \Omega$ for all $t \ge t_0$. On each interval of $\gamma^{-1}(M \setminus \overline{\Omega})$, γ must be a geodesic. But there cannot be such an interval both of whose endpoints lie in $\gamma^{-1}(\partial \Omega)$, because the function

$$\lambda d(\gamma(t), p) - d_{\Omega}(\gamma(t), q) = \lambda d(\gamma(t), p) - \lambda t$$

is convex on such an interval (since γ is a geodesic and M is nonpositively curved), and by Lemma 2.6 it vanishes on both endpoints of the interval and is positive in its interior, a contradiction. Therefore, there is some $t_0 > 0$ such that $\gamma|_{[0,t_0]}$ is a geodesic, while $\gamma(t) \in \partial \Omega$ for all $t \ge t_0$. By Corollary 3.5, if we write in polar coordinates

$$\gamma(t_0) = (r_0, \theta_0) = (f(\theta_0), \theta_0),$$

then there exists some $0 < c_0 < \pi$, depending only on λ, such that

$$\angle(\theta_0, u_0) \le a < \pi - c_0. \tag{3.4}$$

Lemma 2.6 implies that for $t \ge t_0$, the curve γ satisfies

$$d(\gamma(t), p) = \lambda^{-1} d_{\Omega}(\gamma(t), q) = t,$$

and since γ is speed-λ, it follows that in polar coordinates, it takes the form

$$\gamma(t) = (r_0 + (t - t_0), \eta(t)) \qquad t_0 \le t \le T$$

where $\eta : [t_0, T] \to S_p M$ is a Lipschitz path satisfying

$$|\dot{\eta}(t)| = \sqrt{\lambda^2 - 1}/G(r_0 + (t - t_0))$$

(this derivative being defined almost everywhere). Therefore

$$\angle(\eta(T), \eta(t_0)) \le \int_{t_0}^{T} |\dot{\eta}(s)| ds = \sqrt{\lambda^2 - 1} \int_{t_0}^{T} \frac{1}{G(r_0 + (s - t_0))} ds. \tag{3.5}$$

Now, it follows from the triangle inequality and the definition (2.4) of Ω_1 that

$$\mathcal{B}(p, \ell/(\lambda + 1)) \subseteq \Omega_1,$$

and therefore r_0 is bounded from below by $\ell/(\lambda + 1)$. By the assumption $I < \infty$, if ℓ is large enough, the right hand side of (3.5) is strictly smaller than c_0, so by (3.4) and (3.5),

$$\angle(\theta, u_0) \le \angle(\theta, \theta_0) + \angle(\theta_0, u_0) = \angle(\eta(T), \eta(t_0)) + \angle(\theta_0, u_0) < \pi$$

as desired. □

In the case $\dim M = 2$, the surface M is either hyperbolic or parabolic (since it is rotationally-symmetric and non-compact). In the former case $I < \infty$, and in the latter $I = \infty$, see Milnor [8]. Theorem 1.1 now follows.

Remark. The proof of Proposition 3.2 enables us to identify the shape of Ω in the spherically symmetric, nonpositively curved case (both when $I = \infty$ and when $I < \infty$). Suppose for simplicity that $\dim M = 2$, in which case, by passing to normal coordinates, we may assume that $M = (\mathbb{R}^2, g)$ for some rotationally symmetric metric g, and $p = (0, 0)$. There exists $0 \le a < \pi$ such that $(f(\theta)\cos\theta, f(\theta)\sin\theta) \in \partial\Omega$ is visible exactly when $-a \le \theta \le a$, and $\partial\Omega \cap \{|\theta| \le a\}$ coincides with $\partial\Omega_1$. The remainder $\partial\Omega \cap \{a \le |\theta| \le \pi\}$ is the union of two speed-λ curves whose distance from the origin p increases at rate 1. In particular, if M is the Euclidean plane, then $a = \pi/2$, $\partial\Omega \cap \{x \ge 0\} = \partial\Omega_1 \cap \{x \ge 0\}$ is a circular arc, and $\partial\Omega \cap \{x \le 0\}$ is the union of two logarithmic spirals (see Figure 1). If M is the unit disc with the Poincaré metric, then the shape Ω_1 is not a circle, but the two curves are still logarithmic spirals.

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