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Pointwise Multipliers on Weak Morrey Spaces

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Abstract: We consider generalized weak Morrey spaces with variable growth condition on spaces of homogeneous type and characterize the pointwise multipliers from a generalized weak Morrey space to another one. The set of all pointwise multipliers from a weak Lebesgue space to another one is also a weak Lebesgue space. However, we point out that the weak Morrey spaces do not always have this property just as the Morrey spaces not always.

Keywords: Morrey space; weak Morrey space; pointwise multiplier

MSC: 42B35, 46E30

1 Introduction

Let $\Omega = (\Omega, \mu)$ be a complete σ -finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to \mathbb{R} or \mathbb{C} . Then $L^0(\Omega)$ is a linear space under the usual sum and scalar multiplication. Let $E_1, E_2 \subset L^0(\Omega)$ be subspaces. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2 , if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\text{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2 . We abbreviate $\text{PWM}(E, E)$ to $\text{PWM}(E)$. The pointwise multipliers are basic operators on function spaces and thus the characterization of pointwise multipliers is not only interesting itself but also sometimes very useful to other study. Recently, it turned out that the characterization of pointwise multipliers plays key roles in the boundedness of operators and bilinear decompositions, see [1, 3, 16, 36, 37].

For $p \in (0, \infty]$, $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the quasi-norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_{\Omega} |f(x)|^p d\mu(x) \right)^{1/p}, & 0 < p < \infty, \\ \text{ess sup}_{x \in \Omega} |f(x)|, & p = \infty. \end{cases}$$

Then $L^p(\Omega)$ is a complete quasi-normed space (quasi-Banach space). If $p \in [1, \infty]$, then it is a Banach space. It is well known as Hölder's inequality that

$$\|fg\|_{L^{p_2}(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. This shows that

$$\text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega),$$

and

$$\|g\|_{\text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))} \leq \|g\|_{L^{p_3}(\Omega)},$$

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where $\|g\|_{\text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))}$ is the operator norm of $g \in \text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))$. Conversely, we can show the reverse inclusion by using the uniform boundedness theorem or the closed graph theorem. That is,

$$\text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega) \quad \text{and} \quad \|g\|_{\text{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega))} = \|g\|_{L^{p_3}(\Omega)}. \quad (1.1)$$

If $p_1 = p_2 = p$, then

$$\text{PWM}(L^p(\Omega)) = L^\infty(\Omega) \quad \text{and} \quad \|g\|_{\text{PWM}(L^p(\Omega))} = \|g\|_{L^\infty(\Omega)}. \quad (1.2)$$

The proofs of (1.1) and (1.2) are in Maligranda and Persson [19, Proposition 3 and Theorem 1]. See [28] for a survey on pointwise multipliers.

The characterization (1.1) was extended to Lorentz, Orlicz, Musielak-Orlicz spaces, etc, see [2, 14, 15, 17–19, 27, 29] and the references in [28]. For weak Lebesgue spaces we also have

$$\text{PWM}(\text{w}L^{p_1}(\Omega), \text{w}L^{p_2}(\Omega)) = \text{w}L^{p_3}(\Omega) \quad \text{and} \quad \|g\|_{\text{PWM}(\text{w}L^{p_1}(\Omega), \text{w}L^{p_2}(\Omega))} \sim \|g\|_{\text{w}L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$, see [23]. For Morrey spaces the pointwise multipliers were investigated in [24, 25]. In this paper we consider generalized weak Morrey spaces with variable growth condition on spaces of homogeneous type in the sense of Coifman and Weiss [4, 5]. To establish the characterization of pointwise multipliers on them, we first prove a generalized Hölder's inequality for the generalized weak Morrey spaces. Next, to characterize the pointwise multipliers, we use the fact that all pointwise multipliers on the generalized weak Morrey spaces are bounded operators. This fact follows from Theorem 1.1 and Corollary 1.2 below. Moreover, we point out that

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n)) \supsetneq \text{w}L_{p_3, \phi_3}(\mathbb{R}^n)$$

for some cases even if $1/p_2 = 1/p_1 + 1/p_3$ and $\phi_2 = \phi_1 \phi_3$, as well as the Morrey spaces.

We always assume that a function space $E \subset L^0(\Omega)$ has the following property:

If a measurable subset $\Omega_1 \subset \Omega$ satisfies that

$$\mu(\{x \in \Omega : f(x) \neq 0\} \setminus \Omega_1) = 0 \text{ for every } f \in E, \text{ then } \mu(\Omega \setminus \Omega_1) = 0, \quad (1.3)$$

see [13, pages 94] in which this property is referred to as $\text{supp } E = \Omega$. We say that a quasi-normed space $E \subset L^0(\Omega)$ has the lattice property if the following holds:

$$f \in E, h \in L^0(\Omega), |h| \leq |f| \text{ a.e.} \Rightarrow h \in E, \|h\|_E \leq \|f\|_E. \quad (1.4)$$

Then we have the following theorem:

Theorem 1.1 ([28, Theorem 2.7]). *Let a quasi-normed space $E \subset L^0(\Omega)$ have the lattice property (1.4). For any sequence of functions $f_j \in E$, $j = 1, 2, \dots$, if $f_j \rightarrow 0$ in E , then $f_j \rightarrow 0$ in measure on every measurable set with finite measure.*

Using the closed graph theorem, we have the following corollary:

Corollary 1.2 ([28, Corollary 2.8]). *If E_1 and E_2 are complete quasi-normed spaces with the lattice property (1.4), then all $g \in \text{PWM}(E_1, E_2)$ are bounded operators.*

We will show weak Morrey spaces are complete quasi-normed spaces with the lattice property (1.4). Then all pointwise multipliers from a weak Morrey space to another weak Morrey space are bounded operators.

We also use the Fatou property of weak Morrey spaces. Here, a quasi-normed space E has the Fatou property if

$$f_j \in E \ (j = 1, 2, \dots), f_j \geq 0, f_j \nearrow f \text{ a.e. and } \sup_j \|f_j\|_E < \infty \Rightarrow f \in E \text{ and } \|f\|_E \leq \sup_j \|f_j\|_E. \quad (1.5)$$

Morrey spaces are introduced by Morrey [20]. The generalized Morrey and weak Morrey spaces with variable growth condition were introduced in [22]. For the theory of Morrey spaces, see [32, 33] for example. For weak Morrey spaces, see [7, 8, 34, 35], etc. We consider Morrey and weak Morrey spaces on spaces of homogeneous type in the sense of Coifman and Weiss [4, 5]. It is known that the space of homogeneous type provides a natural setting for the study of both function spaces and the boundedness of operators. Many mathematicians have developed the theory of harmonic analysis on spaces of homogeneous type, since Coifman and Weiss defined the Hardy spaces on them by using atoms. For recent developments, see [6, 10–12, 38] for example.

The organization of this paper is as follows. We recall the definitions of the space of homogeneous type $X = (X, d, \mu)$ and generalized weak Morrey spaces with variable growth condition in Section 2. Then we state main results in Section 3 where we give a sufficient condition for the characterization

$$\text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X)) = wL_{p_3, \phi_3}(X). \quad (1.6)$$

Moreover, we give a necessary condition for (1.6) when $X = \mathbb{R}^n$. We prove the sufficient condition in Section 4 and the necessary condition in Section 5. The proof method is almost same as [24, 25]. However we need to investigate the properties of the quasi-norm on the weak Morrey space in the proofs.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p , are dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

2 Morrey and weak Morrey spaces on spaces of homogeneous type

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e., X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$\begin{aligned} d(x, y) &\geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y, \\ d(x, y) &= d(y, x), \\ d(x, y) &\leq K_1 (d(x, z) + d(z, y)), \end{aligned} \quad (2.1)$$

the balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x , μ is defined on a σ -algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty, \quad (2.2)$$

where $K_i \geq 1$ ($i = 1, 2$) are constants independent of $x, y, z \in X$ and $r > 0$.

Next we recall the generalized Morrey and weak Morrey spaces with variable growth condition. For a variable growth function $\phi : X \times (0, \infty) \rightarrow (0, \infty)$ and a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$.

Definition 2.1. For an index $p \in (0, \infty]$, a variable growth function $\phi : X \times (0, \infty) \rightarrow (0, \infty)$ and a ball B , let

$$\|f\|_{p, \phi, B} = \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x)|^p d\mu(x) \right)^{1/p}, \quad 0 < p < \infty, \quad (2.3)$$

$$\|f\|_{p, \phi, B, \text{weak}} = \frac{1}{\phi(B)} \left(\frac{\sup_{t>0} t^p \mu(\{x \in B : |f(x)| > t\})}{\mu(B)} \right)^{1/p}, \quad 0 < p < \infty, \quad (2.4)$$

$$\|f\|_{\infty, \phi, B} = \|f\|_{\infty, \phi, B, \text{weak}} = \frac{1}{\phi(B)} \text{ess sup}_{x \in B} |f(x)|, \quad p = \infty. \quad (2.5)$$

Let $L_{p,\phi}(X)$ and $wL_{p,\phi}(X)$ be the sets of all functions f such that the following functionals are finite:

$$\|f\|_{L_{p,\phi}} = \sup_B \|f\|_{p,\phi,B}, \quad \|f\|_{wL_{p,\phi}} = \sup_B \|f\|_{p,\phi,B,\text{weak}},$$

respectively, where the suprema are taken over all balls B in X .

In 1938 Morrey [20] introduced the case $\phi(r) = r^{(\lambda-n)/p}$ with $\lambda \in (0, n)$ on \mathbb{R}^n . Namely,

$$\|f\|_{L^{p,\lambda}} = \sup_{B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p}. \quad (2.6)$$

The symbol $L^{p,\lambda}$ was used by Peetre [31] in 1966.

Using the L^p and weak L^p quasi-norms on the ball B , we can also write

$$\|f\|_{p,\phi,B} = \frac{\|f\|_{L^p(B)}}{\phi(B)\mu(B)^{1/p}}, \quad \|f\|_{p,\phi,B,\text{weak}} = \frac{\|f\|_{wL^p(B)}}{\phi(B)\mu(B)^{1/p}}, \quad 0 < p \leq \infty.$$

Here and in the sequel, we always regard that

$$\mu(B)^{1/p} = 1 \quad \text{if } p = \infty. \quad (2.7)$$

Then the spaces $L_{p,\phi}(X)$ and $wL_{p,\phi}(X)$ are complete quasi-normed linear spaces with the lattice property (1.4) and the Fatou property (1.5), since $L^p(B)$ and $wL^p(B)$ have these properties for each ball B . Note that

$$\begin{aligned} \|f + g\|_{L_{p,\phi}} &\leq 2^{1/\min(p,1)-1} (\|f\|_{L_{p,\phi}} + \|g\|_{L_{p,\phi}}), \\ \|f + g\|_{wL_{p,\phi}} &\leq 2^{1/\min(p,1)} (\|f\|_{wL_{p,\phi}} + \|g\|_{wL_{p,\phi}}). \end{aligned}$$

If $1 \leq p \leq \infty$, then $\|f\|_{L_{p,\phi}}$ is a norm and thereby $L_{p,\phi}(X)$ is a Banach space. If $\phi(B) = \mu(B)^{-1/p}$, then $L_{p,\phi}(X) = L^p(X)$ and $wL_{p,\phi}(X) = wL^p(X)$.

For two variable growth functions ϕ_1 and ϕ_2 , we write $\phi_1 \sim \phi_2$ if there exists a positive constant C such that, for all balls B ,

$$C^{-1}\phi_1(B) \leq \phi_2(B) \leq C\phi_1(B).$$

In this case, two function spaces defined by ϕ_1 and by ϕ_2 coincide with equivalent quasi-norms.

We consider the following conditions on variable growth function ϕ :

$$\frac{1}{A_1} \leq \frac{\phi(x,s)}{\phi(x,r)} \leq A_1, \quad \text{if } \frac{1}{2} \leq \frac{s}{r} \leq 2, \quad (2.8)$$

$$\frac{1}{A_2} \leq \frac{\phi(x,r)}{\phi(y,r)} \leq A_2, \quad \text{if } |x-y| \leq r, \quad (2.9)$$

where A_1 and A_2 are positive constants independent of $x, y \in X$ and $r, s \in (0, \infty)$. The condition (2.8) is called the doubling condition. The condition (2.9) is introduced in [21] and studied in [30] precisely. In this paper, we call it the nearness condition.

Note that, if $\phi(x,r)\mu(B(x,r))^{1/p} \rightarrow 0$ as $r \rightarrow \infty$ for some $x \in X$, then $L_{p,\phi}(X) = wL_{p,\phi}(X) = \{0\}$. To avoid it we consider the following condition:

$$\phi(x,r)\mu(B(x,r))^{1/p} \leq A_3\phi(x,s)\mu(B(x,s))^{1/p}, \quad \text{if } r < s, \quad (2.10)$$

where A_3 is a positive constant independent of $x \in X$ and $r, s \in (0, \infty)$. We also consider the following condition: For any ball B , there exists a positive constant C_B such that

$$\inf_{B(x,r) \subset B} \phi(x,r) \geq C_B. \quad (2.11)$$

We say that a function $\theta : X \times (0, \infty) \rightarrow (0, \infty)$ is almost increasing (resp. almost decreasing) if there exists a positive constant C such that, for all $x \in X$ and $r, s \in (0, \infty)$,

$$\theta(x, r) \leq C\theta(x, s) \quad (\text{resp. } \theta(x, s) \leq C\theta(x, r)), \quad \text{if } r < s. \quad (2.12)$$

Then the condition (2.10) means that $\phi(x, r)\mu(B(x, r))^{1/p}$ is almost increasing. If ϕ is almost decreasing and satisfies (2.8) and (2.9), then there exists a positive constant C such that, for all balls B ,

$$\inf_{B(x, r) \subset B} \phi(x, r) \geq C\phi(B). \quad (2.13)$$

Remark 2.1. Let $p \in (0, \infty]$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. If ϕ satisfies (2.8), (2.9), (2.10) and (2.11), then, for any ball B , its characteristic function χ_B is in $L_{p, \phi}(X)$ and in $wL_{p, \phi}(X)$, see Lemma 4.4. Consequently, all finitely simple functions are in $L_{p, \phi}(X)$ and $wL_{p, \phi}(X)$. Moreover, since $X = \bigcup_{k=1}^{\infty} B(x, k)$ for every $x \in X$, we see that both $L_{p, \phi}(X)$ and $wL_{p, \phi}(X)$ satisfy (1.3).

3 Main results

We denote by $\|g\|_{\text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X))}$ the operator norm of the pointwise multiplier $g \in \text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X))$. The first result is a generalized Hölder's inequality for weak Morrey spaces.

Theorem 3.1. Let $p_i \in (0, \infty]$ and $\phi_i : X \times (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2, 3$. If $1/p_1 + 1/p_3 \leq 1/p_2$ and $\phi_1\phi_3 \leq C\phi_2$ for some positive constant C , then

$$\|fg\|_{wL_{p_2, \phi_2}} \leq C \frac{q^{1/q} p_1^{1/p_1} p_3^{1/p_3}}{p_2^{1/p_2}} \|f\|_{wL_{p_1, \phi_1}} \|g\|_{wL_{p_3, \phi_3}}, \quad (3.1)$$

where $1/q = 1/p_2 - 1/p_1 - 1/p_3$ with convention $\infty^{1/\infty} = 1$. Consequently,

$$wL_{p_3, \phi_3}(X) \subset \text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X)),$$

and, for all $g \in wL_{p_3, \phi_3}(X)$,

$$\|g\|_{\text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X))} \leq C \frac{q^{1/q} p_1^{1/p_1} p_3^{1/p_3}}{p_2^{1/p_2}} \|g\|_{wL_{p_3, \phi_3}}.$$

Remark 3.1. If $1/p_1 + 1/p_3 \leq 1/p_2$ and $\phi_1\phi_3 \leq C\phi_2$, then

$$\|fg\|_{L_{p_2, \phi_2}} \leq C \|f\|_{L_{p_1, \phi_1}} \|g\|_{L_{p_3, \phi_3}}, \quad (3.2)$$

which is a generalized Hölder's inequality for Morrey spaces, see [24, Lemma 4.1].

Next we state the reverse inclusion.

Theorem 3.2. Let $p_i \in (0, \infty]$ and $\phi_i : X \times (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2, 3$. Assume that p_i and ϕ_i , $i = 1, 2, 3$, satisfy (2.8), (2.9), (2.10) and (2.11), and that $\phi_3^{1/p_1} / \phi_1^{1/p_3}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$. If $1/p_2 \leq 1/p_1 + 1/p_3$ and $\phi_2 \leq C\phi_1\phi_3$ for some positive constant C , then

$$\text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X)) \subset wL_{p_3, \phi_3}(X),$$

and, for all $g \in \text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X))$,

$$\|g\|_{wL_{p_3, \phi_3}} \leq C' \|g\|_{\text{PWM}(wL_{p_1, \phi_1}(X), wL_{p_2, \phi_2}(X))}.$$

Remark 3.2. Under the same assumption as Theorem 3.2, we have

$$\text{PWM}(L_{p_1, \phi_1}(X), L_{p_2, \phi_2}(X)) \subset L_{p_3, \phi_3}(X),$$

$$\|g\|_{L_{p_3, \phi_3}} \leq C' \|g\|_{\text{PWM}(L_{p_1, \phi_1}(X), L_{p_2, \phi_2}(X))},$$

see [24, Section 5].

By the above two theorems we have the following theorem.

Theorem 3.3. Let $p_i \in (0, \infty]$ and $\phi_i : X \times (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2$. Assume that $p_2 \leq p_1$ and that p_i and ϕ_i ($i = 1, 2$) satisfy (2.8), (2.9), (2.10) and (2.11). Assume also that ϕ_2/ϕ_1 satisfies (2.11) and that $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$. Define p_3 and ϕ_3 as $1/p_3 = 1/p_2 - 1/p_1$ and $\phi_3 = \phi_2/\phi_1$. Then

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(X), \text{w}L_{p_2, \phi_2}(X)) = \text{w}L_{p_3, \phi_3}(X),$$

and $\|g\|_{\text{PWM}(\text{w}L_{p_1, \phi_1}(X), \text{w}L_{p_2, \phi_2}(X))}$ is equivalent to $\|g\|_{\text{w}L_{p_3, \phi_3}}$.

Since $\text{w}L_{p, \phi}(X) = L^\infty(X)$ if $p = \infty$ and $\phi \equiv 1$, we have the following corollary.

Corollary 3.4. Let $p \in (0, \infty]$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. Assume that p and ϕ satisfy (2.8), (2.9), (2.10) and (2.11). Then

$$\text{PWM}(\text{w}L_{p, \phi}(X)) = L^\infty(X),$$

and $\|g\|_{\text{PWM}(\text{w}L_{p, \phi}(X))} \sim \|g\|_{L^\infty}$. Moreover,

$$\text{PWM}(L^\infty(X), \text{w}L_{p, \phi}(X)) = \text{w}L_{p, \phi}(X).$$

and $\|g\|_{\text{PWM}(L^\infty(X), \text{w}L_{p, \phi}(X))} \sim \|g\|_{\text{w}L_{p, \phi}}$.

Remark 3.3. Let $0 < p_2 < p_1 \leq \infty$, $1/p_3 = 1/p_2 - 1/p_1$ and $\phi_3 = \phi_2/\phi_1$. Assume that

- (i) p_1 and ϕ_1 satisfy (2.8), (2.9) and (2.10),
- (ii) ϕ_2 satisfies (2.8) and (2.9),
- (iii) ϕ_2/ϕ_1 satisfies (2.11),
- (iv) $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$.

Then p_i and ϕ_i , $i = 1, 2, 3$, satisfy (2.8), (2.9), (2.10) and (2.11). Actually, (i) and (ii) imply that ϕ_3 satisfies (2.8) and (2.9), (i) and (iv) imply that

$$\phi_i(x, r)\mu(B(x, r))^{1/p_i} = \left(\phi_1(x, r)\mu(B(x, r))^{1/p_1}\right)^{p_1/p_i} \left(\phi_2^{1/p_1}/\phi_1^{1/p_2}\right)^{p_1}, \quad i = 2, 3,$$

satisfy (2.10), and, (iii) and (iv) imply that

$$\phi_i = (\phi_2/\phi_1)^{p_i/(p_1-p_2)} \left(\phi_2^{1/p_1}/\phi_1^{1/p_2}\right)^{-p_1 p_2/(p_1-p_2)} \quad i = 1, 2,$$

satisfy (2.11).

For the case $\phi_i : (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2, 3$, we have the following corollary.

Corollary 3.5. Let $p_i \in (0, \infty]$ and $\phi_i : (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2$. Assume that $p_2 \leq p_1$, that ϕ_i is almost decreasing and that $\phi_i(r)\mu(B(x, r))^{1/p_i}$ satisfies (2.10) for $i = 1, 2$. Assume also that $\liminf_{r \rightarrow 0} \phi_2(r)/\phi_1(r) > 0$ and that $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$. Define p_3 and ϕ_3 as $1/p_3 = 1/p_2 - 1/p_1$ and $\phi_3 = \phi_2/\phi_1$. Then

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(X), \text{w}L_{p_2, \phi_2}(X)) = \text{w}L_{p_3, \phi_3}(X),$$

and $\|g\|_{\text{PWM}(\text{w}L_{p_1, \phi_1}(X), \text{w}L_{p_2, \phi_2}(X))}$ is equivalent to $\|g\|_{\text{w}L_{p_3, \phi_3}}$.

Remark 3.4. Under the same assumption as Theorem 3.3, we have

$$\text{PWM}(L_{p_1, \phi_1}(X), L_{p_2, \phi_2}(X)) = L_{p_3, \phi_3}(X),$$

and $\|g\|_{\text{PWM}(L_{p_1, \phi_1}(X), L_{p_2, \phi_2}(X))}$ is equivalent to $\|g\|_{L_{p_3, \phi_3}}$, see [24, Theorem 2.1].

Let X be the n -dimensional Euclidean space \mathbb{R}^n , $d(x, y)$ be the usual distance $|x - y|$ and μ be the Lebesgue measure. Let $p \in (0, \infty)$. For $\lambda(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\lambda^* \in \mathbb{R}$, let

$$\phi(x, r) = \begin{cases} r^{\lambda(x)}, & 0 < r < 1, \\ r^{\lambda^*}, & 1 \leq r < \infty. \end{cases} \quad (3.3)$$

Assume that $-n/p \leq \lambda(x) \leq 0$ and $-n/p \leq \lambda^* \leq 0$ and that $\lambda(\cdot)$ is log-Hölder continuous, that is, there exists a positive constant C such that, for all $x, y \in \mathbb{R}^n$,

$$|\lambda(x) - \lambda(y)| \leq \frac{C}{\log(e/|x - y|)} \quad \text{if } 0 < |x - y| < 1.$$

Then ϕ satisfies (2.8), (2.9), (2.10) and (2.11), see [26, Proposition 3.3].

Corollary 3.6. Let $p_i \in (0, \infty)$, $\lambda_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\lambda_i^* \in \mathbb{R}$, and let ϕ_i be defined as (3.3), $i = 1, 2, 3$. Assume that $-n/p_i \leq \lambda_i(x) \leq 0$ and $-n/p_i \leq \lambda_i^* \leq 0$ and that $\lambda_i(\cdot)$ is log-Hölder continuous, $i = 1, 2, 3$. If $1/p_1 + 1/p_3 = 1/p_2$, $\lambda_1(\cdot) + \lambda_3(\cdot) = \lambda_2(\cdot)$, $\lambda_1^* + \lambda_3^* = \lambda_2^*$, and if $\lambda_2(\cdot)/p_1 - \lambda_1(\cdot)/p_2 \geq 0$ and $\lambda_2^*/p_1 - \lambda_1^*/p_2 \geq 0$, then

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n)) = \text{w}L_{p_3, \phi_3}(\mathbb{R}^n),$$

and $\|g\|_{\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n))}$ is equivalent to $\|g\|_{\text{w}L_{p_3, \phi_3}}$.

Next we consider the necessity of the assumption in Theorem 3.3 in the case of $X = \mathbb{R}^n$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. We denote by $|E|$ the Lebesgue measure of $E \subset \mathbb{R}^n$.

Theorem 3.7. Let $p_i \in (0, \infty]$ and $\phi_i : (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2$. Assume that ϕ_1 is almost decreasing and that $\phi_1(r)r^{n/p_1}$ is almost increasing. If one of the following conditions holds,

- (i) $\liminf_{r \rightarrow 0} \phi_2(r)/\phi_1(r) = 0$,
- (ii) $p_1 < p_2$ and $\lim_{r \rightarrow 0} \phi_1(r) = \infty$,

then

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\}.$$

Theorem 3.8. Let $p_i \in (0, \infty]$ and $\phi_i : (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2$. Assume that $p_2 \leq p_1$, that ϕ_i is almost decreasing and that $\phi_i(r)r^{n/p_i}$ is almost increasing for $i = 1, 2$. Define p_3 and ϕ_3 as $1/p_3 = 1/p_2 - 1/p_1$ and $\phi_3 = \phi_2/\phi_1$. Then

$$\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n)) = \text{w}L_{p_3, \phi_3}(\mathbb{R}^n),$$

if and only if $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$.

In this case, $\|g\|_{\text{PWM}(\text{w}L_{p_1, \phi_1}(\mathbb{R}^n), \text{w}L_{p_2, \phi_2}(\mathbb{R}^n))}$ is equivalent to $\|g\|_{\text{w}L_{p_3, \phi_3}}$.

Following the definition (2.6), for $p \in (0, \infty)$ and $\lambda \in [0, n]$, we define $\text{w}L^{p, \lambda}(\mathbb{R}^n)$ as the set of all functions $f \in L^0(\mathbb{R}^n)$ such that

$$\|f\|_{\text{w}L^{p, \lambda}} = \sup_{B(x, r)} \left(\frac{\sup_{t > 0} t^p |\{y \in B(x, r) : |f(y)| > t\}|}{r^\lambda} \right)^{1/p} \quad (3.4)$$

is finite. Then $\text{w}L^{p, \lambda}(\mathbb{R}^n) = \text{w}L^p(\mathbb{R}^n)$ if $\lambda = 0$ and $\text{w}L^{p, \lambda}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $\lambda = n$.

Corollary 3.9. Let $p_i \in (0, \infty)$ and $\lambda_i \in [0, n]$, $i = 1, 2$. Then

$$\text{PWM}(\text{w}L^{p_1, \lambda_1}(\mathbb{R}^n), \text{w}L^{p_2, \lambda_2}(\mathbb{R}^n)) \begin{cases} = \text{w}L^{p_2, \lambda_2}(\mathbb{R}^n), & \lambda_1 = n, \\ = \{0\}, & p_1 < p_2 \text{ and } 0 \leq \lambda_1 < n, \\ = \{0\}, & p_1 = p_2 \text{ and } 0 \leq \lambda_1 < \lambda_2 \leq n, \\ = L^\infty(\mathbb{R}^n), & p_1 = p_2 \text{ and } 0 \leq \lambda_1 = \lambda_2 \leq n, \\ \supsetneq \{0\}, & p_1 = p_2 \text{ and } 0 \leq \lambda_2 < \lambda_1 < n, \\ = \{0\}, & p_1 > p_2 \text{ and } 0 < n + (\lambda_1 - n)p_2/p_1 < \lambda_2 \leq n, \\ = L^\infty(\mathbb{R}^n), & p_1 > p_2 \text{ and } 0 \leq \lambda_2 = n + (\lambda_1 - n)p_2/p_1 \leq n, \\ = \text{w}L^{p_3, \lambda_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } 0 \leq \lambda_1 \leq \lambda_2 < n + (\lambda_1 - n)p_2/p_1 < n, \\ \supsetneq \text{w}L^{p_3, \lambda_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } 0 \leq \lambda_1 p_2/p_1 < \lambda_2 < \lambda_1 < n, \\ \supsetneq \text{w}L^{p_3}(\mathbb{R}^n), & p_1 > p_2 \text{ and } 0 \leq \lambda_2 = \lambda_1 p_2/p_1 < \lambda_1 < n, \\ \supsetneq \{0\}, & p_1 > p_2 \text{ and } 0 \leq \lambda_2 < \lambda_1 p_2/p_1 < \lambda_1 < n, \end{cases}$$

where $p_3 = p_1 p_2 / (p_1 - p_2)$ and $\lambda_3 = (p_1 \lambda_2 - p_2 \lambda_1) / (p_1 - p_2)$.

For $\text{PWM}(L^{p_1, \lambda_1}(\mathbb{R}^n), L^{p_2, \lambda_2}(\mathbb{R}^n))$, see [25, Corollary 2.4].

4 Proofs of Theorems 3.1–3.3

First we state Hölder's inequality for weak L^p -spaces.

Lemma 4.1. Let (Ω, μ) be a measure space and let $p_i \in (0, \infty]$, $i = 1, 2, 3$. If $1/p_1 + 1/p_3 = 1/p_2$, then

$$\|fg\|_{\text{w}L^{p_2}(\Omega)} \leq \frac{p_1^{1/p_1} p_3^{1/p_3}}{p_2^{1/p_2}} \|f\|_{\text{w}L^{p_1}(\Omega)} \|g\|_{\text{w}L^{p_3}(\Omega)}, \quad (4.1)$$

with convention $\infty^{1/\infty} = 1$.

For the proof, see Grafakos [9, page 16] for example. If $p_1 = \infty$ or $p_3 = \infty$, then the inequality above is clear, since $\|f\|_{\text{w}L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

Lemma 4.2. Let $p, \tilde{p} \in (0, \infty]$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. If $p \leq \tilde{p}$, then $\text{w}L_{p, \phi}(X) \supset \text{w}L_{\tilde{p}, \phi}(X)$ and

$$\|f\|_{\text{w}L_{p, \phi}} \leq \frac{\tilde{p}^{1/\tilde{p}} q^{1/q}}{p^{1/p}} \|f\|_{\text{w}L_{\tilde{p}, \phi}},$$

with convention $\infty^{1/\infty} = 1$, where $1/q = 1/p - 1/\tilde{p}$.

Proof. By Lemma 4.1 we have that, for any ball B ,

$$\|f\|_{\text{w}L^p(B)} \leq \frac{\tilde{p}^{1/\tilde{p}} q^{1/q}}{p^{1/p}} \|f\|_{\text{w}L^{\tilde{p}}(B)} \|1\|_{\text{w}L^q(B)} = \frac{\tilde{p}^{1/\tilde{p}} q^{1/q}}{p^{1/p}} \|f\|_{\text{w}L^{\tilde{p}}(B)} \mu(B)^{1/q}.$$

Then

$$\frac{\|f\|_{\text{w}L^p(B)}}{\phi(B)\mu(B)^{1/p}} \leq \frac{\tilde{p}^{1/\tilde{p}} q^{1/q}}{p^{1/p}} \frac{\|f\|_{\text{w}L^{\tilde{p}}(B)} \mu(B)^{1/q}}{\phi(B)\mu(B)^{1/p}} = \frac{\tilde{p}^{1/\tilde{p}} q^{1/q}}{p^{1/p}} \frac{\|f\|_{\text{w}L^{\tilde{p}}(B)}}{\phi(B)\mu(B)^{1/\tilde{p}}},$$

which shows the conclusion. \square

Proof of Theorem 3.1. Let $1/\tilde{p}_2 = 1/p_1 + 1/p_3$. Then $1/p_2 = 1/\tilde{p}_2 + 1/q$ and $p_2 \leq \tilde{p}_2$. By Lemma 4.2 we have

$$\|fg\|_{\text{w}L_{p_2, \phi}} \leq \frac{\tilde{p}_2^{1/\tilde{p}_2} q^{1/q}}{p_2^{1/p_2}} \|fg\|_{\text{w}L_{\tilde{p}_2, \phi}}. \quad (4.2)$$

By Lemma 4.1 and $\phi_1\phi_3 \leq C\phi_2$, we have that, for any ball B ,

$$\frac{\|fg\|_{\mathrm{WL}^{\tilde{p}_2}(B)}}{\phi_2(B)\mu(B)^{1/\tilde{p}_2}} \leq C \frac{p_1^{1/p_1} p_3^{1/p_3}}{\tilde{p}_2^{1/\tilde{p}_2}} \frac{\|f\|_{\mathrm{WL}^{p_1}(B)} \|g\|_{\mathrm{WL}^{p_3}(B)}}{\phi_1(B)\phi_3(B)\mu(B)^{1/p_1}\mu(B)^{1/p_3}},$$

which shows

$$\|fg\|_{\mathrm{WL}^{\tilde{p}_2}, \phi_2} \leq C \frac{p_1^{1/p_1} p_3^{1/p_3}}{\tilde{p}_2^{1/\tilde{p}_2}} \|f\|_{\mathrm{WL}^{p_1}, \phi_1} \|g\|_{\mathrm{WL}^{p_3}, \phi_3}. \quad (4.3)$$

Combining (4.2) and (4.3), we have the conclusion. \square

Next we give two lemmas and one proposition to prove Theorem 3.2.

Lemma 4.3. *Let $p \in (0, \infty]$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. Suppose that p and ϕ satisfy (2.8), (2.9) and (2.10). If $\operatorname{supp} f$ is in some ball B_0 and if*

$$\sup_{B \subset 3K_1^2 B_0} \|f\|_{p, \phi, B, \text{weak}} \leq M$$

for some positive constant M , then $f \in \mathrm{WL}_{p, \phi}(X)$ and

$$\|f\|_{\mathrm{WL}_{p, \phi}} \leq CM,$$

where the positive constant C depends only on K_1, K_2, A_1, A_2 and A_3 .

Proof. Let $B_0 = B(a, r)$. For any ball $B(b, s)$, we show that

$$\|f\|_{p, \phi, B(b, s), \text{weak}} \leq CM.$$

We may assume that $B(a, r) \cap B(b, s) \neq \emptyset$.

Case 1: Assume that $s \leq r$ and $B(a, r) \cap B(b, s) \neq \emptyset$. Then $d(a, b) \leq 2K_1 r$ and $B(b, s) \subset B(a, 3K_1^2 r)$. Hence

$$\|f\|_{p, \phi, B(b, s), \text{weak}} \leq M.$$

Case 2: Assume that $s > r$ and $B(a, r) \cap B(b, s) \neq \emptyset$. Then $d(a, b) \leq 2K_1 s$ and $B(a, s) \subset B(b, 3K_1^2 s)$. In this case we have

$$\mu(B(a, 3K_1^2 s)) \leq K' \mu(B(b, s)),$$

since $\mu(B(a, 3K_1^2 s)) \lesssim \mu(B(a, s))$ and $\mu(B(b, 3K_1^2 s)) \lesssim \mu(B(b, s))$ by (2.2). By (2.8) and (2.9) we have

$$\phi(a, 3K_1^2 s) \leq A' \phi(b, s).$$

Then, using the above inequalities, $\operatorname{supp} f \subset B(a, r)$ and (2.10), we have

$$\begin{aligned} \|f\|_{p, \phi, B(b, s), \text{weak}} &= \frac{\|f\|_{\mathrm{WL}^p(B(b, s))}}{\phi(b, s)\mu(B(b, s))^{1/p}} \\ &\leq \frac{A' K'^{1/p} \|f\|_{\mathrm{WL}^p(B(a, r))}}{\phi(a, 3K_1^2 s)\mu(B(a, 3K_1^2 s))^{1/p}} \\ &\leq \frac{A_3 A' K'^{1/p} \|f\|_{\mathrm{WL}^p(B(a, r))}}{\phi(a, r)\mu(B(a, r))^{1/p}} \\ &\leq A_3 A' K'^{1/p} M. \end{aligned}$$

The proof is complete. \square

Lemma 4.4. *Let $p \in (0, \infty]$ and $\phi : X \times (0, \infty) \rightarrow (0, \infty)$. If ϕ satisfies (2.8), (2.9), (2.10) and (2.11), then, for any ball B , its characteristic function χ_B is in $L_{p, \phi}(X)$ and $\mathrm{WL}_{p, \phi}(X)$. Moreover, if ϕ is almost decreasing, then there exists a positive constant C such that, for all balls B ,*

$$\|\chi_B\|_{\mathrm{WL}_{p, \phi}} \leq \|\chi_B\|_{L_{p, \phi}} \leq \frac{C}{\phi(B)}.$$

Proof. For any ball $B' \subset 3K_1^2 B$, we have $\phi(B') \geq C_B$, where C_B is a positive constant depending on B .

$$\frac{\|\chi_B\|_{\text{WL}^p(B')}}{\phi(B')\mu(B')^{1/p}} \leq \frac{\|\chi_B\|_{L^p(B')}}{\phi(B')\mu(B')^{1/p}} = \frac{1}{\phi(B')} \leq \frac{1}{C_B}.$$

By Lemma 4.3 we obtain that

$$\|\chi_B\|_{\text{WL}_{p,\phi}(X)} \leq \|\chi_B\|_{L_{p,\phi}(X)} \lesssim \frac{1}{C_B}.$$

Moreover, if ϕ is almost decreasing, then we can take $C\phi(B)$ as C_B , where C is independent of B , see (2.13). \square

Proposition 4.5. Let $p_i \in (0, \infty]$ and $\phi_i : X \times (0, \infty) \rightarrow (0, \infty)$, $i = 1, 2, 3$. Assume that $1/p_2 \leq 1/p_1 + 1/p_3$, that p_i and ϕ_i satisfy (2.8), (2.9), (2.10) and (2.11) for $i = 1, 3$, and that $\phi_2 \leq C\phi_1\phi_3$ for some positive constant C . Assume also that $\phi_3^{1/p_1}/\phi_1^{1/p_3}$ is almost increasing with convention $\phi_i^{1/\infty} = 1$. Then, for any $g \in \text{WL}_{p_3,\phi_3}(X) \cap \text{PWM}(\text{WL}_{p_1,\phi_1}(X), \text{WL}_{p_2,\phi_2}(X))$, there exists $f \in \text{WL}_{p_1,\phi_1}(X)$ such that $f \not\equiv 0$ and

$$\|f\|_{\text{WL}_{p_1,\phi_1}} \|g\|_{\text{WL}_{p_3,\phi_3}} \leq C' \|fg\|_{\text{WL}_{p_2,\phi_2}}, \quad (4.4)$$

where C' is a positive constant independent of f and g .

Proof. We may assume that $g \not\equiv 0$. We may also assume that $1/p_1 + 1/p_3 = 1/p_2$, since $\|fg\|_{\text{WL}_{p_2,\phi_2}} \lesssim \|fg\|_{\text{WL}_{p_2,\phi_2}}$ if $1/p_2 < 1/\tilde{p}_2 = 1/p_1 + 1/p_3$. Then it is enough to prove the following three cases.

- (i) $p_1, p_2, p_3 \in (0, \infty)$.
- (ii) $p_1 = \infty$ and $p_2 = p_3 = p \in (0, \infty)$.
- (iii) $p_3 = \infty$ and $p_1 = p_2 = p \in (0, \infty]$.

Case 1. $p_1, p_2, p_3 \in (0, \infty)$: Let $g \in \text{WL}_{p_3,\phi_3}(X) \setminus \{0\}$. Choose a ball B_0 such that

$$\|g\|_{\text{WL}_{p_3,\phi_3}} \leq \frac{2\|g\|_{\text{WL}^{p_3}(B_0)}}{\phi_3(B_0)\mu(B_0)^{1/p_3}}, \quad (4.5)$$

and let

$$f(x) = \begin{cases} |g(x)|^{p_3/p_1}, & \text{if } x \in B_0, \\ 0, & \text{if } x \notin B_0. \end{cases}$$

Then

$$|f|^{p_1} = |fg|^{p_2} = |g|^{p_3} \quad \text{on } B_0,$$

and

$$\|f\|_{\text{WL}^{p_1}(B_0)}^{p_1} = \|fg\|_{\text{WL}^{p_2}(B_0)}^{p_2} = \|g\|_{\text{WL}^{p_3}(B_0)}^{p_3}, \quad (4.6)$$

which shows

$$\|f\|_{\text{WL}^{p_1}(B_0)} \|g\|_{\text{WL}^{p_3}(B_0)} = \|fg\|_{\text{WL}^{p_2}(B_0)}. \quad (4.7)$$

Moreover, for all balls B ,

$$\|f\|_{\text{WL}^{p_1}(B)}^{p_1} = \|fg\|_{\text{WL}^{p_2}(B)}^{p_2} \leq \|g\|_{\text{WL}^{p_3}(B)}^{p_3}. \quad (4.8)$$

If $B \subset 3K_1^2 B_0$, then

$$\frac{\phi_3(B)^{p_3/p_1}}{\phi_1(B)} \lesssim \frac{\phi_3(3K_1^2 B_0)^{p_3/p_1}}{\phi_1(3K_1^2 B_0)} \sim \frac{\phi_3(B_0)^{p_3/p_1}}{\phi_1(B_0)}, \quad (4.9)$$

since $\phi_3^{p_3/p_1}/\phi_1$ is almost increasing and satisfies (2.8) and (2.9). Hence, by (4.8), (4.9), (4.5) and (4.6) we have

$$\begin{aligned} \sup_{B \subset 3K_1^2 B_0} \frac{\|f\|_{\mathbf{WL}^{p_1}(B)}}{\phi_1(B)\mu(B)^{1/p_1}} &\leq \sup_{B \subset 3K_1^2 B_0} \frac{\phi_3(B)^{p_3/p_1}}{\phi_1(B)} \left(\frac{\|g\|_{\mathbf{WL}^{p_1}(B)}}{\phi_3(B)\mu(B)^{1/p_3}} \right)^{p_3/p_1} \\ &\lesssim \frac{\phi_3(B_0)^{p_3/p_1}}{\phi_1(B_0)} \left(\|g\|_{\mathbf{WL}_{p_3, \phi_3}} \right)^{p_3/p_1} \\ &\lesssim \frac{\phi_3(B_0)^{p_3/p_1}}{\phi_1(B_0)} \left(\frac{\|g\|_{\mathbf{WL}^{p_3}(B_0)}}{\phi_3(B_0)\mu(B_0)^{1/p_3}} \right)^{p_3/p_1} \\ &= \frac{\|f\|_{\mathbf{WL}^{p_1}(B_0)}}{\phi_1(B_0)\mu(B_0)^{1/p_1}}. \end{aligned}$$

Therefore, using Lemma 4.3, we have that $f \in \mathbf{WL}_{p_1, \phi_1}(X)$ and that

$$\|f\|_{\mathbf{WL}_{p_1, \phi_1}} \lesssim \frac{\|f\|_{\mathbf{WL}^{p_1}(B_0)}}{\phi_1(B_0)\mu(B_0)^{1/p_1}}, \quad (4.10)$$

which also shows that $fg \in \mathbf{WL}_{p_2, \phi_2}(X)$, since $g \in \text{PWM}(\mathbf{WL}_{p_1, \phi_1}(X), \mathbf{WL}_{p_2, \phi_2}(X))$. Finally, using (4.10), (4.5), (4.7) and the inequality $\phi_2 \leq C\phi_1\phi_3$, we have

$$\begin{aligned} \|f\|_{\mathbf{WL}_{p_1, \phi_1}} \|g\|_{\mathbf{WL}_{p_3, \phi_3}} &\lesssim \frac{\|f\|_{\mathbf{WL}^{p_1}(B_0)}}{\phi_1(B_0)\mu(B_0)^{1/p_1}} \frac{\|g\|_{\mathbf{WL}^{p_3}(B_0)}}{\phi_3(B_0)\mu(B_0)^{1/p_3}} \\ &\lesssim \frac{\|fg\|_{\mathbf{WL}^{p_2}(B_0)}}{\phi_2(B_0)\mu(B_0)^{1/p_2}} \leq \|fg\|_{\mathbf{WL}_{p_2, \phi_2}}. \end{aligned}$$

Case 2. $p_1 = \infty$ and $p_2 = p_3 = p \in (0, \infty)$: Let $g \in \mathbf{WL}_{p_3, \phi_3}(X) \setminus \{0\}$. Choose a ball B_0 such that

$$\|g\|_{\mathbf{WL}_{p, \phi_3}} \leq \frac{2\|g\|_{\mathbf{WL}^p(B_0)}}{\phi_3(B_0)\mu(B_0)^{1/p}}, \quad (4.11)$$

and let

$$f(x) = \begin{cases} 1, & \text{if } x \in B_0, \\ 0, & \text{if } x \notin B_0. \end{cases}$$

If $B \subset 3K_1^2 B_0$, then

$$\frac{1}{\phi_1(B)} \lesssim \frac{1}{\phi_1(3K_1^2 B_0)} \sim \frac{1}{\phi_1(B_0)},$$

since $1/\phi_1 = (\phi_3^{1/\infty}/\phi_1^{1/p_3})^{p_3}$ is almost increasing and satisfies (2.8) and (2.9), which shows

$$\sup_{B \subset 3K_1^2 B_0} \frac{1}{\phi_1(B)} \operatorname{ess\,sup}_{x \in B} |f(x)| \lesssim \frac{1}{\phi_1(B_0)}.$$

Then, by Lemma 4.3, we have that $f \in \mathbf{WL}_{\infty, \phi_1}(X)$ and that

$$\|f\|_{\mathbf{WL}_{\infty, \phi_1}} \lesssim \frac{1}{\phi_1(B_0)}, \quad (4.12)$$

which also shows that $fg \in \mathbf{WL}_{p, \phi_2}(X)$, since $g \in \text{PWM}(\mathbf{WL}_{\infty, \phi_1}(X), \mathbf{WL}_{p, \phi_2}(X))$. Using (4.11), (4.12) and the inequality $\phi_2 \leq C\phi_1\phi_3$, we have

$$\|f\|_{\mathbf{WL}_{\infty, \phi_1}} \|g\|_{\mathbf{WL}_{p, \phi_3}} \lesssim \frac{1}{\phi_1(B_0)} \frac{\|g\|_{\mathbf{WL}^p(B_0)}}{\phi_3(B_0)\mu(B_0)^{1/p}} \lesssim \frac{\|fg\|_{\mathbf{WL}^p(B_0)}}{\phi_2(B_0)\mu(B_0)^{1/p}} \leq \|fg\|_{\mathbf{WL}_{p, \phi_2}}.$$

Case 3. $p_3 = \infty$ and $p_1 = p_2 = p \in (0, \infty)$: Let $g \in \mathbf{WL}_{\infty, \phi_3}(X) \setminus \{0\}$. Choose a ball B_0 such that

$$\|g\|_{\mathbf{WL}_{\infty, \phi_3}} \leq \frac{2}{\phi_3(B_0)} \operatorname{ess\,sup}_{x \in B_0} |g(x)|, \quad (4.13)$$

and let

$$f(x) = \begin{cases} 1, & \text{if } x \in B_0 \text{ and } |g(x)| > m/2, \\ 0, & \text{if } x \notin B_0 \text{ or } |g(x)| \leq m/2, \end{cases} \quad m = \operatorname{ess\,sup}_{x \in B_0} |g(x)|.$$

Then $f \in \mathbf{w}L_{p,\phi_1}(X)$ by Lemma 4.4, which also shows that $fg \in \mathbf{w}L_{p,\phi_2}(X)$, since $g \in \operatorname{PWM}(\mathbf{w}L_{p,\phi_1}(X), \mathbf{w}L_{p,\phi_2}(X))$. For all balls B , we have

$$\frac{\|f\|_{\mathbf{w}L^p(B)}}{\mu(B)^{1/p}} \times \frac{m}{2} = \frac{\|(m/2)f\|_{\mathbf{w}L^p(B)}}{\mu(B)^{1/p}} \leq \frac{\|fg\|_{\mathbf{w}L^p(B)}}{\mu(B)^{1/p}},$$

where we regard that $\mu(B)^{1/p} = 1$ if $p = \infty$. If $B \subset 3K_1^2 B_0$, then

$$\frac{1}{\phi_1(B)} \leq C \frac{\phi_3(B)}{\phi_2(B)} \lesssim \frac{\phi_3(3K_1^2 B_0)}{\phi_2(B)} \sim \frac{\phi_3(B_0)}{\phi_2(B)}, \quad (4.14)$$

since $\phi_2 \leq C\phi_1\phi_3$ and ϕ_3 satisfies (2.8), (2.9) and (2.10) with $p_3 = \infty$. By Lemma 4.3, we have

$$\begin{aligned} \|f\|_{\mathbf{w}L_{p,\phi_1}} \|g\|_{\mathbf{w}L_{\infty,\phi_3}} &\lesssim \left(\sup_{B \subset 3K_1^2 B_0} \frac{\|f\|_{\mathbf{w}L^p(B)}}{\phi_1(B)\mu(B)^{1/p}} \right) \times \frac{2m}{\phi_3(B_0)} \\ &\lesssim \sup_{B \subset 3K_1^2 B_0} \frac{\|fg\|_{\mathbf{w}L^p(B)}}{\phi_2(B)\mu(B)^{1/p}} \leq \|fg\|_{\mathbf{w}L_{p,\phi_2}}. \end{aligned}$$

The proof is complete. \square

Proof of Theorem 3.2. Let $g \in \operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))$. Then g is a bounded operator by Corollary 1.2. Take a sequence of finitely simple functions $g_j \geq 0$, $j = 1, 2, \dots$, such that $g_j \nearrow |g|$ a.e. Then by Lemma 4.4 and the lattice property of $\mathbf{w}L_{p_2,\phi_2}(X)$ we see that

$$g_j \in \mathbf{w}L_{p_3,\phi_3}(X) \cap \operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))$$

and that

$$\|g_j\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))} \leq \|g\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))}.$$

By Proposition 4.5 we have

$$\|f\|_{\mathbf{w}L_{p_1,\phi_1}} \|g_j\|_{\mathbf{w}L_{p_3,\phi_3}} \lesssim \|fg_j\|_{\mathbf{w}L_{p_2,\phi_2}} \lesssim \|f\|_{\mathbf{w}L_{p_1,\phi_1}} \|g_j\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))},$$

for some $f \in \mathbf{w}L_{p_1,\phi_1}(X)$. That is,

$$\|g_j\|_{\mathbf{w}L_{p_3,\phi_3}} \lesssim \|g\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))}.$$

Then by the Fatou property of $\mathbf{w}L_{p_3,\phi_3}(X)$ we obtain that $g \in \mathbf{w}L_{p_3,\phi_3}(X)$ and that

$$\|g\|_{\mathbf{w}L_{p_3,\phi_3}} \lesssim \|g\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))},$$

which is the conclusion. \square

Proof of Theorem 3.3. By Theorem 3.1 we have

$$\mathbf{w}L_{p_3,\phi_3}(X) \subset \operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X)),$$

and

$$\|g\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))} \lesssim \|g\|_{\mathbf{w}L_{p_3,\phi_3}(X)}.$$

Next, by Remark 3.3 we see that ϕ_3 also satisfies (2.8), (2.9), (2.10) and (2.11). Moreover, by the relation

$$\frac{\phi_3^{1/p_1}}{\phi_1^{1/p_3}} = \frac{(\phi_2/\phi_1)^{1/p_1}}{\phi_1^{1/p_3}} = \frac{\phi_2^{1/p_1}}{\phi_1^{1/p_2}},$$

we see that $\phi_3^{1/p_1}/\phi_1^{1/p_3}$ is almost increasing from the assumption. Then by Theorem 3.2 we have

$$\mathbf{w}L_{p_3,\phi_3}(X) \supset \operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X)),$$

and

$$\|g\|_{\operatorname{PWM}(\mathbf{w}L_{p_1,\phi_1}(X), \mathbf{w}L_{p_2,\phi_2}(X))} \gtrsim \|g\|_{\mathbf{w}L_{p_3,\phi_3}(X)}.$$

Combining these results we have the conclusion. \square

5 Proofs of Theorems 3.7 and 3.8

In this section we always treat $wL_{p,\phi}(\mathbb{R}^n)$ with $\phi : (0, \infty) \rightarrow (0, \infty)$. To prove Theorems 3.7 and 3.8 we first give some lemmas. For the first lemma, see [9, page 14] for example.

Lemma 5.1. *Let $0 < q < p$ and $|E| < \infty$. Then*

$$\int_E |f(y)|^q dy \leq \frac{p}{p-q} |E|^{1-q/p} \|f\|_{wL^p(E)}^q.$$

Lemma 5.2. *Let $p \in (0, \infty]$, and let $f \in wL_{loc}^p(\mathbb{R}^n)$. Assume that, for all $x \in \mathbb{R}^n$, there exists a sequence $\{r_j\}$ of positive real numbers such that $r_j \rightarrow 0$ and*

$$\lim_{j \rightarrow \infty} \frac{\|f\|_{wL^p(B(x, r_j))}}{|B(x, r_j)|^{1/p}} = 0.$$

Then $f = 0$ a.e. \mathbb{R}^n .

Proof. Let $q \in (0, p)$. Then, by Lemma 5.1,

$$\frac{1}{|B(x, r_j)|} \int_{B(x, r_j)} |f(y)|^q dy \leq \frac{p}{p-q} \left(\frac{\|f\|_{wL^p(B(x, r_j))}}{|B(x, r_j)|^{1/p}} \right)^q \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

By Lebesgue's differentiation theorem we have the conclusion. □

The next two lemmas follow from the definition.

Lemma 5.3. *Let $p \in (0, \infty)$ and B be a ball. If $\text{supp } f \cap \text{supp } g = \emptyset$, then*

$$\|f + g\|_{wL^p(B)}^p \leq \|f\|_{wL^p(B)}^p + \|g\|_{wL^p(B)}^p.$$

Lemma 5.4. *Let $p \in (0, \infty]$ and $\phi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$. Then*

$$\left\| \sum_{j=1}^N f_j \right\|_{wL_{p,\phi}} \leq \sum_{j=1}^N (2^{1/\min(p,1)})^j \|f_j\|_{wL_{p,\phi}}.$$

Lemma 5.5. *Let $p \in (0, \infty]$ and $\phi : (0, \infty) \rightarrow (0, \infty)$. Assume that ϕ is almost decreasing and $\phi(r)r^{n/p}$ is almost increasing. Let $B_0 = B(a_0, r_0)$, and let $\{B_j\}_{j=1}^\infty$ be a sequence of balls which are pairwise disjoint. Assume that a sequence $\{f_j\}_{j=1}^\infty$ of functions satisfies*

$$\begin{cases} B_j \subset B_0, \\ \text{supp } f_j \subset (1/3)B_j, \\ \|f_j\|_{wL^p} \leq c_1 \phi(r_0) |B_j|^{1/p}, \\ \|f_j\|_{wL_{p,\phi}} \leq c_2, \end{cases} \quad j = 1, 2, \dots,$$

for some positive constants c_1 and c_2 . Then $f = \sum_{j=1}^\infty f_j$ is in $wL_{p,\phi}(\mathbb{R}^n)$ and $\|f\|_{wL_{p,\phi}} \leq C(c_1 + c_2)$, where the constant C depends only on n and ϕ .

Proof. Let $B_j = B(a_j, r_j)$, $j = 1, 2, \dots$. For any ball $B = B(a, r) \subset 3B_0$, let

$$\begin{aligned} J_1 &= \{j \in \mathbb{N} : B \cap (1/3)B_j \neq \emptyset, r \geq r_j/3\}, \\ J_2 &= \{j \in \mathbb{N} : B \cap (1/3)B_j \neq \emptyset, r < r_j/3\}, \end{aligned}$$

and

$$I_i = \frac{\|\sum_{j \in J_i} f_j(x)\|_{\text{wL}^p(B)}}{\phi(r)|B|^{1/p}}, \quad i = 1, 2.$$

If $j \in J_1$, then $(1/3)B_j \subset 3B$. It follows that

$$\bigcup_{j \in J_1} B(a_j, r_j/3) \subset B(a, 3r) \quad \text{and} \quad \sum_{j \in J_1} (r_j/3)^n \leq (3r)^n.$$

Since $\text{supp } f_j \cap \text{supp } f_k = \emptyset$ for $j \neq k$, by Lemma 5.3,

$$\begin{aligned} I_1 &\leq \frac{1}{\phi(r)|B|^{1/p}} \left(\sum_{j \in J_1} \|f_j\|_{\text{wL}^p((1/3)B_j)}^p \right)^{1/p} \\ &\leq \frac{c_1 \phi(r_0)}{\phi(r)} \left(\frac{1}{|B|} \sum_{j \in J_1} |B_j| \right)^{1/p} \leq Cc_1, \quad \text{if } p \in (0, \infty), \end{aligned}$$

and

$$I_1 = \frac{\|\sum_{j \in J_1} f_j\|_{\text{wL}^\infty(B)}}{\phi(r)} = \frac{\sup_{j \in J_1} \|f_j\|_{\text{wL}^\infty((1/3)B_j)}}{\phi(r)} \leq \frac{c_1 \phi(r_0)}{\phi(r)} \leq Cc_1, \quad \text{if } p = \infty.$$

If $j \in J_2$, then $B \subset B_j$, i.e., J_2 has only one element. Hence

$$I_2 \leq \frac{\|f_j\|_{\text{wL}^p(B)}}{\phi(r)|B|^{1/p}} \leq \|f_j\|_{\text{wL}_{p,\phi}} \leq c_2.$$

By Lemma 4.3 we have $\|f\|_{\text{wL}_{p,\phi}} \leq C(c_1 + c_2)$. □

We denote by $Q(a, r)$ the cube

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_i - a_i| \leq r/2, \quad i = 1, \dots, n\}$$

centered at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and of sidelength $r > 0$.

Proof of Theorem 3.7. Case (i): Let $\liminf_{r \rightarrow 0} \phi_2(r)/\phi_1(r) = 0$. Then there exists a sequence $\{r_j\}_{j=1}^\infty$ of positive numbers such that

$$r_j \rightarrow 0 \quad \text{and} \quad \frac{\phi_2(r_j)}{\phi_1(r_j)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Let $g \in \text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n))$ and $C_g = \|g\|_{\text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n))}$. For any $x \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\|g\|_{\text{wL}^{p_2}(B(x, r_j))}}{|B(x, r_j)|^{1/p_2}} &= \frac{\|g\chi_{B(x, r_j)}\|_{\text{wL}^{p_2}(B(x, r_j))}}{|B(x, r_j)|^{1/p_2}} \leq \phi_2(r_j) \|g\chi_{B(x, r_j)}\|_{\text{wL}_{p_2, \phi_2}} \\ &\leq C_g \phi_2(r_j) \|\chi_{B(x, r_j)}\|_{\text{wL}_{p_1, \phi_1}} \lesssim C_g \frac{\phi_2(r_j)}{\phi_1(r_j)} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

By Lemma 5.2 we get $g = 0$ a.e.

Case (ii): Let $p_1 < p_2$ and $\lim_{r \rightarrow 0} \phi_1(r) = \infty$. Take a sequence $\{s_k\}_{k=1}^\infty$ of positive real numbers such that $s_k > s_{k+1}$, $k = 1, 2, \dots$, and $s_k \rightarrow 0$ as $k \rightarrow \infty$. Let

$$l_k = \left[\left(\phi_1(s_k)^{p_1/n} s_k \right)^{-1} \right] + 1, \quad m_k = \left[\phi_1(s_k)^{p_1/n} \right] + 1,$$

where $[\alpha]$ is the integer part of the positive real number α . Then, by the almost increasingness of $\phi_1(r)r^{n/p_1}$ and the almost decreasingness of ϕ_1 , we have

$$l_k \sim \left(\phi_1(s_k)^{p_1/n} s_k \right)^{-1}, \quad m_k \sim \phi_1(s_k)^{p_1/n}.$$

Hence

$$\frac{\phi_1(1/(l_k m_k))}{(m_k)^{n/p_1}} \sim \phi_1(s_k)(s_k l_k)^{n/p_1} \sim 1, \quad (5.1)$$

$$\frac{\phi_1(1/(l_k m_k))}{(m_k)^{n/p_2}} \sim \phi_1(s_k)(s_k l_k)^{n/p_2} \sim \phi_1(s_k)^{1-p_1/p_2} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (5.2)$$

For any fixed $a_0 \in \mathbb{R}^n$, we divide the cube $Q(a_0, 1)$ into $(l_k)^n$ sub-cubes $Q(b_{k,j}, 1/l_k); j = 1, 2, \dots, (l_k)^n$, i.e.,

$$Q(a_0, 1) = \bigcup_{j=1}^{(l_k)^n} Q(b_{k,j}, 1/l_k),$$

$$Q(b_{k,j}, 1/l_k)^\circ \cap Q(b_{k,j'}, 1/l_k)^\circ = \emptyset \quad \text{for } j \neq j',$$

where Q° is the interior of the cube Q . Let $O \in \mathbb{R}^n$ be the origin. We divide the cube $Q(O, 1/l_k)$ into $(m_k)^n$ sub-cubes $Q(e_{k,i}, 1/(l_k m_k)); i = 1, 2, \dots, (m_k)^n$, i.e.,

$$Q(O, 1/l_k) = \bigcup_{i=1}^{(m_k)^n} Q(e_{k,i}, 1/(l_k m_k))$$

$$Q(e_{k,i}, 1/(l_k m_k))^\circ \cap Q(e_{k,i'}, 1/(l_k m_k))^\circ = \emptyset \quad \text{for } i \neq i'.$$

Then

$$Q(b_{k,j}, 1/l_k) = \bigcup_{i=1}^{(m_k)^n} Q(b_{k,j} + e_{k,i}, 1/(l_k m_k))$$

$$Q(b_{k,j} + e_{k,i}, 1/(l_k m_k))^\circ \cap Q(b_{k,j} + e_{k,i'}, 1/(l_k m_k))^\circ = \emptyset \quad \text{for } i \neq i', \\ j = 1, 2, \dots, l_k^n.$$

Let

$$f_{k,j,i}(x) = \begin{cases} \phi_1(1/(l_k m_k)), & x \in Q(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \\ 0, & x \notin Q(b_{k,j} + e_{k,i}, 1/(l_k m_k)), \end{cases} \\ f_{k,i} = \sum_{j=1}^{(l_k)^n} f_{k,j,i}.$$

First we show that there exists a positive constant C such that, for all $k \in \mathbb{N}$ and $i = 1, 2, \dots, (m_k)^n$,

$$\|f_{k,i}\|_{WL_{p_1, \phi_1}} \leq C. \quad (5.3)$$

We note that

$$B(b_{k,j} + e_{k,i}, 1/(2l_k)) \subset Q(b_{k,j} + e_{k,i}, 1/l_k) \subset Q(a_0, 2) \subset B(a_0, \sqrt{n}), \\ B(b_{k,j} + e_{k,i}, 1/(2l_k)) \cap B(b_{k,j'} + e_{k,i}, 1/(2l_k)) = \emptyset \quad \text{for } j \neq j'.$$

Since $\phi(s_k) \rightarrow \infty$ as $k \rightarrow \infty$, we may assume $m_k \geq 3\sqrt{n}$. Hence

$$\text{supp } f_{k,j,i} = Q(b_{k,j} + e_{k,i}, 1/(l_k m_k)) \\ \subset B(b_{k,j} + e_{k,i}, \sqrt{n}/(2l_k m_k)) \subset B(b_{k,j} + e_{k,i}, 1/(6l_k)).$$

By (5.1) we have

$$\|f_{k,j,i}\|_{WL_{p_1}} \leq \phi_1(1/(l_k m_k))(1/l_k m_k)^{n/p_1} \sim (1/l_k)^{n/p_1}.$$

By Lemma 4.4 we have

$$\|f_{k,j,i}\|_{WL_{p_1, \phi_1}} \leq C.$$

Hence, by Lemma 5.5 we have (5.3).

Now, let $g \in \text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n))$ and $C_g = \|g\|_{\text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n))}$. Then

$$\phi_1(1/l_k m_k) \|g\|_{\text{wL}^{p_2}(\text{supp } f_{k,i})} = \|gf_{k,i}\|_{\text{wL}^{p_2}(B(a_0, \sqrt{n}))} \leq \phi_2(\sqrt{n}) |B(a_0, \sqrt{n})|^{1/p_2} \|gf_{k,i}\|_{\text{wL}^{p_2, \phi_2}} \lesssim C_g.$$

Since

$$\begin{cases} Q(a_0, 1) = \bigcup_{i=1}^{(m_k)^n} \text{supp } f_{k,i}, \\ (\text{supp } f_{k,i})^\circ \cap (\text{supp } f_{k,i'})^\circ = \emptyset \quad \text{for } i \neq i', \end{cases} \quad k = 1, 2, \dots,$$

we have by Lemma 5.3 and (5.2)

$$\begin{aligned} \|g\|_{\text{wL}^{p_2}(Q(a_0, 1))} &\leq \left(\sum_{i=1}^{(m_k)^n} \left(\|g\|_{\text{wL}^{p_2}(\text{supp } f_{k,i})} \right)^{p_2} \right)^{1/p_2} \\ &\lesssim (m_k)^{n/p_2} \phi_1(1/l_k m_k)^{-1} C_g \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad \text{if } p_2 \in (0, \infty), \end{aligned}$$

and

$$\begin{aligned} \|g\|_{\text{wL}^\infty(Q(a_0, 1))} &= \max \{ \|g\|_{\text{wL}^\infty(\text{supp } f_{k,i})} : i = 1, 2, \dots, (m_k)^n \} \\ &\lesssim \phi_1(1/l_k m_k)^{-1} C_g \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad \text{if } p_2 = \infty. \end{aligned}$$

Therefore $g = 0$ a.e. □

Proof of Theorem 3.8. Let $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ be almost increasing. If $\liminf_{r \rightarrow 0} \phi_2(r)/\phi_1(r) > 0$, then Theorem 3.3 shows that

$$\text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n)) = \text{wL}_{p_3, \phi_3}(\mathbb{R}^n).$$

If $\liminf_{r \rightarrow 0} \phi_2(r)/\phi_1(r) = 0$, then $\text{wL}_{p_3, \phi_3}(\mathbb{R}^n) = \{0\}$, since $\phi_3 = \phi_2/\phi_1$. In this case Theorem 3.7 shows that

$$\text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n)) = \{0\} = \text{wL}_{p_3, \phi_3}(\mathbb{R}^n).$$

Next, we assume that $\phi_2^{1/p_1}/\phi_1^{1/p_2}$ is not almost increasing. In this case $p_2 < \infty$, since $p_2 = \infty$ implies $p_1 = \infty$ and $\phi_2^{1/p_1}/\phi_1^{1/p_2} \sim 1$. Then, for all $k \in \mathbb{N}$, there exist positive real numbers r_k and s_k , $k = 1, 2, \dots$, such that

$$s_k < r_k \quad \text{and} \quad \phi_2(s_k)^{1/p_1}/\phi_1(s_k)^{1/p_2} \geq (4C_{p_2})^{k/p_2} \phi_2(r_k)^{1/p_1}/\phi_1(r_k)^{1/p_2},$$

where $C_{p_2} = 2^{1/\min(p_2, 1)}$. We will construct a function

$$g \in \text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n)) \setminus \text{wL}_{p_3, \phi_3}(\mathbb{R}^n).$$

Let

$$m_k = \left\lceil \frac{r_k \phi_2(r_k)^{p_2/n}}{s_k \phi_2(s_k)^{p_2/n}} \right\rceil + 1,$$

where $[\alpha]$ denotes the integer part of the positive real number α . By almost increasingness of $\phi_2(r)r^{n/p_2}$ and by almost decreasing of ϕ_2 , we have

$$m_k \sim \frac{r_k \phi_2(r_k)^{p_2/n}}{s_k \phi_2(s_k)^{p_2/n}}, \quad \frac{r_k}{m_k} > 6c_0 s_k \quad \text{for some } c_0 > 0. \quad (5.4)$$

Let $O \in \mathbb{R}^n$ be the origin. We divide the cube $Q(O, r_k)$ into $(m_k)^n$ sub-cubes $Q(b_{k,j}, r_k/m_k)$; $j = 1, 2, \dots, (m_k)^n$, i.e.,

$$\begin{aligned} Q(O, r_k) &= \bigcup_{j=1}^{(m_k)^n} Q(b_{k,j}, r_k/m_k), \\ Q(b_{k,i}, r_k/m_k)^\circ \cap Q(b_{k,j}, r_k/m_k)^\circ &= \emptyset \quad \text{for } i \neq j, \end{aligned}$$

where Q° is the interior of Q . Let

$$g_{k,j}(x) = \begin{cases} \phi_3(c_0 s_k), & x \in B(b_{k,j}, c_0 s_k) \\ 0, & x \notin B(b_{k,j}, c_0 s_k) \end{cases}$$

$$g_k = \sum_{j=1}^{(m_k)^n} g_{k,j}, \quad g = \sum_{k=1}^{\infty} \frac{1}{(2C_{p_2})^k} g_k.$$

Here we note that

$$\text{supp } g_{k,j} \subset B(b_{k,j}, r_k/(6m_k)), \quad B(b_{k,j}, r_k/(2m_k)) \subset Q(b_{k,j}, r_k/m_k).$$

First we show that

$$g \in \text{PWM}(\text{wL}_{p_1, \phi_1}(\mathbb{R}^n), \text{wL}_{p_2, \phi_2}(\mathbb{R}^n)).$$

For all $f \in \text{wL}_{p_1, \phi_1}(\mathbb{R}^n)$, by Lemma 4.1 we have

$$\begin{aligned} & \|fg_{k,j}\|_{\text{wL}_{p_2}} \\ & \leq \frac{p_1^{1/p_1} p_3^{1/p_3}}{p_2^{1/p_2}} \|f\|_{\text{wL}_{p_1}(B(b_{k,j}, c_0 s_k))} \|g_{k,j}\|_{\text{wL}_{p_3}(B(b_{k,j}, c_0 s_k))} \\ & \leq \frac{p_1^{1/p_1} p_3^{1/p_3}}{p_2^{1/p_2}} \phi_1(c_0 s_k) |B(b_{k,j}, c_0 s_k)|^{1/p_1} \|f\|_{\text{wL}_{p_1, \phi_1}} \phi_3(c_0 s_k) |B(b_{k,j}, c_0 s_k)|^{1/p_3} \\ & \lesssim \phi_2(c_0 s_k) |B(b_{k,j}, c_0 s_k)|^{1/p_2} \|f\|_{\text{wL}_{p_1, \phi_1}}. \end{aligned}$$

Using the doubling condition of ϕ and (5.4), we have

$$\|fg_{k,j}\|_{\text{wL}_{p_2}} \lesssim \phi_2(s_k) (s_k)^{n/p_2} \|f\|_{\text{wL}_{p_1, \phi_1}} \sim \phi_2(r_k) (r_k/m_k)^{n/p_2} \|f\|_{\text{wL}_{p_1, \phi_1}}.$$

By Theorem 3.1 and Lemma 4.4 we have

$$\|fg_{k,j}\|_{\text{wL}_{p_2, \phi_2}} \lesssim \|f\|_{\text{wL}_{p_1, \phi_1}} \|g_{k,j}\|_{\text{wL}_{p_3, \phi_3}} \lesssim \|f\|_{\text{wL}_{p_1, \phi_1}}.$$

Hence, by Lemma 5.5 we conclude that

$$\|fg_k\|_{\text{wL}_{p_2, \phi_2}} = \left\| \sum_{j=1}^{(m_k)^n} fg_{k,j} \right\|_{\text{wL}_{p_2, \phi_2}} \leq C \|f\|_{\text{wL}_{p_1, \phi_1}}.$$

Then by Lemma 5.4 we have

$$\|fg\|_{\text{wL}_{p_2, \phi_2}} \leq \left\| \sum_{k=1}^{\infty} \frac{1}{(2C_{p_2})^k} fg_k \right\|_{\text{wL}_{p_2, \phi_2}} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \|fg_k\|_{\text{wL}_{p_2, \phi_2}} \leq C \|f\|_{\text{wL}_{p_1, \phi_1}},$$

which shows that g is a pointwise multiplier from $\text{wL}_{p_1, \phi_1}(\mathbb{R}^n)$ to $\text{wL}_{p_2, \phi_2}(\mathbb{R}^n)$.

On the other hand, since $\text{supp } g_k \subset Q(O, r_k) \subset B(O, \sqrt{n}r_k/2)$,

$$\begin{aligned} & \frac{\|g\|_{\text{wL}_{p_3}(B(O, \sqrt{n}r_k/2))}}{\phi_3(\sqrt{n}r_k/2) |B(O, \sqrt{n}r_k/2)|^{1/p_3}} \geq \frac{1}{(2C_{p_2})^k} \frac{\|g_k\|_{\text{wL}_{p_3}(B(O, \sqrt{n}r_k/2))}}{\phi_3(\sqrt{n}r_k/2) |B(O, \sqrt{n}r_k/2)|^{1/p_3}} \\ & \sim \frac{1}{(2C_{p_2})^k} \frac{\phi_3(s_k) (m_k s_k)^{n/p_3}}{\phi_3(r_k) (r_k)^{n/p_3}} \\ & = \left(\frac{m_k s_k \phi_2(s_k)^{p_2/n}}{r_k \phi_2(r_k)^{p_2/n}} \right)^{n/p_3} \left(\frac{\phi_2(s_k)^{1/p_1} / \phi_1(s_k)^{1/p_2}}{(2C_{p_2})^k \phi_2(r_k)^{1/p_1} / \phi_1(r_k)^{1/p_2}} \right)^{p_2} \\ & \gtrsim 2^k \quad \text{for } k = 1, 2, \dots, \end{aligned}$$

which shows that $g \notin \text{wL}_{p_3, \phi_3}(\mathbb{R}^n)$. □

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References

- [1] A. Bonami, J. Cao, L. D. Ky, L. Liu, D. Yang and W. Yuan, Multiplication between Hardy spaces and their dual spaces, *J. Math. Pures Appl.* (9) 131 (2019), 130–170.
- [2] J. M. Calabuig, O. Delgado and E. A. Sánchez Pérez, Generalized perfect spaces, *Indag. Math. (N.S.)* 19 (2008), No. 3, 359–378.
- [3] J. Cao, L. D. Ky and D. Yang, Bilinear decompositions of products of local Hardy and Lipschitz or BMO spaces through wavelets, *Commun. Contemp. Math.* 20 (2018), No. 3, 1750025, 30 pp.
- [4] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin and New York, 1971.
- [5] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* 83 (1977), No. 4, 569–645.
- [6] X. Fu, T. Ma and D. Yang, Real-variable characterizations of Musielak-Orlicz Hardy spaces on spaces of homogeneous type, *Ann. Acad. Sci. Fenn. Math.* 45 (2020), 343–410.
- [7] H. Gunawan, D. I. Hakim, K. M. Limanta and A. A. Masta, Inclusion properties of generalized Morrey spaces, *Math. Nachr.* 290 (2017), No. 2-3, 332–340.
- [8] H. Gunawan, D. I. Hakim, E. Nakai and Y. Sawano. On inclusion relation between weak Morrey spaces and Morrey spaces, *Nonlinear Anal.* 168 (2018), 27–31.
- [9] L. Grafakos, *Classical Fourier analysis*, Third edition, Graduate Texts in Mathematics, 249. Springer, New York, 2014.
- [10] Z. He, Y. Han, J. Li, L. Liu, D. Yang and W. Yuan, A complete real-variable theory of Hardy spaces on spaces of homogeneous type, *J. Fourier Anal. Appl.* 25 (2019), No. 5, 2197–2267.
- [11] Z. He, L. Liu, D. Yang and W. Yuan, New Calderón reproducing formulae with exponential decay on spaces of homogeneous type, *Sci. China Math.* 62 (2019), No. 2, 283–350.
- [12] Z. He, D. Yang and W. Yuan, Real-variable characterizations of local Hardy spaces on spaces of homogeneous type, *Math. Nachr.* to appear. DOI:10.1002/mana.201900320.
- [13] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, Translated from the Russian by Howard L. Silcock, Second edition, Pergamon Press, Oxford-Elmsford, N.Y., 1982.
- [14] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise multipliers of Calderón-Lozanovskii spaces, *Math. Nachr.* 286 (2013), No. 8-9, 876–907.
- [15] P. Kolwicz, K. Leśnik and L. Maligranda, Pointwise products of some Banach function spaces and factorization, *J. Funct. Anal.* 266 (2014), No. 2, 616–659.
- [16] L. D. Ky, On the product of functions in BMO and H^1 over spaces of homogeneous type, *J. Math. Anal. Appl.* 425 (2015), No. 2, 807–817.
- [17] K. Lesnik and J. Tomaszewski, Pointwise multipliers of Orlicz function spaces and factorization, *Positivity* 21 (2017), No. 4, 1563–1573.
- [18] L. Maligranda and E. Nakai, Pointwise multipliers of Orlicz spaces, *Arch. Math. (Basel)* 95 (2010), No. 3, 251–256.
- [19] L. Maligranda and L. E. Persson, Generalized duality of some Banach function spaces, *Indag. Math.* 51 (1989), No. 3, 323–338.
- [20] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* 43 (1938), No. 1, 126–166.
- [21] E. Nakai, Pointwise multipliers for functions of weighted bounded mean oscillation, *Studia Math.* 105 (1993), No. 2, 105–119.
- [22] E. Nakai, Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.* 166 (1994), 95–103.
- [23] E. Nakai, Pointwise multipliers on the Lorentz spaces, *Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci.* 45 (1996), No. 1, 1–7.
- [24] E. Nakai, Pointwise multipliers on the Morrey spaces, *Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci.* 46 (1997), 1–11.
- [25] E. Nakai, A characterization of pointwise multipliers on the Morrey spaces, *Sci. Math.* 3 (2000), No. 3, 445–454.
- [26] E. Nakai, Singular and fractional integral operators on Campanato spaces with variable growth conditions, *Rev. Mat. Complut.* 23 (2010), No. 2, 355–381.
- [27] E. Nakai, Pointwise multipliers on Musielak-Orlicz spaces, *Nihonkai Math. J.* 27 (2016), 135–146.
- [28] E. Nakai, Pointwise multipliers on several functions spaces – a survey –. *Linear Nonlinear Anal.* 3 (2017), No. 1, 27–59.
- [29] E. Nakai, Pointwise multipliers on Musielak-Orlicz-Morrey spaces. *Function spaces and inequalities*, 257–281, Springer Proc. Math. Stat., 206, Springer, Singapore, 2017.
- [30] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, *J. Funct. Anal.* 262, (2012), No. 9, 3665–3748.
- [31] J. Peetre, On convolution operators leaving $L^{p,\lambda}$ spaces invariant, *Ann. Mat. Pura Appl.* (4) 72 (1966), 295–304.
- [32] Y. Sawano, G. Di Fazio and D. I. Hakim, *Morrey Spaces, Introduction and Applications to Integral Operators and PDE's*, Volume I, Chapman and Hall/CRC, 2020. ISBN 9781498765510

- [33] Y. Sawano, G. Di Fazio and D. I. Hakim, *Morrey Spaces, Introduction and Applications to Integral Operators and PDE's*, Volume II, Chapman and Hall/CRC, 2020. ISBN 9780367459154
- [34] Y. Sawano and S. R. El-Shabrawy, Weak Morrey spaces with applications, *Math. Nachr.* 291 (2018), No. 1, 178–186.
- [35] N. K. Tumulun and H. Gunawan, Morrey spaces are embedded between weak Morrey spaces and Stummel classes, *J. Indones. Math. Soc.* 25 (2019), No. 3, 203–209.
- [36] Y. Zhang, D. Yang and W. Yuan, Real-variable characterizations of local Orlicz-slice Hardy spaces with application to bilinear decompositions, *Commun. Contemp. Math.* (to appear).
- [37] Y. Zhang, D. Yang and W. Yuan, Bilinear decomposition and divergence-curl estimates on products related to local Hardy spaces and their dual spaces, *J. Funct. Anal.* 280 (2021), 108796, 74 pp.
- [38] X. Zhou, Z. He and D. Yang, Real-variable characterizations of Hardy-Lorentz spaces on spaces of homogeneous type with applications to real interpolation and boundedness of Calderón-Zygmund operators, *Anal. Geom. Metr. Spaces* 8 (2020), No. 1, 182–260.