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Commutators on Weighted Morrey Spaces on Spaces of Homogeneous Type

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Abstract: In this paper, we study the boundedness and compactness of the commutator of Calderón–Zygmund operators T on spaces of homogeneous type (X, d, μ) in the sense of Coifman and Weiss. More precisely, we show that the commutator $[b, T]$ is bounded on the weighted Morrey space $L^{p,\kappa}_{\omega}(X)$ with $\kappa \in (0, 1)$ and $\omega \in A_p(X)$, $1 < p < \infty$, if and only if b is in the BMO space. We also prove that the commutator $[b, T]$ is compact on the same weighted Morrey space if and only if b belongs to the VMO space. We note that there is no extra assumptions on the quasimetric d and the doubling measure μ.

Keywords: commutator; compact operator; BMO space; VMO space; weighted Morrey space; space of homogeneous type

MSC: 42B20, 43A80

1 Introduction

It is well-known that the boundedness and compactness of the commutator of Calderón–Zygmund operators on certain function spaces and their characterizations play an important role in various areas, such as harmonic analysis, complex analysis, (nonlinear) PDE, etc. See for example [3, 9, 10, 13, 18–20, 22, 24, 25] and the references therein. Recently, equivalent characterizations of the boundedness and the compactness of commutators were further extended to Morrey spaces over the Euclidean space by Di Fazio and Ragusa [16] and Chen et al. [5], and to weighted Morrey spaces by Komori and Shirai [30] for commutators of Calderón–Zygmund operator and by Tao, Da. Yang and Do. Yang [34, 35] for the commutator of the Cauchy integral and Beurling-Ahlfors transformation, respectively. For more results on the boundedness of operators on Morrey spaces in different settings, we refer the reader to other studies as in [1, 15, 17, 26–28, 33, 37, 38] for instance.

Thus, along this literature, it is natural to study the boundedness and compactness of commutators of Calderón–Zygmund operators on weighted Morrey spaces in a more general setting: spaces of homogeneous type in the sense of Coifman and Weiss [8], as Yves Meyer remarked in his preface to [11], "One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis."

We say that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss if d is a quasimetric on X and μ is a nonzero measure satisfying the doubling condition. A *quasi-metric* d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if

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$x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) \le A_0[d(x, z) + d(z, y)]. \tag{1.1}$$

We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C_μ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \le C_\mu \mu(B(x, r)) < \infty, \tag{1.2}$$

where $B(x, r)$ is the quasi-metric ball by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$. We point out that the doubling condition (1.2) implies that there exists a positive constant n (the *upper dimension* of μ) such that for all $x \in X$, $\lambda \ge 1$ and $r > 0$,

$$\mu(B(x, \lambda r)) \le C_\mu \lambda^n \mu(B(x, r)). \tag{1.3}$$

Throughout this paper we assume that $\mu(X) = \infty$ and that $\mu(\{x_0\}) = 0$ for every $x_0 \in X$. We now recall the definition of Calderón–Zygmund operators on spaces of homogeneous type.

Definition 1.1. *We say that T is a Calderón–Zygmund operator on (X, d, μ) if T is bounded on $L^2(X)$ and has an associated kernel $K(x, y)$ such that $T(f)(x) = \int_X K(x, y) f(y) d\mu(y)$ for any $x \notin \operatorname{supp} f$, and $K(x, y)$ satisfies the following estimates: for all $x \neq y$,*

$$|K(x, y)| \le \frac{C}{V(x, y)}, \tag{1.4}$$

and for $d(x, x') \le (2A_0)^{-1} d(x, y)$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le \frac{C}{V(x, y)} \beta\left(\frac{d(x, x')}{d(x, y)}\right), \tag{1.5}$$

where $V(x, y) = \mu(B(x, d(x, y)))$, $\beta : [0, 1] \to [0, \infty)$ is continuous, increasing, subadditive, and $\omega(0) = 0$. Throughout this paper we assume that $\beta(t) = t^{\sigma_0}$, for some $\sigma_0 > 0$.

Note that by the doubling condition there exist two constants C_1 and C_2 such that $C_1 V(y, x) \le V(x, y) \le C_2 V(y, x)$. As in [12], we assume that for any Calderón–Zygmund operator T as in Definition 1.1 with $\beta(t) \to 0$ as $t \to 0$, the following "non-degenerate" condition holds: there exist positive constants c_0 and $\bar{A}$ such that for every $x \in X$ and $r > 0$, there exists $y \in B(x, \bar{A}r) \setminus B(x, r)$, satisfying

$$|K(x, y)| \ge \frac{1}{c_0 \mu(B(x, r))}. \tag{1.6}$$

This condition gives a lower bound on the kernel and in $\mathbb{R}^n$. This "non-degenerate" condition was first introduced in [22]. This is a natural assumption on the kernel of the singular integrals, since it is obviously true for Hilbert transform and Riesz transforms in the Euclidean setting, and for the Beurling-Ahlfors transformation in the complex setting. Beyond these, we note that, for example, on stratified Lie groups, a similar condition of the Riesz transform kernel lower bound was shown to be true in [13].

Let T be a Calderón–Zygmund operator on X. Suppose that $b \in L^1_{\mathrm{loc}}(X)$ and $f \in L^p(X)$. Let $[b, T]$ be the commutator defined by

$$[b, T]f(x) := b(x)T(f)(x) - T(bf)(x).$$

Let $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $\omega \in A_p(X)$. The *weighted Morrey space* $L^{p,\kappa}_\omega(X)$ is defined by

$$L^{p,\kappa}_\omega(X) := \{f \in L^p_{loc}(X) : \|f\|_{L^{p,\kappa}_\omega(X)} < \infty\},$$

where

$$\|f\|_{L^{p,\kappa}_\omega(X)} := \sup_B \left\{ \frac{1}{\omega(B)^\kappa} \int_B |f(x)|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}}.$$

Our main results are the following:

Theorem 1.2. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$ and $\omega \in A_p(X)$. Suppose $b \in L^1_{\mathrm{loc}}(X)$ and that T is a Calderón–Zygmund operator as in Definition 1.1 . Then the commutator $[b,T]$ has the following boundedness characterization:*

(i) *If $b \in BMO(X)$, then $[b,T]$ is bounded on $L^{p,\kappa}_\omega(X)$.*
(ii) *Suppose T also satisfies the non-degenerate condition* (1.6). *If b is real valued and $[b,T]$ is bounded on $L^{p,\kappa}_\omega(X)$, then $b \in BMO(X)$.*

Theorem 1.3. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$ and $\omega \in A_p(X)$. Suppose $b \in L^1_{\mathrm{loc}}(X)$ and that T is a Calderón–Zygmund operator as in Definition 1.1. Then the commutator $[b,T]$ has the following compactness characterization:*

(i) *If $b \in VMO(X)$, then $[b,T]$ is compact on $L^{p,\kappa}_\omega(X)$.*
(ii) *Suppose T also satisfies the non-degenerate condition* (1.6). *If b is real valued and $[b,T]$ is compact on $L^{p,\kappa}_\omega(X)$, then $b \in VMO(X)$.*

We mainly combine the ideas in [12] and [35] to prove our main result. We also point out that to obtain the above theorem, we provide an equivalent characterisation of $VMO(X)$, which is stated in Lemma 2.4 below, and is of independent interest.

Throughout the paper, we denote by C and $\widetilde{C}$ *positive constants* which are independent of the main parameters, but they may vary from line to line. For every $p \in (1,\infty)$, we denote by p' the conjugate of p, i.e., $\frac{1}{p'} + \frac{1}{p} = 1$. If $f \le Cg$ or $f \ge Cg$, we then write $f \lesssim g$ or $f \gtrsim g$; and if $f \lesssim g \lesssim f$, we write $f \approx g$.

2 Preliminaries on Spaces of Homogeneous Type

Let (X,d,μ) be a space of homogeneous type as mentioned in Section 1. We now recall the definition of the BMO and VMO spaces.

Definition 2.1. *A function $b \in L^1_{\mathrm{loc}}(X)$ belongs to the BMO space $BMO(X)$ if*

$$\|b\|_{BMO(X)} := \sup_B M(b,B) := \sup_B \frac{1}{\mu(B)} \int_B \left|b(x) - b_B\right| \, d\mu(x) < \infty,$$

where the sup is taken over all quasi-metric balls $B \subset X$ and

$$b_B = \frac{1}{\mu(B)} \int_B b(y) d\mu(y).$$

The following John-Nirenberg inequalities on spaces of homogeneous type come from [29].

Lemma 2.2 ([29]). *If $f \in \mathrm{BMO}(X)$, then there exist positive constants C_1 and C_2 such that for every ball $B \subset X$ and every $\alpha > 0$, we have*

$$\mu(\{x \in B : |f(x) - f_B| > \alpha\}) \le C_1 \lambda(B) \exp\Big\{ - \frac{C_2}{\|f\|_{\mathrm{BMO}(X)}} \alpha \Big\}.$$

We recall the median value $\alpha_B(f)$ (see [4]): for any real valued function $f \in L^1_{\mathrm{loc}}(X)$ and $B \subset X$, $\alpha_B(f)$ is the real number such that

$$\inf_{c \in \mathbb{R}} \frac{1}{\mu(B)} \int_B |f(x) - c| d\mu(x) = \frac{1}{\mu(B)} \int_B |f(x) - \alpha_B(f)| d\mu(x).$$

Moreover, it is known that $\alpha_B(f)$ satisfies that

$$\mu(\{x \in B : f(x) > \alpha_B(f)\}) \le \frac{\mu(B)}{2} \tag{2.1}$$

and

$$\mu(\{x \in B : f(x) < \alpha_B(f)\}) \le \frac{\mu(B)}{2}. \tag{2.2}$$

And it is easy to see that for any ball $B \subset X$,

$$M(b, B) \approx \frac{1}{\mu(B)} \int_B \left| b(x) - \alpha_B(b) \right| d\mu(x), \tag{2.3}$$

where the implicit constants are independent of the function b and the ball B.

By $\mathrm{Lip}(\beta)$, $0 < \beta < \infty$, we denote the set of all functions $\phi(x)$ defined on X such that there exists a finite constant C satisfying

$$|\phi(x) - \phi(y)| \le C d(x, y)^\beta$$

for every x and y in X. $\|\phi\|_\beta$ will stand for the least constant C satisfying the condition above. By $\mathrm{Lip}_c(\beta)$, we denote the set of all $\mathrm{Lip}(\beta)$ functions with bounded support on X.

Definition 2.3. *We define* $\mathrm{VMO}(X)$ *as the closure of the* $\mathrm{Lip}_c(\beta)$ *functions X under the norm of the BMO space.*

We will make use of the following characterization of $\mathrm{VMO}(X)$ whose proof is given in the Appendix. An equivalent characterization exists for the Euclidean and the stratified Lie groups case; one can refer to [36] and [4].

Lemma 2.4. *Let* $f \in \mathrm{BMO}(X)$. *Then* $f \in \mathrm{VMO}(X)$ *if and only if f satisfies the following three conditions:*

(i) $\lim\limits_{a \to 0} \sup\limits_{r_B = a} M(f, B) = 0$;
(ii) $\lim\limits_{a \to \infty} \sup\limits_{r_B = a} M(f, B) = 0$;
(iii) $\lim\limits_{r \to \infty} \sup\limits_{B \subset X \setminus B(x_0, r)} M(f, B) = 0$,

where r_B *is the radius of the ball B and* x_0 *is a fixed point in X.*

To this end, we recall the definition of A_p weights.

Definition 2.5. *Let* $\omega(x)$ *be a nonnegative locally integrable function on X. For* $1 < p < \infty$, *we say* ω *is an* A_p weight, *written* $\omega \in A_p$, *if*

$$[\omega]_{A_p} := \sup_B \left(\fint_B \omega \right) \left(\fint_B \left(\frac{1}{\omega} \right)^{1/(p-1)} \right)^{p-1} < \infty.$$

Here the supremum is taken over all balls $B \subset X$. *The quantity* $[\omega]_{A_p}$ *is called the* A_p constant of ω. *For* $p = 1$, *we say* ω *is an* A_1 weight, *written* $\omega \in A_1$, *if* $M(\omega)(x) \le \omega(x)$ *for* μ*-almost every* $x \in X$, *and for* $p = \infty$, *let* $A_\infty := \cup_{1 \le p < \infty} A_p$ *and we have* $[\omega]_{A_\infty} := \sup_B \left(\fint_B \omega \right) \exp\left(\fint_B \log\left(\frac{1}{\omega} \right) \right) < \infty$.

Note that for $\omega \in A_p$ the measure $\omega(x) d\mu(x)$ is a doubling measure on X. To be more precise, we have that for all $\lambda > 1$ and all balls $B \subset X$,

$$\omega(\lambda B) \le \lambda^{np} [\omega]_{A_p} \omega(B), \tag{2.4}$$

where n is the upper dimension of the measure μ, as in (1.3). We also point out that for $\omega \in A_\infty$, there exist $\gamma > 0$ such that for every ball B,

$$\mu\left(\left\{ x \in B : \ \omega(x) \ge \gamma \fint_B \omega \right\} \right) \ge \frac{1}{2} \mu(B).$$

This implies that for every ball B and for all $\delta \in (0,1)$,

$$\fint_B \omega \le C \left(\fint_B \omega^\delta \right)^{1/\delta}; \tag{2.5}$$

see also [25].

By the definition of A_p weight and Hölder's inequality, we can easily obtain the following standard properties.

Lemma 2.6. *Let $\omega \in A_p(X)$,$p \ge 1$. Then there exists constants $\hat{C}_1, \hat{C}_2 > 0$ and $\sigma \in (0,1)$ such that*

$$\hat{C}_1 \left(\frac{\mu(E)}{\mu(B)} \right)^p \le \frac{\omega(E)}{\omega(B)} \le \hat{C}_2 \left(\frac{\mu(E)}{\mu(B)} \right)^\sigma$$

for any measurable subset E of a quasi-metric ball B.

According to [2, Theorem 5.5], we have the following result for BMO functions on X.

Lemma 2.7. *Let $0 < p < \infty$, $v \in A_\infty(X)$ and $f \in \mathrm{BMO}(X)$. Then*

$$\|f\|_{\mathrm{BMO}(X)} \approx \sup_{B \subset X} \left\{ \frac{1}{v(B)} \int_B |f(x) - f_{B,v}|^p v(x) d\mu(x) \right\}^{\frac{1}{p}},$$

where $f_{B,v} = \frac{1}{v(B)} \int_B f(y) v(y) d\mu(y)$.

3 Characterization of Boundedness for Commutators

In this section, we give the proof of Theorem 1.2.

3.1 Proof of Theorem 1.2(i).

In order to prove Theorem 1.2(i), we need the following lemma.

Lemma 3.1 ([12]). *Let $b \in BMO(X)$ and T be a Calderón–Zygmund operator on (X, d, μ) a space of homogeneous type. If $1 < p < \infty$ and $\omega \in A_p(X)$, then $[b, T]$ is bounded on $L^p_\omega(X)$.*

Proof of Theorem 1.2(i). Let $1 < p < \infty$. It is sufficient to prove that

$$\left(\frac{1}{[\omega(B)]^\kappa} \int_B |[b,T](x)|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \lesssim \|b\|_{\mathrm{BMO}(X)} \|f\|_{L^{p,\kappa}_\omega(X)},$$

holds for any ball B. Now fix a ball $B = B(x_0, r)$ and decompose $f = f\chi_{2A_0B} + f\chi_{X\setminus 2A_0B} =: f_1 + f_2$. Without loss of generality we assume thorough out the proof that constant $A_0 := 1$ for the quasi-triangle inequality. Then

$$\frac{1}{\omega(B)^\kappa} \int_B |[b,T]f(x)|^p \omega(x) d\mu(x) \lesssim \frac{1}{\omega(B)^\kappa} \int_B |[b,T]f_1(x)|^p \omega(x) d\mu(x) + \frac{1}{\omega(B)^\kappa} \int_B |[b,T]f_2(x)|^p \omega(x) d\mu(x) =: I + II.$$

For the first term I, by Lemma 3.1 one has

$$\frac{1}{\omega(B)^\kappa} \int_B |[b,T]f_1(x)|^p \omega(x) d\mu(x) \le \frac{1}{\omega(B)^\kappa} \int_X |[b,T]f_1(x)|^p \omega(x) d\mu(x)$$

$$\lesssim \|b\|_{\mathrm{BMO}(X)}^p \frac{1}{\omega(B)^\kappa} \int_{2B} |f(x)|^p \omega(x) d\mu(x)$$
$$\lesssim \|b\|_{\mathrm{BMO}(X)}^p \|f\|_{L^{p,\kappa}_\omega(X)}^p.$$

We thus obtain

$$\|[b, T]f_1\|_{L^{p,\kappa}_\omega(X)} \lesssim \|b\|_{\mathrm{BMO}(X)} \|f\|_{L^{p,\kappa}_\omega(X)}.$$

Now for the second term II, observe that for $x \in B$, by (1.4), we have

$$\begin{aligned}
|[b, T]f_2(x)|^p &\le \left(\int_X |b(x) - b(y)||K(x, y)||f_2(y)| d\mu(y) \right)^p \\
&\lesssim \left(\int_{X\setminus 2B} \frac{|b(x) - b(y)|}{V(x, y)} |f(y)| d\mu(y) \right)^p \\
&\lesssim \left(\int_{X\setminus 2B} \frac{|f(y)|}{V(x_0, y)} (|b(x) - b_{B,\omega}| + |b_{B,\omega} - b(y)|) d\mu(y) \right)^p \\
&\lesssim \left(\int_{X\setminus 2B} \frac{|f(y)|}{V(x_0, y)} d\mu(y) \right)^p |b(x) - b_{B,\omega}|^p + \left(\int_{X\setminus 2B} \frac{|f(y)|}{V(x_0, y)} |b_{B,\omega} - b(y)| d\mu(y) \right)^p,
\end{aligned}$$

where $b_{B,\omega} = \frac{1}{\omega(B)} \int_B b(y)\omega(y) d\mu(y)$. Hence we get that

$$\begin{aligned}
\frac{1}{\omega(B)^\kappa} \int_B |[b, T]f_2(x)|^p \omega(x) d\mu(x) &\lesssim \frac{1}{\omega(B)^\kappa} \left(\int_{X\setminus 2B} \frac{|f(y)|}{V(x_0, y)} d\mu(y) \right)^p \int_B |b(x) - b_{B,\omega}|^p \omega(x) d\mu(x) \\
&\quad + \left(\int_{X\setminus 2B} \frac{|f(y)|}{V(x_0, y)} |b_{B,\omega} - b(y)| d\mu(y) \right)^p \omega(B)^{1-\kappa} \\
&=: III + IV.
\end{aligned}$$

Next, to estimate III and IV, we need to decompose $X \setminus 2B$ into suitable annuli. By Noting that $2^k B \to X$, as $k \to \infty$, we see that

$$\lim_{k\to\infty} \mu(2^k B) = \infty. \tag{3.1}$$

Then we choose a smallest $j_1 \ge 1$ such that,

$$\mu(2^{j_1} B) \ge 2\mu(B). \tag{3.2}$$

We claim that such j_1 exist, since otherwise, for all $j_1 \ge 1$, we have $\mu(2^{j_1} B) \le 2\mu(B)$. Then it contradicts (3.1). We further point out that, since j_1 is the smallest that satisfies the criteria (3.3), we get that

$$\mu(2^{j_1-1} B) < 2\mu(B).$$

Then, from the doubling property, we also have

$$\mu(2^{j_1} B) \le C_\mu \mu(2^{j_1-1} B) \le 2C_\mu \mu(B). \tag{3.3}$$

Next we choose a smallest $j_2 \ge j_1 + 1$ such that

$$\mu(2^{j_2} B) \ge 2\mu(2^{j_1} B).$$

and that

$$\mu(2^{j_2}B) \le 2C_\mu\mu(2^{j_1}B).$$

Similarly we see that such j_2 exists. By induction, there exists a sequence $\{j_k\}_{k=1}^{\infty}$ such that

$$2C_\mu\mu(2^{j_k}B) \ge \mu(2^{j_{k+1}}B) \ge 2\mu(2^{j_k}B), \qquad k \ge 1. \tag{3.4}$$

For III, by using Hölder's inequality, and using Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned}
III &\lesssim \frac{1}{\omega(B)^\kappa}\left(\sum_{k=0}^{\infty}\int\limits_{2^{j_{k+1}}B\setminus 2^{j_k}B}\frac{|f(y)|}{V(x_0,y)}d\mu(y)\right)^p\int\limits_B|b(x)-b_{B,\omega}|^p\omega(x)d\mu(x)\\
&\lesssim \|f\|^p_{L^{p,\kappa}_\omega(X)}\frac{1}{\omega(B)^\kappa}\left(\sum_{k=0}^{\infty}\frac{1}{\omega(2^{j_{k+1}}B)^{\frac{1-\kappa}{p}}}\right)^p\int\limits_B|b(x)-b_{B,\omega}|^p\omega(x)d\mu(x)\\
&\lesssim \|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}\left(\sum_{k=0}^{\infty}\left(\frac{\omega(B)}{\omega(2^{j_{k+1}}B)}\right)^{\frac{1-\kappa}{p}}\right)^p\\
&\lesssim \|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}\left(\sum_{k=0}^{\infty}2^{-k\sigma\frac{1-\kappa}{p}}\right)^p\\
&\lesssim \|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}.
\end{aligned}$$

For the term IV, using Hölder's inequality and the decomposition for $X\setminus B$ as above, we get

$$\begin{aligned}
IV &\lesssim \left(\sum_{k=0}^{\infty}\frac{1}{\mu(2^{j_k}B)}\int\limits_{2^{j_{k+1}}B}|f(y)||b_{B,\omega}-b(y)|d\mu(y)\right)^p\omega(B)^{1-\kappa}\\
&\lesssim \Bigg(\sum_{k=0}^{\infty}\frac{1}{\mu(2^{j_k}B)}\left(\int\limits_{2^{j_{k+1}}B}|f(y)|^p\omega(y)d\mu(y)\right)^{\frac{1}{p}}\\
&\quad\times\left(\int\limits_{2^{j_{k+1}}B}|b_{B,\omega}-b(y)|^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}\Bigg)^p\omega(B)^{1-\kappa}\\
&\lesssim \|f\|^p_{L^{p,\kappa}_\omega(X)}\left\{\sum_{k=0}^{\infty}\frac{\omega(2^{j_{k+1}}B)^{\frac{\kappa}{p}}}{\mu(2^{j_k}B)}\left(\int\limits_{2^{j_{k+1}}B}|b_{B,\omega}-b(y)|^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}\right\}^p\omega(B)^{1-\kappa}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\left(\int\limits_{2^{j_{k+1}}B}|b_{B,\omega}-b(y)|^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}\\
&\le\left(\int\limits_{2^{j_{k+1}}B}\left(|b(y)-b_{2^{j_{k+1}}B,\omega^{1-p'}}|+|b_{2^{j_{k+1}}B,\omega^{1-p'}}-b_{B,\omega}|\right)^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}\\
&\le\left(\int\limits_{2^{j_{k+1}}B}\left(|b(y)-b_{2^{j_{k+1}}B,\omega^{1-p'}}|\right)^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}
\end{aligned}$$

$$+\left(\int_{2^{j_{k+1}}B}\left(\left|b_{2^{j_{k+1}}B,\omega^{1-p'}}-b_{B,\omega}\right|\right)^{p'}\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}$$
$$=: V+VI.$$

We have $\omega^{1-p'}\in A_{p'}(X)$ since $\omega\in A_p(X)$. So we obtain

$$V\lesssim \|b\|_{\mathrm{BMO(X)}}\omega^{1-p'}(2^{j_{k+1}}B)^{\frac{1}{p'}}.$$

For VI, we have

$$\begin{aligned}\left|b_{2^{j_{k+1}}B,\omega^{1-p'}}-b_{B,\omega}\right|&\le\left|b_{2^{j_{k+1}}B,\omega^{1-p'}}-b_{2^{j_{k+1}}B}\right|+\left|b_{2^{j_{k+1}}B}-b_B\right|+\left|b_B-b_{B,\omega}\right|\\&\lesssim\frac{1}{\omega^{1-p'}(2^{j_{k+1}}B)}\int_{2^{j_{k+1}}B}\left|b(y)-b_{2^{j_{k+1}}B}\right|\omega(y)^{1-p'}d\mu(y)\\&\quad+(k+1)\|b\|_{\mathrm{BMO(X)}}+\frac{1}{\omega(B)}\int_B\left|b(y)-b_B\right|\omega(y)d\mu(y).\end{aligned}$$

Since $b\in\mathrm{BMO}(X)$, by Lemma 2.2, there exist some constants $C_1>0$ and $C_2>0$ such that for any ball B and $\alpha>0$

$$\mu(\{x\in B:|b(x)-b_B|>\alpha\})\le C_1\mu(B)e^{-\frac{C_2\alpha}{\|b\|_{\mathrm{BMO(X)}}}}.$$

Then by Lemma 2.6, we have

$$\omega(\{x\in B:\left|b(x)-b_B\right|>\alpha\})\le C_1\omega(B)e^{-\frac{C_2\alpha\sigma}{\|b\|_{\mathrm{BMO(X)}}}}$$

for some $\sigma\in(0,1)$. Hence we have

$$\begin{aligned}\int_B\left|b(y)-b_B\right|\omega(y)d\mu(y)&=\int_0^\infty\omega(\{y\in B:\left|b(y)-b_B\right|>\alpha\})d\alpha\\&\lesssim\omega(B)\int_0^\infty e^{-\frac{C_2\alpha\sigma}{\|b\|_{\mathrm{BMO(X)}}}}d\alpha\\&\lesssim\omega(B)\|b\|_{\mathrm{BMO(X)}}.\end{aligned}$$

Similarly, we have

$$\left(\int_{2^{j_{k+1}}B}\left|b(y)-b_{2^{j_{k+1}}B}\right|\omega(y)^{1-p'}d\mu(y)\right)^{\frac{1}{p'}}\lesssim(k+1)\|b\|_{\mathrm{BMO(X)}}\omega^{1-p'}(2^{j_{k+1}}B)^{1/p'}.$$

Together with Lemma 2.6, we have

$$\begin{aligned}IV&\lesssim\|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}\left[\sum_{k=0}^\infty\frac{\omega(2^{j_{k+1}}B)^{\frac{\kappa}{p}}}{\mu(2^{j_k}B)}(k+1)\omega^{1-p'}(2^{j_{k+1}}B)^{1/p'}\right]^p\omega(B)^{1-\kappa}\\&\lesssim\|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}\left[\sum_{k=0}^\infty\frac{(k+1)\omega(B)^{\frac{1-\kappa}{p}}}{\omega(2^{j_{k+1}}B)^{\frac{1-\kappa}{p}}}\right]^p\\&\lesssim\|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}\left[\sum_{k=0}^\infty(k+1)2^{-\frac{(k+1)(1-\kappa)\sigma}{p}}\right]^p\\&\lesssim\|f\|^p_{L^{p,\kappa}_\omega(X)}\|b\|^p_{\mathrm{BMO(X)}}.\end{aligned}$$

Therefore we have

$$\left\|[b,T]f_2\right\|_{L^{p,\kappa}_\omega(X)}\lesssim\|f\|_{L^{p,\kappa}_\omega(X)}\|b\|_{\mathrm{BMO(X)}}.$$

This completes the proof. □

3.2 Proof of Theorem 1.2(ii).

We first recall another version of the homogeneous condition (formulated in [12]): there exist positive constants $3 \le A_1 \le A_2$ such that for any ball $B := B(x_0, r) \subset X$, there exist balls $\widetilde{B} := B(y_0, r)$ such that $A_1 r \le d(x_0, y_0) \le A_2 r$, and for all $(x, y) \in (B \times \widetilde{B})$, $K(x, y)$ does not change sign and

$$|K(x, y)| \gtrsim \frac{1}{\mu(B)}. \tag{3.5}$$

If the kernel $K(x, y) := K_1(x, y) + iK_2(x, y)$ is complex-valued, where $i^2 = -1$, then at least one of K_i satisfies (3.5).

Then we first point out that the homogeneous condition (1.6) implies (3.5).

Lemma 3.2 ([12]). *Let T be the Calderón–Zygmund operator as in Definition 1.1 and satisfy the homogeneous condition as in* (1.6). *Then T satisfies* (3.5).

Proof of Theorem 1.2(ii). To prove $b \in BMO(X)$, it suffices to show for any ball $B \subset X$, we have $M(b, B) \lesssim 1$. Let $B = B(x_0, r)$ be a quasi metric ball in X. Let $\widetilde{B} := B(y_0, r) \subset X$ be the measurable set in (3.5). Following [12], we take

$$E_1 := \{x \in B : b(x) \ge \alpha_{\tilde{B}}(b)\} \quad E_2 := \{x \in B : b(x) < \alpha_{\tilde{B}}(b)\};$$
$$F_1 \subset \{y \in \tilde{B} : b(y) \le \alpha_{\tilde{B}}(b)\} \quad F_2 \subset \{y \in \tilde{B} : b(y) \ge \alpha_{\tilde{B}}(b)\},$$

with $\alpha_{\tilde{B}}(b)$ the median value of b over $\tilde{B}$, such that $\mu(F_1) = \mu(F_2) = \frac{1}{2}\mu(\tilde{B})$ and $F_1 \cap F_2 = \emptyset$. For any $(x, y) \in E_j \times F_j$, $j \in \{1, 2\}$, we have

$$|b(x) - b(y)| = |b(x) - \alpha_{\tilde{B}}(b)| + |\alpha_{\tilde{B}}(b) - b(y)| \ge |b(x) - \alpha_{\tilde{B}}(b)|.$$

Since b is real valued, using Lemma 2.6, Hölder's inequality, boundedness of $[b, T]$ on $L^{p,\kappa}_\omega(X)$ and (3.5), we get

$$\begin{aligned}
M(b, B) &\lesssim \frac{1}{\mu(B)} \int_B \left|b(x) - \alpha_{\tilde{B}}(b)\right| d\mu(x) \approx \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} |b(x) - \alpha_{\tilde{B}}(b)| d\mu(x) \\
&\lesssim \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \int_{F_j} \frac{\left|b(x) - \alpha_{\tilde{B}}(b)\right|}{\mu(B)} d\mu(y) d\mu(x) \\
&\approx \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \int_{F_j} \frac{\left|b(x) - \alpha_{\tilde{B}}(b)\right|}{V(x, y)} d\mu(y) d\mu(x) \\
&\lesssim \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \int_{F_j} \left|b(x) - b(y)\right| \frac{1}{V(x, y)} d\mu(y) d\mu(x) \\
&\lesssim \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \left| \int_{F_j} \left|b(x) - b(y)\right| K(x, y) d\mu(y) \right| d\mu(x) \\
&\lesssim \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \left|[b, T]\chi_{F_j}(x)\right| d\mu(x) \\
&\lesssim \sum_{j=1}^{2} \frac{1}{\mu(B)} \int_{E_j} \left\|[b, T]\chi_{F_j}\right\|_{L^{p,\kappa}_\omega(X)} [\omega(B)]^{\frac{\kappa-1}{p}} \mu(B) \\
&\lesssim \sum_{j=1}^{2} \left\|[b, T]\right\|_{L^{p,\kappa}_\omega(X) \to L^{p,\kappa}_\omega(X)} \left\|\chi_{F_j}\right\|_{L^{p,\kappa}_\omega(X)} [\omega(B)]^{\frac{\kappa-1}{p}}
\end{aligned}$$

$$\lesssim \|[b, T]\|_{L^{p,\kappa}_{\omega}(X)\to L^{p,\kappa}_{\omega}(X)} [\omega(\tilde{B})]^{\frac{1-\kappa}{p}} [\omega(B)]^{\frac{\kappa-1}{p}}$$
$$\lesssim \|[b, T]\|_{L^{p,\kappa}_{\omega}(X)\to L^{p,\kappa}_{\omega}(X)}.$$

This finishes the proof of Theorem 1.2(ii). □

4 Compactness Characterization of the Commutator

Now we will prove Theorem 1.3.

4.1 Proof of Theorem 1.3(i).

We will first give a sufficient condition for subsets of weighted Morrey spaces to be relatively compact. Recall that a subset $\mathcal{F}$ of $L^{p,\kappa}_{\omega}(X)$ is said to be totally bounded (relatively compact) if the $L^{p,\kappa}_{\omega}(X)$ closure of $\mathcal{F}$ is compact.

Lemma 4.1. *For any $p \in (1, \infty)$, $\kappa \in (0, 1)$ and $\omega \in A_p(X)$, a subset $\mathcal{F}$ of $L^{p,\kappa}_{\omega}(X)$ is totally bounded if the set $\mathcal{F}$ satisfies the following three conditions:*

(i) $\mathcal{F}$ is bounded, namely,

$$\sup_{f\in\mathcal{F}} \|f\|_{L^{p,\kappa}_{\omega}(X)} < \infty;$$

(ii) $\mathcal{F}$ uniformly vanishes at infinity, namely, for any $\epsilon \in (0, \infty)$, there exists some positive constant M such that, for any $f \in \mathcal{F}$,

$$\|f\chi_{\{x\in X: d(x_0,x)>M\}}\|_{L^{p,\kappa}_{\omega}(X)} < \epsilon,$$

where x_0 is a fixed point in X;

(iii) $\mathcal{F}$ is uniformly equicontinuous, namely,

$$\lim_{r\to 0} \|f(x) - f_{B(x,r)}\|_{L^{p,\kappa}_{\omega}(X)} = 0$$

uniformly for $f \in \mathcal{F}$.

The proof of this lemma follows from [32] using a minor modification from Euclidean setting to space of homogeneous type, since it only requires following properties of underlying space: metric and doubling measure.

We will now establish the boundedness of maximal operator $T_\star$ of a family of smooth truncated operators $\{T_\eta\}_{\eta\in(0,\infty)}$ as follows. For $\eta \in (0, \infty)$, let

$$T_\eta f(x) := \int_X K_\eta(x, y) f(y) d\mu(y),$$

where the kernel $K_\eta := K(x, y)\varphi(\frac{d(x,y)}{\eta})$ with $\varphi \in C^\infty(\mathbf{R})$ satisfying that

$$\varphi(t) = \begin{cases} \varphi(t) \equiv 0, & \text{if } t \in \left(-\infty, \frac{1}{2}\right) \\ \varphi(t) \in [0, 1], & \text{if } t \in \left[\frac{1}{2}, 1\right] \\ \varphi(t) \equiv 1, & \text{if } t \in (1, \infty). \end{cases}$$

Let

$$[b, T_\eta] f(x) := \int_X [b(x) - b(y)] K_\eta(x, y) f(y) d\mu(y).$$

The maximal operator $T_\star$ is defined as

$$T_\star f(x) := \sup_{\eta\in(0,\infty)} \left| \int_X K_\eta(x,y) f(y) d\mu(y) \right|.$$

Recall the *Hardy-Littlewood maximal Operator* $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y)$$

for any $f \in L^1_{\mathrm{loc}}(X)$ and $x \in X$, where the supremum is taken over all balls B of X that contain x.

Then we have the following lemmas.

Lemma 4.2. *There exists a positive constant C such that, for any $b \in Lip(\beta)$, $0 < \beta < \infty$, $f \in L^1_{\mathrm{loc}}(X)$ and $x \in X$*

$$\big|[b, T_\eta]f(x) - [b, T]f(x)\big| \le C\eta^\beta \mathcal{M}f(x).$$

Proof. Let $f \in L^1_{\mathrm{loc}}(X)$. For any $x \in X$, we have

$$\begin{aligned}
&|[b, T_\eta]f(x) - [b, T]f(x)| \\
&= \left| \int_{\eta/2<d(x,y)\le\eta} [b(x) - b(y)] K_\eta(x,y) f(y) d\mu(y) - \int_{d(x,y)\le\eta} [b(x) - b(y)] K(x,y) f(y) d\mu(y) \right| \\
&\lesssim \int_{d(x,y)\le\eta} |b(x) - b(y)||K(x,y)||f(y)| d\mu(y).
\end{aligned}$$

From $b \in \mathrm{Lip}(\beta)$ and (1.4), we have

$$\begin{aligned}
&\int_{d(x,y)\le\eta} |b(x) - b(y)||K(x,y)||f(y)| d\mu(y) \\
&\lesssim C \sum_{j=0}^{\infty} \int_{\eta 2^{-(j+1)}<d(x,y)\le\eta 2^{-j}} \frac{d(x,y)^\beta}{V(x,y)} |f(y)| d\mu(y) \\
&\lesssim C\eta^\beta \mathcal{M}f(x),
\end{aligned}$$

which completes the proof of the Lemma 4.2. □

Lemma 4.3. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$ and $\omega \in A_p(X)$. Then there exists a positive constant C such that, for any $f \in L^{p,\kappa}_\omega(X)$,*

$$\|T_\star\|_{L^{p,\kappa}_\omega(X)} + \|\mathcal{M}f\|_{L^{p,\kappa}_\omega(X)} \le C\|f\|_{L^{p,\kappa}_\omega(X)}.$$

Proof. For the boundedness of $\mathcal{M}$ on $L^{p,\kappa}_\omega(X)$ one can refer to [2]. We only consider the boundedness of $T_\star$. For any fixed ball $B \subset X$ and $f \in L^{p,\kappa}_\omega(X)$, we write

$$f := f_1 + f_2 := f\chi_{2B} + f\chi_{X\setminus 2B}.$$

Again, following the argument in (3.4), there exist $j_k \in \mathbb{N}$ such that

$$2C_\mu \mu(2^{j_k}B) \ge \mu(2^{j_{k+1}}B) \ge 2\mu(2^{j_k}B), \qquad \forall k \ge 1.$$

Observe $f_1 \in L^p_\omega(X)$. Then, from the boundedness of $T_\star$ on $L^p_\omega(X)$ (see, for example, [23, Theorem 1.1]), the Hölder inequality, size and smoothness of Kernel, we deduce that

$$\left[\int_B |T_\star f(x)|^p \omega(x) d\mu(x)\right]^{\frac{1}{p}} \lesssim \left[\int_B |T_\star f_1(x)|^p \omega(x) d\mu(x)\right]^{\frac{1}{p}} + \sum_{k=0}^{\infty} \left\{ \int_B \left[\int_{2^{j_{k+1}}B\setminus 2^{j_k}B} \frac{|f(y)|}{V(x,y)} d\mu(y) \right]^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}}$$

$$\begin{aligned}
&\lesssim \left[\int_{2B} |f(x)|^p \omega(x)\,d\mu(x)\right]^{\frac{1}{p}} + \sum_{k=0}^{\infty}\left[\frac{\omega(B)}{\mu(2^{j_k}B)^p}\left\{\int_{2^{j_{k+1}}B} |f(y)|[\omega(y)]^{\frac{1}{p}}[\omega(y)]^{-\frac{1}{p}}\,d\mu(y)\right\}^p\right]^{\frac{1}{p}}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}[\omega(B)]^{\frac{\kappa}{p}} + \sum_{k=0}^{\infty}\left\{\omega(B)\left[\omega\left(2^{j_k}B\right)\right]^{\kappa-1}\|f\|^p_{L^{p,\kappa}_{\omega}(X)}\right\}^{\frac{1}{p}}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}\omega(B)^{\frac{\kappa}{p}} + \sum_{k=0}^{\infty}\left\{[\omega(B)]^{\kappa}2^{-k\sigma(1-\kappa)}\|f\|^p_{L^{p,\kappa}_{\omega}(X)}\right\}^{\frac{1}{p}}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}[\omega(B)]^{\frac{\kappa}{p}},
\end{aligned}$$

where, in the fourth inequality, we used Lemma 2.6 with some $\sigma \in (0,1)$. This finishes the proof of Lemma 4.3. □

Proof of Theorem 1.3(*i*) . When $b \in \mathrm{VMO}(X)$, for any $\varepsilon \in (0,\infty)$, there exists $b^{(\varepsilon)} \in \mathrm{Lip}_c(\beta)$, $0<\beta<\infty$ such that $\left\|b-b^{(\varepsilon)}\right\|_{\mathrm{BMO}(X)} < \varepsilon$. Then, from the boundedness of the commutator $[b,T]$ on $L^{p,\kappa}_{\omega}(X)$, we obtain

$$\begin{aligned}
\left\|[b,T]f-[b^{(\varepsilon)},T]f\right\|_{L^{p,\kappa}_{\omega}(X)} &= \left\|[b-b^{(\varepsilon)},T]f\right\|_{L^{p,\kappa}_{\omega}(X)}\\
&\lesssim \left\|b-b^{(\varepsilon)}\right\|_{\mathrm{BMO}(X)}\|f\|_{L^{p,\kappa}_{\omega}(X)}\\
&\lesssim \varepsilon\|f\|_{L^{p,\kappa}_{\omega}(X)}.
\end{aligned}$$

Moreover, by using Lemmas 4.2 and 4.3, we get

$$\lim_{\eta\to 0}\left\|[b,T_\eta]-[b,T]\right\|_{L^{p,\kappa}_{\omega}(X)\to L^{p,\kappa}_{\omega}(X)} = 0.$$

Now it suffices to show that, for any $b \in \mathrm{Lip}_c(\beta)$, $0<\beta<\infty$ and $\eta \in (0,\infty)$ small enough, $[b,T_\eta]$ is a compact operator on $L^{p,\kappa}_{\omega}(X)$, which is equivalent to show that, for any bounded subset $\mathcal{F} \subset L^{p,\kappa}_{\omega}(X)$, $[b,T_\eta]\mathcal{F}$ is relatively compact. That is, we need to verify $[b,T_\eta]$ satisfies the conditions (i) through (iii) of Lemma 4.1.

Observe by [30, Theorem 3.4] and the fact that $b \in \mathrm{BMO}(X)$, we know that $[b,T_\eta]$ is bounded on $L^{p,\kappa}_{\omega}(X)$ for the given $p \in (1,\infty)$, $\kappa \in (0,1)$ and $\omega \in A_p(X)$, which implies that $[b,T_\eta]\mathcal{F}$ satisfies condition (i) of Lemma 4.1.

Next, let x_0 be a fixed point in X. Since $b \in \mathrm{Lip}_c(\beta)$, we may further assume $\|b\|_{L^\infty} = 1$. Observe that there exists a positive constant R_0 such that $\mathrm{supp}\,(b) \subset B(x_0,R_0)$. Let $M \in (10R_0,\infty)$. Thus, for any $y \in B(x_0,R_0)$ and $x \in X$ with $d(x_0,x) > M$, $d(x,y) \sim d(x_0,x)$. Then, for $x \in X$ with $d(x_0,x) > M$, by Hölder inequality and using that $V(x,y) \sim \mu(B(x_0,d(x,x_0)))$ we conclude that

$$\begin{aligned}
\left|[b,T_\eta]f(x)\right| &\le \int_X |b(x)-b(y)|\left|K_\eta(x,y)\right||f(y)|\,d\mu(y)\\
&\le \int_X |b(y)|\left|K(x,y)\right||f(y)|\,d\mu(y)\\
&\lesssim \int_{B(x_0,R_0)}\frac{|f(y)|}{V(x,y)}\,d\mu(y)\\
&\lesssim \int_{B(x_0,R_0)}\frac{|f(y)|}{\mu(B(x_0,d(x,x_0)))}\,d\mu(y)\\
&\lesssim \frac{1}{\mu(B(x_0,d(x,x_0)))}\left[\int_{B(x_0,R_0)}|f(y)|^p\omega(y)\,d\mu(y)\right]^{\frac{1}{p}}\left\{\int_{B(x_0,R_0)}[\omega(y)]^{-\frac{p'}{p}}\,d\mu(y)\right\}^{\frac{1}{p'}}\\
&\lesssim \frac{\mu(B(x_0,R_0))}{\mu(B(x_0,d(x,x_0)))}\left[\omega\left(B(x_0,R_0)\right)\right]^{\frac{\kappa-1}{p}}\|f\|_{L^{p,\kappa}_{\omega}(X)}.
\end{aligned}$$

From $\lim_{k\to\infty}\mu(B(x_0,kM))=\infty$, we have that there exist $j_k\in\mathbb{N}$ such that

$$\mu(B(x_0,2^{j_1}M))\geq 2\mu(B(x_0,M)) \text{ and } \mu(B(x_0,2^{j_{k+1}}M))\geq 2\mu(B(x_0,2^{j_k}M)).$$

Therefore, for any fixed ball $B:=B(\widetilde{x},\widetilde{r})\subset X$, by Lemma 2.6 , we have

$$\begin{aligned}
&\frac{1}{[\omega(B)]^{\kappa}}\int\limits_{B\cap\{x\in X:d(x_0,x)>M\}}|[b,T_\eta]f(x)|^p\omega(x)d\mu(x)\\
&\lesssim \mu(B(x_0,R_0))^p\frac{[\omega(B(x_0,R_0))]^{(\kappa-1)}}{[\omega(B)]^{\kappa}}\|f\|_{L^{p,\kappa}_{\omega}(X)}\sum_{k=0}^{\infty}\frac{\omega\left(B\cap\left\{x\in X:2^{j_k}M<d(x_0,x)\leq 2^{j_{k+1}}M\right\}\right)}{\mu(B(x_0,2^{j_k}M))^p}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}\sum_{k=0}^{\infty}\frac{\omega\left(B(x_0,2^{j_{k+1}}M)\right)^{1-\kappa}}{\omega(B(x_0,R_0))^{1-\kappa}}\frac{\mu(B(x_0,R_0))^p}{\mu(B(x_0,2^{j_k}M))^p}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}\sum_{k=0}^{\infty}\frac{\mu(B(x_0,R_0))^{p\kappa}}{\mu(B(x_0,2^{j_k}M))^{p\kappa}}\\
&\lesssim \|f\|_{L^{p,\kappa}_{\omega}(X)}\sum_{k=0}^{\infty}2^{-k}\frac{\mu(B(x_0,R_0))^{p\kappa}}{\mu(B(x_0,M))^{p\kappa}}\\
&\lesssim \frac{\mu(B(x_0,R_0))^{p\kappa}}{\mu(B(x_0,M))^{p\kappa}}\|f\|^p_{L^{p,\kappa}_{\omega}(X)}.
\end{aligned}$$

Therefore the condition (ii) of Lemma 4.1 holds for $[b,T_\eta]\mathcal{F}$ with large M.

Now we will prove $[b,T_\eta]\mathcal{F}$ also satisfies (iii) of Lemma 4.1. Let η be a fixed positive constant small enough and $r<\frac{\eta}{8A_0^2}$. Then, for any $x\in X$, we have

$$[b,T_\eta]f(x)-\left([b,T_\eta]f\right)_{B(x,r)}=\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)}[b,T_\eta]f(x)-[b,T_\eta]f(y)d\mu(y).$$

Note that

$$\begin{aligned}
&[b,T_\eta]f(x)-[b,T_\eta]f(y)\\
&\quad=[b(x)-b(y)]\int\limits_X K_\eta(x,z)f(z)d\mu(z)+\int\limits_X\left[K_\eta(x,z)-K_\eta(y,z)\right][b(y)-b(z)]f(z)d\mu(z)\\
&\quad=:L_1(x,y)+L_2(x,y).
\end{aligned}$$

As $b\in\mathrm{Lip}_c(\beta)$, it follows that, for any $y\in B(x,r)$

$$\left|L_1(x,y)\right|=|b(x)-b(y)|\left|\int\limits_X K_\eta(x,z)f(z)d\mu(z)\right|\lesssim r^\beta T_\star(f)(x).$$

To estimate $L_2(x,y)$, we first observe that $K_\eta(x,z)=0$, $K_\eta(y,z)=0$ for any $y\in B(x,r)$, $d(x,z)\leq\frac{\eta}{4A_0}$ and $r<\frac{\eta}{8A_0^2}$. Moreover, by the definition of K_η we know that, for any $y\in B(x,r)$, $d(x,z)>\frac{\eta}{4A_0}$ and $r<\frac{\eta}{8A_0^2}$,

$$\left|K_\eta(x,z)-K_\eta(y,z)\right|\lesssim\frac{1}{V(x,z)}\frac{d(x,y)^{\sigma_0}}{d(x,z)^{\sigma_0}}.$$

This in turn implies that, for any $y\in B(x,r)$

$$\begin{aligned}
\left|L_2(x,y)\right|&\lesssim\int\limits_{d(x,z)>\frac{\eta}{4A_0}}\frac{|f(z)|}{V(x,z)}\frac{d(x,y)^{\sigma_0}}{d(x,z)^{\sigma_0}}d\mu(z)\\
&\lesssim\sum_{k=0}^{\infty}\frac{r^{\sigma_0}}{(2^k\eta)^{\sigma_0}}\frac{1}{\mu(B(x,\frac{2^k\eta}{4A_0}))}\int\limits_{\frac{2^k\eta}{4A_0}<d(x,z)\leq\frac{2^{k+1}\eta}{4A_0}}|f(z)|d\mu(z)\\
&\lesssim\frac{r^{\sigma_0}}{\eta^{\sigma_0}}\mathcal{M}f(x).
\end{aligned}$$

Using the estimates of $L_1(x,y)$ and $L_2(x,y)$, we have

$$\left|[b,T_\eta]f(x)-\left([b,T_\eta]f\right)_{B(x,r)}\right|\lesssim r^\beta T_\star(f)(x)+\frac{r^{\sigma_0}}{\eta^{\sigma_0}}\mathcal{M}f(x).$$

Then, by Lemma 4.3 and the boundedness of $\mathcal{M}$ on $L^{p,\kappa}_\omega(X)$, we obtain

$$\|[b,T_\eta]f(x)-\left([b,T_\eta]f\right)_{B(x,r)}\|_{L^{p,\kappa}_\omega}\lesssim (r^\beta+\frac{r^{\sigma_0}}{\eta^{\sigma_0}})\|f\|_{L^{p,\kappa}_\omega}.$$

Consequently, $[b,T_\eta]\mathcal{F}$ satisfies condition (iii) of Lemma 4.1. Thus, $[b,T_\eta]$ is a compact operator for any $b\in \mathrm{Lip}_c(\beta)$. This finishes the proof of Theorem 1.3(i). □

4.2 Proof of Theorem 1.3(ii).

Next, we establish a lemma for the upper and the lower bounds of integrals of $[b,T]f_j$ on certain balls B_j in X for any $j\in\mathbb{N}$.

Lemma 4.4. *Let $p\in(1,\infty)$, $\kappa\in(0,1)$ and $\omega\in A_p(X)$. Suppose that $b\in \mathrm{BMO}(X)$ is a real-valued function with $\|b\|_{\mathrm{BMO}(X)}=1$ and there exists $\gamma\in(0,\infty)$ and a sequence $\{B_j\}_{j\in\mathbb{N}}:=\{B(x_j,r_j)\}_{j\in\mathbb{N}}$ of balls in X, with $\{x_j\}_{j\in\mathbb{N}}\subset X$ and $\{r_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ such that, for any $j\in\mathbb{N}$*

$$M(b,B_j)>\gamma. \tag{4.1}$$

Then there exist real-valued functions $\{f_j\}_{j\in\mathbb{N}}\subset L^{p,\kappa}_\omega(X)$, positive constants K_0 large enough, $\widetilde{C}_0$, $\widetilde{C}_1$ and $\widetilde{C}_2$ such that, for any $j\in\mathbb{N}$ and integer $k\geq K_0$, $\|f_j\|_{L^{p,\kappa}_\omega(X)}\leq\tilde{C}_0$,

$$\int_{B_j^k}|[b,T]f_j(x)|^p\,\omega(x)d\mu(x)\geq\tilde{C}_1\frac{\gamma^p\mu(B_j)^p}{\mu(A_2^kB_j)^p}\left[\omega(B_j)\right]^{\kappa-1}\omega\left(A_2^kB_j\right), \tag{4.2}$$

where $B_j^k:=\widetilde{A_2^{k-1}B_j}$ is the ball associates with $A_2^{k-1}B_j$ in (3.5) and

$$\int_{A_2^{k+1}B_j\setminus A_2^kB_j}|[b,T]f_j(x)|^p\,\omega(x)d\mu(x)\leq\tilde{C}_2\frac{\mu(B_j)^p}{\mu(A_2^kB_j)^p}\left[\omega(B_j)\right]^{\kappa-1}\omega\left(A_2^kB_j\right). \tag{4.3}$$

Proof. For each $j\in\mathbb{N}$, we define function f_j as follows:

$$f_j^{(1)}:=\chi_{B_{j,1}}-\chi_{B_{j,2}}:=\chi_{\left\{x\in B_j;b(x)>\alpha_{B_j}(b)\right\}}-\chi_{\left\{x\in B_j;b(x)<\alpha_{B_j}(b)\right\}},\quad f_j^{(2)}:=a_j\chi_{B_j}$$

and

$$f_j:=\left[\omega(B_j)\right]^{\frac{\kappa-1}{p}}\left(f_j^{(1)}-f_j^{(2)}\right),$$

where B_j is as in the assumption of Lemma 4.4 and $a_j\in\mathbb{R}$ is a constant such that

$$\int_X f_j(x)d\mu(x)=0. \tag{4.4}$$

Then, by the definition of a_j, (2.1) and (2.2) we have $|a_j|\leq 1/2$, $\mathrm{supp}\,(f_j)\subset B_j$ and, for any $x\in B_j$,

$$f_j(x)\left(b(x)-\alpha_{B_j}(b)\right)\geq 0. \tag{4.5}$$

Moreover, since $|a_j|\leq 1/2$, we can obtain that, for any $x\in\left(B_{j,1}\cup B_{j,2}\right)$,

$$|f_j(x)|\sim\left[\omega(B_j)\right]^{\frac{\kappa-1}{p}} \tag{4.6}$$

and hence

$$\begin{aligned}\|f_j\|_{L_w^{p,\kappa}(X)} &\lesssim \sup_{B\subset X}\left\{\frac{\omega\left(B\cap B_j\right)}{[\omega(B)]^\kappa}\right\}^{\frac{1}{p}}\left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}}\\ &\lesssim \sup_{B\subset X}\left[\omega\left(B\cap B_j\right)\right]^{\frac{1-\kappa}{p}}\left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}}\lesssim 1.\end{aligned}$$

Observe that, for any $k\in\mathbb{N}$, we have

$$A_2^{k-1}B_j\subset(A_2+1)\,B_j^k\subset A_2^{k+1}B_j \tag{4.7}$$

hence we have

$$\omega\left(B_j^k\right)\sim\omega\left(A_2^kB_j\right) \tag{4.8}$$

Observe that

$$[b,T](f)=\left[b-\alpha_B(b)\right]T(f)-T([b-\alpha_{B_j}(b)]f). \tag{4.9}$$

Using Kernel estimates, (4.4), (4.6) and the fact that $d\left(x,x_j\right)\sim d(x,\xi)$ for any $x\in B_j^k$ with integer $k\geq 2$ and $\xi\in B_j$, we have, for any $x\in B_j^k$,

$$\begin{aligned}\left|\left[b(x)-\alpha_{B_j}(b)\right]T\left(f_j\right)(x)\right| &= \left|b(x)-\alpha_{B_j}(b)\right|\left|\int_{B_j}\left[K(x,\xi)-K\left(x,x_j\right)\right]f_j(\xi)d\mu(\xi)\right| \\ &\leq \left|b(x)-\alpha_{B_j}(b)\right|\int_{B_j}\left|K(x,\xi)-K\left(x,x_j\right)\right|\left|f_j(\xi)\right|d\mu(\xi)\\ &\lesssim \left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}}\left|b(x)-\alpha_{B_j}(b)\right|\int_{B_j}\frac{1}{V(x,x_j)}\left(\frac{d\left(\xi,x_j\right)}{d\left(x,x_j\right)}\right)^{\sigma_0}d\mu(\xi)\\ &\lesssim \frac{\left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}}}{A_2^{k\sigma_0}}\frac{\mu(B_j)}{\mu(A_2^kB_j)}\left|b(x)-\alpha_{B_j}(b)\right|.\end{aligned} \tag{4.10}$$

As $\|b\|_{\mathrm{BMO}(X)}=1$ by John-Nirenberg inequality(c.f.[6]), for each $k\in\mathbb{N}$ and ball $B\subset X$, we have

$$\begin{aligned}\int_{A_2^{k+1}B}\left|b(x)-\alpha_B(b)\right|^p d\mu(x) &\lesssim \int_{A_2^{k+1}B}\left|b(x)-\alpha_{A_2^{k+1}B}(b)\right|^p d\mu(x)+\mu(A_2^{k+1}B)\left|\alpha_{A_2^{k+1}B}(b)-\alpha_B(b)\right|^p\\ &\lesssim k^p\mu(A_2^kB),\end{aligned} \tag{4.11}$$

where the last inequality is due to the fact that

$$\left|\alpha_{A_2^{k+1}B}(b)-\alpha_B(b)\right|\lesssim\left|\alpha_{A_2^{k+1}B}(b)-b_{A_2^{k+1}B}\right|+\left|b_{A_2^{k+1}B}-b_B\right|+\left|b_B-\alpha_B(b)\right|\lesssim k.$$

Since $\omega\in A_p\left(X\right)$, there exists $\epsilon\in(0,\infty)$ such that the reverse Hölder inequality

$$\left[\frac{1}{\mu(B)}\int_B\omega(x)^{1+\epsilon}d\mu(x)\right]^{\frac{1}{1+\epsilon}}\lesssim\frac{1}{\mu(B)}\int_B\omega(x)d\mu(x)$$

holds for any ball $B\subset X$. Then by the Hölder inequality, (4.11), (4.7) and (4.10) we can deduce that there exists a positive constant $\widetilde{C}_3$ such that, for any $k\in\mathbb{N}$

$$\int_{B_j^k}\left|\left[b(x)-\alpha_{B_j}(b)\right]T\left(f_j\right)(x)\right|^p\omega(x)d\mu(x) \tag{4.12}$$

$$
\begin{aligned}
&\lesssim \frac{\left[\omega\left(B_j\right)\right]^{\kappa-1}}{A_2^{k\sigma_0 p}} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p} \int\limits_{A_2^{k+1}B_j} \left|b(x)-\alpha_{B_j}(b)\right|^p \omega(x)d\mu(x)\\
&\lesssim \frac{\left[\omega\left(B_j\right)\right]^{\kappa-1}}{A_2^{k\sigma_0 p}} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^{p-1}} \left[\frac{1}{\mu(A_2^{k+1}B_j)} \int\limits_{A_2^{k+1}B_j} \left|b(x)-\alpha_{B_j}(b)\right|^{p(1+\epsilon)'} d\mu(x)\right]^{\frac{1}{(1+\epsilon)'}}\\
&\times \left[\frac{1}{\mu(A_2^{k+1}B_j)} \int\limits_{A_2^{k+1}B_j} \omega(x)^{1+\epsilon} d\mu(x)\right]^{\frac{1}{1+\epsilon}}\\
&\le \widetilde{C}_3 \frac{k^p}{A_2^{k\sigma_0 p}} \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p} \left[\omega\left(B_j\right)\right]^{\kappa-1} \omega\left(A_2^k B_j\right).
\end{aligned}
$$

By Lemma 3.1, (4.5), (4.6), (2.3), (4.1) and (1.6) for any $x \in B_j^k$, we have

$$
\begin{aligned}
\left|T\left(\left[b-\alpha_{B_j}(b)\right]f_j\right)(x)\right| &= \left|\int\limits_{B_{j,1}\cup B_{j,2}} K(x,\xi)\left[b(\xi)-\alpha_{B_j}(b)\right]f_j(\xi)d\xi\right|\\
&\gtrsim \int\limits_{B_{j,1}\cup B_{j,2}} \frac{\left|\left[b(\xi)-\alpha_{B_j}(b)\right]f_j(\xi)\right|}{\mu(B(x,d(x,\xi)))} d\mu(\xi)\\
&\gtrsim \frac{1}{\mu(A_2^k B_j)}\left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}} \int\limits_{B_j} \left|b(\xi)-\alpha_{B_j}(b)\right| d\mu(\xi)\\
&\gtrsim \frac{\gamma\mu(B_j)}{\mu(A_2^k B_j)}\left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}}.
\end{aligned}
$$

Then together with (4.8) we obtain that there exists a positive constant $\widetilde{C}_4$ such that

$$
\begin{aligned}
\int\limits_{B_j^k} \left|T\left(\left[b-\alpha_{B_j}(b)\right]f_j\right)(x)\right|^p \omega(x)d\mu(x) &\ge \frac{\gamma^p\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(B_j^k\right)\\
&\ge \widetilde{C}_4\frac{\gamma^p\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right).
\end{aligned}
\tag{4.13}
$$

Now we take $K_0 \in (0,\infty)$ large enough such that, for any integer $k \ge K_0$

$$
\tilde{C}_4\frac{\gamma^p}{2^{p-1}} - \widetilde{C}_3\frac{k^p}{A_2^{k\sigma_0 p}} \ge \widetilde{C}_4\frac{\gamma^p}{2^p}.
$$

From this and (4.9), (4.12) and (4.13), we have

$$
\begin{aligned}
&\int\limits_{B_j^k} \left|[b,T]f_j(x)\right|^p \omega(x)d\mu(x)\\
&\quad\ge \frac{1}{2^{p-1}}\int\limits_{B_j^k}\left|T\left(\left[b-\alpha_{B_j}(b)\right]f_j\right)(x)\right|^p\omega(x)d\mu(x) - \int\limits_{B_j^k}\left|\left[b(x)-\alpha_{B_j}(b)\right]T\left(f_j\right)(x)\right|^p\omega(x)d\mu(x)\\
&\quad\ge \left(\tilde{C}_4\frac{\gamma^p}{2^{p-1}} - \widetilde{C}_3\frac{k^p}{A_2^{k\sigma_0 p}}\right)\frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right)\\
&\quad\ge \widetilde{C}_4\frac{\gamma^p}{2^p}\frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right).
\end{aligned}
$$

This implies (4.2).

On the other hand, since $\operatorname{supp}(f_j) \subset B_j$, by (4.6) and (2.3) and $\|b\|_{\mathrm{BMO}(X)} = 1$, we obtain that, for any $x \in A_2^{k+1}B_j \setminus A_2^k B_j$

$$\left|T\left(\left[b-\alpha_{B_j}(b)\right]f_j\right)(x)\right| \lesssim \left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}} \int_{B_j} \frac{\left|b(\xi)-\alpha_{B_j}(b)\right|}{V(x,\xi)}\,d\mu(\xi) \lesssim \left[\omega\left(B_j\right)\right]^{\frac{\kappa-1}{p}} \frac{\mu(B_j)}{\mu(A_2^k B_j)}.$$

Therefore, by (4.12) with B_j^k replaced by $A_2^{k+1}B_j \setminus A_2^k B_j$, we can deduce that, for any integer $k \geq K_0$

$$\begin{aligned}
&\int_{A_2^{k+1}B_j\setminus A_2^k B_j} \left|\left[b,T\right]f_j(x)\right|^p \omega(x)d\mu(x)\\
&\lesssim \int_{A_2^{k+1}B_j\setminus A_2^k B_j} \left|T\left(\left[b-\alpha_{B_j}(b)\right]f_j\right)(x)\right|^p \omega(x)d\mu(x)\\
&\quad+\int_{A_2^{k+1}B_j\setminus A_2^k B_j} \left|\left[b(x)-\alpha_{B_j}(b)\right]T\left(f_j\right)(x)\right|^p \omega(x)d\mu(x)\\
&\lesssim \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right)+\frac{k^p}{A_2^{k\sigma_0 p}}\frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right)\\
&\lesssim \frac{\mu(B_j)^p}{\mu(A_2^k B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^k B_j\right).
\end{aligned}$$

This completes the proof of Lemma 4.4. □

We also need the following technical result to handle the weighted estimate for the necessity of the compactness of the commutators.

Lemma 4.5. *Let $1 < p < \infty$, $0 < \kappa < 1$, $\omega \in A_p(X)$, $b \in \mathrm{BMO}(X)$, $\gamma, K_0 > 0$, $\{f_j\}_{j\in\mathbb{N}}$ and $\{B_j\}_{j\in\mathbb{N}}$ be as in Lemma 4.4. Assume that $\{B_j\}_{j\in\mathbb{N}} := \{B(x_j, r_j)\}_{j\in\mathbb{N}}$ also satisfies the following two conditions:*

(i) $\forall \ell, m \in \mathbb{N}$ and $\ell \neq m$

$$A_2C_1B_\ell \bigcap A_2C_1B_m = \emptyset, \tag{4.14}$$

where $C_1 := A_2^{K_1} > C_2 := A_2^{K_0}$ for some $K_1 \in \mathbb{N}$ large enough.

(ii) $\{r_j\}_{j\in\mathbb{N}}$ is either non-increasing or non-decreasing in j, or there exist positive constants $C_{\min}$ and $C_{\max}$ such that, for any $j \in \mathbb{N}$

$$C_{\min} \leq r_j \leq C_{\max}.$$

Then there exists a positive constant C such that, for any $j, m \in \mathbb{N}$

$$\left\|[b,T]f_j - [b,T]f_{j+m}\right\|_{L^{p,\kappa}_\omega(X)} \geq C.$$

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(X)} = 1$ and $\{r_j\}_{j\in\mathbb{N}}$ is non-increasing. Let $\{f_j\}_{j\in\mathbb{N}}$, $\widetilde{C}_1$, $\widetilde{C}_2$ be as in Lemma 4.4 associated with $\{B_j\}_{j\in\mathbb{N}}$.

By (4.2), (4.8), Lemma 2.6 with $\omega \in A_p(X)$, we find that, for any $j \in \mathbb{N}$,

$$\begin{aligned}
&\left[\int_{A_2^{K_0}B_j} |[b,T]f_j(x)|^p \omega(x)d\mu(x)\right]^{1/p} \left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p} \qquad (4.15)\\
&\geq \left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\left\{\int_{A_2^{K_0-1}B_j} \left|\left[b,T\right]f_j(x)\right|^p \omega(x)d\mu(x)\right\}^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\geq \left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\left\{\widetilde{C}_1\gamma^p\frac{\mu(B_j)^p}{\mu(A_2^{K_0-1}B_j)^p}\left[\omega\left(B_j\right)\right]^{\kappa-1}\omega\left(A_2^{K_0-1}B_j\right)\right\}^{1/p}\\
&\gtrsim \left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\left\{\gamma^p\frac{\left[\omega\left(B_j\right)\right]^{\kappa}}{A_2^{np(K_0-1)}}\right\}^{1/p}\\
&\geq C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}\left[\omega\left(B_j\right)\right]^{-\kappa/p}\left[\omega\left(B_j\right)\right]^{\kappa/p}\\
&= C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}
\end{aligned}
$$

for some positive constant C_3 independent of γ and A_2. We next prove that, for any $j, m\in\mathbb{N}$,

$$
\left[\int\limits_{A_2^{K_0}B_j}\left|[b,T]f_{j+m}(x)\right|^p\omega(x)d\mu(x)\right]^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\leq\frac{1}{2}C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}. \tag{4.16}
$$

Since $\operatorname{supp}\left(f_{j+m}\right)\subset B_{j+m}$, from (2.3), (4.6), (4.14) and $\|b\|_{\mathrm{BMO}(X)}=1$, it follows that, for any $x\in A_2^{K_0}B_j$

$$
\begin{aligned}
\left|T\left(\left[b-\alpha_{B_{j+m}}(b)\right]f_{j+m}\right)(x)\right| &\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\int_{B_{j+m}}|K(x,\xi)|\left|b(x)-\alpha_{B_{j+m}}(b)\right|d\mu(\xi)\\
&\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V\left(x_j,x_{j+m}\right)}.
\end{aligned}
$$

And hence we have

$$
\begin{aligned}
&\left\{\int\limits_{A_2^{K_0}B_j}\left|T\left(\left[b-\alpha_{B_{j+m}}(b)\right]f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{\frac{1}{p}}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\frac{\kappa}{p}} \qquad (4.17)\\
&\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V\left(x_j,x_{j+m}\right)}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{\frac{1-\kappa}{p}}.
\end{aligned}
$$

Moreover, from (4.6) we deduce that, for any $x\in A_2^{K_0}B_j$

$$
\begin{aligned}
\left|T\left(f_{j+m}\right)(x)\right| &\leq \int\limits_{B_{j+m}}\left|K(x,\xi)-K\left(x,x_{j+m}\right)\right|\left|f_{j+m}(\xi)\right|d\mu(\xi)\\
&\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\frac{r_{j+m}^{\sigma_0}}{d\left(x_j,x_{j+m}\right)^{\sigma_0}}.
\end{aligned} \tag{4.18}
$$

By using (4.18), the fact $\{r_j\}_{j\in\mathbb{N}}$ is non-increasing in j and Hölder's and reverse Hölder's inequalities we have

$$
\begin{aligned}
&\left\{\int\limits_{A_2^{K_0}B_j}\left|\left[b(x)-\alpha_{B_{j+m}}(b)\right]T\left(f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\\
&\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\frac{r_{j+m}^{\sigma_0}}{d\left(x_j,x_{j+m}\right)^{\sigma_0}}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\\
&\quad\times\left[\int\limits_{A_2^{K_0}B_j}\left|b(x)-\alpha_{B_{j+m}}(b)\right|^p\omega(x)d\mu(x)\right]^{1/p}\\
&\lesssim \left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\frac{r_{j+m}^{\sigma_0}}{d\left(x_j,x_{j+m}\right)^{\sigma_0}}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{\frac{1-\kappa}{p}}
\end{aligned}
$$

$$\times\left(\log\frac{d\left(x_j,x_{j+m}\right)}{r_{j+m}}+\log\frac{d\left(x_j,x_{j+m}\right)}{r_j}\right)$$
$$\lesssim\left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{\frac{1-\kappa}{p}}\frac{r_{j+m}^{\sigma_0}}{d\left(x_j,x_{j+m}\right)^{\sigma_0}}\log\frac{d\left(x_j,x_{j+m}\right)}{r_{j+m}}.$$

Notice that, for C_1 large enough, by (4.14) we know that $d\left(x_j,x_{j+m}\right)$ is also large enough and hence

$$\left(\frac{d\left(x_j,x_{j+m}\right)}{r_{j+m}}\right)^{-\sigma_0}\log\frac{d\left(x_j,x_{j+m}\right)}{r_{j+m}}\lesssim 1. \tag{4.19}$$

Using (4.17), (4.18) and (4.19), we deduce that

$$\left\{\int\limits_{A_2^{K_0}B_j}\left|[b,T]\left(f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}$$
$$\le\left\{\int\limits_{A_2^{K_0}B_j}\left|T\left(\left[b-\alpha_{B_{j+m}}(b)\right]f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}$$
$$+\left\{\int\limits_{A_2^{K_0}B_j}\left|\left[b(x)-\alpha_{B_{j+m}}(b)\right]T\left(f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}$$
$$\lesssim\left[\omega\left(B_{j+m}\right)\right]^{\frac{\kappa-1}{p}}\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{\frac{1-\kappa}{p}}$$
$$\lesssim\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\left[\frac{\omega\left(B(x_j,d(x_j,x_{j+m}))\right)}{\omega\left(B_{j+m}\right)}\right]^{\frac{1-\kappa}{p}}$$
$$\le C'\left[\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\right]^{\kappa}.$$

Note that $\lim\limits_{k\to\infty}\mu(A_2^kB_{j+m})=\infty$. Then for C_1 large enough, we have

$$\mu(C_1B_{j+m})\ge\left(\frac{2C'}{C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}}\right)^{\frac{1}{\kappa}}\mu(B_{j+m}).$$

This implies that $C'\left[\frac{\mu(B_{j+m})}{V(x_j,x_{j+m})}\right]^{\kappa}\le C'\left[\frac{\mu(B_{j+m})}{\mu(C_1B_{j+m})}\right]^{\kappa}\le\frac{1}{2}C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}$. This finishes the proof of (4.16). By (4.15) and (4.16) we know that, for any $j,m\in\mathbb{N}$ and C_1 large enough

$$\left\{\int\limits_{A_2^{K_0}B_j}\left|[b,T]\left(f_j\right)(x)-[b,T]\left(f_{j+m}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}$$
$$\ge\left\{\int\limits_{A_2^{K_0}B_j}\left|[b,T]\left(f_j\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}$$
$$-\left\{\int\limits_{A_2^{K_0}B_j}\left|[b,T]\left(f_{j+m}\right)(x)\right|^p\omega(x)d\mu(g)\right\}^{1/p}\left[\omega\left(A_2^{K_0}B_j\right)\right]^{-\kappa/p}\ge\frac{1}{2}C_3\gamma A_2^{-n(\kappa K_0+K_0-1)}.$$

This finishes the proof of Lemma 4.7. □

Proof of Theorem 1.3(ii). Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(X)} = 1$. To show $b \in \mathrm{VMO}(X)$, noticing that $b \in \mathrm{BMO}(X)$ is a real-valued function, we can use a contradiction argument via Lemmas 2.4, 4.4 and 4.5. Now observe that, if $b \notin \mathrm{VMO}(X)$, then b does not satisfy at least one of (i) through (iii) of Lemma 2.4. We show that $[b, T]$ is not compact on $L^{p,\kappa}_{\omega}(X)$ in any of the following three cases.

Case (i) b does not satisfy condition (i) Lemma 2.4. Then there exist $\gamma \in (0, \infty)$ and a sequence

$$\left\{B_j^{(1)}\right\}_{j\in\mathbb{N}} := \left\{B(x_j^{(1)}, r_j^{(1)})\right\}_{j\in\mathbb{N}}$$

of balls in X satisfying (4.1) and that $r_j^{(1)} \to 0$ as $j \to \infty$. Let x_0 be a fixed point in X. We will now consider the following two subcases.

Subcase (i) There exists a positive constant M such that $0 \le d(x_0, x_j^{(1)}) < M$ for all $x_j^{(1)}, j \in \mathbb{N}$. That is, $x_j^{(1)} \in B_0 := B(x_0, M), \forall j \in \mathbb{N}$. Let $\{f_j\}_{j\in\mathbb{N}}$ be associated with $\{B_j\}_{j\in\mathbb{N}}$, $\tilde{C}_1$ $\tilde{C}_2$, K_0 and C_2 be as in Lemmas 4.4 and 4.5. Let $p_0 \in (1, p)$ be such that $\omega \in A_{p_0}(X)$ and $C_4 := A_2^{K_2} > C_2 = A_2^{K_0}$ for $K_2 \in \mathbb{N}$ large enough such that

$$C_5 := \frac{\widetilde{C}_1 \hat{C}_2 \gamma^p}{C_\mu} A_2{}^{nK_0(\sigma-p)} > 2\frac{\widetilde{C}_2}{\hat{C}_1}\frac{A_2^{K_2(p_0-p)}}{1 - A_2^{K_2(p_0-p)}}, \tag{4.20}$$

where $\hat{C}_1$ and $\hat{C}_2$ are as in Lemma 2.6. As we know $\left|r_j^{(1)}\right| \to 0$ as $j \to \infty$ and $\left\{x_j^{(1)}\right\}_{j\in\mathbb{N}} \subset B_0$, we may choose a subsequence $\left\{B_{j_\ell}^{(1)}\right\}_{\ell\in\mathbb{N}}$ of $\left\{B_j^{(1)}\right\}_{j\in\mathbb{N}}$ such that, for any $j \in \mathbb{N}$,

$$\frac{\mu\left(B_{j_{\ell+1}}^{(1)}\right)}{\mu\left(B_{j_\ell}^{(1)}\right)} < \frac{1}{C_4^n} \text{ and } \omega\left(B_{j_{\ell+1}}^{(1)}\right) \le \omega\left(B_{j_\ell}^{(1)}\right). \tag{4.21}$$

For fixed $\ell, m \in \mathbb{N}$, define

$$\mathcal{J} := C_4 B_{j_\ell}^{(1)} \backslash C_2 B_{j_\ell}^{(1)}, \quad \mathcal{J}_1 := \mathcal{J} \backslash C_4 B_{j_{\ell+m}}^{(1)} \quad \text{and} \quad \mathcal{J}_2 := X \backslash C_4 B_{j_{\ell+m}}^{(1)}.$$

Notice that

$$\mathcal{J}_1 \subset \left[\left(C_4 B_{j\ell}^{(1)}\right) \cap \mathcal{J}_2\right] \quad \text{and} \quad \mathcal{J}_1 = \mathcal{J} \cap \mathcal{J}_2.$$

We then have

$$\begin{aligned}
&\left\{\int_{C_4 B_{j_\ell}^{(1)}} \left|[b, T]\left(f_{j_\ell}\right)(x) - [b, T]\left(f_{j_{i+m}}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p} \qquad (4.22)\\
&\ge \left\{\int_{\mathcal{J}_1} \left|[b, T]\left(f_{j_\ell}\right)(x) - [b, T]\left(f_{j_{\ell+m}}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p}\\
&\ge \left\{\int_{\mathcal{J}_1} \left|[b, T]\left(f_{j_\ell}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p} - \left\{\int_{\mathcal{J}_2} \left|[b, T]\left(f_{j_{\ell+m}}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p}\\
&= \left\{\int_{\mathcal{J}\cap\mathcal{J}_2} \left|[b, T]\left(f_{j_\ell}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p} - \left\{\int_{\mathcal{J}_2} \left|[b, T]\left(f_{j_{\ell+m}}\right)(x)\right|^p \omega(x) d\mu(x)\right\}^{1/p}\\
&=: \mathrm{F}_1 - \mathrm{F}_2.
\end{aligned}$$

We will first consider the term F_1. Assume that $E_{j_\ell} := \mathcal{J}\backslash\mathcal{J}_2 \neq \emptyset$. Then $E_{j_\ell} \subset C_4 B^{(1)}_{j_{\ell+m}}$ by (4.21) we have

$$\mu\left(E_{j_\ell}\right) \leq C_4^n \mu\left(B^{(1)}_{j_{\ell+m}}\right) < \mu\left(B^{(1)}_{j_\ell}\right). \tag{4.23}$$

Now let

$$B^{(1)}_{j_{\ell,k}} := \widetilde{A_2^{k-1} B^{(1)}_{j_\ell}},$$

be the ball associates with $A_2^{k-1}B^{(1)}_{j_\ell}$ in (3.5). Then using (4.23), we have

$$\mu\left(B^{(1)}_{j_{\ell,k}}\right) = \mu\left(A_2^{k-1}B^{(1)}_{j_\ell}\right) > \mu(E_{j_\ell}).$$

By this, we further know that there exist finite mutually disjoint $\left\{B^{(1)}_{j_{\ell,k}}\right\}_{k=K_0}^{K_2-2}$ intersecting E_{j_ℓ}. By (4.2) and Lemma 2.6, we conclude that

$$\begin{aligned}
\mathrm{F}_1^p &\geq \sum_{k=K_0, B^{(1)}_{j_{\ell,k}}\cap E_{j_\ell}=\emptyset}^{K_2-2} \int_{B^{(1)}_{j_{\ell,k}}} \left|[b,T]\left(f_{j_\ell}\right)(x)\right|^p \omega(x)d\mu(x) \qquad (4.24)\\
&\geq \widetilde{C}_1\gamma^p \sum_{k=K_0, B^{(1)}_{j_{\ell,k}}\cap E_{j_\ell}=\emptyset}^{K_2-2} \frac{\mu\left(B^{(1)}_{j_\ell}\right)^p}{\mu\left(A_2^k B^{(1)}_{j_\ell}\right)^p} \omega\left(B^{(1)}_{j_\ell}\right)^{\kappa-1} \omega\left(A_2^k B^{(1)}_{j_\ell}\right)\\
&\geq \sum_{k=K_0, B^{(1)}_{j_{\ell,k}}\cap E_{j_\ell}=\emptyset}^{K_2-2} \widetilde{C}_1\hat{C}_2\gamma^p \frac{\mu\left(B^{(1)}_{j_\ell}\right)^p}{\mu\left(A_2^k B^{(1)}_{j_\ell}\right)^p} A_2^{nk\sigma} \omega\left(B^{(1)}_{j_\ell}\right)^{\kappa}\\
&\geq \frac{\widetilde{C}_1\hat{C}_2\gamma^p}{C_\mu} A_2{}^{nK_0(\sigma-p)} \omega\left(B^{(1)}_{j_\ell}\right)^{\kappa} = C_5 \omega\left(B^{(1)}_{j_\ell}\right)^{\kappa}.
\end{aligned}$$

If $E_{j_l} := \mathcal{J} \setminus \mathcal{J}_2 = \emptyset$, the inequality is still true.
Note that $\lim_{k\to\infty} \mu(A_2^k B^{(1)}_{j_{l+m}}) = \infty$. Then there exist $j_k \in \mathbb{N}$ such that

$$\mu(A_2^{j_1} B^{(1)}_{j_{l+m}}) \geq A_2^{K_2}\mu(A_2^{K_2} B^{(1)}_{j_{l+m}}) \text{ and } \mu(A_2^{j_{k+1}} B^{(1)}_{j_{l+m}}) \geq A_2^{K_2}\mu(A_2^{j_k} B^{(1)}_{j_{l+m}}).$$

Moreover, from the proof of (4.3), Lemma 4.4, (4.20) and (4.21), we deduce that

$$\begin{aligned}
\mathrm{F}_2^p &\leq \sum_{k=0}^{\infty} \int_{A_2^{j_{k+1}} B^{(1)}_{j_{l+m}} \backslash A_2^{j_k} B^{(1)}_{j_{l+m}}} |[b,T](f_{j_{l+m}})(x)|^p \omega(x)d\mu(x) \qquad (4.25)\\
&\leq \widetilde{C}_2 \sum_{k=0}^{\infty} \frac{\mu\left(B^{(1)}_{j_{\ell+m}}\right)^p}{\mu\left(A_2^{j_k} B^{(1)}_{j_{\ell+m}}\right)^p} \left[\omega\left(B^{(1)}_{j_{\ell+m}}\right)\right]^{\kappa-1} \omega\left(A_2^{j_k} B^{(1)}_{j_{\ell+m}}\right)\\
&\leq \widetilde{C}_2 \sum_{k=0}^{\infty} \frac{\mu\left(B^{(1)}_{j_{\ell+m}}\right)^p}{A_2^{(k+1)K_2 p}\mu\left(A_2^{K_2} B^{(1)}_{j_{\ell+m}}\right)^p} \left[\omega\left(B^{(1)}_{j_{\ell+m}}\right)\right]^{\kappa-1} \frac{1}{\hat{C}_1} \frac{A_2^{(k+1)K_2p_0}\mu\left(A_2^{K_2} B^{(1)}_{j_{\ell+m}}\right)^{p_0}}{\mu\left(B^{(1)}_{j_{\ell+m}}\right)^{p_0}} \omega\left(B^{(1)}_{j_{\ell+m}}\right)\\
&\leq \frac{\widetilde{C}_2}{\hat{C}_1} \sum_{k=0}^{\infty} A_2^{(k+1)K_2(p_0-p)} \frac{\mu\left(B^{(1)}_{j_{\ell+m}}\right)^{p-p_0}}{\mu\left(A_2^{K_2} B^{(1)}_{j_{\ell+m}}\right)^{p-p_0}} \left[\omega\left(B^{(1)}_{j_{\ell+m}}\right)\right]^{\kappa}\\
&\leq \frac{\widetilde{C}_2}{\hat{C}_1} \frac{A_2^{K_2(p_0-p)}}{1-A_2^{K_2(p_0-p)}} \left[\omega\left(B^{(1)}_{j_{\ell+m}}\right)\right]^{\kappa}.
\end{aligned}$$

By (4.21), (4.22),(4.24) and (4.25) we obtain

$$\left\{\int_{C_4B_{j_\ell}^{(1)}}\left|[b,T]\left(f_{j_\ell}\right)(x)-[b,T]\left(f_{j_{\ell+m}}\right)(x)\right|^p\omega(x)d\mu(x)\right\}^{1/p}$$
$$\geq C_5^{1/p}\left[\omega\left(B_{j_\ell}^{(1)}\right)\right]^{\kappa/p}-\left(\tfrac{C_5}{2}\right)^{1/p}\left[\omega\left(B_{j_\ell}^{(1)}\right)\right]^{\kappa/p}\gtrsim\left[\omega\left(B_{j_\ell}^{(1)}\right)\right]^{\kappa/p}.$$

Thus, $\{[b,T]f_j\}_{j\in\mathbb{N}}$ is not relatively compact in $L_\omega^{p,\kappa}(X)$, which implies that $[b,T]$ is not compact on $L_\omega^{p,\kappa}(X)$. Therefore, b satisfies condition (i) of Lemma 2.4.

Subcase (ii) There exists a subsequence $\left\{B_{j_e}^{(1)}\right\}_{\ell\in\mathbb{N}}:=\left\{B\left(x_{j_\ell}^{(1)},r_{j_\ell}^{(1)}\right)\right\}_{\ell\in\mathbb{N}}$ of $\left\{B_j^{(1)}\right\}_{j\in\mathbb{N}}$ such that $d(x_0,x_{j_\ell}^{(1)})\to\infty$ as $\ell\to\infty$. In this subcase, by $\mu\left(B_{j\ell}^{(1)}\right)\to 0$ as $\ell\to\infty$, we can take a mutually disjoint subsequence of $\left\{B_{j_\ell}^{(1)}\right\}_{\ell\in\mathbb{N}}$, still denoted by $\left\{B_{j_i}^{(1)}\right\}_{\ell\in\mathbb{N}}$, satisfying (4.14) as well. This, via Lemma 4.5 implies that $[b,T]$ is not compact on $L_\omega^{p,\kappa}(X)$, which is a contradiction to our assumption. Thus, b satisfies condition (i) of Lemma 2.4.

Case (ii) If b does not satisfy condition (ii) of Lemma 2.4. In this case, there exist $\gamma\in(0,\infty)$ and a sequence $\left\{B_j^{(2)}\right\}_{j\in\mathbb{N}}$ of balls in X satisfying (4.1) and that $|r_{B_j^{(2)}}|\to\infty$ as $j\to\infty$. We further consider the following two subcases as well.

Subcase (i) There exists an infinite subsequence $\left\{B_{j\ell}^{(2)}\right\}_{\ell\in\mathbb{N}}$ of $\left\{B_j^{(2)}\right\}_{j\in\mathbb{N}}$ and a point $x_0\in X$ such that, for any $\ell\in\mathbb{N}$, $x_0\in A_2C_1B_{j_\ell}^{(2)}$. As $|r_{B_{j_\ell}^{(2)}}|\to\infty$ as $\ell\to\infty$, it follows that there exists a subsequence, denoted as earlier by $\left\{B_{j\ell}^{(2)}\right\}_{\ell\in\mathbb{N}}$, such that, for any $\ell\in\mathbb{N}$

$$\frac{\mu\left(B_{j_e}^{(2)}\right)}{\mu\left(B_{j_\ell+1}^{(2)}\right)}<\frac{1}{C_4^n}. \tag{4.26}$$

Observe that $2A_2C_1B_{j_\ell}^{(2)}\subset 2A_2C_1B_{j_{\ell+1}}^{(2)}$ for any $j_\ell\in\mathbb{N}$ and hence

$$\omega\left(2A_2C_1B_{j_{\ell+1}}^{(2)}\right)\geq\omega\left(2A_2C_1B_{j_\ell}^{(2)}\right),\quad M\left(b,2A_2C_1B_{j_\ell}\right)>\frac{\gamma}{8A_2^2C_1^2}. \tag{4.27}$$

We can use a similar method as that used in Subcase (i) of Case (i) and redefine our sets in a reversed order. That is, for any fixed $\ell,k\in\mathbb{N}$, let

$$\widetilde{\mathcal{J}}:=2A_2C_4C_1B_{\ell+k}^{(2)}\backslash 2A_2C_2C_1B_{\ell+k}^{(2)},$$
$$\widetilde{\mathcal{J}}_1:=\widetilde{\mathcal{J}}\backslash 2A_2C_4C_1B_{j_l}^{(2)},$$
$$\widetilde{\mathcal{J}}_2:=X\backslash 2A_2C_4C_1B_{j_\ell}^{(2)}.$$

As in Case (i), by Lemma 4.4, (4.26) and (4.27), we conclude that the commutator $[b,T]$ is not compact on $L_\omega^{p,\kappa}(X)$. This contradiction implies that b satisfies condition (ii) of Lemma 4.4.

Subcase (ii) For any $z\in X$ the number of $\left\{A_2C_1B_j^{(2)}\right\}_{j\in\mathbb{N}}$ containing z is finite. In this subcase, for each square $B_{j_0}^{(2)}\in\left\{B_j^{(2)}\right\}_{j\in\mathbb{N}}$, the number of $\left\{A_2C_1B_j^{(2)}\right\}_{j\in\mathbb{N}}$ intersecting $A_2C_1B_{j_0}^{(2)}$ is finite. Then we take a mutually disjoint subsequence $\left\{B_{j_\ell}^{(2)}\right\}_{\ell\in\mathbb{N}}$ satisfying (4.1) and (4.14). From Lemma 4.5, we can deduce that $[b,T]$ is not compact on $L_\omega^{p,\kappa}(X)$. Thus, b satisfies condition (ii) of Lemma 2.4.

Case (iii) Condition (iii) of Lemma 2.4 does not hold for b. Then there exists $\gamma>0$ such that for any $r>0$, there exists $B\subset X\backslash B(x_0,r)$ with $M(b,B)>\gamma$. As in [4] for the γ above, there exists a sequence $\left\{B_j^{(3)}\right\}_j$ of balls such that for any j,

$$M\left(b,B_j^{(3)}\right)>\gamma, \tag{4.28}$$

and for any $i\neq m$,

$$\gamma_1B_i^{(3)}\cap\gamma_1B_m^{(3)}=\emptyset, \tag{4.29}$$

for sufficiently large γ_1 since, by Case (i) and (ii), $\left\{B_j^{(3)}\right\}_{j\in\mathbb{N}}$ satisfies the conditions (i) and (ii) of Lemma 2.4, it follows that there exist positive constants $C_{\min}$ and $C_{\max}$ such that

$$C_{\min} \le r_j \le C_{\max}, \quad \forall j \in \mathbb{N}.$$

By this and Lemma 4.5 we conclude that, if $[b, T]$ is compact on $L^{p,\kappa}_{\omega}(X)$, then b also satisfies condition (iii) of Lemma 2.4. This finishes the proof of Theorem 1.3(ii) and hence of Theorem 1.3. □

5 Appendix: Characterisation of VMO(X)

In this section, we provide a characterisation of the VMO space on X by giving the proof of Lemma 2.4.

Proof of Lemma 2.4. In the following, for any integer m, we use B^m to denote the ball $B(x_0, 2^m)$, where x_0 is a fixed point in X.

Necessary condition: Assume that $f \in \mathrm{VMO}(X)$. If $f \in \mathrm{Lip}_c(\beta)$, then (i)-(iii) hold. In fact, by the uniform continuity, f satisfies (i). Since $f \in L^1(X)$, f satisfies (ii). By the fact that f is compactly supported, f satisfies (iii). If $f \in \mathrm{VMO}(X)\backslash\mathrm{Lip}_c(\beta)$, by definition, for any given $\varepsilon > 0$, there exists $f_\varepsilon \in \mathrm{Lip}_c(\beta)$ such that $\|f-f_\varepsilon\|_{\mathrm{BMO}(X)} < \varepsilon$. Since f_ε satisfies (i)-(iii), by the triangle inequality of $\mathrm{BMO}(X)$ norm, we can see (i)-(iii) hold for f.

Sufficient condition: In this proof for $j = 1, 2, \cdots, 8$, the value α_j is a positive constant depending only on n and α_i for $1 \le i < j$. Assume that $f \in \mathrm{BMO}(X)$ and satisfies (i)-(iii). To prove that $f \in \mathrm{VMO}(X)$, it suffices to show that there exist positive constants α_1, α_2 such that, for any $\varepsilon > 0$, there exists $\phi_\varepsilon \in \mathrm{BMO}(X)$ satisfying

$$\inf_{h\in\mathrm{Lip}_c(\beta)} \|\phi_\varepsilon - h\|_{\mathrm{BMO}(X)} < \alpha_1\varepsilon, \tag{5.1}$$

and

$$\|\phi_\varepsilon - f\|_{\mathrm{BMO}(X)} < \alpha_2\varepsilon. \tag{5.2}$$

By (i), there exists $i_\varepsilon \in \mathbb{N}$ such that

$$\sup\left\{M(f, B) : r_B \le 2^{-i_\varepsilon+4}\right\} < \varepsilon. \tag{5.3}$$

By (iii), there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\sup\left\{M(f, B) : B \cap B^{j_\varepsilon} = \emptyset\right\} < \varepsilon. \tag{5.4}$$

We first establish a cover of X. Observe that

$$B^{j_\varepsilon} = B^{-i_\varepsilon} \bigcup \left(\bigcup_{\nu=1}^{2^{j_\varepsilon+i_\varepsilon}-1} B\left(x_0, (\nu+1)2^{-i_\varepsilon}\right) \setminus B\left(x_0, \nu 2^{-i_\varepsilon}\right) \right) =: \bigcup_{\nu=0}^{2^{j_\varepsilon+i_\varepsilon}-1} \mathcal{R}^{j_\varepsilon}_{\nu,-i_\varepsilon}$$

For $m > j_\varepsilon$,

$$\begin{aligned} B^m \setminus B^{m-1} &= \bigcup_{\nu=0}^{2^{j_\varepsilon+i_\varepsilon-1}-1} B\left(x_0, 2^{m-1} + (\nu+1)2^{m-j_\varepsilon-i_\varepsilon}\right) \setminus B\left(x_0, 2^{m-1} + \nu 2^{m-j_\varepsilon-i_\varepsilon}\right) \\ &=: \bigcup_{\nu=0}^{2^{j_\varepsilon+i_\varepsilon-1}-1} \mathcal{R}^{m}_{\nu,m-j_\varepsilon-i_\varepsilon}. \end{aligned}$$

For each $\mathcal{R}^{j_\varepsilon}_{\nu,-i_\varepsilon}$, $\nu = 1, 2, \cdots, 2^{j_\varepsilon+i_\varepsilon}-1$, let $\tilde{\mathcal{B}}^{j_\varepsilon}_{\nu,-i_\varepsilon}$ be an open cover of $\mathcal{R}^{j_\varepsilon}_{\nu,-i_\varepsilon}$ consisting of open balls with radius 2^{-i_ε} and center on the sphere $S(x_0, (\nu + 2^{-1})2^{-i_\varepsilon})$. Let $\mathcal{B}^{j_\varepsilon}_{0,-i_\varepsilon} = \{B(x_0, 2^{-i_\varepsilon})\}$ and $\mathcal{B}^{j_\varepsilon}_{\nu,-i_\varepsilon}$ be the finite subcover of $\tilde{\mathcal{B}}^{j_\varepsilon}_{\nu,-i_\varepsilon}$. Similarly, for each $m > j_\varepsilon$ and $\nu = 0, 1, \cdots, 2^{j_\varepsilon+i_\varepsilon-1} - 1$, let $\mathcal{B}^{m}_{\nu,m-j_\varepsilon-i_\varepsilon}$ be the finite cover of $\mathcal{R}^{m}_{\nu,m-j_\varepsilon-i_\varepsilon}$ consisting of open balls with radius $2^{m-j_\varepsilon-i_\varepsilon}$ and center on the sphere $S(x_0, (2^{m-1} + (\nu + 2^{-1})2^{m-j_\varepsilon-i_\varepsilon})$.

We define B_x as follows. If $x \in B^{j_\varepsilon}$, then there is $\nu \in \{0, 1, \cdots, 2^{j_\varepsilon+i_\varepsilon} - 1\}$ such that $x \in \mathcal{R}^{j_\varepsilon}_{\nu,-i_\varepsilon}$, let B_x be a ball in $\mathcal{B}^{j_\varepsilon}_{\nu,-i_\varepsilon}$ that contains x. If $x \in B^m \setminus B^{m-1}$, $m > j_\varepsilon$, then there is $\nu \in \{0, 1, \cdots, 2^{j_\varepsilon+i_{\varepsilon-1}} - 1\}$ such that $x \in \mathcal{R}^m_{\nu,m-j_\varepsilon-i_\varepsilon}$, let B_x be a ball in $\mathcal{B}^m_{\nu,m-j_\varepsilon-i_\varepsilon}$ that contains x. We can see that if $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$, then

$$\text{either } r_{B_x} \le 2\, r_{B_{x'}} \text{ or } r_{B_{x'}} \le 2\, r_{B_x}. \tag{5.5}$$

In fact, if $r_{B_x} > 2r_{B_{x'}}$, then there is $m_0 \in \mathbb{N}$ such that $x \in B^{m_0+2} \setminus B^{m_0+1}$ and $x' \in B^{m_0}$, thus

$$d(x, x') \ge d(x_0, x) - d(x_0, x') \ge 2^{m_0+1} - 2^{m_0} > 2^{m_0+2-j_\varepsilon-i_\varepsilon} + 2^{m_0-j_\varepsilon-i_\varepsilon} = r_{B_x} + r_{B_{x'}},$$

which is contradict to the fact that $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$ (Without loss of generality, here we assume that $A_0 = 1$ in the quasi-triangle inequality. Otherwise, we just need to take $r_{B^m} = ([2A_0] + 1)^m$ and make some modifications).

Now we define ϕ_ε. By (ii), there exists $m_\varepsilon > j_\varepsilon$ large enough such that when $r_B > 2^{m_\varepsilon-i_\varepsilon-j_\varepsilon}$, we have

$$M(f, B) < 2^{n(-i_\varepsilon-j_\varepsilon-1)-1}\varepsilon. \tag{5.6}$$

Define

$$\phi_\varepsilon(x) = \begin{cases} f_{B_x}, & \text{if } x \in B^{m_\varepsilon}, \\ f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon-1}}, & \text{if } x \in X \setminus B^{m_\varepsilon}. \end{cases}$$

We claim that there exist positive constants α_3, α_4 such that if $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$ or $x, x' \in X \setminus B^{m_\varepsilon-1}$, then

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| < \alpha_3\varepsilon. \tag{5.7}$$

And if $2B_x \cap 2B_{x'} \neq \emptyset$, then for any $x_1 \in B_x$, $x_2 \in B_{x'}$, we have

$$\left|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)\right| < \alpha_4\varepsilon. \tag{5.8}$$

Assume (5.7) and (5.8) at the moment, we now continue to prove the sufficiency of Lemma 2.4.

Now we show (5.1). Let $\tilde{h}_\varepsilon(x) := \phi_\varepsilon(x) - f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon-1}}$. By definition of ϕ_ε, we can see that $\tilde{h}_\varepsilon(x) = 0$ for $x \in X \setminus B^{m_\varepsilon}$ and $\|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{\mathrm{BMO}(X)} = 0$.

Observe that $\operatorname{supp}(\tilde{h}_\varepsilon) \subset B^{m_\varepsilon}$ and there exists a function $h_\varepsilon \in C_c(X)$ such that for any $x \in X$, $|\tilde{h}_\varepsilon(x) - h_\varepsilon(x)| < \varepsilon$. Let $\eta(s)$ be an infinitely differentiable function defined on $[0, \infty)$ such that $0 \le \eta(s) \le 1$, $\eta(s) = 1$ for $0 \le s \le 1$ and $\eta(s) = 0$ for $s \ge 2$. And let

$$\rho(x, y, t) = \Big(\int_X \eta(d(x, z)/t)d\mu(z)\Big)^{-1} \eta(d(x, y)/t)$$

and

$$h^t_\varepsilon(x) = \int_X \rho(x, y, t)h_\varepsilon(y)d\mu(y).$$

Then by [31, Lemmas 3.15 and 3.23], $h^t_\varepsilon(x)$ approaches to $h_\varepsilon(x)$ uniformly for $x \in X$ as t goes to 0 and $h^t_\varepsilon \in \mathrm{Lip}_c(\beta)$ for $\beta > 0$. Since

$$\begin{aligned} \|h^t_\varepsilon - \phi_\varepsilon\|_{\mathrm{BMO}(X)} &\le \|h^t_\varepsilon - h_\varepsilon\|_{\mathrm{BMO}(X)} + \|h_\varepsilon - \tilde{h}_\varepsilon\|_{\mathrm{BMO}(X)} + \|\tilde{h}_\varepsilon - \phi_\varepsilon\|_{\mathrm{BMO}(X)} \\ &\le \|h^t_\varepsilon - h_\varepsilon\|_{\mathrm{BMO}(X)} + 2\varepsilon, \end{aligned}$$

we can obtain (5.1) by letting t go to 0 and by taking $\alpha_1 = 2$.

Now we show (5.2). To this end, we only need to prove that for any ball $B \subset X$,

$$M(f - \phi_\varepsilon, B) < \alpha_2\varepsilon.$$

We first prove that for every B_x with $x \in B^{m_\varepsilon}$,

$$\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') \le \alpha_5\varepsilon\mu(B_x). \tag{5.9}$$

In fact,

$$\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') = \int_{B_x \cap B^{m_\varepsilon}} |f(x') - f_{B_{x'}}| d\mu(x') + \int_{B_x \cap (X \setminus B^{m_\varepsilon})} |f(x') - f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon - 1}}| d\mu(x').$$

When $x \in B(x_0, 2^{m_\varepsilon} - 2^{m_\varepsilon - i_\varepsilon - j_\varepsilon})$, then $B_x \subset B^{m_\varepsilon}$, thus

$$\begin{aligned}\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') &= \int_{B_x} |f(x') - f_{B_{x'}}| d\mu(x') \\ &\le \int_{B_x} |f(x') - f_{B_x}| d\mu(x') + \int_{B_x} |f_{B_x} - f_{B_{x'}}| d\mu(x') \\ &= \mu(B_x) M(f, B_x) + \int_{B_x} |f_{B_x} - f_{B_{x'}}| d\mu(x').\end{aligned}$$

Note that if $x' \in B_x$, then $B_x \cap B_{x'} \neq \emptyset$. Therefore, If $B_x \cap B^{j_\varepsilon} = \emptyset$, by (5.4) and (5.7), we have

$$\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') < (\varepsilon + \alpha_3 \varepsilon) \mu(B_x).$$

If $B_x \cap B^{j_\varepsilon} \neq \emptyset$, then $r_{B_x} \le 2^{-i_\varepsilon + 1}$, then by (5.3) and (5.7),

$$\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') < (\varepsilon + \alpha_3 \varepsilon) \mu(B_x).$$

When $x \in B^{m_\varepsilon} \setminus B(x_0, 2^{m_\varepsilon} - 2^{m_\varepsilon - j_\varepsilon - i_\varepsilon})$, it is clear that $B_x \cap B^{j_\varepsilon} = \emptyset$, then by (5.4), (5.6) and (5.7), we have

$$\begin{aligned}&\int_{B_x} \left|f(x') - \phi_\varepsilon(x')\right| d\mu(x') \\ &\le \int_{B_x \cap B^{m_\varepsilon}} |f(x') - f_{B_x}| d\mu(x') + \int_{B_x \cap B^{m_\varepsilon}} |f_{B_x} - f_{B_{x'}}| d\mu(x') \\ &\quad + \int_{B_x \cap (X \setminus B^{m_\varepsilon})} |f(x') - f_{B^{m_\varepsilon + 1}}| d\mu(x') + \int_{B_x \cap (X \setminus B^{m_\varepsilon})} |f_{B^{m_\varepsilon + 1}} - f_{B^{m_\varepsilon} \setminus B^{m_\varepsilon - 1}}| d\mu(x') \\ &\le \mu(B_x) M(f, B_x) + \alpha_3 \varepsilon \mu(B_x) + \mu(B^{m_\varepsilon + 1}) M(f, B^{m_\varepsilon + 1}) + \frac{\mu(B^{m_\varepsilon + 1}) \mu(B_x)}{\mu(B^{m_\varepsilon} \setminus B^{m_\varepsilon - 1})} M(f, B^{m_\varepsilon + 1}) \\ &< (C_1 \varepsilon + \alpha_3 \varepsilon) \mu(B_x).\end{aligned}$$

Then (5.9) holds by taking $\alpha_5 = (C_1 + \alpha_3)$.

Let B be an arbitrary ball in X, then $M(f - \phi_\varepsilon, B) \le M(f, B) + M(\phi_\varepsilon, B)$. If $B \subset B^{m_\varepsilon}$ and $\max\{r_{B_x} : B_x \cap B \neq \emptyset\} > 8 r_B$, then

$$\min\{r_{B_x} : B_x \cap B \neq \emptyset\} > 2 r_B. \tag{5.10}$$

In fact, assume that $r_{B_{\widehat{x}}} = \max\{r_{B_x} : B_x \cap B \neq \emptyset\}$ and $\widehat{x} \in B^{l_0} \setminus B^{l_0 - 1}$ for some $l_0 \in \mathbb{Z}$. Then $B \subset B^{l_0} \cap \frac{3}{2} B_{\widehat{x}}$. If $l_0 \le j_\varepsilon$, then (5.10) holds. If $l_0 > j_\varepsilon$, then $r_{B_{\widehat{x}}} = 2^{l_0 - j_\varepsilon - i_\varepsilon}$, and

$$r_B < \frac{1}{8} r_{B_{\widehat{x}}} = 2^{l_0 - j_\varepsilon - i_\varepsilon - 3}.$$

Since for any $x' \in \frac{3}{2} B_{\widehat{x}}$,

$$d(x_0, x') \ge d(x_0, \widehat{x}) - d(\widehat{x}, x') \ge 2^{l_0 - 1} - \frac{3}{2} 2^{l_0 - j_\varepsilon - i_\varepsilon} > 2^{l_0 - 1} - 2^{l_0 - j_\varepsilon - i_\varepsilon + 1},$$

we have

$$\operatorname{dist}(x_0, \frac{3}{2}B_{\widehat{x}}) := \inf_{x' \in \frac{3}{2}B_{\widehat{x}}} d(x_0, x') > 2^{l_0-1} - 2^{l_0-j_\varepsilon-i_\varepsilon+1}.$$

Thus $B \subset B^{l_0} \setminus \frac{3}{2}B^{l_0-2}$. Therefore, if $B_x \cap B \neq \emptyset$, then $x \in B^{l_0} \setminus B^{l_0-2}$, which implies that $r_{B_x} \geq 2^{l_0-2-j_\varepsilon-i_\varepsilon} > 2r_B$.

From (5.10) we can see that if $B_{x_i} \cap B \neq \emptyset$ and $B_{x_j} \cap B \neq \emptyset$, then $2B_{x_i} \cap 2B_{x_j} \neq \emptyset$. Then by (5.8), we can get

$$\begin{aligned} M(\phi_\varepsilon, B) &\leq \frac{1}{\mu(B)} \int_B \frac{1}{\mu(B)} \int_B \left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| d\mu(x')d\mu(x) \\ &= \frac{1}{\mu(B)^2} \sum_{i:B_{x_i}\cap B\neq\emptyset} \int_{B_{x_i}\cap B} \sum_{j:B_{x_j}\cap B\neq\emptyset} \int_{B_{x_j}\cap B} \left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| d\mu(x')d\mu(x) \\ &< \alpha_4\varepsilon\frac{1}{\mu(B)^2} \left(\sum_{i:B_{x_i}\cap B\neq\emptyset} \mu(B_{x_i} \cap B) \right) \left(\sum_{i:B_{x_j}\cap B\neq\emptyset} \mu(B_{x_j} \cap B) \right) < \alpha_4\alpha_6^2\varepsilon. \end{aligned}$$

Moreover, if $B \cap B^{j_\varepsilon} \neq \emptyset$, then by (5.10), $r_B < 2^{-i_\varepsilon}$, thus by (5.3), we have $M(f, B) < \varepsilon$. If $B \cap B^{j_\varepsilon} = \emptyset$, then by (5.4), $M(f, B) < \varepsilon$. Consequently,

$$M(f - \phi_\varepsilon, B) \leq M(f, B) + M(\phi_\varepsilon, B) < \left(1 + \alpha_4\alpha_6^2\right)\varepsilon.$$

If $B \subset B^{m_\varepsilon}$ and $\max\{r_{B_x} : B_x \cap B \neq \emptyset\} \leq 8r_B$, since the number of B_x with $x \in B^{m_\varepsilon}$ that covers B is bounded by α_7, by (5.9), we have

$$\begin{aligned} M(f - \phi_\varepsilon, B) &\leq \frac{2}{\mu(B)} \int_B \left|f(x) - \phi_\varepsilon(x)\right| d\mu(x) \leq \frac{2}{\mu(B)} \sum_{i:B_{x_i}\cap B\neq\emptyset} \int_{B_{x_i}} \left|f(x) - \phi_\varepsilon(x)\right| d\mu(x) \\ &\leq \frac{2}{\mu(B)}\alpha_5\varepsilon \sum_{i:B_{x_i}\cap B\neq\emptyset} \mu(B_{x_i}) \leq \frac{2}{\mu(B)}\alpha_5\alpha_7\varepsilon\mu(8B) \leq C_2\alpha_5\alpha_7\varepsilon. \end{aligned}$$

If $B \subset X \setminus B^{m_\varepsilon-1}$, then $B \cap B^{j_\varepsilon} = \emptyset$, from (5.4) we can see $M(f, B) < \varepsilon$. By (5.7),

$$M(\phi_\varepsilon, B) \leq \frac{1}{\mu(B)} \int_B \frac{1}{\mu(B)} \int_B \left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| d\mu(x')d\mu(x) < \alpha_3\varepsilon.$$

Therefore,

$$M(f - \phi_\varepsilon, B) \leq M(f, B) + M(\phi_\varepsilon, B) < (1 + \alpha_3)\varepsilon.$$

If $B \cap (X \setminus B^{m_\varepsilon}) \neq \emptyset$ and $B \cap B^{m_\varepsilon-1} \neq \emptyset$. Let p_B be the smallest integer such that $B \subset B^{p_B}$, then $p_B > m_\varepsilon$. If $p_B = m_\varepsilon + 1$, then $r_B > \frac{1}{2}(2^{m_\varepsilon} - 2^{m_\varepsilon-1}) = 2^{m_\varepsilon-2}$. If $p_B > m_\varepsilon + 1$, then $r_B > \frac{1}{2}(2^{p_B-1} - 2^{m_\varepsilon-1})$. Thus

$$\frac{\mu(B^{p_B})}{\mu(B)} \leq \frac{C_3}{2}.$$

Therefore,

$$\begin{aligned} M(f - \phi_\varepsilon, B) &\leq \frac{1}{\mu(B)} \int_B \left|f(x) - \phi_\varepsilon(x) - (f - \phi_\varepsilon)_{B^{p_B}}\right| d\mu(x) + \left|(f - \phi_\varepsilon)_{B^{p_B}} - (f - \phi_\varepsilon)_B\right| \\ &\leq 2\frac{\mu(B^{p_B})}{\mu(B)} \frac{1}{\mu(B^{p_B})} \int_{B^{p_B}} \left|f(x) - \phi_\varepsilon(x) - (f - \phi_\varepsilon)_{B^{p_B}}\right| d\mu(x) \\ &\leq C_3 \left(M(f, B^{p_B}) + M(\phi_\varepsilon, B^{p_B})\right) \leq C_3 \left(\varepsilon + M(\phi_\varepsilon, B^{p_B})\right), \end{aligned}$$

where the last inequality comes from (5.6). By definition,

$$M(\phi_\varepsilon, B^{p_B}) \leq \frac{1}{\mu(B^{p_B})} \int_{B^{p_B}} \left|\phi_\varepsilon(x) - (\phi_\varepsilon)_{B^{p_B}\setminus B^{m_\varepsilon}}\right| d\mu(x) + \left|(\phi_\varepsilon)_{B^{p_B}\setminus B^{m_\varepsilon}} - (\phi_\varepsilon)_{B^{p_B}}\right|$$

$$\leq \frac{2}{\mu(B^{p_B})} \int\limits_{B^{p_B}} \left|\phi_\varepsilon(x) - (\phi_\varepsilon)_{B^{p_B}\setminus B^{m_\varepsilon}}\right| d\mu(x).$$

By (5.4), (5.9) and the fact that $\phi_\varepsilon(x) = f_{B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}}$ if $x \in X \setminus B^{m_\varepsilon}$, we have

$$\begin{aligned}
&\int\limits_{B^{p_B}} \left|\phi_\varepsilon(x) - (\phi_\varepsilon)_{B^{p_B}\setminus B^{m_\varepsilon}}\right| d\mu(x) \leq \int\limits_{B^{p_B}} \frac{1}{\mu(B^{p_B}\setminus B^{m_\varepsilon})} \int\limits_{B^{p_B}\setminus B^{m_\varepsilon}} |\phi_\varepsilon(x) - \phi_\varepsilon(x')| d\mu(x') d\mu(x) \\
&= \int\limits_{B^{m_\varepsilon}} |\phi_\varepsilon(x) - f_{B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}}| d\mu(x) \\
&\leq \int\limits_{B^{m_\varepsilon}} |\phi_\varepsilon(x) - f(x)| d\mu(x) + \int\limits_{B^{m_\varepsilon}} |f(x) - f_{B^{m_\varepsilon}}| dx + \mu(B^{m_\varepsilon}) |f_{B^{m_\varepsilon}} - f_{B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}}| \\
&\leq \sum_{i: B_{x_i}\cap B^{m_\varepsilon}\neq\emptyset, x_i\in B^{m_\varepsilon}} \int\limits_{B_{x_i}} |\phi_\varepsilon(x) - f(x)| d\mu(x) + \left(\mu(B^{m_\varepsilon}) + \frac{\mu(B^{m_\varepsilon})^2}{\mu(B^{m_\varepsilon}\setminus B^{m_\varepsilon-1})}\right) M(f, B^{m_\varepsilon}) \\
&< \alpha_5\varepsilon \sum_{i: B_{x_i}\cap B^{m_\varepsilon}\neq\emptyset, x_i\in B^{m_\varepsilon}} \mu(B_{x_i}) + 3\varepsilon\mu(B^{m_\varepsilon}) < (\alpha_5\alpha_8 + 3)\varepsilon\mu(B^{m_\varepsilon}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M(f - \phi_\varepsilon, B) &\leq C_3 \left(\varepsilon + M(\phi_\varepsilon, B^{p_B})\right) \leq C_3 \left(\varepsilon + \frac{2\mu(B^{m_\varepsilon})}{\mu(B^{p_B})}(\alpha_5\alpha_8 + 3)\varepsilon\right) \\
&< C_4 (\alpha_5\alpha_8 + 3)\varepsilon.
\end{aligned}$$

Then (5.2) holds by taking $\alpha_2 = \max\{1 + \alpha_4\alpha_6^2, 1 + \alpha_3, C_4(\alpha_5\alpha_8 + 3)\}$. This finishes the proof of Lemma 2.4.

Proof of (5.7)*:*

We first claim that

$$\sup\left\{\left|f_{B_x} - f_{B_{x'}}\right| : x, x' \in B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}\right\} < C_5\varepsilon. \tag{5.11}$$

By (5.6), for any $x \in B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}$, we have

$$\begin{aligned}
\left|f_{B_x} - f_{B^{m_\varepsilon+1}}\right| &\leq \frac{\mu(B^{m_\varepsilon+1})}{\mu(B_x)} \frac{1}{\mu(B^{m_\varepsilon+1})} \int\limits_{B^{m_\varepsilon+1}} \left|f(x') - f_{B^{m_\varepsilon+1}}\right| d\mu(x') \\
&= \frac{\mu(B^{m_\varepsilon+1})}{\mu(B_x)} M(f, B^{m_\varepsilon+1}) < \frac{C_5}{2}\varepsilon.
\end{aligned}$$

Similarly, for any $x' \in B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}$, $|f_{B_{x'}} - f_{B^{m_\varepsilon+1}}| < \frac{C_5}{2}\varepsilon$. Consequently, (5.11) holds.

For the case $x, x' \in X \setminus B^{m_\varepsilon-1}$, firstly, if $x, x' \in X \setminus B^{m_\varepsilon}$, then by definition

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| = 0.$$

Secondly, if $x, x' \in B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}$, then by (5.11), we have

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| < C_5\varepsilon.$$

Thirdly, without loss of generality, we may assume that $x \in B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}$ and $x' \in X \setminus B^{m_\varepsilon}$, then by (5.6), we have

$$\begin{aligned}
\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| &= \left|f_{B_x} - f_{B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}}\right| \leq \left|f_{B_x} - f_{B^{m_\varepsilon+1}}\right| + \left|f_{B^{m_\varepsilon+1}} - f_{B^{m_\varepsilon}\setminus B^{m_\varepsilon-1}}\right| \\
&\leq \frac{\mu(B^{m_\varepsilon+1})}{\mu(B_x)} M(f, B_{m_{\varepsilon+1}}) + \frac{\mu(B^{m_\varepsilon+1})}{\mu(B^{m_\varepsilon}\setminus B^{m_\varepsilon-1})} M(f, B_{m_{\varepsilon+1}}) \\
&\leq \left(\frac{\mu(B^{m_\varepsilon+1})}{\mu(B_x)} + \frac{\mu(B^{m_\varepsilon+1})}{\mu(B^{m_\varepsilon}\setminus B^{m_\varepsilon-1})}\right) M(f, B_{m_{\varepsilon+1}})
\end{aligned}$$

$$< C_6\varepsilon.$$

For the case $\overline{B}_x \cap \overline{B}_{x'} \neq \emptyset$ and $x, x' \in B^{m_\varepsilon - 1}$, we may assume $B_x \neq B_{x'}$ and $r_{B_x} \leq r_{B_{x'}}$. By (5.5), $B_{x'} \subset 5B_x \subset 15B_{x'}$. If $x' \in B^{j_\varepsilon+1}$, then by (5.3), we have

$$\begin{aligned}\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| &= \left|f_{B_x} - f_{B_{x'}}\right| \leq \left|f_{B_x} - f_{3B_{x'}}\right| + \left|f_{B_{x'}} - f_{3B_{x'}}\right| \\ &\leq \left(\frac{\mu(3B_{x'})}{\mu(B_x)} + \frac{\mu(3B_{x'})}{\mu(B_{x'})}\right) M(f, 3B_{x'}) \\ &\leq C_7\varepsilon.\end{aligned}$$

If $x' \notin B^{j_\varepsilon+1}$, then $3B_{x'} \cap B^{j_\varepsilon} = \emptyset$, by (5.4), we have

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| \leq C_7 M(f, 3B_{x'}) \leq C_7\varepsilon.$$

Therefore, (5.7) holds by taking $\alpha_3 = \max\{C_5, C_6, C_7\}$.

Proof of (5.8)*:*

Since $x_1 \in B_x$, $x_2 \in B_{x'}$, we have $B_{x_1} \cap B_x \neq \emptyset$ and $B_{x_2} \cap B_{x'} \neq \emptyset$, by (5.7),

$$\begin{aligned}\left|\phi_\varepsilon(x_1) - \phi_\varepsilon(x_2)\right| &\leq \left|\phi_\varepsilon(x_1) - \phi_\varepsilon(x)\right| + \left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| + \left|\phi_\varepsilon(x') - \phi_\varepsilon(x_2)\right| \\ &\leq 2\alpha_3\varepsilon + \left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right|.\end{aligned}$$

We may assume $B_x \neq B_{x'}$ and $r_{B_x} \leq r_{B_{x'}}$. If $x, x' \in X \setminus B^{m_\varepsilon - 1}$, then (5.8) follows from (5.7). If $x, x' \in B^{m_\varepsilon - 1}$, when $x' \in B^{j_\varepsilon+1}$, then $2^{-i_\varepsilon} \leq r_{B_x} \leq r_{B_{x'}} \leq 2^{-i_\varepsilon+1}$, thus $B_{x'} \subset 10B_x \subset 60B_{x'}$, by (5.3), we have

$$\begin{aligned}\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| &\leq \left|f_{B_x} - f_{6B_{x'}}\right| + \left|f_{B_{x'}} - f_{6B_{x'}}\right| = \left(\frac{\mu(6B_{x'})}{\mu(B_x)} + \frac{\mu(6B_{x'})}{\mu(B_{x'})}\right) M(f, 6B_{x'}) \\ &\leq C_9\varepsilon.\end{aligned}$$

When $x' \notin B^{j_\varepsilon+1}$, then there exist $\tilde{m}_0 \in \mathbb{N}$ and $\tilde{m}_0 \geq j_\varepsilon + 2$ such that $x' \in B^{\tilde{m}_0} \setminus B^{\tilde{m}_0 - 1}$. Since $2B_x \cap 2B_{x'} \neq \emptyset$, we have $B_x \subset 6B_{x'}$. Note that $6B_{x'} \cap B^{\tilde{m}_0 - 2} = \emptyset$, (in fact, for any $\tilde{x} \in 6B_{x'}$, $d(x_0, \tilde{x}) \geq d(x_0, x') - d(x', \tilde{x}) \geq 2^{\tilde{m}_0 - 1} - 6 \cdot 2^{\tilde{m}_0 - j_\varepsilon - i_\varepsilon} > 2^{\tilde{m}_0 - 2}$), thus $B_x \cap B^{\tilde{m}_0 - 2} = \emptyset$ and then $\frac{1}{2} r_{B_{x'}} = 2^{\tilde{m}_0 - 1 - j_\varepsilon - i_\varepsilon} \leq r_{B_x} \leq 2^{\tilde{m}_0 - j_\varepsilon - i_\varepsilon} = r_{B_{x'}}$. Therefore, $B_{x'} \subset 10B_x$. Then by (5.4), we have

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| \leq C_9 M(f, 6B_{x'}) < C_9\varepsilon.$$

If $x \in B^{m_\varepsilon - 1}$ and $x' \in X \setminus B^{m_\varepsilon - 1}$, since $2B_x \cap 2B_{x'} \neq \emptyset$, by the construction of B_x we can see that $x \in B^{m_\varepsilon - 1} \setminus B^{m_\varepsilon - 2}$ and $x' \in B^{m_\varepsilon} \setminus B^{m_\varepsilon - 1}$. Thus, $B_{x'} \subset 10B_x \subset 40B_{x'}$. Then by (5.6), we have

$$\left|\phi_\varepsilon(x) - \phi_\varepsilon(x')\right| < C_{10} M(f, 4B_{x'}) < C_{10}\varepsilon.$$

Taking $\alpha_4 = C_9 + C_{10} + 2\alpha_3$, then (5.8) holds. □

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