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A Weak Type Vector-Valued Inequality for the Modified Hardy–Littlewood Maximal Operator for General Radon Measure on $\mathbb{R}^n$

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Abstract: The aim of this paper is to prove the weak type vector-valued inequality for the modified Hardy–Littlewood maximal operator for general Radon measure on $\mathbb{R}^n$. Earlier, the strong type vector-valued inequality for the same operator and the weak type vector-valued inequality for the dyadic maximal operator were obtained. This paper will supplement these existing results by proving a weak type counterpart.

Keywords: maximal operator; dyadic cube; Morrey space

MSC: 42B25, 42B35

1 Introduction

The conclusion of this paper is that we can readily transplant the boundedness of the dyadic maximal operator to modified uncentered maximal operators. As a typical case, we prove the following vector-valued inequality to supplement results in [2, 5]. Here and below, by a Radon measure, we mean a measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on open sets. Here and below we use the symbol $\mathcal{Q}$ to denote the set of all cubes whose edges are parallel to coordinate axes. The symbol $\mathcal{Q}(\mu)$ stands for the subset which consists of all cubes $Q \in \mathcal{Q}$ with $\mu(Q) > 0$.

Theorem 1.1. *Let $1 < k, q < \infty$ and μ be a Radon measure such that $\mu(B) > 0$ for any open ball B in $\mathbb{R}^n$ with positive radius. For a Borel measurable function f, define the modified uncentered maximal operator $M_{k,\mu}$ by*

$$M_{k,\mu}f(x) = \sup_{Q \in \mathcal{Q}} \frac{\chi_Q(x)}{\mu(kQ)} \int_Q |f(y)| d\mu(y) \quad (x \in \mathbb{R}^n),$$

where kQ, which is concentric to Q, is the k-times expansion of a cube Q. Then there exists a constant $C > 0$ which depends only on k and q such that

$$\mu \left\{ x \in \mathbb{R}^n \, : \, \left(\sum_{j=1}^{\infty} M_{k,\mu} f_j(x)^q \right)^{\frac{1}{q}} > \lambda \right\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{q}} d\mu(x)$$

for any sequence of Borel measurable functions $\{f_j\}_{j=1}^{\infty}$ and any $\lambda > 0$.

The assumption on μ is not so strong. It is postulated so that we can justify $\frac{\chi_Q(x)}{\mu(kQ)}$ in the above. What is important here is that the constant C does not depend on μ. In this sense, Theorem 1.1 is a universal estimate.

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The operator $M_{k,\mu}$ is called the uncentered maximal operator and it is essentially different from the centered maximal operator when μ is a general Radon measure. Theorem 2.1 will complement what is known for $M_{k,\mu}$. First of all, the usual boundedness of $M_{k,\mu}$ is known in [7, p. 129]. In fact, Tolsa proved the following estimate:

Proposition 1.2. *Let $1 < k, p < \infty$ and μ be a Radon measure. Then there exist constants $C_1 = C_1(k)$, $C_p = C_p(k) > 0$ such that*

$$\mu\left\{x \in \mathbb{R}^n : M_{k,\mu}f(x) > \lambda\right\} \le \frac{C_1}{\lambda}\int_{\mathbb{R}^n}|f(x)|d\mu(x) \quad (\lambda > 0) \tag{1.1}$$

and

$$\|M_{k,\mu}f\|_{L^p(\mu)} \le C_p\|f\|_{L^p(\mu)} \tag{1.2}$$

for any Borel measurable function f.

In [5, Theorem 1.7], the present author passed (1.2) to the vector-valued case:

Proposition 1.3. *Let $1 < k, p, q < \infty$ and μ be a Radon measure. Then there exists a constant $C_{p,q} = C_{p,q}(k) > 0$ such that*

$$\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty} M_{k,\mu}f_j(x)^q\right)^{\frac{p}{q}} d\mu(x) \le C_{p,q}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^q\right)^{\frac{p}{q}} d\mu(x).$$

for any sequence of Borel measurable functions $\{f_j\}_{j=1}^{\infty}$.

Notice that a counterpart of Proposition 1.3 with $q = \infty$ is already included in (1.2). An important idea shared strongly by many recent researchers is that the dyadic maximal operators are much easier to handle than the usual maximal operator. Here we recall some notions. For $j \in \mathbb{Z}$ and $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ define $Q_{jk} \in \mathcal{Q}$ by

$$Q_{jk} \equiv \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_n}{2^j}, \frac{k_n+1}{2^j}\right) = \prod_{l=1}^{n}\left[\frac{k_l}{2^j}, \frac{k_l+1}{2^j}\right).$$

A dyadic cube is a set of the form Q_{jk} for some $j \in \mathbb{Z}$, $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$. For each $j \in \mathbb{Z}$, $\mathcal{D}_j(\mathbb{R}^n)$ stands for the set of all dyadic cubes with volume 2^{-jn}. Finally, $\mathcal{D}(\mathbb{R}^n)$ stands for the set of all dyadic cubes. The dyadid maximal operator $M_\mu^{\mathcal{D}}$ is given by

$$M_\mu^{\mathcal{D}}f(x) = \sup_{Q\in\mathcal{D}}\frac{\chi_Q(x)}{\mu(Q)}\int_Q|f(y)|d\mu(y).$$

As for the dyadic maximal operator $M_\mu^{\mathcal{D}}$, we have the following estimates:

Theorem 1.4. *Let $1 < p, q < \infty$.*

1. *For all Borel measurable functions $\{f_j\}_{j=1}^{\infty}$,*

$$\left\|\left(\sum_{j=1}^{\infty}(M_\mu^{\mathcal{D}}f_j)^q\right)^{\frac{1}{q}}\right\|_{L^p(\mu)} \le C_{p,q}\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mu)}.$$

   Here $C_{p,q}$ is a constant that is independent of $\{f_j\}_{j=1}^{\infty}$.

2. *For all Borel measurable functions $\{f_j\}_{j=1}^{\infty}$ and $\lambda > 0$,*

$$\mu\left\{x \in \mathbb{R}^n : \left(\sum_{j=1}^{\infty}M_\mu^{\mathcal{D}}f_j(x)^q\right)^{\frac{1}{q}} > \lambda\right\} \le \frac{C_{1,q}}{\lambda}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^q\right)^{\frac{1}{q}} d\mu(x).$$

   Here $C_{1,q}$ is a constant that is independent of $\{f_j\}_{j=1}^{\infty}$ and $\lambda > 0$.

Theorem 1.4(2) is contained in [2, A.2], while we can argue as in [5, Theorem 1.7] for the proof of Theorem 1.4(1) by using the estimate

$$\int_{\mathbb{R}^n} M_\mu^{\mathcal{D}} f(x)^p |g(x)| d\mu(x) \le p' 2^p \int_{\mathbb{R}^n} |f(x)|^p M_\mu^{\mathcal{D}} g(x) d\mu(x), \tag{1.3}$$

which is valid for all Borel measurable functions f and g, where p' denotes the conjugate index. If we go through a similar argument to [5, Corollary 2.2], then we obtain (1.3).

See [8] for more about the analysis on Euclidean space with a general Radon measure.

Thus, we can say that among these estimates, Theorem 1.1 is essentially new in this paper. The main tool to prove Theorem 1.1 is to reduce the matters to maximal operators generated by a more general family $\mathfrak{D}$ than the family of dyadic cubes. When we are given such a family $\mathfrak{D}$, we define

$$M_\mu^{\mathfrak{D}} f(x) = \sup_{Q \in \mathfrak{D}} \frac{\chi_Q(x)}{\mu(Q)} \int_Q |f(y)| d\mu(y).$$

To describe the property of $\mathfrak{D}$ we desire, we set up notation. Let Q be a (right-open) cube. A (dyadic) child of a cube Q is any of the 2^n (right-open) cubes obtained by partitioning Q by the hyperplanes parallel to the faces of Q and dividing each edge into 2 equal parts. Define inductively

$$\mathcal{D}_0(Q) = \{Q\}, \quad \mathcal{D}_1(Q) = \{R \,:\, R \text{ is a dyadic child of } Q\}, \quad \mathcal{D}_k(Q) = \bigcup_{R \in \mathcal{D}_1(Q)} \mathcal{D}_{k-1}(R).$$

We also set

$$\mathcal{D}(Q) = \bigcup_{k=1}^{\infty} \mathcal{D}_k(Q).$$

We will be interested in families $\mathfrak{D}$ which enjoy the following properties:

1. There exists $a > 0$ such that
$$\sum_{Q \in \mathfrak{D}, |Q| = a^n 2^{jn}} \chi_Q = 1$$
for any $j \in \mathbb{Z}$.
2. For any $Q \in \mathcal{D}$, $\mathcal{D}(Q) \subset \mathfrak{D}$.

This generalized dyadic grid includes the notion of dyadic grids in [4]. We will call such $\mathfrak{D}$ satisfying 1. and 2. a *generalized dyadic grid* in this paper. To prove Thoerem 1.1, we will construct a family of generalized dyadic grids that is adapted to $M_{\mu,k}$, which will be done in Definition 2.5.

As an application, we obtain the weak vector-valued maximal inequality for the Morrrey space $\mathcal{M}_q^p(k,\mu)$ for $k > 1$ and $1 \le q \le p < \infty$. Define the Morrey space $\mathcal{M}_q^p(k,\mu)$ by $\mathcal{M}_q^p(k,\mu) \equiv \left\{ f \in L^q_{\rm loc}(\mu) : \|f\|_{\mathcal{M}_q^p(k,\mu)} < \infty \right\}$, where

$$\|f\|_{\mathcal{M}_q^p(k,\mu)} \equiv \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(z)|^q \mathrm{d}\mu(z) \right)^{\frac{1}{q}}. \tag{1.4}$$

An important observation made in [6, Proposition 1.1] is that $\mathcal{M}_q^p(k,\mu)$ and $\mathcal{M}_q^p(2,\mu)$ are isomorphic as long as $k > 1$. We have the following vector-valued maximal inequality for the modified maximal operator.

Corollary 1.5. *If* $k, \kappa > 1$, $1 < q \le p < \infty$ *and* $1 < r \le \infty$, *then*

$$\sup_{\lambda > 0} \lambda \left\| \chi_{(\lambda,\infty]} \left(\left(\sum_{j=1}^{\infty} M_{\kappa,\mu} f_j{}^r \right)^{\frac{1}{r}} \right) \right\|_{\mathcal{M}_q^p(k,\mu)} \lesssim_{n,p,q,r,\kappa,k} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p(k,\mu)}.$$

for all sequences $\{f_j\}_{j=1}^{\infty} \subset L^0(\mu)$.

Let

$$k' \equiv \frac{4\kappa(\kappa+7)}{(3\kappa+1)(\kappa-1)}.$$

The proof of Corollary 1.5 is analogous to [6, Theorem 2.4], where we showed

$$\left\| \left(\sum_{j=1}^{\infty} M_{\kappa,\mu} f_j{}^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q\left(k',\mu\right)} \lesssim_{n,p,q,r,\kappa,k} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q\left(\frac{4\kappa}{3\kappa+1},\mu\right)}$$

under the additional assumption $q > 1$. If we modify the proof of [6, Theorem 2.4] suitably, then thanks to Theorem 1.1, we obtain

$$\sup_{\lambda>0} \lambda \left\| \chi_{(\lambda,\infty]} \left(\left(\sum_{j=1}^{\infty} M_{\kappa,\mu} f_j{}^r \right)^{\frac{1}{r}} \right) \right\|_{\mathcal{M}^p_q(k',\mu)} \lesssim_{n,p,q,r,\kappa,k} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}^p_q(k,\mu)}.$$

Since $\mathcal{M}^p_q(k,\mu)$ and $\mathcal{M}^p_q(k',\mu)$ are isomorphic, we obtain the desired result.

The remaining part of this paper is devoted to the proof of Theorem 1.1.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 hinges on the following result in [2] as well as a construction of generalized dyadic grids.

Lemma 2.1. *Let $\mathfrak{D}$ be a generalized dyadic grid and let $1 < q < \infty$. Then, for all Borel measurable functions $\{f_j\}_{j=1}^{\infty}$ and $\lambda > 0$,*

$$\mu\left\{ x \in \mathbb{R}^n : \left(\sum_{j=1}^{\infty} M_\mu^{\mathfrak{D}} f_j(x)^q \right)^{\frac{1}{q}} > \lambda \right\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{q}} d\mu(x),$$

where C depends only on q.

Proof. As we mentioned, when $\mathfrak{D} = \mathcal{D}$, this is Theorem 1.4, which was proved in [2, A.2]. By a slight adaptation of the proof there, we still have the same result for the case where

$$\mathfrak{D} = \mathcal{D}(a,b) := \{Q : \{ay+b : y \in Q\} \in \mathcal{D}\}$$

for some $a > 0$ and $b \in \mathbb{R}^n$.

For the proof of Lemma 2.1 in the general case, we let

$$\mathfrak{D}_{\le R} = \{Q \in \mathfrak{D} : |Q| \le R^n\}.$$

Since $\mathfrak{D}$ is a generalized dyadic grid, thanks to the properties of 1. and 2. of generalized dyadic cubes there exists $a > 0$ and $b \in \mathbb{R}^n$ such that

$$\mathfrak{D}_{\le R} = \mathcal{D}(a,b) \cap \{Q \in \mathfrak{D} : |Q| \le R^n\}.$$

Thus, we can reduce the matters to the maximal operator generated by $\mathcal{D}(a,b)$; there exists $C > 0$ independent of R such that

$$\mu\left\{ x \in \mathbb{R}^n : \left(\sum_{j=1}^{\infty} M_\mu^{\mathfrak{D}_{\le R}} f_j(x)^q \right)^{\frac{1}{q}} > \lambda \right\} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{q}} d\mu(x).$$

Letting $R \to \infty$, we obtain the desired result. □

We will move on to the construction of generalized dyadic grids.

Definition 2.2. Let $N \in \mathbb{N}$ be an odd integer.

1. Let $j \in \mathbb{Z}$. Set
$$\mathcal{D}_j^\dagger(\mathbb{R}) \equiv \left\{2^{-j}[m, m+1) \,:\, m \in N^{-1}\mathbb{Z}\right\}$$
and
$$\mathcal{D}_j^\dagger(\mathbb{R}^n) \equiv \{Q_1 \times Q_2 \times \cdots \times Q_n \,:\, Q_1, Q_2, \ldots, Q_n \in \mathcal{D}_j^\dagger(\mathbb{R})\}.$$
2. Set $\mathcal{D}^\dagger(\mathbb{R}^n) \equiv \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j^\dagger(\mathbb{R}^n)$.
3. Set $\mathcal{D}^\dagger(b)(\mathbb{R}^n) \equiv \{Q \,:\, \{b^{-1}y \,:\, y \in Q\} \in \mathcal{D}^\dagger(\mathbb{R}^n)\}$ for $b \geq 1$.

We write
$$Q_m^{(N)} \equiv N^{-1}[m_1, m_1+N) \times N^{-1}[m_2, m_2+N) \times \cdots \times N^{-1}[m_n, m_n+N)$$
for $m \in \mathbb{Z}^n$ and $N \in \mathbb{N}$.

Lemma 2.3. *For any $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, any odd integer $N > 0$ and $j \in \mathbb{N}$, there uniquely exists $R \in \mathcal{D}^\dagger(\mathbb{R}^n)$ such that $Q_m^{(N)} \in \mathcal{D}_j(R)$.*

Proof. We will consider the case $n = 1$; a passage to higher dimensions can be achieved by means of the product. We write $Q_m^{(N)} = [N^{-1}m, N^{-1}(m+N))$ with $m \in \mathbb{Z}$.

We will induct on j. We start with the base case $j = 1$. If m is even, then $R = [N^{-1}m, N^{-1}(m+2N))$ is the only cube in $\mathcal{D}^\dagger(\mathbb{R})$ that contains $Q_m^{(N)}$ and satisfies $|R| = 2|Q_m^{(N)}|$. If m is odd, then $R = [N^{-1}(m-N), N^{-1}(m+N))$ is the only cube in $\mathcal{D}^\dagger(\mathbb{R})$.

We will move on to the general case. Let $j \geq 2$. So far, by the induction assumption, we know that there uniquely exists $S \in \mathcal{D}^\dagger(\mathbb{R})$ such that $Q_m^{(N)} \in \mathcal{D}_{j-1}(S)$. As we have shown, there exists a cube R such that $S \in \mathcal{D}_1(R)$. Thus, $Q_m^{(N)} \in \mathcal{D}_j(R)$. Let us discuss the uniqueness. If R' is another cube such that $Q_m^{(N)} \in \mathcal{D}_j(R')$. Let $T \in \mathcal{D}_{j-1}(R)$ be such that $Q_m^{(N)} \in \mathcal{D}_1(T)$, and let $T' \in \mathcal{D}_{j-1}(R')$ be such that $Q_m^{(N)} \in \mathcal{D}_1(T')$. Since we have already proved the assertion for the case where $j = 1$, we have $T' = T$. Thus, $T \in \mathcal{D}_{j-1}(R) \cap \mathcal{D}_{j-1}(R')$. By the induction assumption, $R = R'$, proving the uniqueness of R. □

Remark 2.4. For any $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, $N > 0$ and $j \in \mathbb{N}$, one has
$$\sum_{\tilde{m}\in m+N\mathbb{Z}^n} \sum_{R: Q_{\tilde{m}}^{(N)} \in \mathcal{D}_j(R)} \chi_R = 1.$$
In fact, a simple induction argument reduces matters to the case where $j = 1$. In that case, one can reexamine the argument above to have the desired equality.

Using the cube
$$Q_m^{(N)} \equiv N^{-1}[m_1, m_1+N) \times N^{-1}[m_2, m_2+N) \times \cdots \times N^{-1}[m_n, m_n+N)$$
above, we define the set $\mathcal{D}_m$ as follows:

Definition 2.5. Let N be an odd integer and $m = (m_1, m_2, \ldots, m_n) \in \{0, 1, 2, \ldots, N-1\}^n$. Then define
$$\mathfrak{D}_m(\mathbb{R}^n) \equiv \bigcup_{\tilde{m}\in m+N\mathbb{Z}^n} \mathcal{D}(Q_{\tilde{m}}^{(N)}) \cup \bigcup_{\tilde{m}\in m+N\mathbb{Z}^n} \{R \in \mathcal{D}^\dagger(\mathbb{R}^n) \,:\, Q_{\tilde{m}}^{(N)} \in \mathcal{D}(R)\}.$$
See [1, 3] for the prototype results. In particular, when $N = 3$, our grid goes back to [3, Theorem 1.7].

Theorem 2.6. *Let N be an odd integer.*

1. *For each $m = (m_1, m_2, \ldots, m_n) \in \{0, 1, 2, \ldots, N-1\}^n$, if $Q \in \mathfrak{D}_m(\mathbb{R}^n)$, then $\mathcal{D}(Q) \subset \mathfrak{D}_m(\mathbb{R}^n)$.*

2. *We have*

$$\sum_{Q\in\mathfrak{D}_m(\mathbb{R}^n),\ell(Q)=2^{-j}} \chi_Q = 1.$$

3. *We have a partition of* $\mathcal{D}^\dagger(\mathbb{R}^n)$:

$$\mathcal{D}^\dagger(\mathbb{R}^n) = \bigcup_{m=(m_1,m_2,\dots,m_n)\in\{0,1,2,\dots,N-1\}^n} \mathfrak{D}_m(\mathbb{R}^n).$$

Proof.

1. If $Q \in \mathfrak{D}_m(\mathbb{R}^n)$, then there exists $\tilde{m} \in m+N\mathbb{Z}^n$ such that $Q \in \mathcal{D}(Q^{(N)}_{\tilde{m}})$ or that $Q^{(N)}_{\tilde{m}} \in \mathcal{D}(Q)$. If $Q \in \mathcal{D}(Q^{(N)}_{\tilde{m}})$, then $\mathcal{D}(Q) \subset \mathcal{D}(Q^{(N)}_{\tilde{m}})$ and hence $\mathcal{D}(Q) \subset \mathfrak{D}_m(\mathbb{R}^n)$. If $Q^{(N)}_{\tilde{m}} \in \mathcal{D}(Q)$, then for any cube $R \in \mathcal{D}(Q)$ there exists $m^\star \in m + N\mathbb{Z}^n = \tilde{m} + N\mathbb{Z}^n$ such that $Q^{(N)}_{m^\star} \subset R$ or that $R \subset Q^{(N)}_{m^\star}$, or equivalently $Q^{(N)}_{m^\star} \in \mathcal{D}(R)$ or that $R \in \mathcal{D}(Q^{(N)}_{m^\star})$. Thus, $R \in \mathfrak{D}_m(\mathbb{R}^n)$. Since R is arbitrary, it follows that $\mathcal{D}(Q) \subset \mathfrak{D}_m(\mathbb{R}^n)$.
2. We will use the backward induction on j because (2) is true for $j = 0$ and this also shows that (2) is also true for $j \geq 0$. Suppose that (2) is true for $j = j_0 \leq 0$. We will prove

$$\sum_{Q\in\mathfrak{D}_m(\mathbb{R}^n)\cap\mathcal{D}^\dagger_{j_0-1}(\mathbb{R}^n)} \chi_Q = 1.$$

Since for any cube $R \in \mathfrak{D}_m(\mathbb{R}^n)\cap\mathcal{D}^\dagger_{j_0}(\mathbb{R}^n)$, there uniquely exists $Q \in \mathfrak{D}_m(\mathbb{R}^n)$ such that $R \in \mathcal{D}_1(Q)$ thanks to Lemma 2.7, we have

$$1 = \sum_{R\in\mathfrak{D}_m(\mathbb{R}^n)\cap\mathcal{D}^\dagger_{j_0}(\mathbb{R}^n)} \chi_R \leq \sum_{Q\in\mathfrak{D}_m(\mathbb{R}^n)\cap\mathcal{D}^\dagger_{j_0-1}(\mathbb{R}^n)} \chi_Q.$$

It thus remains to show that

$$\sum_{Q\in\mathfrak{D}_m(\mathbb{R}^n)\cap\mathcal{D}^\dagger_{j_0-1}(\mathbb{R}^n)} \chi_Q \leq 1.$$

If two cubes $Q_1, Q_2 \in \mathfrak{D}_m(\mathbb{R}^n) \cap \mathcal{D}^\dagger_{j_0-1}(\mathbb{R}^n)$ meet at a point x, then there uniquely exists a cube $R \in \mathfrak{D}_m(\mathbb{R}^n) \cap \mathcal{D}^\dagger_{j_0}(\mathbb{R}^n)$ such that $x \in R$ by the induction assumption. Since $\mathcal{D}_1(Q_1) \cup \mathcal{D}_1(Q_2) \subset \mathfrak{D}_m(\mathbb{R}^n) \cap \mathcal{D}^\dagger_{j_0}(\mathbb{R}^n)$ and $x \in R \cap Q_1 \cap Q_2$, it follows from Remark 2.4 that $R \in \mathcal{D}_1(Q_1) \cup \mathcal{D}_1(Q_2)$. Thus, by Lemma 2.3, $Q_1 = Q_2$.
3. Let $Q \in \mathcal{D}^\dagger(\mathbb{R}^n)$. If $|Q| < 1$, then there uniquely exists $\tilde{m} \in \mathbb{Z}^n$ such that $Q \in \mathcal{D}(Q^{(N)}_{\tilde{m}})$. If $|Q| \geq 1$, then there uniquely exists $\tilde{m} \in \mathbb{Z}^n$ such that $Q^{(N)}_{\tilde{m}} \in \mathcal{D}(Q)$. In any case, if we choose $m = (m_1, m_2, \dots, m_n) \in \{0, 1, 2, \dots, N-1\}^n$ so that $\tilde{m} \in m + N\mathbb{Z}^n$, then $Q \in \mathcal{D}^\dagger_m(\mathbb{R}^n)$.

□

Lemma 2.7. *Suppose that we have positive parameters* $k, a > 1$ *and an odd integer* $N \in \mathbb{N}$ *such that*

$$a^{-1} + N^{-1} < 1, \quad N^{-1} < \frac{k-1}{2}a^{-2}, \quad a < \sqrt{\frac{k+1}{2}}. \tag{2.1}$$

For each cube Q *and* $k > 1$, *such that* $a^{-2} < \ell(Q) \leq a^{-1}$ *there exists* $R \in \mathcal{D}^\dagger(\mathbb{R}^n)$ *such that* $Q \subset R \subset kQ$.

Proof. It suffices to consider the case where $n = 1$; a passage to higher dimensions can be achieved by the tensor product. Choose a cube $R \in \mathcal{D}^\dagger_0(\mathbb{R})$ so that $-N^{-1} < \inf R - \inf Q \leq 0$. Since $\ell(Q) < a^{-1} < 1 - N^{-1}$, it follows that $Q \subset R$.

Meanwhile, since $\sup Q = \ell(Q) + \inf Q > a^{-2} + \inf Q$,

$$\sup R - \sup Q = \inf R - \sup Q + 1 < \inf R - \inf Q + 1 - a^{-2} \leq 1 - a^{-2}.$$

Since kQ is obtained by adding two intervals of length $\frac{k-1}{2}\ell(Q)$ to both sides of Q, it follows that $R \subset kQ$. □

Note that we can choose $a > 1$ and $N \in \mathbb{N}$ so that (2.1) holds and that $(\log_2 a)^{-1} \in \mathbb{N}$. In fact, we choose $a < \sqrt{\frac{k+1}{2}}$ slightly greater than 1 so that $S \equiv (\log_2 a)^{-1} \in \mathbb{N}$. If we choose $N \gg 1$, then the first two conditions in (2.1) are also satisfied.

Here and below we fix $a > 1$ and $N \in \mathbb{N}$ so that (2.1) are also satisfied and that $S = (\log_2 a)^{-1} \in \mathbb{N}$.

By a scaling, we transform Lemma 2.7 to the form which we use.

Corollary 2.8. *For each cube Q and $k > 1$, there exists $l = 0, 1, \dots, S-1$ and $R \in \mathcal{D}^{\dagger}(2^{\frac{l}{S}})(\mathbb{R}^n)$ such that $Q \subset R \subset kQ$.*

Proof. Since there exists $l = 0, 1, \dots, S-1$ and $j \in \mathbb{Z}$ such that $a^{-2} < 2^{j-\frac{l}{S}}\ell(R) \le a^{-1}$, we can find $R \in \mathcal{D}^{\dagger}(2^{\frac{l}{S}})(\mathbb{R}^n)$ such that $Q \subset R \subset kQ$ from Lemma 2.7 □

We conclude the proof of Theorem 1.1. To prove Theorem 1.1, we observe

$$M_{k,\mu}\varphi \le \sum_{l=1}^{S-1} M_{\mu}^{\mathcal{D}(2^{\frac{l}{S}})}\varphi$$

thanks to Corollary 2.8. This means that we can reduce the matters to $M_{\mu}^{\mathcal{D}^{\dagger}(S^{\frac{l}{S}})}$, which is a consequence of Lemma 2.1.

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