

Research Article

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Ultradiversification of Diversities

<https://doi.org/10.1515/agms-2020-0100>

Received March 27, 2020; accepted December 8, 2020

Abstract: In this paper, using the idea of ultrametrization of metric spaces we introduce ultradiversification of diversities. We show that every diversity has an ultradiversification which is the greatest nonexpansive ultradiversity image of it. We also investigate a Hausdorff-Bayod type problem in the setting of diversities, namely, determining what diversities admit a subdominant ultradiversity. This gives a description of all diversities which can be mapped onto ultradiversities by an injective nonexpansive map. The given results generalize similar results in the setting of metric spaces.

Keywords: Ultrametric space; diversity; ultradiversity; ultrametrization; ultradiversification

MSC: 26E30, 54E40, 54E99

1 Introduction and Preliminaries

An ultrametric space is a metric space (X, d) in which the distance function d satisfies the strong triangle inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, for all $x, y, z \in X$. A description of all metric spaces which can be mapped onto ultrametric spaces by an injective nonexpansive map is given in [7]. Indeed, it is shown that for any metric space (X, d) there exists an ultrametrization of X which is the greatest nonexpansive ultrametric image of (X, d) . This, in particular, determines that the category of ultrametric spaces and nonexpansive maps is a reflective subcategory in the category of all metric spaces and the nonexpansive maps. Moreover, a complete solution of the Hausdorff-Bayod problem, namely, determining what metric spaces admit a subdominant ultrametric is given in [7]. In fact, the Hausdorff-Bayod problem for nonexpansive injective maps of metric spaces is that “For what metric spaces (X, d) does there exist an ultrametric Δ on X such that the identity map $i : (X, d) \rightarrow (X, \Delta)$ is nonexpansive?” (see [8] and references therein).

On the other hand, diversities were introduced in [2] as a generalization of metric spaces and tight span of metric spaces was developed by diversities. Recently, some other aspects of metric space theory carried over to diversities (see e.g., [4, 6]). In addition, a diversity counterpart of ultrametric spaces was introduced in [9] under the name “Ultradiversity”.

In this paper, inspired by the ultrametrization method of metric spaces given in [7], we show that for any diversity (X, δ) there exists an ultradiversification of X which is the greatest nonexpansive ultradiversity image of the diversity (X, δ) (Theorem 2.1). In addition, the question that whether for any diversity there exists an ultradiversity smaller than it leads us to investigate a Hausdorff-Bayod type problem in the setting of diversities, i.e., determining that what diversities admit a subdominant ultradiversity (Theorem 2.2).

In order to introduce the ultradiversification of diversities, an analogous notion to ultrametrization of metric spaces, we need to review some notions. We start with some definitions and preliminaries regarding diversities and ultrametrization of metric spaces.

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Definition 1.1 [9] An ultradiversity is a pair (X, δ) in which X is a nonempty set and $\delta : \langle X \rangle \rightarrow \mathbb{R}$ is a real function on the set of all finite subsets $\langle X \rangle$ of X satisfying:

(UD1) $\delta(A) \geq 0$ and $\delta(A) = 0$ if and only if $|A| \leq 1$,

(UD2) If $B \neq \emptyset$, then

$$\delta(A \cup C) \leq \max \{ \delta(A \cup B), \delta(B \cup C) \},$$

for all $A, B, C \in \langle X \rangle$.

Notice that each ultradiversity (X, δ) is also a diversity, i.e., in addition to (UD1) and (UD2) it satisfies the condition: If $B \neq \emptyset$, then $\delta(A \cup C) \leq \delta(A \cup B) + \delta(B \cup C)$, for all $A, B, C \in \langle X \rangle$ (see [2]). For recent works on diversities we also refer to [3–6].

It is worth mentioning that for every ultradiversity (diversity) (X, δ) , the function $d : X \times X \rightarrow \mathbb{R}$ defined as $d(x, y) = \delta(\{x, y\})$, for all $x, y \in X$, is an ultrametric (a metric), called the induced ultrametric (metric) for (X, δ) . Furthermore, every diversity (and therefore ultradiversity) δ enjoys the monotonicity property, i.e., $A \subseteq B$ implies $\delta(A) \leq \delta(B)$. From (UD2) and the monotonicity of the ultradiversity δ , it is easy to see that if $A \cap B \neq \emptyset$, then

$$\delta(A \cup B) = \max \{ \delta(A), \delta(B) \}. \quad (1.1)$$

Example 1.1 let (X, d) be an ultrametric space. Define

$$\delta(A) = \text{diam}_d(A) = \max \{ d(a, b) : a, b \in A \},$$

where $A \in \langle X \rangle$. Then (X, δ) is an ultradiversity which is called the induced diameter ultradiversity for the ultrametric space (X, d) (or briefly, for the ultrametric d). Furthermore, it can be seen that every $a \in A$ is a diametral point of A , i.e., $d(a, b) = \text{diam}_d(A)$, for some $b \in A$.

Example 1.2 Let G be a finite connected weighted graph with positive weights and A be a subset of the vertices of G . Define

$$\delta(A) = \min \{ \ell_T : T \text{ is a tree containing } A \}, \quad (1.2)$$

where ℓ_T is the maximum edge weight along T . Then δ is an ultradiversity on vertices of G (see Figure 1). Indeed, without loss of generality suppose that $\delta(A \cup B) \leq \delta(B \cup C)$ and let T be a tree containing $B \cup C$. Thus $\ell_S \leq \ell_T$, for some tree S containing $A \cup B$. There obviously exists a tree R containing $A \cup C$ consisting of edges of P and T with $\ell_R = \ell_T$. Therefore $\delta(A \cup C) \leq \delta(B \cup C)$ which shows δ satisfies (UD2).

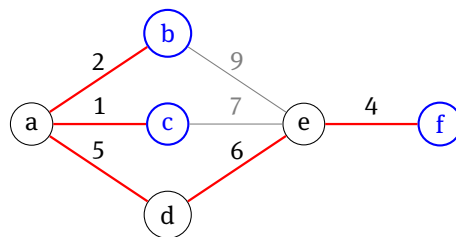


Figure 1: In the finite connected graph G with edge weights indicated by the numbers near the edges, the blue vertices b, c and f indicate a subset A of the set of all vertices $\{a, b, c, d, e, f\}$. The red tree S spans A and has maximum edge weight 6, while any other spanning tree over A has maximum edge weight greater than 6.

Example 1.3 In the taxonomic hierarchy of organisms, taxonomic ranks from the first and smallest to the more inclusive ones are species, genus, family, order, class, phylum, kingdom, domain, etc. For every finite

set of organisms A define

$$\delta(A) = \begin{cases} 0 & |A| \leq 1 \\ 1 & \text{and all organisms of } A \text{ belong to the same species} \\ n & \text{all organisms of } A \text{ belong to a same } n^{\text{th}} \text{ taxonomic} \\ & \text{rank, but not to a same } n-1^{\text{th}} \text{ one.} \end{cases}$$

Then δ is an ultradiversity on the set of all organisms.

Example 1.4 Let X be a normed space. Define

$$\delta(A) = \begin{cases} 0 & |A| \leq 1 \\ \frac{1}{1 + \min_{x \in A} \|x\|} & |A| > 1, \end{cases}$$

for all $A \in \langle X \rangle$. Then δ is an ultradiversity on X .

The next example is in a more general form than the previous example.

Example 1.5 Let X be a nonempty set. If $f : X \rightarrow (0, \infty)$ is an arbitrary function and $g : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function, then the real function δ defined by

$$\delta(A) = \begin{cases} 0 & |A| \leq 1 \\ g(\min f(A)) & |A| > 1, \end{cases}$$

where $A \in \langle X \rangle$ is an ultradiversity on X .

Now, we review some concepts given in [7]. We recall that a map $f : (X, d_1) \rightarrow (Y, d_2)$ of metric spaces is nonexpansive if $d_2(f(x), f(y)) \leq d_1(x, y)$, for all $x, y \in X$. Let (X, d) be a metric space. By [7, Theorem 5], there are an ultrametric space (uX, d_u) and a nonexpansive surjection $u : (X, d) \rightarrow (uX, d_u)$ such that for any nonexpansive map $f : (X, d) \rightarrow (Y, r)$, where (Y, r) is an arbitrary ultrametric space, there exists a unique nonexpansive map $uf : (uX, d_u) \rightarrow (Y, r)$ that commutes the following diagram, i.e., $uf \circ u = f$:

$$\begin{array}{ccc} (uX, d_u) & \xrightarrow{uf} & (Y, r) \\ u \uparrow & \nearrow f & \\ (X, d) & & \end{array}$$

Then, the ultrametric space (uX, d_u) is called an ultrametrization of the metric space (X, d) .

For $\varepsilon > 0$ two elements a and b of X are called ε -linkable if there is a finite sequence $(x_n)_{n=1}^N$ of elements of X with $x_1 = a$ and $x_N = b$ such that $d(x_n, x_{n+1}) \leq \varepsilon$, for all $n < N$. The function $\Delta : X \times X \rightarrow [0, \infty)$ defined by $\Delta(x, y) = \inf\{\varepsilon : x \text{ and } y \text{ are } \varepsilon\text{-linkable}\}$, for all $x, y \in X$ enjoys the strong triangle inequality, while the property that $\Delta(x, y) = 0$ implies $x = y$ may not be valid generally. Consider the equivalence relation \sim on X given by “ $x \sim y$ if and only if x and y are ε -linkable, for every $\varepsilon > 0$ ”. Let $[x]$ be the equivalence class of a point x , uX be the quotient set X/\sim , and u be the canonical projection map. Then the function d_u defined as

$$d_u([x], [y]) = \Delta(x, y), \quad (x, y \in X) \quad (1.3)$$

is an ultrametric on uX , and $u : (X, d) \rightarrow (uX, d_u)$ is a nonexpansive surjection (since every pair (x, y) is obviously $d(x, y)$ -linkable). If $f : (X, d) \rightarrow (Y, r)$ is a nonexpansive map, where (Y, r) is an ultrametric space, then the map $uf : (uX, d_u) \rightarrow (Y, r)$ defined as

$$uf([x]) = f(x), \quad (x \in X) \quad (1.4)$$

is a nonexpansive map which is clearly unique with the property that $uf \circ u = f$. Thus, every metric space has an ultrametrization.

In the next section, we introduce the ultradiversification of diversities. The given results generalize similar results of [7].

2 Ultradiversification

A map $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ of diversities is called nonexpansive if $\delta_Y(f(A)) \leq \delta_X(A)$, for all $A \in \langle X \rangle$ (see [2]). Notice that for any nonexpansive map $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ of diversities the map $f : (X, d_X) \rightarrow (Y, d_Y)$ is also nonexpansive, where d_X and d_Y are the metrics induced by δ_X and δ_Y , respectively. Moreover, two diversities (X, δ_X) and (Y, δ_Y) are said to be isomorphic if there exists a bijective map $f : (X, \delta_X) \rightarrow (Y, \delta_Y)$ such that $\delta_X(A) = \delta_Y(f(A))$, for all $A \in \langle X \rangle$. We say that a finite subset A of a diversity (X, δ) is ε -linkable if each two elements a and b of A are ε -linkable with respect to the induced metric of δ (or equivalently, if there exists an ε -tree T containing A , i.e., a tree $T = (V, E)$ on the underlying set X with $\delta(\{u, v\}) \leq \varepsilon$, for every edge $\{u, v\} \in E$, and $A \subseteq V$). Moreover (X, δ) is said to be totally unlinked if its induced metric is so, i.e., each two elements x and y of X are not ε -linkable, for some positive number ε (see [7] and [8]).

Example 1.1 shows that any ultrametric space induces an ultradiversity, namely, the diameter ultradiversity. Unlike the variety of diversities (see the diversities in [2–6]), ultradiversities have a common intrinsic form. The following result allows us to consider every ultradiversity as a diameter ultradiversity.

Proposition 2.1 *Let (X, δ) be an ultradiversity with induced metric space (X, d) . Then (X, δ) is the induced diameter ultradiversity (X, diam_d) .*

Proof. Let $A = \{a_1, a_2, \dots, a_n\}$ be any finite subset of X and $\text{diam}_d(A) = d(a_i, a_j)$, for some i and j . From the monotonicity of δ and (1.1) we have

$$\begin{aligned} \text{diam}_d(A) &= \delta(\{a_i, a_j\}) \\ &\leq \delta(A) \\ &= \max\{\delta(\{a_1, a_2\}), \delta(\{a_2, \dots, a_n\})\} \\ &= \max\{\delta(\{a_1, a_2\}), \delta(\{a_2, a_3\}), \delta(\{a_3, \dots, a_n\})\} \\ &\vdots \\ &= \max\{\delta(\{a_1, a_2\}), \delta(\{a_2, a_3\}), \dots, \delta(\{a_{n-1}, a_n\})\} \\ &\leq \text{diam}_d(A). \end{aligned}$$

□

Theorem 2.1 *Let (X, δ) be a diversity. Then, there exists a unique ultradiversity (uX, δ_u) up to isomorphism having the following property: There is a nonexpansive surjection $u : (X, \delta) \rightarrow (uX, \delta_u)$ such that for any nonexpansive map $f : (X, \delta) \rightarrow (Y, \sigma)$ where (Y, σ) is an arbitrary ultradiversity, there exists a unique nonexpansive map $uf : (uX, \delta_u) \rightarrow (Y, \sigma)$ that commutes the following diagram, i.e., $uf \circ u = f$:*

$$\begin{array}{ccc} (uX, \delta_u) & \xrightarrow{uf} & (Y, \sigma) \\ \uparrow u & \nearrow f & \\ (X, \delta) & & \end{array}$$

Proof. Let (X, d) be the induced metric space of (X, δ) . Let (uX, d_u) be the ultrametrization of (X, d) defined as (1.3), and $u : X \rightarrow uX$ be the canonical projection map. If $A = \{a_1, a_2, \dots, a_n\}$ is a finite subset of X , then

$$d_u(u(a_{i_0}), u(a_{j_0})) = \max_{1 \leq i, j \leq n} d_u(u(a_i), u(a_j)),$$

for some i_0 and j_0 . Let δ_u be the diameter diversity of d_u . Since u is nonexpansive in the sense of metrics and δ is monotone, we have

$$d_u(u(a_{i_0}), u(a_{j_0})) \leq d(a_{i_0}, a_{j_0}) = \delta(\{a_{i_0}, a_{j_0}\}) \leq \delta(A).$$

This implies that $u : (X, \delta) \rightarrow (uX, \delta_u)$ is also nonexpansive in the sense of diversities. Let $f : (X, \delta) \rightarrow (Y, \sigma)$ be any nonexpansive map where (Y, σ) is an ultradiversity, and r be the induced metric of σ . Since the map $f : (X, d) \rightarrow (Y, r)$ is nonexpansive, so is the map $uf : (uX, d_u) \rightarrow (Y, r)$ defined as (1.4). Note that it is also unique with the property $uf \circ u = f$. For any finite subset $\mathcal{A} = \{[a_1], [a_2], \dots, [a_n]\}$ of $\langle uX \rangle$ we have

$$\begin{aligned} \sigma(uf(\mathcal{A})) &= \text{diam}_r(uf(\mathcal{A})) \\ &= r(uf([a_{i_0}]), uf([a_{j_0}])) \quad (\text{for some } i_0 \text{ and } j_0) \\ &\leq d_u([a_{i_0}], [a_{j_0}]) \\ &\leq \delta_u(\mathcal{A}). \end{aligned}$$

Moreover, if (vX, d_v) is another ultradiversity which has this property with the corresponding nonexpansive surjection $v : (X, d) \rightarrow (vX, d_v)$, then there are nonexpansive maps $uf : (uX, d_u) \rightarrow (vX, d_v)$ and $vf : (vX, d_v) \rightarrow (uX, d_u)$ such that $uf \circ vf = 1_{(vX, d_v)}$ and $vf \circ uf = 1_{(uX, d_u)}$. Thus uf is an isomorphism. \square

We call the ultradiversity (uX, δ_u) given in Theorem 2.1 an ultradiversification of the diversity (X, δ) . In fact, it can also be considered as the greatest nonexpansive ultradiversity image of (X, δ) . To see this, let (\mathcal{X}, Δ) be such an ultradiversity with a corresponding surjection nonexpansive map $u : (X, \delta) \rightarrow (\mathcal{X}, \Delta)$, i.e., for every nonexpansive map f from (X, δ) to an arbitrary ultradiversity (Y, σ) we have

$$\sigma(f(A)) \leq \Delta(u(A)) \quad (A \in \langle X \rangle). \quad (2.1)$$

Define $g : (\mathcal{X}, \Delta) \rightarrow (Y, \sigma)$ by $g(\eta) = f(x)$ for some x with $u(x) = \eta$. If $u(x) = u(y)$ for some $x, y \in X$, then $\sigma(f(\{x, y\})) \leq \Delta(u(\{x, y\})) (= 0)$. Thus $f(x) = f(y)$ and so g is well-defined. The nonexpansivity of g can be easily seen from (2.1) and g is clearly the unique map with the property $g \circ u = f$. Thus (\mathcal{X}, Δ) is an ultradiversification of (X, δ) . On the other hand, every ultradiversification (\mathcal{X}, Δ) of (X, δ) has obviously the property (2.1).

Remark 2.1 According to the method given in [7], to reach an ultrametrization of a metric space, an alternative way can also be used to identify the ultradiversification diam_{d_u} (Theorem 2.1). Indeed, define

$$\widehat{\Delta}(A) = \inf\{\varepsilon : A \text{ is } \varepsilon\text{-linkable}\} \quad (A \in \langle X \rangle). \quad (2.2)$$

Then, $\text{diam}_{d_u}(A) = \widehat{\Delta}(A)$ in which $A = u(A)$. To see this, we first show that for every $A \in \langle X \rangle$ we have

$$\widehat{\Delta}(A) = \max\{\Delta(a, b) : a, b \in A\}.$$

By definition, we see that if A is ε -linkable, then $\Delta(a, b) \leq \widehat{\Delta}(A)$ for every $a, b \in A$. Hence

$$\max\{\Delta(a, b) : a, b \in A\} \leq \widehat{\Delta}(A).$$

Conversely, suppose that a_0 and b_0 are two elements of A which maximize $\Delta(a, b)$ and ε is a positive real number. Since each two elements a and b in A are $(\Delta(a, b) + \varepsilon)$ -linkable, A is $(\Delta(a_0, b_0) + \varepsilon)$ -linkable. Therefore, $\widehat{\Delta}(A) \leq \Delta(a_0, b_0) + \varepsilon$. Since ε was arbitrary, we have

$$\widehat{\Delta}(A) \leq \max\{\Delta(a, b) : a, b \in A\}.$$

Now, for every $A \in \langle uX \rangle$ with $u(A) = A$ we have

$$\text{diam}_{d_u}(A) = \max\{d_u([a], [b]) : a, b \in A\} = \max\{\Delta(a, b) : a, b \in A\} = \widehat{\Delta}(A). \quad (2.3)$$

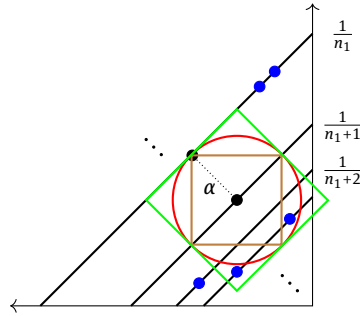


Figure 2: The blue points indicate a finite subset A of X . Having noted to the balls drawn, one can see that if α is the minimum distance between points of the n_1^{th} line and the $n_1 + 1^{\text{th}}$ line, which is equal to the distance between the two black points, then there obviously exists an α -tree containing A , while for no ε smaller than this distance A is ε -linkable.

It is clear that every ultradiversity is an ultradiversification of itself. In addition, it is not hard to see that any two diversities on a set X with equivalent induced metrics can have the same uX . To illustrate it more, let us see the diversities given in the following example.

Example 2.1 Let $X = \{(x, x + \frac{1}{n}) : x \in \mathbb{R}, n \in \mathbb{N}\}$. Suppose that $(x, x + \frac{1}{n}) \sim (y, y + \frac{1}{m})$ if and only if $n = m$. Then, each class of the form $[(x, x + \frac{1}{n})]$ can be identified with n and therefore uX with \mathbb{N} . In addition, for every finite subset $\mathcal{A} = \left\{[(x_1, x_1 + \frac{1}{n_1})], \dots, [(x_k, x_k + \frac{1}{n_k})]\right\}$ of uX , without loss of generality we can assume that $n_1 < \dots < n_k$. Then,

1. For any diversity δ_E on X which has Euclidean metric as its induced metric, we have

$$(\delta_E)_u(\mathcal{A}) = \frac{\sqrt{2}}{2 n_1(n_1 + 1)}.$$

2. For the ℓ_1 -diversity δ_1 given by

$$\delta_1(A) = \sum_{i=1}^2 \max_{a, b \in A} |a_i - b_i|,$$

where $A \in \langle X \rangle$ (see [4]), we have

$$(\delta_1)_u(\mathcal{A}) = \frac{1}{n_1(n_1 + 1)}.$$

3. For the ℓ_∞ -diversity δ_∞ which is in fact the diameter diversity of the supremum metric d_∞ , i.e.,

$$\delta_\infty(A) = \max_{\substack{a, b \in A \\ 1 \leq i \leq 2}} |a_i - b_i|,$$

where $A \in \langle X \rangle$, we have

$$(\delta_\infty)_u(\mathcal{A}) = \frac{1}{2 n_1(n_1 + 1)}.$$

An intuition of the ultradiversification δ_u of the diversities (X, δ) given in Example 2.1 can be seen in Figure 2.

Example 2.2 Let δ be any diversity in \mathbb{R}^k which induces the d_p -metric (the standard metric of the classical space ℓ^p) on \mathbb{R}^k , for some $p \in [1, \infty]$. Since each two elements x and y of \mathbb{R}^k are ε -linkable for any positive real number ε , the trivial diversity on any singleton can be considered as an ultradiversification of (\mathbb{R}^k, δ) .

Example 2.3 Let (X, σ) be a finite diversity and G be the complete graph on vertices X with edge weight $\sigma(\{x, y\})$ for every edge $\{x, y\}$. Let δ be defined as in (1.2). Then (X, δ) is an ultradiversification for (X, σ) .

The previous example and the fact that every ultradiversity is an ultradiversification of itself allow us to consider every finite ultradiversity as that of given in Example 1.2.

By the next example it is seen that different diversities can have a same ultradiversification.

Example 2.4 Let

$$D = (-\infty, 0) \cup \{\dots, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \dots\}.$$

Let $X = D \times \mathbb{R}$ and δ be a diversity on X with induced metric d_p , where $(1 \leq p \leq \infty)$. Then

$$uX = \{\bar{0}\} \cup \{\dots, \frac{1}{3}, \frac{1}{2}, \bar{1}, \bar{2}, \bar{3}, \dots\},$$

where $\bar{0} = (-\infty, 0) \times \mathbb{R} = [(x, y)]$ and $\frac{1}{n} = \{\frac{1}{n}\} \times \mathbb{R} = [(\frac{1}{n}, y)]$, for all $x \in (-\infty, 0)$, $y \in \mathbb{R}$, and $n \in \mathbb{N}$. Note also that $\bar{n} = \{n\} \times \mathbb{R} = [(n, y)]$, for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. The canonical projection map u is

$$u((x, y)) = \begin{cases} \bar{0} & x < 0 \\ \frac{1}{n} & x = \frac{1}{n} \\ \bar{n} & x = n, \end{cases}$$

for all $(x, y) \in X$. Now an ultradiversification δ_u is given as

$$\delta_u(A) = m - \sup(D \setminus [m, \infty)), \quad (A \subseteq \langle uX \rangle)$$

in which $m = \sup\{x : x \in \mathbb{R} \text{ and } u((x, 0)) \in A\}$.

While the following proposition gives a characterization of ultradiversities, it is also a generalization of Lemma 6 in [7].

Proposition 2.2 *A diversity (X, δ) is an ultradiversity if and only if no finite subset A of X is ε -linkable for any $\varepsilon < \delta(A)$. In particular, every ultradiversity is totally unlinked.*

Proof. Let (X, δ) be an ultradiversity and A be a finite subset of X . By induction on the cardinal number of A we show that if A is ε -linkable, then $\delta(A) \leq \varepsilon$. This is trivial when $|A| = 1$. We also assume that this is true for every n -point subset of X . If A is an ε -linkable subset of X with $|A| = n + 1$, then there is an ε -tree $T = (V, E)$ containing A . Let u be an arbitrary leaf of T and v be the vertex for which $\{u, v\} \in E$. Since $\delta(\{u, v\}) \leq \varepsilon$ and $A \setminus \{u\}$ is an n -point ε -linkable subset of X , from equation (1.1) we have

$$\delta(A) = \max\{\delta(\{u, v\}), \delta(A \setminus \{u\})\} \leq \varepsilon.$$

Conversely, if (X, δ) is not an ultradiversity, then there exist $A, B, C \in \langle X \rangle$ such that $\delta(A \cup C) > \max\{\delta(A \cup B), \delta(B \cup C)\}$. Since $A \cup B$ and $B \cup C$ are $\delta(A \cup B)$ -linkable and $\delta(B \cup C)$ -linkable, respectively, we have that $A \cup C$ is $\max\{\delta(A \cup B), \delta(B \cup C)\}$ -linkable. In fact, if A consists of n points a_1, \dots, a_n , then the graph (A, E) where $E = \{\{a_i, a_{i+1}\}\}_{i=1}^{n-1}$ is obviously a $\delta(A)$ -tree, since $\delta(\{a_i, a_{i+1}\}) \leq \delta(A)$ for each i , and therefore A is $\delta(A)$ -linkable. \square

Let (X, d) be a metric space. Given the collection $\{d_i : i \in I\}$ of ultrametrics on X dominated by d , the ultrametric

$$s(x, y) = \sup\{d_i(x, y) : i \in I\}, \quad (x, y \in X)$$

is the largest ultrametric on X dominated by d which is called the subdominant ultrametric of the metric d (see [1]). The following result is a Hausdorff-Bayod type problem (see [8, Problem 1]) in the setting of diversities. It determines that what diversities admit a subdominant ultradiversity.

Theorem 2.2 *Let (X, δ) be a diversity and d be its induced metric. Then (X, δ) is totally unlinked if and only if there exists an ultradiversity Δ dominated by δ , i.e., $\Delta(A) \leq \delta(A)$ for every $A \in \langle X \rangle$. In addition, for every arbitrary diversity δ if ψ is defined as*

$$\psi(A) = \sup \Delta(A), \quad (A \in \langle X \rangle)$$

in which supremum is taken over all ultradiversities Δ on X dominated by δ , then (X, ψ) is the induced diameter diversity for the subdominant ultrametric s of d . Furthermore, (X, ψ) is an ultradiversification of (X, δ) provided that (X, δ) is totally unlinked.

Proof. Suppose that δ is totally unlinked. If a finite subset A of X is ε -linkable for any $\varepsilon > 0$, then it does not have more than one point. Therefore, the function $\hat{\Delta}$ defined as (2.2) satisfies (UD1). Further, since by (2.3) we have $\text{diam}_{d_u}(u(A)) = \hat{\Delta}(A)$, it also satisfies (UD2). Thus $\hat{\Delta}$ is an ultradiversity which is clearly dominated by δ since every $A \in \langle X \rangle$ is $\delta(A)$ -linkable.

Conversely, let $\Delta \leq \delta$ for some ultradiversity Δ . By the fact that every ultradiversity is totally unlinked and the fact that for two diversities δ_1 and δ_2 on X such that $\delta_1 \leq \delta_2$, if δ_1 is totally unlinked, so is δ_2 we imply that δ is totally unlinked. Now, let Δ be an ultradiversity dominated by δ and d_Δ be the induced metric for Δ . Let A be a finite subset of X , and a_0 and b_0 in A be such that $\text{diam}_s(A) = s(a_0, b_0)$. Since $d_\Delta \leq d$, we have $d_\Delta \leq s$ and therefore $d_\Delta(a, b) \leq s(a, b) \leq s(a_0, b_0)$, for all $a, b \in A$. Thus, by Proposition 2.1, $\Delta(A) = \text{diam}_{d_\Delta}(A) \leq \text{diam}_s(A)$. Hence, $\psi \leq \text{diam}_s$. On the other hand, since $\text{diam}_s(A) \leq \text{diam}_\Delta(A) \leq \delta(A)$, for any finite subset A of X , we have $\psi = \text{diam}_s$.

Now we show that $\hat{\Delta} = \psi$. By the definition of ψ , it is obvious that $\hat{\Delta} \leq \psi$. For the reverse, suppose that Δ is an ultradiversity on X dominated by δ . Also, suppose that A is a finite subset of X which is ε -linkable for some ε , and $\Delta(A) = \Delta(\{a, b\})$ for some $a, b \in A$. There is a finite sequence $(x_n)_{n=1}^N$ of elements of X with $x_1 = a$ and $x_N = b$ such that $d(x_n, x_{n+1}) \leq \varepsilon$, for all $n < N$. By (1.1) we have

$$\Delta(\{a, b\}) \leq \max_{n < N} \Delta(\{x_n, x_{n+1}\}) \leq \max_{n < N} \delta(\{x_n, x_{n+1}\}) \leq \varepsilon.$$

This implies that $\psi \leq \hat{\Delta}$. Finally, as (X, δ) is totally unlinked, we have $[x] = \{x\}$, for all $x \in X$ and therefore two ultradiversities (uX, diam_{d_u}) and $(X, \hat{\Delta})$ are obviously isomorphic (see (2.3)). Thus, (X, ψ) is an ultradiversification of (X, δ) . \square

Theorem 2.2 describes all diversities (X, δ) which can be mapped onto an ultradiversity (X, Δ) by an injective nonexpansive map f . In particular, if such $f : (X, \delta) \rightarrow (X, \Delta)$ exists, then the identity map $i : (X, \delta) \rightarrow (X, \Delta')$ is nonexpansive in which Δ' is an ultradiversity defined as $\Delta'(A) = \Delta(f(A))$, for all $A \in \langle X \rangle$. We also have the following.

Corollary 2.1 The following statements are equivalent:

- i) The diversity (X, δ) is totally unlinked;
- ii) The identity map $i : (X, \delta) \rightarrow (X, \Delta)$ is nonexpansive, for some ultradiversity Δ ;
- iii) There exists an injective nonexpansive map $f : (X, \delta) \rightarrow (X, \Delta)$, for some ultradiversity Δ .

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