Anal. Geom. Metr. Spaces 2016; 4:104–159 DE GRUYTER OPEN

Research Article **Open Access**

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Infinitesimal Structure of Differentiability Spaces, and Metric Differentiation

DOI 10.1515/agms-2016-0005
Received October 30, 2015; accepted July 18, 2016

Abstract: We prove metric differentiation for differentiability spaces in the sense of Cheeger [10, 14, 27]. As corollaries we give a new proof of one of the main results of [14], a proof that the Lip-lip constant of any Lip-lip space in the sense of Keith [27] is equal to 1, and new nonembeddability results.

Keywords: Metric measure space; bi-Lipschitz embedding; measurable differentiable structure; differentiability space; metric differentiation

MSC: 30L99, 30L05

1 Introduction

In this paper we study the metric geometry of differentiability spaces in the sense of Cheeger [10, 14, 27]. We develop the infinitesimal geometry of Lipschitz curves and Lipschitz functions, generalizing and refining earlier work on spaces satisfying Poincaré inequalities and differentiability spaces; using this we formulate and establish metric differentiation for differentiability spaces. We then give several applications of these results. They include a new proof that the minimal generalized upper gradient of a Lipschitz function is its pointwise upper Lipschitz constant, which is one of the main results of [14], an alternate proof that the Lip-lip constant of any differentiability space is equal to 1 [42], and new nonembeddability results.

In order to motivate the theory and place it in context, we begin with some background. We will make some additional historical comments at the conclusion of the introduction, after stating our results.

Metric differentiation for $\mathbb{R}^n$

The first instance of metric differentiation was for Lipschitz maps $F : \mathbb{R}^n \to Z$, where Z is an arbitrary metric space; this is due to Ambrosio in the $n = 1$ case and Kirchheim for general n [4, 29]. Although Rademacher's differentiability theorem for Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^m$ does not apply in this situation, and in fact the usual notion of differentiability does not even make sense since Z has no linear structure, Ambrosio and Kirchheim introduced a new kind of differentiation —metric differentiation— and proved that it always holds. Metric differentiation associates to the map F a measurable Finsler metric, i.e. a measurable assignment $x_0 \mapsto \|\cdot\|_F(x_0)$ of a seminorm (here we identify the tangent space $T_{x_0}\mathbb{R}^n$ with $\mathbb{R}^n$ itself), which captures the geometry of the pullback distance function

$$\varrho_F(x_1, x_2) = d_Z(F(x_1), F(x_2)) \tag{1.1}$$

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in the sense that for almost every $x_0 \in \mathbb{R}^n$, the pseudodistance ϱ_F satisfies

$$\varrho_F(x, x_0) = \|x - x_0\|_F (x_0) + o(\|x - x_0\|_{\mathbb{R}^N}). \tag{1.2}$$

A slightly different (and stronger) way to express metric differentiation is in terms of the family of pseudodistances $\{\varrho_F^\lambda(x_0) : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)\}_{\lambda \in (0,\infty)}$ obtained by rescaling ϱ_F centered at x_0:

$$\varrho_F^\lambda(x_0)(x_1, x_2) = \lambda \cdot \varrho_F(x_0 + \lambda^{-1}(x_1), x_0 + \lambda^{-1}(x_2)). \tag{1.3}$$

For almost every x_0, as $\lambda \to \infty$ the pseudodistance $\varrho_F^\lambda(x_0)$ converges uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$ to the pseudodistance associated with the seminorm $\|\cdot\|_F(x_0)$. An additional aspect of metric differentiation is that for a Lipschitz curve $\gamma : I \to \mathbb{R}^n$, the length of the path $F \circ \gamma : I \to Z$ is given by integrating the speed of γ with respect to the Finsler metric $\|\cdot\|_F$,

$$\text{length}(F \circ \gamma) = \int_I \|\gamma'(t)\|_F(\gamma(t))\, dt, \tag{1.4}$$

provided that for a.e. $t \in I$, the norm $\|\cdot\|_F$ is defined at $\gamma(t)$, and (1.2) holds with $x_0 = \gamma(t)$. Such curves γ exist in abundance by Fubini's theorem.

Like Rademacher's theorem for Lipschitz maps $\mathbb{R}^n \to \mathbb{R}^m$, metric differentiation for maps $\mathbb{R}^n \to Z$ as above can be proved by reducing to the $n = 1$ case. Likewise, one ingredient in our approach to metric differentiation for differentiability spaces is a specific form of the 1-dimensional case of metric differentiation due to Ambrosio-Kirchheim, [8].

The $\mathbb{R}^n$ version of metric differentiation has been applied to the theory of rectifiable sets and currents in metric spaces [7, 8, 29], to the theory of Sobolev spaces with metric space targets [34], and in geometric group theory [31, 45, 46]. As an historical note, we mention that metric differentiation was discovered independently in conversations between Korevaar-Schoen and the second author in 1992-93, who were unaware of Kirchheim's work at the time [34].

Metric differentiation for Carnot groups

A generalization of metric differentiation to Carnot groups was established by Pauls [37]. If $F : \mathbb{G} \to Z$ is a Lipschitz map from a Carnot group $\mathbb{G}$ equipped with a Carnot-Caratheodory metric to a metric space Z, then for any $x_0 \in \mathbb{G}$ one can apply the canonical rescaling of $\mathbb{G}$ to the pseudodistance ϱ_F to produce a family of rescaled pseudodistances

$$\{\varrho_F^\lambda(x_0) : \mathbb{G} \times \mathbb{G} \to [0, \infty)\}_{\lambda \in (0,\infty)}$$

analogous to (1.3). Pauls showed that there is a measurable assignment $x_0 \mapsto \|\cdot\|(x_0)$ of seminorms to the horizontal subbundle of $\mathbb{G}$, such that for almost every $x_0 \in \mathbb{G}$ with respect to Haar measure, as $\lambda \to \infty$, the rescalings $\varrho_F^\lambda(x_0)$ converge on compact subsets of $\mathbb{G}\times\mathbb{G}$ to the Carnot-Caratheodory pseudodistance associated with $\|\cdot\|_F(x_0)$; however, this convergence is only asserted to hold on the subset of pairs $(x_1, x_2) \in \mathbb{G} \times \mathbb{G}$ lying on horizontal geodesics. This restriction to special pairs is necessary even in the case of the Heisenberg group, as was shown in [30]. Pauls used his metric differentiation theorem to prove that nonabelian Carnot groups cannot be bilipschitz embedded in Alexandrov spaces, generalizing an earlier result of Semmes [21] (which was based on Pansu's version of Rademacher's theorem for mappings between Carnot groups). Another application was a second proof [18] of the fact that the Heisenberg group cannot be biLipschitz embedded in L_1 (originally proved in [17]).

Differentiability spaces

The main goal in this paper is to generalize metric differentiation to a large class of metric measure spaces, namely differentiability spaces. These were first introduced and studied in [14] without being given a name;

see in particular, Theorem 4.38, Definition 4.42 and the surrounding discussion. There it was shown that PI spaces —metric measure spaces that are doubling and satisfy a Poincaré inequality in the sense of Heinonen-Koskela [24]— are differentiability spaces. Differentiability spaces were further studied in [10, 27] (under slightly different hypotheses), where they were called spaces with a strong measurable differentiable structure, and Lipschitz differentiability spaces, respectively. Examples of differentiability spaces include PI spaces such as Carnot groups with Carnot-Caratheodory metrics, and more generally Borel subsets of PI spaces, with the restricted measures. We recall (see Section 2) that a differentiability space (X, μ) has a countable collection $\{(U_i, \varphi_i)\}$ of charts, where $\cup_i U_i$ has full measure in X. Also, there are canonically defined measurable tangent and cotangent bundles TX, T^*X, and for any Lipschitz function $u : X \to \mathbb{R}$, there is a well-defined differential du, which is a measurable section of T^*X.

Remark 1.5. We emphasize that the cotangent and tangent bundles are not on the same footing: the existence of the cotangent bundle follows quite directly from definition of differentiability space, whereas the tangent bundle is defined as the dual of the cotangent bundle i.e. $TX = (TX^*)^*$. It was observed in [16] that for PI spaces, given a Lipschitz curve γ, for certain parameter values, one can define a velocity vector $\gamma'(t) \in T_{\gamma(t)}X$ and that such velocity vectors span the tangent space almost everywhere; in [15] "span" was upgraded to "are dense". As will be seen below, this new geometric characterization of tangent vectors was crucial to subsequent developments including the papers [15], [11] and the main results of the present paper, a first example being Theorem 1.7.

Remark 1.6. It is interesting to contrast the paper [6], which proves some of the results in [14] in greater generality, with some of the results proven here. While the contexts are quite different, since [6] is concerned with Sobolev spaces whereas we consider differentibility spaces, generalizations of the norm of the gradient play a role both papers. Also, in both papers, some notion of curve —rectifiable curves in [6] and curve fragments here— as well as related notions of "directional derivative" appear. Of course the actual statements and techniques are of necessity different in spirit. This is illustrated by the example of a totally disconnected subset $S \subset [0, 1]^n$ with positive Lebesgue measure. In this case the metric measure space $(S, \mathcal{L} \llcorner S)$ has completely degenerate Sobolev spaces (the Sobolev norm reduces to the L^p-norm, and the notions of gradient they consider are all identically zero) whereas the norms considered here are simply the restrictions of the usual norms on $\mathbb{R}^n$.

For a Carnot group $\mathbb{G}$ with a Carnot-Caratheodory metric, the horizontal bundle can be canonically identified with the tangent bundle $T\mathbb{G}$ of $\mathbb{G}$ viewed as a PI space. This example indicates that in order to formulate a version of metric differentiation for a differentiability space (X, μ), one needs to identify a measurable seminorm on the tangent bundle TX and a family of geodesics that will play the role of the family of horizontal geodesics. We first discuss these in the case of the identity map $X \to X$, initially focussing on the measurable seminorm on TX; the treatment in this special case may be viewed as part of the intrinsic structure theory of X itself.

For the remainder of the introduction (X, μ) will denote a differentiability space.

The canonical norm on *TX*

We now consider several ways of defining a seminorm on the tangent bundle TX; as indicated above, these will be used in the formulation of metric differentiation in the special case of the identity map $X \to X$. In the first, we choose a countable dense set $\{x_i\} \subset X$, and let $u_i : X \to \mathbb{R}$ be distance function $u_i(x) = d(x, x_i)$. For every i, since the differential du_i is a measurable section of the cotangent bundle, by duality it defines a measurable family of linear functions on the tangent spaces, and therefore $|du_i(\cdot)|$ defines a measurable family of seminorms on TX; taking supremum we may define

$$\|v\|_1 = \sup_i |du_i(v)| .$$

As a variations on this, we may define $\|\cdot\|_2$ and $\|\cdot\|_3$ by replacing the collection of distance functions $\{u_i\}$ with the collections of all distance functions and all 1-Lipschitz functions, respectively; note that this requires a little care since these collections are uncountable, see Lemma 2.33. Finally, it was observed in [14] that the pointwise upper Lipschitz constant induces a canonical measurable norm on the cotangent bundle T^*X, and by duality this yields a norm $\|\cdot\|_4$ on TX.

Theorem 1.7 (See Section 6)**.** *The seminorms described above agree almost everywhere; more precisely, any two instances of any one of the four constructions above agree almost everywhere. In particular, they are all norms, and $\|\cdot\|_1$ is independent of the choice of the countable dense subset.*

We will henceforth use $\|\cdot\|$ denote the norms $\|\cdot\|_i$, $1 \le i \le 4$ on the full measure set where they are well-defined and agree.

Generic curves and pairs

We now discuss the role of curves in differentiability spaces. For this we fix a particular choice of charts $\{(U_i, \varphi_i)\}$ as above. If $\gamma : I \to X$ is a Lipschitz curve, then one would like to make sense, for almost every $t \in I$, of the velocity $\gamma'(t)$ and its norm $\|\gamma'(t)\|$, where $\|\cdot\|$ is the norm from Theorem 1.7 (compare (1.4)). Clearly this is impossible for an arbitrary curve γ, since it could lie entirely in the complement of the set where the tangent bundle TX and the norm are well-defined. To address this, we work with generic curves, and generic pairs. Roughly speaking (see Section 3 for the precise definition) if $\gamma : I \to X$ is Lipschitz curve and $t \in I$, then the pair (γ, t) is *generic* if for some chart (U_i, φ_i) of the differentiable structure, the time t is:

- A Lebesgue density point of the inverse image $\gamma^{-1}(U_i)$.
- An approximate continuity point of the measurable function $(\varphi_i \circ \gamma)' : I \to \mathbb{R}^{n_i}$.
- A density point of $\gamma^{-1}(Y)$, where $Y \subset X$ is a full measure subset of $\cup_i U_i$ where the norm $\|\cdot\|$ is well-defined.

The curve γ is *generic* if the pair (γ, t) is generic for almost every $t \in I$. It follows readily from the definitions that for any generic pair (γ, t), both the velocity vector $\gamma'(t) \in TX$ and its norm $\|\gamma'(t)\|$ are well-defined. More generally, we may use essentially the same notions when γ is a *curve fragment* rather than a curve, i.e. a Lipschitz map $\gamma : C \to X$, where $C \subset \mathbb{R}$ is closed subset; this additional generality is essential because a differentiability space might have no nonconstant Lipschitz curves. Also, if $\mathcal{F}$ and $\mathcal{C}$ are countable collections of Lipschitz functions and bounded Borel functions respectively, we may impose the additional requirement that t is an approximate continuity point of $(f \circ \gamma)'$ and $u \circ \gamma$ for all $f \in \mathcal{F}$, $u \in \mathcal{C}$.

Metric differentiation along curves

Using the notions of genericity above, we can formulate one aspect of metric differentiation, which is a statement about curve fragments. This uses the concept of the length of a curve fragment, which is straightforward extension of the length of a curve.

Theorem 1.8. *Suppose $\gamma : C \to X$ is a curve fragment.*

(1) If (γ, t) is a generic pair, then t is a point of metric differentiability of γ in the sense that (1.3) holds with $F = \gamma$, $x_0 = t$, and for pairs of points x_1, x_2 where the right-hand side is defined, and moreover $\|\gamma'(t)\| = \|\frac{\partial}{\partial t}\|_\gamma(t)$.

(2) If γ is generic, then the length of γ is given by

$$\text{length}(\gamma) = \int_C \|\gamma'(t)\| d\mathcal{L} ,$$

where $\|\cdot\|$ is the norm of Theorem 1.7.

Theorem 1.8 is essentially just an application of Theorem 1.7, and the method of proof of the 1-dimensional version of metric differentiation given in [8], which exploits a countable collection of distance functions as in the definition of $\|\cdot\|_1$; See (2.5) and Theorem 4.3.

Remark 1.9. We point out that unlike in the Carnot group case (and in particular $\mathbb{R}^n$), in a differentiability space (for instance the Laakso spaces [35]) one can have, for a full measure set of points $x \in X$, two generic pairs (γ_1, t_1), (γ_2, t_2) such that $\gamma_i(t_i) = x$, the velocity vectors $\gamma_1'(t_1)$, $\gamma_2'(t_2)$ coincide, but the curves are not tangent to first order in the sense that $\limsup_{s\to 0} \frac{d(\gamma_1(t_1+s),\gamma_2(t_2+s))}{s} > 0$. Thus it somewhat surprising that the tangent vector alone controls the speed of the curve.

The density of generic velocities in TX, and consequences

While the definition of genericity is convenient for stating results about individual curve fragments, in order to use it in statements about (X, μ) that hold at almost every point, such as Theorem 1.8, it is crucial to know that generic curve fragments exist in abundance. This is not at all obvious because the definition of a differentiability space is based on the behavior of Lipschitz functions and does not involve curves explicitly; in particular it is not even clear why X should contain any curve fragments with positive length. To deduce the needed abundance, we invoke Bate's fundamental work on differentiability spaces and Alberti representations (see Section 2.3 for the definition). Bate's work shows that one can characterize differentiability spaces by means of differentiability of Lipschitz functions along curve fragments. The main consequence that we will use here is that for μ-a.e. $p \in X$, the set of generic velocity vectors is dense in T_pX (see Theorem 5.3). Here a *generic velocity vector* is the velocity vector $\gamma'(t) \in T_{\gamma(t)}X$ of a generic pair (γ, t).

Theorem 1.7 and the density of velocity vectors leads directly to the following:

Corollary 1.10. *If $u : X \to \mathbb{R}$ is a Lipschitz function, then for μ-a.e. $p \in X$, the pointwise upper Lipschitz constant* $\operatorname{Lip} u(p)$ *is the supremal normalized directional derivative of u over generic pairs (γ, t) with $\gamma(t) = p$:*

$$\operatorname{Lip} u(p) = \sup \left\{ \frac{(u\circ\gamma)'(t)}{\|\gamma'(t)\|} = \frac{(u\circ\gamma)'(t)}{\|\frac{\partial}{\partial t}\|_{\gamma}(t)} \;\middle|\; (\gamma, t) \text{ generic},\ \gamma(t) = p,\ \gamma'(t) \neq 0 \right\}.$$

Corollary 1.10 has two further consequences. The first is a new proof of the characterization of the minimal generalized upper gradient in PI spaces as the pointwise Lipschitz constant (see Section 6.3); this was one of the main results in [14]. The second is a new proof of the following recent result of the third author (see Section 6.2).

Theorem 1.11 ([42]). *If (X, μ) is a differentiability space, and $u : X \to \mathbb{R}$ is a Lipschitz function, then for μ-a.e. $p \in X$ we have* $\operatorname{Lip} u(p) = \operatorname{lip} u(p)$. *Here* $\operatorname{lip} u(p)$ *is the pointwise lower Lipschitz constant (Definition 2.25).*

We recall that [27] introduced the Lip-lip-condition for a metric measure space, which says that for some $C \in \mathbb{R}$, and every Lipschitz function $u : X \to \mathbb{R}$, the upper and lower pointwise Lipschitz constants satisfy $\operatorname{Lip} u \le C \operatorname{lip} u$ almost everywhere. Keith showed that under mild assumptions on the measure, a metric measure space satisfying a Lip-lip-condition is a differentiability space. Combining this with Theorem 1.11, it follows that one may always take $C = 1$. We note that when (X, μ) is PI space, or more generally a Borel subset of a PI space with the restricted measure, it followed from the earlier work [14] that $\operatorname{Lip} u = \operatorname{lip} u$ almost everywhere. These results indicate a strong similarity between PI spaces and differentiability spaces.

For more discussion of these results we refer the reader to the corresponding Sections.

The structure of blow-ups

For a general differentiability space, there is no natural rescaling as in the Carnot group case, so to formulate an analog of the convergence of the rescaled pseudodistances (1.3), we consider sequences of rescalings of X

with the measure μ suitably renormalized, and take pointed Gromov-Hausdorff limits of the metric measure spaces, as well as the chart functions and Alberti representations. We give a brief and informal account of this here, and refer the reader to Section 7 for more discussion. For simplicity, in the following statement we assume in addition that X is a doubling metric space.

Theorem 1.12. *For μ-a.e. $x \in X$, if $\{\lambda_j\}$ is any sequence of scale factors with $\lambda_j \to \infty$, and $x \in U_i$, then there is a sequence $\{\lambda'_j\}$ such that the sequence*

$$\{(\lambda_j X, \lambda'_j \mu, x) \xrightarrow{\varphi_i} (\lambda_j \mathbb{R}^{n_i}, \varphi_i(x))\}$$

of pointed rescalings of the chart $\varphi_i : X \to \mathbb{R}^{n_i}$ subconverges in the pointed measured Gromov-Hausdorff sense to a pointed blow-up map

$$\hat{\varphi}_i : (\hat{X}, \hat{\mu}, \star) \to (T_x X, 0)\,,$$

where $(\hat{X}, \hat{\mu})$ is a doubling metric measure space. Moreover:

(1) When $T_x X$ is equipped with the norm $\|\cdot\|$ of Theorem 1.7, then the map $\hat{\varphi}_i : \hat{X} \to T_x X$ becomes a metric submersion (see Definition 1.13 below).

(2) For every unit vector v in the normed space $(T_x X, \|\cdot\|)$, there is an Alberti representation of $\hat{\mu}$ whose support is contained in the collection of unit speed geodesics $\gamma : \mathbb{R} \to \hat{X}$ with the property that $\hat{\varphi}_i \circ \gamma : \mathbb{R} \to T_x X$ has constant velocity v; furthermore, the measure associated to each such γ is just arclength. This Alberti representation is obtained by blowing-up suitable Alberti representations in X.

Definition 1.13. A map $f : Y \to Z$ between metric spaces is a metric submersion if it is a 1-Lipschitz surjection, and for every $y_1 \in Y$, $z_2 \in Z$, there is a $y_2 \in f^{-1}(z_2)$ such that $d(y_1, y_2) = d(f(y_1), z_2)$. Equivalently, given any two fibers $f^{-1}(z_1)$, $f^{-1}(z_2) \subset Y$, the distance function from the fiber $f^{-1}(z_1)$ is constant and equal to $d(z_1, z_2)$ on the other fiber $f^{-1}(z_2)$.

To aid the reader's intuition, it might be helpful to look at the example $(\mathbb{R}^2, \mathcal{L}^2)$, where on $\mathbb{R}^2$ we consider the l^1-norm; as this norm is not strictly convex, one can obtain an Alberti representation of $\mathcal{L}^2$ by using unit-speed geodesics in $\mathcal{L}^2$ with corners, i.e. geodesics which do not lie in straight lines. Blowing-up such representations at a generic point, one obtains an Alberti representation of $\mathcal{L}^2$ whose transverse measure is concentrated on the set of straight lines in $\mathbb{R}^2$.

There are precursors to Theorem 1.12 in [14] in the case of PI spaces. In that case the blow-ups (tangent cones) are also PI spaces, the coordinate functions blow-up to generalized linear functions, and [14] proved the surjectivity of the canonical map $Y \to T_x X$. Distinguished geodesics of a different sort were discussed in [14], namely the gradient lines of generalized linear functions; however, unlike the curves in the support of the Alberti representations of Theorem 1.12 (2), these need not be affine with respect to the blow-up chart $\hat{\varphi}_i$.

Remark 1.14. The third author [42] and David [22] also have results related to Theorem 1.12 (1). They show that certain blow-up maps are Lipschitz quotient maps, which is a weaker version of the metric submersion property. The paper [42] is concerned with the relationship between Weaver derivations [44] and Alberti representations without the assumption that one has a differentiability space, so the setup there is much more general than the one considered here. We point out that our results in Section 7 have natural counterparts in that general context, under the assumption that μ is asymptotically doubling. We note that one of the main ingredients in Theorem 1.12 is a procedure for blowing-up Alberti representations, which has other applications. In particular, it allows one to blow-up Weaver derivations under the assumption that the background measure is asymptotically doubling. We point out that, as the metric measure space X does not need to possess a group of dilations, it is not trivial to find a correct way to rescale derivations and pass to a limit; however, by taking advantage of the representation of Weaver derivations in terms of Alberti representations proven in [42], one can use Theorem 7.18 to blow-up a derivation at a generic point. Moreover, as the blown-up Alberti representation is concentrated on the set of geodesic lines, the blown-up derivation corresponds to a 1-normal current (in the sense of Lang) without boundary. We refer the reader to Section 7 (in particular Theorem 7.22 and Remark 7.2) for more details.

Theorem 1.12 implies that the blow-up of any Lipschitz function at a generic point is harmonic, in the following sense.

Definition 1.15. Suppose (W, ζ) is proper metric measure space, where ζ is a locally finite Borel measure. Then a Lipschitz function $u : W \to \mathbb{R}$ is p-Lip-*harmonic* if for every ball $B(x, r) \subset W$, and every Lipschitz function $v : W \to \mathbb{R}$ that agrees with u outside $B(x, r)$, we have

$$\int_{B(x,r)} (\mathrm{Lip}\, v)^p \, d\hat{\mu} \geq \int_{B(x,r)} (\mathrm{Lip}\, u)^p \, d\hat{\mu} \, .$$

Theorem 1.12 yields:

Corollary 1.16. *Suppose $u : X \to \mathbb{R}$ is a Lipschitz function. Then for μ a.e. $x \in X$, for any blow-up sequence as in Theorem 1.12 there is a blow-up limit $\hat{u} : Y \to \mathbb{R}$ such that:*

(1) *$\hat{u}$ is* p-Lip-harmonic *for all $p \geq 1$.*
(2) *For any $y \in Y$, $r \in [0, \infty)$ we have* $\mathrm{var}(u, y, r) = r \cdot \mathrm{Lip}(u)(x)$, *where* $\mathrm{var}(u, y, r)$ *is the variation of $\hat{u}$ over $B(y, r)$:*

$$\mathrm{var}(u, y, r) = \sup\{|u(z) - u(y)| \mid z \in B(y, r)\} \, .$$

In particular, $\mathrm{Lip}(\hat{u})(y) = \mathrm{lip}(\hat{u})(y) = \mathrm{Lip}(u)(x)$ *for all $y \in Y$, and* $\mathrm{Lip}(u)(x)$ *is also the global Lipschitz constant of $\hat{u}$.*

We remark that in the terminology of [27, Sec. 6], part (2) of the corollary says that blow-ups are 1-quasilinear; this refines [27, Sec. 6], where it was shown that blow-ups are K-quasilinear for some K.

It is an open question whether a blow-up of a differentiability space must be a PI space, or even a differentiability space. Corollary 1.16 may be compared with the result from [14], which asserts that blow-ups of Lipschitz functions are generalized linear functions —p-harmonic functions with constant norm gradient. The proof in [14] is quite different however —it is based on asymptotic harmonicity and breaks down in differentiability spaces.

The results above all speak to the broader topic of the infinitesimal structure of differentiability spaces. There are a number of open questions here. The present state of knowledge makes it difficult to formulate compelling conjectures or questions in a precise form, but one may ask the following:

Question 1.17. If (X, μ) is a differentiability space, is there a countable collection $\{U_i\}$ of Borel subsets of X, such that $\mu(X \setminus \cup_i U_i) = 0$ and every U_i admits a measure-preserving isometric embedding in a PI space?

If the answer is yes, then blow-ups of differentiability spaces at generic points will also be PI spaces, so one may approach this question by trying to verify that blow-ups have various properties of PI spaces, such as quasiconvexity, a differentiable structure, etc. It is of independent interest to gain a better understanding of the structure of blow-ups in the PI space case. Known examples suggest that the blown-up Alberti representations may have accessibility properties similar to the accessibility one has in Carnot groups.

Remark 1.18. Since the preprint version of this paper was posted, there has been significant progress on these issues. It was shown in [43] that at almost every point, any blow-up of a differentiability space is again a differentiability space. It was shown in [12] that at almost every point, every blow-up of an RNP differentiability space —a metric measure space satisfying a variant of the differentiability space condition for Lipschitz functions taking values in Banach spaces with the Radon-Nikodym Property— satisfies a non-homogeneous Poincaré inequality, and consequently is a quasiconvex RNP differentiability space; it was also shown [12] that RNP differentiability spaces may be characterized by quantitative connectedness properties of universal Alberti representations, and by asymptotic non-homogeneous Poincaré inqualities.

The infinitesimal geometry of Lipschitz maps

We now return to the general case of metric differentiation. Consider a Lipschitz map $F : X \to Z$, where Z is any metric space, and let $\varrho = \varrho_F$ be the pullback distance function (1.1). Our results in this case parallel what has been discussed above for the special case of the identity map $\mathrm{id}_X : X \to X$, so we will be brief and focus on the novel features; see Section 8 for the details.

The map F gives rise to a distinguished subset of the Lipschitz functions on X, namely the set of pullbacks $u \circ F$, where $u : Z \to \mathbb{R}$ is Lipschitz, or equivalently, the set of functions $v : X \to \mathbb{R}$ that are Lipschitz with respect to the pseudodistance ϱ.

Theorem 1.19 (Theorem 8.6). *There is a canonical subbundle $\mathcal{W}_\varrho \subset T^*X$ such that the differential of any ϱ-Lipschitz function $v : X \to \mathbb{R}$ belongs to $\mathcal{W}_\varrho$ μ-almost everywhere. Moreover, for any countable dense subset $D_X \subset X$, the set of differentials of the corresponding ϱ-distance functions span $\mathcal{W}_\varrho$.*

One may construct several seminorms on TX analogous to the seminorms $\|\cdot\|_j$, $1 \le j \le 4$, of Theorem 1.7. For instance, given a countable dense subset $D_X \subset X$, we may define a seminorm by

$$\|\cdot\|_{1,\varrho} = \sup\{|d\rho_x| \mid x \in D_X\}\,,$$

where ϱ_x is the ϱ-distance from x; analogs of the other three seminorms are defined similarly, using the pseudodistance ϱ instead of the distance function d_X.

Theorem 1.20 (Theorem 8.24). *The seminorms agree almost everywhere, giving rise to a canonical seminorm $\|\cdot\|_\varrho$ on TX.*

Unlike in the case of the identity map, when $\varrho = d_X$, the canonical seminorm need not be a norm. Instead it induces a norm on the quotient bundle $TX/\mathcal{W}_\varrho^\perp$ and a dual norm $\|\cdot\|_\varrho^*$ on the canonical subbundle $\mathcal{W}_\varrho \subset T^*X$; here $\mathcal{W}_\varrho^\perp \subset TX$ is the annihilator of the $\mathcal{W}_\varrho \subset T^*X$, .

There are two different ways to formulate metric differentiation in terms of blow-ups. In the first, we refine Theorem 1.12 by bringing in the sequence of rescaled pseudodistances $\{\lambda_j\varrho\}$ as well. After passing to a subsequence, these will Gromov-Hausdorff converge (in a natural sense) to a limiting pseudodistance $\hat{\varrho}$ on Y. Then in addition to conclusions (1) and (2) of Theorem 1.12, we have:

(3) When Y and T_pX are equipped with the pseudodistance $\hat{\varrho}$ and the seminorm $\|\cdot\|_\varrho$ of Theorem 1.20 respectively, the map $\hat{\varphi}_i : Y \to T_pX$ is a metric submersion.
(4) For every unit vector v in the normed space $(T_pX, \|\cdot\|)$, there is an Alberti representation of $\hat{\mu}$ whose support is contained in the collection of curves $\gamma : \mathbb{R} \to Y$ with the property that $\hat{\varphi}_i \circ \gamma : \mathbb{R} \to T_pX$ has constant velocity v, γ is a unit speed d_Y-geodesic, and a constant $\|v\|_\varrho$-speed $\hat{\varrho}$-geodesic.

A second way to formulate the blow-up assertion is to take an ultralimit of the map $F : X \to Z$. We refer the reader to Section 8.3 for the statements.

One consequence of (4) is that the blown-up Alberti representations appearing in Theorem 1.12 (2) may be viewed as invariants of the differentiability space structure, in the following way. The definitions readily imply that if (X,μ) is a differentiability space, (Z,ν) is a metric measure space, and $F : (X,\mu) \to (Z,\nu)$ is a bilipschitz homeomorphism that is also measure class preserving in the sense that pushforward measure $F_*\mu$ and ν are mutually absolutely continuous, then (Z,ν) is also a differentiability space. When X is doubling, for almost every $p \in X$, we can then take a Gromov-Hausdorff limit of the the sequence of rescalings of F as in Theorem 1.12, to obtain a bilipschitz homeomorphism

$$\hat{F} : (\hat{X}, \hat{p}) \longrightarrow (\hat{Z}, \hat{F}(\hat{p}))\,.$$

This blow-up map $\hat{F}$ will preserve the blow-up measures up to scale, and will preserve the blow-up Alberti representations from Theorem 1.12(2) up to a change of speed that depends only on the choice of tangent vector v.

Applications to embedding

In Section 9 we apply metric differentiation to Lipschitz maps between Carnot groups, Alexandrov spaces with curvature bounded above or below, and the inverse limit spaces in [20], showing that such maps are strongly constrained on an infinitesimal level.

Further discussion

We now make some remarks about the evolution of some of the main ideas in this paper —generic velocities, the proof of abundance, the structure of blow-ups, and their distinguished geodesics.

While [14] clarified many points at the foundation of PI spaces, the role of curves remained somewhat mysterious, and in particular velocity vectors to curves were not considered there. In fact, although Lipschitz curves were used in the original definition of a PI space by Heinonen-Koskela (which is based on upper gradients) there is an equivalent definition in which curves do not appear at all [26].

The first appearance of tangent vectors to curves in the context of PI spaces was in [16]. There a notion similar to generic velocity vectors was introduced, and it was shown that they span the tangent space at a typical point; in addition, there was a new characterization of the minimal generalized upper gradient, which may be viewed as a precursor to Corollary 1.10. Metric differentiation for PI spaces was announced in [16, p.1020]. This was work of the first two authors, which led to an unpublished account of metric differentiation [15] that was similar in several respects to the present paper. For instance, it used a notion of generic velocity vectors, and contained a blown-up statement like Theorem 1.12 involving a distinguished family of geodesics with constant velocity in the blown-up chart; however, it did not use Alberti representations. We mention that is easy to see that the collection of nongeneric Lipschitz curves $\gamma : I \to X$ has zero p-modulus, for every p. This yields a weak form of abundance of generic curves in the PI space case. A key ingredient in [15] was a proof of the density of the directions of generic velocity vectors based on a much deeper argument that borrowed ideas —a renorming argument and the equality $\operatorname{Lip} u$ and the minimal generalized upper gradient— from [14].

Bate's beautiful work on Alberti representations [10, 11] greatly strengthened the connection between curves and differentiability, providing several different alternate characterizations of differentiability spaces in terms of Alberti representations. His approach was partly motivated by the work of Alberti-Csornyei-Preiss on differentiability for subsets of $\mathbb{R}^n$, and an observation of Preiss that the characterization of the minimal generalized upper gradient in [16] implied the existence of Alberti representations for PI spaces [11, Sec. 10].

When [10] appeared, the third author used it to give a proof of Lip = lip based on a renorming construction, without being aware of the contents of [15]. Independently, the first two authors recognized that [10] could be used to give a stronger and more general treatment of metric differentiation, and proposed writing the present paper.

2 Preliminaries

2.1 Standing assumptions and review of differentiability spaces

Throughout this paper, the pair (X, μ) will denote a **differentiability space**; this means that (X, d_X) is a complete, separable metric space, μ is a Radon measure, and the pair (X, μ) admits a measurable differentiable structure as recalled below, cf. [14, 27].

We briefly highlight the main features of a differentiability space, see below for more discussion:

(1) There is a countable collection of charts $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, where $U_\alpha \subset X$ is measurable and φ_α is Lipschitz, such that $X \setminus (\cup_\alpha U_\alpha)$ is μ-null, and each real-valued Lipschitz function f admits a first order Taylor expansion with respect to the components of $\varphi_\alpha : X \to \mathbb{R}^{N_\alpha}$ at generic points of U_α, i.e. there exist a.e. unique

measurable functions $\frac{\partial f}{\partial \varphi_\alpha^i}$ on U_α such that:

$$f(x) = f(x_0) + \sum_{i=1}^{N_\alpha} \frac{\partial f}{\partial \varphi_\alpha^i}(x_0)\left(\varphi_\alpha^i(x) - \varphi_\alpha^i(x_0)\right) + o\left(d_X(x, x_0)\right) \quad (\text{for } \mu\text{-a.e. } x_0 \in U_\alpha). \tag{2.1}$$

(2) There are measurable cotangent and tangent bundles T^*X and TX (see also subsection 2.5). The fibres of T^*X are generated by the differentials of Lipschitz functions, and the tangent bundle of TX is defined *formally* by duality: part of the motivation of the present work is to give a concrete description of TX by using velocity vectors of Lipschitz curves.

(3) Natural dual norms $\|\cdot\|_{\mathrm{Lip}}$ and $\|\cdot\|_{\mathrm{Lip}}^*$ on T^*X and TX respectively. The norm $\|\cdot\|_{\mathrm{Lip}}$ is induced by the pointwise upper Lipschitz constant, i.e. for any real-valued Lipschitz functions f we have $\|df\|_{\mathrm{Lip}} = \mathrm{Lip}\, f(x)$ for μ-a.e. $x \in X$.

We recall that $\mathrm{Lip}\, f(x)$ denotes the **(upper) pointwise Lipschitz constant of** f **at** x, that is:

$$\mathrm{Lip}\, f(x) = \limsup_{r \searrow 0} \sup \left\{ \frac{|f(y) - f(x)|}{r} : d_X(x, y) \le r \right\}. \tag{2.2}$$

We now give a brief review of some definitions from [14, 27]; an exposition can be found in [33]. Let (Z, ν) be a metric measure space. Let $\mathcal{U}$ be a (countable) collection of Lipschitz functions on Z. Then $\mathcal{U}$ is **dependent at** $x \in Z$ if some finite nontrivial linear combination ν of elements of $\mathcal{U}$ is constant to first order at x, i.e. $|\nu(y) - \nu(x)| = o(d_X(x, y)))$. Alternatively, one can say that the pointwise upper Lipschitz constant $\mathrm{Lip}\, \nu$ of ν vanishes at x. The **dimension of** $\mathcal{U}$ **at** x is the supremal cardinality of a subset that is linearly independent at x; the dimension function $\dim_{\mathcal{U}} : Z \to \mathbb{N} \cup \{\infty\}$ is Borel whenever $\mathcal{U}$ is a countable collection. Suppose that $U \subset Z$ is a Borel set with positive ν-measure, and that $\varphi : U \to \mathbb{R}^n$ is Lipschitz. The pair (U, φ) is a **differentiability chart** (or simply **chart**) if the component functions $\varphi_1, \dots, \varphi_n$ of φ are independent at ν-a.e. $x \in U$, and if for each real-valued Lipschitz function f, the $(n+1)$-tuple $(\varphi_1, \dots, \varphi_n, f)$ is dependent at ν-a.e. $x \in U$. In particular, there are, unique up to ν-null sets, Borel functions $\frac{\partial f}{\partial \varphi_i} : U \to \mathbb{R}$ such that the Taylor expansion (2.1) holds for ν-a.e. $x_0 \in U$; in this case we also say that f is **differentiable at** x_0 **with respect to the** $\{\varphi_i\}_{i=1}^n$.

A metric measure space (Z, ν) **admits a measurable differentiable structure** if there exists an countable collection of charts $\left\{(U_\alpha, \varphi_\alpha)\right\}_\alpha$ such that $Z \setminus (\cup_\alpha U_\alpha)$ is ν-null. Without loss of generality, we will always assume that for each pair (α, β), at each point of $U_\alpha \cap U_\beta$ the functions φ_α are differentiable with respect to the functions φ_β.

One says that a metric measure space (Z, ν) is **(almost everywhere) finite dimensional** if for any countable collection $\mathcal{U}$ of Lipschitz functions, the dimension $\dim_{\mathcal{U}}$ is finite almost everywhere. It follows from a selection argument [14, 27] that when ν is σ-finite, then (Z, ν) admits a measurable differentiable structure if and only if it is finite dimensional. Thus, apart from being a standard condition on a measure, σ-finiteness is a natural assumption in the present topic. As the measure ν only enters through its sets of measure zero, one really only cares about the *measure class* of ν; hence if ν is σ-finite, then without loss of generality one may take ν to be finite.

We finally give a brief justification of why we assume X to be complete in the definition of a differentiability space, which was also a working assumption in [10, 13]. Suppose (Z, ν) is a metric measure space, where Z is not necessarily complete. Denote by $\bar{Z}$ its completion, and let $\bar{\nu}$ be the pushforward of ν under the inclusion $Z \to \bar{Z}$. Then any Lipschitz function $u \in \mathrm{Lip}(Z)$ extends uniquely to $\bar{Z}$, and since Z is dense in $\bar{Z}$, the notions of dependence and dimension for a collection $\mathcal{U} \subset \mathrm{Lip}(Z)$ at any $x \in Z$ agree with the notions for the corresponding collection $\bar{\mathcal{U}} \subset \mathrm{Lip}(\bar{Z})$. Hence (Z, ν) has a measurable differentiable structure if and only if $(\bar{Z}, \bar{\nu})$ has a measurable differentiable structure.

2.2 The metric derivative for 1-rectifiable sets

Let Z be a separable metric space and denote by d_Z the metric on Z. We say that a pseudometric ϱ on Z is **Lipschitz compatible** if there is a nonnegative constant C such that:

$$\varrho \leq C d_Z; \tag{2.3}$$

we say that a function $f : Z \to W$ is ϱ-Lipschitz if there is a nonnegative C such that:

$$d_W(f(z_1), f(z_2)) \leq C\varrho(z_1, z_2) \quad (\forall z_1, z_2 \in Z). \tag{2.4}$$

Note that ϱ-Lipschitz functions are necessarily d_Z-Lipschitz; when referring to the background metric d_Z we will simply use the term Lipschitz. We denote by $\mathscr{H}^1$ the 1-dimensional Hausdorff measure on Z and by $\mathscr{H}^1_\varrho$ the 1-dimensional Hausdorff measure associated to the pseudometric ϱ.

We now recall metric differentiation results of [8, 9, 29] in the case of 1-rectifiable sets.

Let Y be a Lebesgue measurable subset of $\mathbb{R}$ and let $\gamma : Y \to Z$ be a Lipschitz map. We fix a countable dense subset $\{z_i\}$ of Z, and let u_i be the pullback of the pseudodistance function $\varrho_{z_i}(\cdot) = \varrho(\cdot, z_i)$ by the map γ. Then γ has a **ϱ-metric differential** $\varrho\text{-md}\gamma : Y \to [0, \infty)$, which is uniquely determined for $\mathcal{L}^1$ a.e. $t \in Y$, and which has the following properties:

(MD1) Rescalings of the pullback pseudometric $\gamma^*\varrho$ at t converge uniformly on compact sets to $\varrho\text{-md}\gamma(t)\, d_{\mathbb{R}}$, that is, the Euclidean distance scaled by the factor $\varrho\text{-md}\gamma(t)$.

(MD2) Consider a point $t \in Y$ such that:

(1) The point t is a Lebesgue density point of Y;
(2) The derivatives of the functions $\{u_i\}_i$ exist at t;
(3) The derivatives $\{u_i'\}_i$ are approximately continuous at t;
(4) The function $\sup_i |u_i'|$ is approximately continuous at t.

Then the ϱ-metric differential exists at t and is given by:

$$\varrho\text{-md}\gamma(t) = \sup_i \left|u_i'(t)\right|. \tag{2.5}$$

(MD3) One has an area formula [29, Thm. 7]:

$$\int_Z \#\left\{t \in Y : \gamma(t) = z\right\}\, d\mathscr{H}^1_\varrho(z) = \int_Y \varrho\text{-md}\gamma(t)\, d\mathcal{L}^1(t). \tag{2.6}$$

In the case in which the metric differential refers to the metric d_Z we will use the symbol $\text{md}\,\gamma$ instead of $d_Z\text{-md}\gamma$.

2.3 Alberti representations

Alberti representations were introduced in [1] to prove the so-called rank-one property for BV functions; they were later applied to study the differentiability properties of Lipschitz functions $f : \mathbb{R}^N \to \mathbb{R}$ [2, 3] and have recently been used to obtain a description of measures in differentiability spaces [10]. We first give an informal definition.

An **Alberti representation** of a Radon measure μ is a generalized Lebesgue decomposition of μ in terms of 1-rectifiable measures: i.e. one writes μ as an integral:

$$\mu = \int \nu_\gamma\, dP(\gamma), \tag{2.7}$$

where $\{\nu_\gamma\}$ is a family of 1-rectifiable measures. The standard example is offered by Fubini's Theorem; given $x \in \mathbb{R}^{N-1}$, denote by $\gamma(x)$ the parametrized line in $\mathbb{R}^N$ given by $\gamma(x)(t) = x + te_N$; then an Alberti representation of the Lebesgue measure $\mathcal{L}^N$ is given by:

$$\mathcal{L}^N = \int_{\mathbb{R}^{N-1}} \mathscr{H}^1_{\gamma(x)}\, d\mathcal{L}^{N-1}(x). \tag{2.8}$$

To make the previous account more precise we introduce more terminology. For more details we refer the reader to [10] and [42, Sec. 2.1]; note however, that we slightly diverge from the treatments in [10, 42] because we discuss also unbounded 1-rectifiable sets: the need to do so becomes apparent in Section 7.

Definition 2.9. A **fragment in** X is a Lipschitz map $\gamma : C \to X$, where $C \subset \mathbb{R}$ is closed. The set of fragments in X will be denoted by $\mathrm{Frag}(X)$.

We need to topologize $\mathrm{Frag}(X)$; let $F(\mathbb{R}\times X)$ denote the set of closed subsets of $\mathbb{R}\times X$ with the Fell topology [25, (12.7)]; we recall that a basis of the Fell topology consists those sets of the form:

$$\left\{F \in F(\mathbb{R}\times X) : F \cap K = \emptyset, F \cap U_i \neq \emptyset \text{ for } i = 1, \dots, n\right\}, \tag{2.10}$$

where K is a compact subset of $\mathbb{R}\times X$, and $\{U_i\}_{i=1}^n$ is a finite collection of open subsets of $\mathbb{R}\times X$. Note that the empty set $\emptyset$ is included in $F(\mathbb{R}\times X)$ and that, if X is locally compact, the topological space $F(\mathbb{R}\times X)$ is compact. We now consider the set $F_c(\mathbb{R}\times X) = F(\mathbb{R}\times X) \setminus \{\emptyset\}$ which is, if X is locally compact, a K_σ, i.e. a countable union of compact sets. Each fragment γ can be identified with an element of $F_c(\mathbb{R}\times X)$ and thus $\mathrm{Frag}(X)$ will be topologized as a subset of $F_c(\mathbb{R}\times X)$. We will use fragments to parametrize 1-rectifiable subsets of X.

We now briefly discuss the topology on Radon measures that allows to make sense of an integral like (2.7). Let $C_c(X)$ denote the set of continuous function defined on X with compact support; recall that the set $C_c(X)$ is a Fréchet space. We denote by $\mathrm{Rad}(X)$ the set of (nonnegative) Radon measures on X; as $\mathrm{Rad}(X)$ can be identified with a subset of the dual of $C_c(X)$, we will topologize it with the restriction of the weak* topology. In particular, when we assert that a map $\psi : Z \to \mathrm{Rad}(X)$ is Borel, we mean that for each $g \in C_c(X)$, the map:

$$z \mapsto \int_X g(x)\, d\left(\psi(z)\right)(x) \tag{2.11}$$

is Borel.

Definition 2.12. An Alberti representation of the measure μ is a pair (P, ν) such that:

- **(Alb1)** P is a Radon measure on $\mathrm{Frag}(X)$;
- **(Alb2)** The map $\nu : \mathrm{Frag}(X) \to \mathrm{Rad}(X)$ is Borel and, for each $\gamma \in \mathrm{Frag}(X)$, we have $\nu_\gamma \ll \mathscr{H}^1_\gamma$, where $\mathscr{H}^1_\gamma$ denotes the 1-dimensional Hausdorff measure on the image of γ;
- **(Alb3)** The measure μ can be represented as $\mu = \int_{\mathrm{Frag}(X)} \nu_\gamma\, dP(\gamma)$;
- **(Alb4)** For each Borel set $A \subset X$ and all real numbers $b \geq a$, the map $\gamma \mapsto \nu_\gamma\left(A \cap \gamma(\mathrm{Dom}\,\gamma \cap [a, b])\right)$ is Borel.

We now recall some definitions regarding additional properties of Alberti representations.

Definition 2.13. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be C**-Lipschitz (resp.** (C, D)**-biLipschitz)** if P-a.e. γ is C-Lipschitz (resp. (C, D)-biLipschitz).

Definition 2.14. Let $\sigma : X \to [0, \infty)$ be Borel and $f : X \to \mathbb{R}$ be Lipschitz. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **have f-speed** $\geq \sigma$ **(resp.** $> \sigma$**)** if for P-a.e. $\gamma \in \mathrm{Frag}(X)$ and $\mathcal{L}^1 \llcorner \mathrm{Dom}\,\gamma$-a.e. t one has $(f\circ\gamma)'(t) \geq \sigma(\gamma(t))\,\mathrm{md}\,\gamma(t)$ (resp. $(f\circ\gamma)'(t) > \sigma(\gamma(t))\,\mathrm{md}\,\gamma(t)$).

Another property regards the *direction*, with respect to a finite tuple of Lipschitz functions, of the fragments used in an Alberti representation. To measure the direction one can use the notion of Euclidean cone:

Definition 2.15. Let $\theta \in (0, \pi/2)$, $v \in \mathbb{S}^{n-1}$; the **open cone** $\mathrm{Cone}(v, \theta) \subset \mathbb{R}^n$ with axis v and opening angle θ is:

$$\mathrm{Cone}(v, \theta) = \{u \in \mathbb{R}^n : \tan\theta\langle v, u\rangle > \|\pi_v^\perp u\|_2\}, \tag{2.16}$$

where $\pi_v^\perp$ denotes the orthogonal projection on the orthogonal complement of the line $\mathbb{R}v$.

Definition 2.17. Given a Lipschitz function $f : X \to \mathbb{R}^n$, an Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **in the f-direction of the open cone** $\mathrm{Cone}(v, \theta)$ if for P-a.e. $\gamma \in \mathrm{Frag}(X)$ and $\mathcal{L}^1 \llcorner \mathrm{Dom}\,\gamma$-a.e. t one has $(f \circ \gamma)'(t) \in \mathrm{Cone}(v, \theta)$.

For the purpose of this paper it will be convenient to obtain Alberti representations with biLipschitz constants close to 1. We will thus use the following result [42, Thm. 2.64]:

Theorem 2.18. *Let X be a complete separable metric space and μ a Radon measure on X. Then the following are equivalent:*

(1) The measure μ admits an Alberti representation $\mathcal{A}$ in the f-direction of $\mathrm{Cone}(v, \theta)$ *with g-speed $> \sigma$;*
(2) For each $\varepsilon > 0$ the measure μ admits a $(1, 1+\varepsilon)$-biLipschitz Alberti representation $\mathcal{A}$ in the f-direction of $\mathrm{Cone}(v, \theta)$ *with g-speed $> \sigma$.*

Moreover, one can always assume that the Alberti representation is of the form $\mathcal{A} = (P, \nu)$, where P is a finite Radon measure concentrated on the set of fragments with compact domain. Additionally, one can assume that $\nu = h\Psi$ where h is a nonnegative Borel function of X and:

$$\Psi_\gamma = \gamma_\sharp \left(\mathcal{L}^1 \llcorner \mathrm{Dom}\,\gamma\right), \tag{2.19}$$

i.e. the push-forward of the restriction of the Lebesgue measure to the domain of γ.

Sometimes we will find it useful to **restrict an Alberti representation** $\mathcal{A} = (P, \nu)$ to a Borel set $U \subset X$ by letting $\mathcal{A} \llcorner U = (P, \nu \llcorner U)$. Other times one knows the existence of Alberti representations on subsets $\{U_\alpha\}_\alpha$ and would like to glue them together. This is accomplished by the following *gluing principle* [42, Thm. 2.67]:

Theorem 2.20. *Let $\{U_\alpha\}_\alpha$ be Borel subsets and suppose that for each α the measure $\mu \llcorner U_\alpha$ admits a (C, D)-biLipschitz Alberti representation in the f-direction of* $\mathrm{Cone}(v, \theta)$ *with f-speed $\geq \sigma$ (or $> \sigma$); then the measure $\mu \llcorner \bigcup_\alpha U_\alpha$ also admits a (C, D)-biLipschitz Alberti representation in the f-direction of* $\mathrm{Cone}(v, \theta)$ *with f-speed $\geq \sigma$ (or $> \sigma$).*

2.4 Results from Bate and Speight

We now recall some results [10, 13] on the structure of measures in differentiability spaces. The original Theorems [14, 27] on the existence of differentiable structures required the measure μ to be doubling. Bate and Speight [13] found a partial converse of this:

Theorem 2.21. *If (X, μ) is a differentiability space, then:*

- *The measure μ is **asymptotically doubling**, i.e. for μ-a.e. x there are $(C_x, r_x) \in (0, \infty)^2$ such that:*

$$\mu\left(B(x, 2r)\right) \leq C_x \mu\left(B(x, r)\right) \quad (\forall r \leq r_x). \tag{2.22}$$

  As a consequence, (X, μ) is a Vitali space, *i.e. the Vitali Covering Theorem holds in (X, μ), and thus also Lebesgue's Differentiation Theorem holds for μ.*
- *Every porous subset is μ-null.*

It was shown in [11, Lemma 8.3] that if if (X, μ) is asymptotically doubling, there are countably many Borel sets $\{U_\alpha\}_\alpha$ such that $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$ and such that each U_α is doubling as a metric space. Moreover, the sets $\{U_\alpha\}_\alpha$ might be assumed to be closed or compact.

Remark 2.23. In particular, at generic points of each U_α, one can obtain blow-ups/tangent cones of (X, μ) by using Gromov's Compactness Theorem (see Section 7). In fact, as porous sets are μ-null, blowing-up

$(U_\alpha, \mu \llcorner U_\alpha)$ at a point p of U_α which is a Lebesgue density point for μ, and is also a point at which U_α is not porous in the ambient space X, will yield the same metric measure spaces as blowing-up (X, μ).

Recently Bate [10] made a deep study of the structure of measures in differentiability spaces by using Alberti representations; in particular, he was able to obtain several characterizations of these spaces. For the sake of brevity we just summarize one characterization as follows:

Theorem 2.24. *The metric measure space (X, μ) is a differentiability space if and only if:*

(1) The measure μ is asymptotically doubling and porous sets are μ-null;
(2) There is a Borel function $\tau : X \to (0, \infty)$ such that, for each real-valued Lipschitz function f, the measure μ admits an Alberti representation with f-speed $\geq \tau \operatorname{Lip} f$.

In [42] it was shown that one may take $\tau = 1$: in subsection 6.2 we provide a proof of this fact which is independent of the results in [42]. To put this in perspective we recall the following definition:

Definition 2.25. Let $f : X \to \mathbb{R}$ be Lipschitz. The **lower pointwise Lipschitz constant of** f **at** x is:

$$\operatorname{lip} f(x) = \liminf_{r \searrow 0} \sup \left\{ \frac{|f(y) - f(x)|}{r} : d_X(x, y) \leq r \right\}. \tag{2.26}$$

In [27] it was shown that the existence of a measurable differentiable structure follows under the assumption that (X, μ) satisfies a **Lip-lip inequality**: this means that there is a $K \geq 1$ such that, for each real-valued Lipschitz function f, one has:

$$\operatorname{Lip} f(x) \leq K \operatorname{lip} f(x) \quad (\text{for } \mu\text{-a.e. } x). \tag{2.27}$$

In particular, Theorem 2.24 implies that in a differentiability space the Lip-lip inequality holds by replacing the constant K with the function τ; thus, showing that one can take $\tau = 1$ implies that the Lip-lip inequality self-improves to an equality. For the case of PI-spaces, the Lip-lip equality was a main result of [14], which followed from the more general result that, for $p > 1$, $\operatorname{Lip} f$ is a representative of the *minimal generalized upper gradient* of f.

The result of [10] that we will mainly use is the existence of Alberti representations in the directions of arbitrary cones:

Theorem 2.28. *Let (U, ψ) be an N-dimensional differentiability chart for the differentiability space (X, μ); then for each $v \in \mathbb{S}^{N-1}$ and each $\theta \in (0, \pi/2)$, the measure $\mu \llcorner U$ admits an Alberti representation in the ψ-direction of* $\operatorname{Cone}(v, \theta)$.

2.5 Measurable Vector Bundles

In this paper we will work with measurable subbundles of the tangent and cotangent bundles associated to a differentiability space. Since we deal with different (measurable) seminorms on these subbundles, we need to introduce a bit of terminology to make the treatment precise. Let (Ω, Σ) be a measure space; a **Σ-measurable vector bundle over** Ω is a quadruple $\mathcal{V} = (I_{\mathcal{V}}, \{N_\alpha\}_{\alpha \in I_{\mathcal{V}}}, \{U_\alpha\}_{\alpha \in I_{\mathcal{V}}}, \{g_{\alpha,\beta}\}_{(\alpha,\beta) \in I_{\mathcal{V},\cap}})$ such that:

(1) The index set $I_{\mathcal{V}}$ is countable and $\{U_\alpha\}_{\alpha \in I_{\mathcal{V}}}$ is a cover of Ω consisting of Σ-measurable sets;
(2) Each N_α is a nonnegative integer and if $U_\alpha \cap U_\beta \neq \emptyset$, then $N_\alpha = N_\beta$;
(3) The (possibly empty set) $I_{\mathcal{V},\cap}$ consists of those pairs $(\alpha, \beta) \in I_{\mathcal{V}} \times I_{\mathcal{V}}$ such that $U_\alpha \cap U_\beta \neq \emptyset$;
(4) Each $g_{\alpha,\beta}$ is a Σ-measurable map $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}\left(\mathbb{R}^{N_\alpha}\right)$.
(5) The collection $\{g_{\alpha,\beta}\}$ satisfies the cocycle condition, i.e. if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then $g_{\alpha,\gamma} = g_{\beta,\gamma} \circ g_{\alpha,\beta}$ Σ-a.e.

If $N = \sup_\alpha N_\alpha < \infty$ the bundle $\mathcal{V}$ is said **to have finite dimension** N.

A **section** σ **of** $\mathcal{V}$ is a collection $\{\sigma_\alpha\}_{I_{\mathcal{V}}}$ of Σ-measurable maps $\sigma_\alpha : U_\alpha \to \mathbb{R}^{N_\alpha}$ such that:

$$g_{\alpha,\beta} \circ \sigma_\alpha = \sigma_\beta. \tag{2.29}$$

A measurable subbundle of $\mathcal{V}$ is a measurable choice of a hyperplane in each fibre. More precisely, let $\mathrm{Gr}\left(\mathbb{R}^N, k\right)$ denote the Grassmanian of unoriented k-dimensional planes in $\mathbb{R}^N$; then a **subbundle** $\mathcal{W}$ of $\mathcal{V}$ is a pair $(\{M_\alpha\}_{I_\mathcal{V}}, \{\varphi_\alpha\}_{I_\mathcal{V}})$ such that:

(1) Each nonnegative integer M_α satisfies $M_\alpha \le N_\alpha$ and if $(\alpha, \beta) \in I_{\mathcal{V},\cap}$, then $M_\alpha = M_\beta$;
(2) Each φ_α is a Σ-measurable map $\varphi_\alpha : U_\alpha \to \mathrm{Gr}\left(\mathbb{R}^{N_\alpha}, M_\alpha\right)$;
(3) For each pair $(\alpha, \beta) \in I_{\mathcal{V},\cap}$ the following compatibility condition holds:

$$g_{\alpha,\beta}\left(\varphi_\alpha(x)\right) = \varphi_\beta(x) \quad (\forall x \in U_\alpha \cap U_\beta). \tag{2.30}$$

We now turn to the construction of seminorms on $\mathcal{V}$ (or on a subbundle).

We define a **generalized seminorm** on $\mathbb{R}^n$ to be a function $\|\cdot\| : \mathbb{R}^N \to \mathbb{R}_+ \cup \{\infty\}$ satisfying the obvious variants of the usual homogeneity and subadditivity conditions:

- $\|0\| = 0$.
- For all $x \in \mathbb{R}^N$, $a \in (0, \infty)$ we have $\|ax\| = a\|x\|$.
- For all $x, y \in \mathbb{R}^N$ we have $\|x + y\| \le \|x\| + \|y\|$ whenever $\|x\|$ and $\|y\|$ are both finite.

For every generalized seminorm $\|\cdot\|$ on $\mathbb{R}^N$, we have the associated supergraph

$$SG(\|\cdot\|) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R} \mid y \ge \|x\|\}\,.$$

The supergraph is a closed convex subset of $\mathbb{R}^{N+1}$ which determines the generalized seminorm uniquely. Note that for any collection $\{\|\cdot\|_\tau\}_{\tau\in\mathcal{T}}$ of generalized seminorms on $\mathbb{R}^N$, the pointwise supremum $\sup_{\tau\in\mathcal{T}} \|\cdot\|_\tau$ defines a generalized seminorm, which we denote by $\|\cdot\|_\mathcal{T}$; if the family is bounded above in the sense that there is a seminorm $\|\cdot\|_0$ such that $\|\cdot\|_\tau \le \|\cdot\|_0$ for all $\tau \in \mathcal{T}$, then the supremum $\|\cdot\|_\mathcal{T}$ is a seminorm. The supergraph $SG(\|\cdot\|_\mathcal{T})$ is the intersection of the supergraphs $\{SG(\|\cdot\|_\tau)\}_{\tau\in\mathcal{T}}$.

Let $\mathrm{Sem}(\mathbb{R}^N)$ and $\mathrm{Sem}_{+\infty}(\mathbb{R}^N)$ denote the collections of seminorms and generalized seminorms on $\mathbb{R}^N$, respectively. Taking supergraphs defines an injection ι from $Sem_{+\infty}(\mathbb{R}^N)$ to the collection of closed subsets of $\mathbb{R}^{N+1}$; we topologize $Sem_{+\infty}(\mathbb{R}^N)$ with the topology induced by ι from the pointed Hausdorff topology on the collection of closed subsets of $\mathbb{R}^{N+1}$. Note that the subspace topology on $\mathrm{Sem}(\mathbb{R}^N) \subset \mathrm{Sem}_{+\infty}(\mathbb{R}^N)$ agrees with the compact-open topology on $\mathrm{Sem}(\mathbb{R}^N)$.

A seminorm (resp. a generalized seminorm) $\|\cdot\|$ on $\mathcal{V}$ is a collection $\{\|\cdot\|_\alpha\}_{\alpha\in I_\mathcal{V}}$ of Σ-measurable maps $\|\cdot\|_\alpha : U_\alpha \to \mathrm{Sem}\left(\mathbb{R}^{N_\alpha}\right)$ (resp. $\mathrm{Sem}_{+\infty}\left(\mathbb{R}^{N_\alpha}\right)$) which satisfy, for each $(\alpha, \beta) \in I_{\mathcal{V},\cap}$, a.e. $x \in U_\alpha \cap U_\beta$, and each $v \in \mathbb{R}^{N_\alpha}$, the following compatibility condition:

$$\|v\|_\alpha(x) = \left\|g_{\alpha,\beta}(v)\right\|_\beta(x) \quad (\forall x \in U_\alpha \cap U_\beta). \tag{2.31}$$

We will essentially work with measurable bundles where $\Omega = X$, a complete separable metric space, and where Σ is the Borel σ-algebra. However, in the case of a metric measure space (X, μ), we implicitly identify vector bundles, sections and seminorms which agree μ-a.e. For example, consider two Borel vector bundles $\mathcal{V} = (I_\mathcal{V}, \{N_\alpha\}_{\alpha\in I_\mathcal{V}}, \{U_\alpha\}_{\alpha\in I_\mathcal{V}}, \{g_{\alpha,\beta}\}_{(\alpha,\beta)\in I_{\mathcal{V},\cap}})$ and $\mathcal{V}' = (I'_\mathcal{V}, \{N'_{\alpha'}\}_{\alpha'\in I'_\mathcal{V}}, \{U'_{\alpha'}\}_{\alpha'\in I'_\mathcal{V}}, \{g'_{\alpha,\beta}\}_{(\alpha',\beta')\in I'_{\mathcal{V},\cap}})$ over X; we identify them if:

(1) Whenever $\mu(U_\alpha \cap U'_{\alpha'}) > 0$ one has $N_\alpha = N'_{\alpha'}$;
(2) Whenever $\mu(U_\alpha \cap U'_{\alpha'}) > 0$ there are a μ-full measure subset $V_{\alpha,\alpha'} \subset U_\alpha \cap U'_{\alpha'}$ and a Borel map $G_{\alpha,\alpha'} : V_{\alpha,\alpha'} \to \mathrm{GL}\left(\mathbb{R}^{N_\alpha}\right)$, such that, if $\mu(U_\beta \cap U'_{\beta'}) > 0$, one has:

$$G_{\beta,\beta'} \circ g_{\alpha,\beta}(x) = g_{\alpha',\beta'} \circ G_{\alpha,\alpha'}(x) \quad (\text{for } \mu\text{-a.e. } x \in V_{\alpha,\alpha'} \cap V_{\beta,\beta'}). \tag{2.32}$$

To construct seminorms on measurable vector bundles we will use often the following lemma.

Lemma 2.33. *Let $\mathcal{V}$ be a measurable vector bundle over X and let $\left\{\|\cdot\|_\tau\right\}_{\tau\in T}$ be a countable collection of generalized seminorms on $\mathcal{V}$.*

We may define a collection of measurable maps $\{\|\cdot\|_{T,\alpha} : U_\alpha \to \mathrm{Sem}_{+\infty}(\mathbb{R}^{N_\alpha})\}_{\alpha\in I_\alpha}$ *by taking the pointwise supremum of seminorms:*

$$\|\cdot\|_{T,\alpha}(x) = \sup_{\tau\in\mathcal{T}} \|\cdot\|_{\tau,\alpha}(x) = \|\cdot\|_{T,\alpha}(x). \tag{2.34}$$

Then $\{\|\cdot\|_{T,\alpha}\}_{\alpha\in I_{\mathcal{V}}}$ *defines a generalized seminorm* $\|\cdot\|_T$ *on* $\mathcal{V}$*, which we call the* ***supremum of the seminorms*** $\{\|\cdot\|_T\}_{\tau\in T}$*. If there is seminorm* $\|\cdot\|_0$ *on* $\mathcal{V}$ *such that*

$$\|\cdot\|_\tau \le \|\cdot\|_0 \tag{2.35}$$

holds almost everywhere, then

$$\|\cdot\|_{\mathcal{T}} \le \|\cdot\|_0 \tag{2.36}$$

almost everywhere, and $\|\cdot\|_{\mathcal{T}}$ *is a seminorm (almost everywhere).*

Suppose now that μ *is a* σ*-finite Borel measure on* X *and let* $\{\|\cdot\|_{\omega}\ .\}_{\omega\in\Omega}$ *be a collection of seminorms on* $\mathcal{V}$ *which is allowed to be uncountable. Then there is a* μ*-a.e. unique generalized seminorm* $\|\cdot\|_\Omega$*, called the* ***essential supremum*** *of the collection* $\{\|\cdot\|_{\omega}\ .\}_{\omega\in\Omega}$*, which satisfies the following properties:*

(Ess-sup1) *For each section* σ *of* $\mathcal{V}$ *and each* $\omega\in\Omega$ *one has:*

$$\|\sigma\|_\Omega \ge \|\sigma\|_\omega \quad \mu\text{-a.e.}; \tag{2.37}$$

(Ess-sup2) *If* $\|\cdot\|'_\Omega$ *is another generalized seminorm satisfying (2.37), then one has:*

$$\|\cdot\|'_\Omega \ge \|\cdot\|_\Omega \quad \mu\text{-a.e.} \tag{2.38}$$

(Ess-sup3) *There is a countable subcollection* $\Omega'\subset\Omega$ *such that the countable supremum* $\|\cdot\|_{\Omega'}$ *as defined after (2.34) agrees with* $\|\cdot\|_\Omega$*.*

Moreover, if there are a seminorm $\|\cdot\|$ *on* $\mathcal{V}$ *and a* $C\ge 0$ *such that:*

$$\|\cdot\|_\omega \le C\,\|\cdot\| \tag{2.39}$$

holds μ*-a.e. and uniformly in* ω*, then* $\|\cdot\|_\Omega$ *can be taken to be a seminorm satisfying:*

$$\|\cdot\|_\Omega \le C\,\|\cdot\| \quad \mu\text{-a.e.} \tag{2.40}$$

Proof. The proof that $\|\cdot\|_T$ defines a generalized seminorm, which is also a norm under the additional assumption (2.35), follows by unwinding the definition of a measurable vector bundle. To prove the second part of Lemma 2.33 we use the approach of [23, Prop. 5.4.7].

We first observe that **(Ess-sup1)** and **(Ess-sup2)** are properties that hold up to μ-null sets, and thus we can construct $\|\cdot\|_\Omega$ independently on each $\mathcal{V}|U_\alpha$, where $\mathcal{V}|U_\alpha$ denotes the union of the fibres of $\mathcal{V}$ over the points $x\in U_\alpha$. We can therefore assume that the cardinality of $I_{\mathcal{V}}$ is one and identify $\mathcal{V}$ with the product $U\times\mathbb{R}^N$. Without loss of generality we can also assume that μ is a probability measure on U. We take a norm $\|\cdot\|$ on $\mathbb{R}^N$, and denote by S^{N-1} and $\mathscr{H}^{N-1}$ the corresponding unit sphere and $(N-1)$-dimensional Hausdorff measure. We finally let π be the probability measure

$$\mu\otimes\mathscr{H}^{N-1}\llcorner S^{N-1}/\mathscr{H}^{N-1}(S^{N-1}). \tag{2.41}$$

Let $\{\|\cdot\|_{\hat\omega}\}_{\hat\omega\in\hat\Omega}$ be the collection of supremums of all finite subcollections of $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$, i.e. the collection of all generalized seminorms of the form

$$\|\cdot\|_{\hat\omega} = \sup\{\|\cdot\|_{\omega_1},\dots,\|\cdot\|_{\omega_k}\}$$

where $\{\omega_1,\dots,\omega_k\}$ ranges over all finite subsets of the collection $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$.

Then $\{\|\cdot\|_\omega\}_{\omega\in\hat\Omega}$ is *upward-filtering*, i.e. for all pairs $(\omega,\omega')\in\hat\Omega^2$ there is an $\omega''\in\hat\Omega$ satisfying:

$$\|\cdot\|_{\omega''} = \max\left\{\|\cdot\|_\omega,\|\cdot\|_{\omega'}\right\}. \tag{2.42}$$

We now consider the increasing homeomorphism:

$$\begin{aligned} \zeta &: \mathbb{R} \to (0,1) \\ t &\mapsto \frac{e^t}{e^t+1}, \end{aligned} \tag{2.43}$$

and observe that the random variables $\left\{\zeta\left(\|\cdot\|_\omega\right)\right\}_{\omega\in\hat{\Omega}}$ are all nonnegative and have π-expectations satisfying:

$$E\left[\zeta\left(\|\cdot\|_\omega\right)\right] \le 1. \tag{2.44}$$

Thus the supremum:

$$q = \sup\left[E\left[\zeta\left(\|\cdot\|_\omega\right)\right] : \|\cdot\|_\omega \in \hat{\Omega}\right\} \tag{2.45}$$

is finite, and we let $T = \{\|\cdot\|_{\omega_n}\}$ denote a maximizing sequence:

$$\lim_{n\to\infty} E\left[\zeta\left(\|\cdot\|_{\omega_n}\right)\right] = q. \tag{2.46}$$

The proof is completed by showing that the countable supremum $\|\cdot\|_T$ satisfies **(Ess-sup1)**, **(Ess-sup2)**, and **(Ess-sup3)**.

We first address **(Ess-sup1)**: suppose that one has $\|\cdot\|_\omega > \|\cdot\|_T$ on a set of positive measure. Then, considering the sequence of norms $\left\{\max\left\{\|\cdot\|_{\omega_n}, \|\cdot\|_\omega\right\}\right\} \subset \hat{\Omega}$ one contradicts (2.46).

We now address **(Ess-sup2)** and take a norm $\|\cdot\|'_\Omega$ satisfying (2.37). Let $A \subset U$ be a set of positive μ-measure: we claim that one has:

$$\lim_{n\to\infty} E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|_{\omega_n}\right)\right] = \sup\left\{E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|_\omega\right)\right] : \omega \in \hat{\Omega}\right\} = E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|_T\right)\right]. \tag{2.47}$$

In fact, if any of the equalities in (2.47) failed, using that ζ is positive and that the collection $\{\|\cdot\|_\omega \cdot\}_{\omega\in\hat{\Omega}}$ is upward-filtering, one would contradict (2.46). As ζ is increasing, we have

$$E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|'_\Omega\right)\right] \ge E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|_{\omega_n}\right)\right], \tag{2.48}$$

and from (2.47) it follows that:

$$E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|'_\Omega\right)\right] \ge E\left[\chi_{A\times\mathbb{R}^N}\zeta\left(\|\cdot\|_T\right)\right], \tag{2.49}$$

from which (2.38) follows.

Note that **(Ess-sup3)** holds, since $\|\cdot\|_T$ is the countable supremum of the the seminorms $\|\cdot\|_{\omega_n}$, each of which is a finite supremum of seminorms from the collection $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$. ☐

3 Generic points and generic velocities

In this Section we fix a complete separable metric space X and introduce a notion of genericity for pairs $(\gamma, t) \in \mathrm{Frag}(X) \times \mathbb{R}$; this notion of genericity will be specified in terms of a quadruple $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ such that: $\mathcal{F}$ is a countable collection of real-valued Lipschitz functions defined on X, $\mathcal{C}$ is a countable collection of real-valued bounded Borel functions defined on X, D_X is a countable dense subset of X, and $\mathcal{S}$ is a countable collection of Lipschitz compatible pseudometrics on X which will always include the metric d_X.

Definition 3.1. We say that the pair (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-**generic** if:

(Gen1) The point t is a Lebesgue density point of $\mathrm{Dom}(\gamma)$;
(Gen2) For each $f \in \mathcal{F}$ the derivative $(f \circ \gamma)'$ exists and is approximately continuous at t;
(Gen3) For each $u \in \mathcal{C}$ the function $u \circ \gamma$ is approximately continuous at t;
(Gen4) For each $x \in D_X$ and each $\varrho \in \mathcal{S}$ the derivative $(\varrho_x \circ \gamma)'$ exists and is approximately continuous at t;
(Gen5) For each $\varrho \in \mathcal{S}$ the function $\sup_{x\in D_X} \left|(\varrho_x \circ \gamma)'(t)\right|$ is approximately continuous at t.

In the case in which $\mathcal{S}$ consists only of d_X we will just write $(\mathcal{F}, \mathcal{C}, D_X)$. Whenever a default choice of the set D_X is assumed, we will omit D_X from the notation.

Remark 3.2. We remark that the proof of [9, Thm. 4.1.6] shows that at a point t where **(Gen1)**, **(Gen4)** and **(Gen5)** hold, the ϱ-metric derivative $\varrho\text{-md}\gamma(t)$ exists and equals $\sup_{x\in D_X}\left|(\varrho_x\circ\gamma)'(t)\right|$. Thus, if (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic, for each $\varrho \in \mathcal{S}$ the ϱ-metric derivative exists and is approximately continuous at t.

We point out that, in the case of a differentiability space (X, μ), Definition 3.1 has a natural interpretation in terms of the μ-tangent bundle TX. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas for (X, μ) and suppose that $\mathcal{F}$ contains the components of all the coordinate functions $\{\varphi_\alpha\}$, and that $\mathcal{C}$ contains all the characteristic functions $\{\chi_{U_\alpha}\}$. Suppose now that (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic and that $\gamma(t) \in \bigcup_\alpha U_\alpha$; then $\gamma'(t)$ is a well-defined element of TX. We are thus led to the following definition.

Definition 3.3. A $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-**generic velocity vector** is an element of TX of the form $\gamma'(t)$, where:

- (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic;
- $\gamma(t) \in \cup_\alpha U_\alpha$;
- For all α, $\mathcal{F}$ contains the components of φ_α and $\mathcal{C}$ contains χ_{U_α}.

As above, in the case in which $\mathcal{S}$ consists only of d_X, we will just write $(\mathcal{F}, \mathcal{C}, D_X)$, and we will omit D_X from the notation if a default choice of the set D_X is assumed.

We now establish measurability for generic pairs.

Lemma 3.4. *The set*

$$G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X) = \left\{(\gamma, t) : (\gamma, t) \text{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)\text{-generic}\right\} \tag{3.5}$$

is a Borel subset of $\mathrm{Frag}(X) \times \mathbb{R}$.

Proof. We prove that $G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ is Borel by showing that certain sets are Borel. Let DOM denote the set of pairs (γ, t) such that $t \in \mathrm{Dom}\,\gamma$:

$$\mathrm{DOM} = \left\{(\gamma, t) \in \mathrm{Frag}(X) \times \mathbb{R} : t \in \mathrm{Dom}\,\gamma\right\}; \tag{3.6}$$

then DOM is closed. Fix $\delta > 0$ and consider the set of pairs (γ, t) where t becomes isolated below scale δ:

$$\mathrm{ISOL}(\delta) = \left\{(\gamma, t) \in \mathrm{DOM} : \mathrm{Dom}\,\gamma \cap (t-\delta, t+\delta) \text{ contains only one point}\right\}; \tag{3.7}$$

then $\mathrm{ISOL}(\delta)$ is closed and $\mathrm{ISOL} = \bigcup_{\delta\in\mathbb{Q}_{>0}} \mathrm{ISOL}(\delta)$ is Borel and consists of the pairs (γ, t) where t is an isolated point of $\mathrm{Dom}\,\gamma$. We can thus attempt to define, for a Lipschitz compatible pseudometric ϱ, the ϱ-metric derivative and, for f Lipschitz, the derivative of f at pairs in $\mathrm{DOM} \setminus \mathrm{ISOL}$. Consider the set:

$$\begin{aligned}\mathrm{MDIFF}(\varrho) &= \left\{(\gamma, t) \in \mathrm{DOM}\setminus\mathrm{ISOL} : \varrho\text{-md}\gamma(t) \text{ exists}\right\}\\ &= \bigcap_{\varepsilon\in\mathbb{Q}_{>0}} \bigcup_{(\delta,\theta)\in\mathbb{Q}_{>0}\times\mathbb{Q}_{\geq 0}} \Big\{(\gamma, t) \in \mathrm{DOM}\setminus\mathrm{ISOL} : \forall s_1, s_2 \in (t-\delta, t+\delta)\cap\mathrm{Dom}\,\gamma,\\ &\qquad \left|\varrho(\gamma(s_1), \gamma(s_2)) - \theta|s_1 - s_2|\right| \leq \varepsilon|s_1 - s_2|\Big\};\end{aligned} \tag{3.8}$$

this set is Borel as all the sets in the curly brackets are closed in $\mathrm{DOM}\setminus\mathrm{ISOL}$. Modifying the definition of MDIFF by constraining θ to lie in a specified interval we also conclude that the map:

$$\begin{aligned}\mathrm{MDer}(\varrho) : \mathrm{MDIFF} &\to [0, \infty)\\ (\gamma, t) &\mapsto \varrho\text{-md}\gamma(t)\end{aligned} \tag{3.9}$$

is Borel. Consider now a real-valued Lipschitz function f defined on X; the set DIFF(f) where $(f\circ\gamma)'(t)$ exists is Borel because we can write it as:

$$\begin{aligned}\mathrm{DIFF}(f) &= \left\{(\gamma,t)\in \mathrm{DOM}\setminus\mathrm{ISOL}:\ f\circ\gamma \text{ is differentiable at } t\right\}\\ &= \bigcap_{\varepsilon\in\mathbb{Q}_{>0}}\ \bigcup_{(\delta,\theta)\in\mathbb{Q}_{>0}\times\mathbb{Q}} \Big\{(\gamma,t)\in\mathrm{DOM}\setminus\mathrm{ISOL}: \forall s_1,s_2\in(t-\delta,t+\delta)\cap\mathrm{Dom}\,\gamma,\\ &\qquad \left|f\circ\gamma(s_1)-f\circ\gamma(s_2)-\theta(s_1-s_2)\right|\le\varepsilon|s_1-s_2|\Big\}\end{aligned}\tag{3.10}$$

where the sets in curly brackets are closed in DOM \ ISOL. Constraining the θ appearing in the definition of DIFF(f) to lie in a given interval we conclude that the map:

$$\begin{aligned}\mathrm{Der}(f):\mathrm{DIFF}(f)&\to\mathbb{R}\\ (\gamma,t)&\mapsto(f\circ\gamma)'(t)\end{aligned}\tag{3.11}$$

is Borel. Regarding condition **(Gen5)** we need also to take a sup of derivatives when they exist; so let Ω be a countable set of Lipschitz functions; then the set:

$$\mathrm{DIFF}(\Omega)=\bigcap_{f\in\Omega}\mathrm{DIFF}(f)\tag{3.12}$$

and the map:

$$\begin{aligned}|\mathrm{Der}(\Omega)|:\mathrm{DIFF}(\Omega)&\to\mathbb{R}\\ (\gamma,t)&\mapsto\sup_{f\in\Omega}\left|(f\circ\gamma)'(t)\right|\end{aligned}\tag{3.13}$$

are Borel.

We now turn to questions pertaining to the approximate continuity of a function at a point in the domain of a fragment. For $L\ge 0$ we will denote by SUB(L) the closed set of those fragments whose domain lies in $[-L,L]$. Suppose now that we are given a Borel set $B\subset\mathrm{DOM}$ and a Borel map $\psi:B\to\mathbb{R}$. For $(\varepsilon,\delta,L)\in(\mathbb{Q}_{>0})^3$ let:

$$\begin{aligned}\tilde\Psi(\varepsilon,\delta,L,B,\psi)=\Big\{&(\gamma,t,s)\in\mathrm{Frag}(X)\times\mathbb{R}^2: s,t\in[-L,L],\\ &(\gamma,t),(\gamma,s)\in B,\ |t-s|\le\delta \text{ and } \left|\psi(\gamma,t)-\psi(\gamma,s)\right|\le\varepsilon\Big\};\end{aligned}\tag{3.14}$$

the set $\tilde\Psi(\varepsilon,\delta,L,B,\psi)$ is Borel and [25, Thm. 17.25] shows that the map:

$$\begin{aligned}\mathrm{Leb}(\varepsilon,\delta,L,B,\psi):B&\to\mathbb{R}\\ (\gamma,t)&\mapsto\mathcal{L}^1\left(\left(\tilde\Psi(\varepsilon,\delta,L,B,\psi)\right)_{(\gamma,t)}\right)\end{aligned}\tag{3.15}$$

is Borel. It is then easy to prove that the sets of pairs $(\gamma,t)\in B$ where some map is approximately continuous at t is Borel; in fact, first define:

$$\begin{aligned}\mathrm{ACONT}(\psi)=\bigcup_{L\in\mathbb{Q}_{>0}}\bigcap_{\varepsilon\in\mathbb{Q}_{>0}}\bigcup_{\delta\in\mathbb{Q}_{>0}}\bigcap_{r\in\mathbb{Q}_{>0}}\Big\{&(\gamma,t)\in B: t\in[-L,L],\\ &\text{and for each } r\le\delta \text{ one has } \mathrm{Leb}(\varepsilon,r,L,B,\psi)(\gamma,t)\ge 2(1-\varepsilon)r\Big\},\end{aligned}\tag{3.16}$$

which is a Borel set; then, for example, ACONT(MDer(d_X)) consists of the pairs (γ,t) where $\mathrm{md}\,\gamma$ exists and is approximately continuous at t. In order to handle the approximate continuity for a Borel map $u:X\to\mathbb{R}$ we introduce the notation Ev(u) to denote the Borel map which evaluates u at $\gamma(t)$:

$$\begin{aligned}\mathrm{Ev}(u):\mathrm{DOM}&\to\mathbb{R}\\ (\gamma,t)&\mapsto u\circ\gamma(t).\end{aligned}\tag{3.17}$$

Letting $\psi : \mathrm{DOM} \to \mathbb{R}$ to be the function which trivially maps each pair (γ, t) to 0, we see that $\mathrm{ACONT}(\psi) = \mathrm{LEBDENS}$ is the set of pairs (γ, t) where t is a Lebesgue density point of $\mathrm{Dom}\,\gamma$.

We finally conclude that $G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ is Borel by observing that, when we use $B = \mathrm{DOM}$ to define ACONT in (3.16), we have:

$$\begin{aligned} G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X) = & \bigcap_{\varrho\in\mathcal{S}} \mathrm{ACONT}(\mathrm{MDer}(\varrho)) \cap \bigcap_{f\in\mathcal{F}} \mathrm{ACONT}(\mathrm{Der}(f)) \\ & \cap \bigcap_{u\in\mathcal{C}} \mathrm{ACONT}(\mathrm{Ev}(u)) \cap \bigcap_{x\in D_X, \varrho\in\mathcal{S}} \mathrm{ACONT}(\mathrm{Der}(\varrho_x)) \\ & \cap \bigcap_{\varrho\in\mathcal{S}} \mathrm{ACONT}\left(|\mathrm{Der}|(\{\varrho_x\}_{x\in D_X})\right) \cap \mathrm{LEBDENS}. \end{aligned} \tag{3.18}$$

□

4 Metric differentials and seminorms on TX

In this section we discuss the first instance of metric differentiation, Theorem 4.3. The point is that in the presence of a differentiable structure, the $\mathscr{H}^1_\varrho$-measure of a fragment γ can be recovered using a seminorm (canonically associated to ϱ) on the tangent bundle TX associated to the differentiable structure. Let (X, μ) be a differentiability space with atlas $\{(U_\alpha, \varphi_\alpha)\}$, and fix a countable dense set $D_X \subset X$.

Definition 4.1. Let Φ be a countable collection of Lipschitz functions on X; we say that a Borel subset $V \subset \bigcup_\alpha U_\alpha$ is a **Φ-differentiability** set if:

(Diff1) The set V has full μ-measure: $\mu\left(\bigcup_\alpha U_\alpha \setminus V\right) = 0$;

(Diff2) For each $(x, f) \in V \times \Phi$, if $x \in U_\alpha$, then f is differentiable at x with respect to the coordinate functions φ_α.

Let $\Phi_{D_X,\varrho} = \{\varrho_x : x \in D_X\}$ and let V be a $\Phi_{D_X,\varrho}$-differentiability set. Using Lemma 2.33, we obtain a seminorm $\|\cdot\|_{D_X,\varrho}$ on TX by defining, for $y \in V$ and $v \in T_yV$:

$$\|v\|_{D_X,\varrho} = \sup_{x\in D_X} \left| d\varrho_x \rvert_y (v)\right|. \tag{4.2}$$

Theorem 4.3. *Let $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ be as in* Section 3 *and let V be a $\Phi_{D_X,\varrho}$-differentiability set. Assume that $\mathcal{F}$ contains all the components of the coordinate functions φ_α, that $\mathcal{C}$ contains the characteristic functions $\{\chi_{U_\alpha}\}_\alpha \cup \{\chi_V\}$, and that $\varrho \in \mathcal{S}$. If $\gamma'(t)$ is an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vector and if $\gamma(t) \in V$, then the metric differential* ϱ-md$\gamma(t)$ *exists and equals $\|\gamma'(t)\|_{D_X,\varrho}$. In particular, if a fragment γ lies in V, we have:*

$$\mathscr{H}^1_\varrho(\mathrm{Im}\,\gamma) = \int_{\mathrm{Dom}\,\gamma} \|\gamma'(t)\|_{D_X,\varrho}\, dt. \tag{4.4}$$

Proof. To fix the ideas suppose that $\gamma(t) \in U_\alpha$. Because of conditions **(Gen4)**, **(Gen5)**, the argument in [9, Thm. 4.1.6] implies that

$$\varrho\text{-md}\gamma(t) = \sup_{x\in D_X} \left|(\varrho_x \circ \gamma)'(t)\right|; \tag{4.5}$$

as $\gamma(t) \in V$, for each $x \in D_X$ the pseudodistance function ϱ_x is differentiable at $\gamma(t)$ with respect to the coordinate functions φ_α; note also that $\varphi_\alpha \circ \gamma$ is differentiable at t by condition **(Gen2)**. Thus,

$$(\varrho_x \circ \gamma)'(t) = \sum_i \frac{\partial \varrho_x}{\partial \varphi^i_\alpha}(\gamma(t)) \left(\varphi^i_\alpha \circ \gamma\right)'(t) = d\varrho_x \rvert_{\gamma(t)} (\gamma'(t)), \tag{4.6}$$

which implies

$$\sup_{x\in D_X} \left|(\varrho_x \circ \gamma)'(t)\right| = \|\gamma'(t)\|_{D_X,\varrho}. \tag{4.7}$$

Formula (4.4) follows from the area formula (2.6) for the pseudometric ϱ by observing that for a fragment γ which lies in V, for $\mathcal{L}^1$-a.e. $t \in \mathrm{Dom}\,\gamma$, the velocity vector $\gamma'(t)$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic. □

In Section 8 (Theorem 8.24) we will show that for different choices D_X and $\tilde{D}_X$ of the countable dense set, the seminorms $\|\cdot\|_{D_X,\varrho}$ and $\|\cdot\|_{\tilde{D}_X,\varrho}$ are the same. The proof uses the density of directions at generic points which is discussed in the next Section. For the case in which $\varrho = d_X$ this follows from Theorem 6.1.

5 Density of generic directions at generic points

In this Section we show that for μ-a.e. $x \in X$ the set of vectors in T_xX which can be represented by $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vectors contains a dense set of "directions" in T_xX. We make this idea precise with the following definition:

Definition 5.1. If V is a finite-dimensional vector space, we say that a subset $W \subset V$ **contains a dense set of directions** if:

$$\overline{[0,\infty)W} = \overline{\{tw \mid t \in [0,\infty),\ w \in W\}} = V\,. \tag{5.2}$$

We now fix an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ for the differentiability space (X, μ) and let N_α denote the dimension of the chart $(U_\alpha, \varphi_\alpha)$. For each α let $\{\mathrm{Cone}(v_{\alpha,k}, \theta_{\alpha,k})\}_{k\in\mathbb{N}}$ denote a collection of open cones with $\{v_{\alpha,k}\} \subset \mathbb{S}^{N_\alpha-1}$ dense in the unit sphere and $\lim_{k\to\infty} \theta_{\alpha,k} = 0$. Using Theorem 2.28, we find Alberti representations $\mathcal{A}_k = (P_k, \nu_k)$ of μ such that, for each α, the restriction $\mathcal{A}_k \llcorner U_\alpha$ is in the φ_α-direction of $\mathrm{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$.

Theorem 5.3. *Let $\Gamma_0 \subset \mathrm{Frag}(X)$ be a Borel set such that, for each k one has $P_k\left(\mathrm{Frag}(X) \setminus \Gamma_0\right) = 0$; and let $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ be as in Definition 3.3. Then there is a μ-measurable subset $Y \subset X$ with full μ-measure such that, for each $x \in Y$, the set of velocity vectors*

$$G_x = \left\{ v \in T_xX \mid v = \gamma'(t) \text{ for } \gamma \in \Gamma_0 \textit{ such that } \gamma'(t) \textit{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)\textit{-generic} \right\}, \tag{5.4}$$

contains a dense set of directions in T_xX.

Proof. Let $Z_k \subset X \times \mathrm{Frag}(X) \times \mathbb{R}$ consist of those triples (x, γ, t) satisfying:

(1) $\gamma'(t)$ is an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vector;
(2) $\gamma(t) = x$ and $\gamma \in \Gamma_0$;
(3) If $\gamma(t) \in U_\alpha$, then $(\varphi_\alpha \circ \gamma)'(t) \in \mathrm{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$.

Using Lemma 3.4 we conclude that Z_k is Borel, and therefore its projection $Y_k \subset U$ on X is Suslin [25], and hence μ-measurable. Note that for each $\gamma \in \Gamma_0$, as $\nu_k(\gamma)$ is absolutely continuous with respect to $\mathscr{H}^1_\gamma$, one has $\nu_k(\gamma)(X \setminus Y_k) = 0$, and therefore $\mu(X \setminus Y_k) = 0$. We conclude that $Y = \bigcap_k Y_k$ is a μ-full measure μ-measurable subset of X. Let $x \in Y \cap U_\alpha$, and let $v \in T_xX$; then for each $\varepsilon > 0$ we can find a k such that, for each $w \in \mathrm{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$, there is a $t_w \in [0, \infty)$ with:

$$\|v - t_w w\|_{l^2} \le \varepsilon \|v\|_{l^2}; \tag{5.5}$$

but as $x \in Y \cap U_\alpha$, there are a fragment $\gamma_k \in \Gamma_0$ and a $t_k \in \mathbb{R}$ such that the vector $\gamma_k'(t_k) \in T_xX$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic and $(\varphi_\alpha \circ \gamma_k)'(t_k) \in \mathrm{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$; thus there is an $s_k \in [0, \infty)$ with

$$\|v - s_k(\varphi_\alpha \circ \gamma_k)'(t_k)\|_{l^2} \le \varepsilon \|v\|_{l^2}, \tag{5.6}$$

which implies $\overline{[0,\infty)G_x} = T_xX$. □

6 Consequences of density of generic directions

In this section we prove the equality of various seminorms on TX (Theorem 1.7), the equality $\mathrm{Lip}\, u = \mathrm{lip}\, u$ a.e. (Theorem 1.11), and give a new proof that in PI spaces the minimal generalized upper gradient agrees with the pointwise Lipschitz constant.

6.1 Equality of natural seminorms on TX

The main result in this subsection is the proof of Theorem 1.7, which is based on the following result.

Theorem 6.1. *Let (X, μ) be a differentiability space and $D_X \subset X$ a countable dense set. Then the seminorm $\|\cdot\|_{D_X,d_X}$ on TX provided by (4.2) (taking $\varrho = d_X$) coincides with the norm $\|\cdot\|^{\star}_{\mathrm{Lip}}$ (see Section 2.1); in particular, the norm $\|\cdot\|_{D_X,d_X}$ does not depend on the choice of D_X.*

Notation: After proving Theorem 1.7, we will change to the notation $\|\cdot\|_{TX}$, $\|\cdot\|_{T^\star X}$ or simply $\|\cdot\|$ to denote the canonical norms on TX and $T^\star X$.

Theorem 6.1 can be regarded as an infinitesimal version of metric differentiation for the identity map $\mathrm{id} : X \to X$; its proof uses the following lemma:

Lemma 6.2. *Suppose that $(V, \|\cdot\|)$ is a finite dimensional normed vector space, with dual space $(V^\star, \|\cdot\|^\star)$. Let W be a subset of the closed unit ball $\overline{B(\|\cdot\|)} \subset V$, such that:*

(H1) *For every $w \in W$, there is a linear functional $\alpha_w \in V^\star$ with $\|\alpha_w\|^\star \le 1$, such that $\alpha_w(w) = 1$;*
(H2) *The set W contains a dense set of directions.*

Then:

(1) For all $w \in W$ one has $\|\alpha_w\|^\star = 1$;
(2) The set W is a dense subset of the unit sphere $S(\|\cdot\|)$;
(3) The seminorm on V defined by $\sup_{w\in W} |\alpha_w(\cdot)|$ agrees with $\|\cdot\|$.

Proof. Note that by **(H1)** each α_w has unit norm (which implies (1)) and that each vector $w \in W$ has unit norm, which implies that $W \subset S(\|\cdot\|)$. Let $v \in S(\|\cdot\|)$; by **(H2)**, for each $\varepsilon > 0$ there are a $w_\varepsilon \in W$ and a $t_\varepsilon \in [0, \infty)$:

$$\|v - t_\varepsilon w_\varepsilon\| \le \varepsilon; \tag{6.3}$$

let β_v a unit norm functional on V assuming the norm at v. Then (6.3) implies:

$$|1 - t_\varepsilon \beta_v(w_\varepsilon)| \le \varepsilon; \tag{6.4}$$

as $|\beta_v(w_\varepsilon)| \le 1$, the previous equation implies $t_\varepsilon \ge 1 - \varepsilon$. On the other hand, evaluating with α_{w_ε}, (6.3) gives

$$\left|\alpha_{w_\varepsilon}(v) - t_\varepsilon\right| \le \varepsilon; \tag{6.5}$$

as the functional α_{w_ε} has unit norm, $t_\varepsilon \le 1 + \varepsilon$. We thus conclude that

$$\|v - w_\varepsilon\| \le \|v - t_\varepsilon w_\varepsilon\| + \|(1 - t_\varepsilon) w_\varepsilon\| \le 2\varepsilon, \tag{6.6}$$

implying (2). Note that, as the functionals α_w have unit norm,

$$\sup_{w\in W} \left\|\alpha_w(\cdot)\right\| \le \|\cdot\|. \tag{6.7}$$

On the other hand, for each $v \in V \setminus \{0\}$ and each $\varepsilon > 0$, choose $w_\varepsilon \in W$ with

$$\left\|\frac{v}{\|v\|} - w_\varepsilon\right\| \le \varepsilon; \tag{6.8}$$

then

$$\alpha_{w_\varepsilon}\left(\frac{v}{\|v\|}\right) \geq 1-\varepsilon, \tag{6.9}$$

implying that:

$$\sup_{w\in W}\left\|\alpha_w(\cdot)\right\| \geq (1-\varepsilon)\|\cdot\|, \tag{6.10}$$

from which (3) follows. □

Proof of Theorem 6.1. Let V be a differentiability set for the countable collection of Lipschitz functions $\{d_X(\cdot,x) : x\in D_X\}$. We let $\mathcal{F}$ contain the components of the coordinate functions, $\mathcal{C}$ contain the characteristic functions of the charts, and $\mathcal{S} = \{d_X\}$. Let Y be the set provided by Theorem 5.3; we will show that for each $p \in Y\cap V$ the norm $\|\cdot\|^\star_{\mathrm{Lip}}$ and the seminorm $\|\cdot\|_{D_X,d_X}$ coincide on the fibre T_pX. Let $\gamma'(t)\in G_p(\mathcal{F},\mathcal{C},D_X)$ with $\gamma'(t)\neq 0$; then $\mathrm{md}\,\gamma(t)\neq 0$. Without loss of generality we assume that p belongs to the chart U_α and we consider a functional $\sum_{i=1}^{N_\alpha} a_i\, d\varphi^i_\alpha\,|_p\in T^\star_pX$; then:

$$\left|\left\langle \sum_{i=1}^{N_\alpha} a_i\, d\varphi^i_\alpha\,|_p, \frac{\gamma'(t)}{\mathrm{md}\,\gamma(t)}\right\rangle\right| = \frac{\left|\sum_{i=1}^{N_\alpha} a_i\left(\varphi^i_\alpha\circ\gamma\right)'(t)\right|}{\mathrm{md}\,\gamma(t)}; \tag{6.11}$$

choose $s_n\searrow 0$ such that $t+s_n\in\mathrm{Dom}\,\gamma$ and note that

$$\begin{aligned}\left|\sum_{i=1}^{N_\alpha} a_i\left(\varphi^i_\alpha\circ\gamma\right)'(t)\right| &= \lim_{n\to\infty}\frac{\left|\sum_{i=1}^{N_\alpha} a_i\left(\left(\varphi^i_\alpha\circ\gamma\right)(t+s_n)-\left(\varphi^i_\alpha\circ\gamma\right)(t)\right)\right|}{s_n}\\ &\leq \limsup_{n\to\infty}\frac{\left|\sum_{i=1}^{N_\alpha} a_i\left(\left(\varphi^i_\alpha\circ\gamma\right)(t+s_n)-\left(\varphi^i_\alpha\circ\gamma\right)(t)\right)\right|}{d_X(\gamma(t+s_n),\gamma(t))}\limsup_{n\to\infty}\frac{d_X(\gamma(t+s_n),\gamma(t))}{s_n}\\ &\leq \left\|\sum_{i=1}^{N_\alpha} a_i\, d\varphi^i_\alpha\,|_p\right\|_{T^*X}\mathrm{md}\,\gamma(t);\end{aligned} \tag{6.12}$$

we thus conclude that:

$$\left|\left\langle \sum_{i=1}^{N_\alpha} a_i\, d\varphi^i_\alpha\,|_p, \frac{\gamma'(t)}{\mathrm{md}\,\gamma(t)}\right\rangle\right| \leq \left\|\sum_{i=1}^{N_\alpha} a_i\, d\varphi^i_\alpha\,|_p\right\|_{\mathrm{Lip}}, \tag{6.13}$$

which implies $\frac{\gamma'(t)}{\mathrm{md}\,\gamma(t)}\in\overline{B(\|\cdot\|^\star_{\mathrm{Lip}}(x))}$.

Let

$$W_p = \left\{\frac{\gamma'(t)}{\mathrm{md}\,\gamma(t)} : \gamma'(t)\neq 0 \text{ and } \gamma'(t)\in G_p(\mathcal{F},\mathcal{C},D_X)\right\}; \tag{6.14}$$

by Theorem 5.3 the set W_p contains a dense set of directions in T_pX. Theorem 4.3 implies then

$$\left\|\gamma'(t)\right\|_{D_X,d_X} = \mathrm{md}\,\gamma(t), \tag{6.15}$$

and so we can find a sequence $\left\{\sum_{i=1}^{N_\alpha} a_{i,k}\, d\varphi^i_\alpha\,|_p\right\}\subset\overline{B\left(\|\cdot\|_{\mathrm{Lip}}\right)}$ such that:

(1) We have:

$$\lim_{k\to\infty}\left\langle\sum_{i=1}^{N_\alpha} a_{i,k}\, d\varphi^i_\alpha\,|_p, \gamma'(t)\right\rangle = \mathrm{md}\,\gamma(t); \tag{6.16}$$

(2) For each k there is an $x_k\in D_X$ with:

$$d\left(d_X(\cdot,x_k)\right)|_p = \sum_{i=1}^{N_\alpha} a_{i,k}\, d\varphi^i_\alpha\,|_p\,. \tag{6.17}$$

By compactness we can find a subsequence of $\left\{\sum_{i=1}^{N_\alpha} a_{i,k}\, d\varphi_\alpha^i \,|_p\right\}$ converging to $\omega_{\gamma'(t)} \in \overline{B\left(\|\cdot\|_{\text{Lip}}\right)}$. Now, (6.16) implies that

$$\omega_{\gamma'(t)}\left(\frac{\gamma'(t)}{\operatorname{md}\gamma(t)}\right) = 1; \tag{6.18}$$

so for $w \in W_p$ of the form $\frac{\gamma'(t)}{\operatorname{md}\gamma(t)}$ let $\alpha_w = \omega_{\gamma'(t)}$; applying Lemma 6.2 we conclude that:

$$\|\cdot\|_{\text{Lip}}^{\star}(p) = \sup_{w\in W_p}\left|\alpha_w(\cdot)\right|; \tag{6.19}$$

but Lemma 6.2 implies also that W_p is dense in $S\left(\|\cdot\|_{\text{Lip}}^{\star}\right)$ and by (6.18) we conclude that for $w' \in W_p$:

$$\sup_{w\in W_p}\left|\alpha_w(w')\right| = 1 = \left\|w'\right\|_{D_X,d_X}, \tag{6.20}$$

from which we have $\|\cdot\|_{\text{Lip}}^{\star}(p) = \|\cdot\|_{D_X,d_X}(p)$. □

Proof of Theorem 1.7. Let $\|\cdot\|_1$–$\|\cdot\|_3$ be the seminorms as in Theorem 1.7, constructed using Lemma 2.33, and let $\|\cdot\|_4$ be the dual Lip norm $\|\cdot\|_{\text{Lip}}^{\star}$. Clearly we have $\|\cdot\|_1 \le \|\cdot\|_2 \le \|\cdot\|_3$.

We claim that

$$\|\cdot\|_3 \le \|\cdot\|_{\text{Lip}}^{\star} \quad \mu-a.e. \tag{6.21}$$

To see this, recall that by Lemma 2.33 there is a countable collection $\{f_i\}$ of 1-Lipschitz functions such that for μ-a.e. $p \in X$, the differentials $df_i(p) \in T_p^{\star}X$ are well-defined, and

$$\|\cdot\|_3(p) = \sup_i |df_i(p)|.$$

Recalling that for μ a.e. $p \in X$ we have $\|df_i(p)\|_{\text{Lip}} = \operatorname{Lip} f_i(p)$, we get that for μ a.e. $p \in X$, every i, and every $v \in T_pX$,

$$|df_i(v)| \le \|df_i\|_{\text{Lip}} \cdot \|v\|_{\text{Lip}}^{\star} = \operatorname{Lip} f_i(p) \cdot \|v\|_{\text{Lip}}^{\star} \le \|v\|_{\text{Lip}}^{\star}$$

since f_i is 1-Lipschitz. Taking supremum gives (6.21).

By Theorem 6.1 we have $\|\cdot\|_1 = \|\cdot\|_{\text{Lip}}^{\star}$ μ-a.e., so Theorem 1.7 follows. □

6.2 A new proof of lip f = Lip f in differentiability spaces

In this subsection we provide a proof, independent of the one given in [42], of the following result:

Theorem 6.22. *Let (X,μ) be a differentiability space and $f : X \to \mathbb{R}$ Lipschitz. The for all $(\varepsilon,\sigma) \in (0,1)^2$ there is a $(1, 1+\varepsilon)$-biLipschitz Alberti representation of μ with f-speed $\ge \sigma \operatorname{Lip} f$. In particular,*

$$\operatorname{Lip} f(x) = \operatorname{lip} f(x) \quad \textit{for } \mu\textit{-a.e. } x. \tag{6.23}$$

The equality (6.23) generalizes one of the main results in [14, Thm. 6.1], which is a consequence of the fact that, in a PI-space (X,μ), the function $\operatorname{Lip} f$ is a representative of the minimal generalized upper gradient g_f of f. This last statement does not make sense in a general differentiability space as one might have $g_f < \operatorname{Lip} f$ on a positive measure set, e.g. because X might not contain enough curves and one might then have $g_f = 0$. However, in a differentiability space the concept of the *maximal slope of f along fragments passing at time $t = 0$ through x* and having 0 as a density point of their domain, remains useful and can be interpreted as *the size of the gradient of f*. The result (6.23) is also proven in [42] in a conceptually different way, and there it is also shown that in a differentiability space one has $\operatorname{Lip} f = |df|$, where $|df|$ is the *local norm* of the form df, which is the Weaver differential form associated to the function f. The proof of Theorem 6.22 relies on the following lemma.

Lemma 6.24. *Let $\|\cdot\|_{l^2}$ denote the standard l^2-norm on $\mathbb{R}^N$, and let $\|\cdot\|'$ denote another norm on $\mathbb{R}^N$ satisfying:*

$$\frac{1}{C}\|\cdot\|_{l^2} \le \|\cdot\|' \le C\|\cdot\|_{l^2}. \tag{6.25}$$

Then the diameter of the set $\mathrm{Cone}(v,\theta)\cap S(\|\cdot\|')$, *with respect to the norm* $\|\cdot\|'$, *is at most*

$$4C^2(1-\cos\theta+\sin\theta). \tag{6.26}$$

Proof. Let $v_1, v_2 \in \mathrm{Cone}(v,\theta)\cap S(\|\cdot\|')$; then we can find $u_1, u_2 \in S(\|\cdot\|_{l^2})$ such that: $v_i = \frac{u_i}{\|u_i\|'}$; now

$$\|u_1-u_2\|_{l^2} \le 2(1-\cos\theta+\sin\theta) \tag{6.27}$$

by using the definition of Euclidean cone. Observe also that (6.25) implies:

$$\left|\|u_1\|'-\|u_2\|'\right| \le \|u_1-u_2\|' \le 2C(1-\cos\theta+\sin\theta); \tag{6.28}$$

thus

$$\begin{aligned}\left\|\frac{u_1}{\|u_1\|'}-\frac{u_2}{\|u_2\|'}\right\|' &= \left\|\frac{u_1}{\|u_1\|'}-\frac{u_2}{\|u_1\|'}+\frac{u_2}{\|u_1\|'}-\frac{u_2}{\|u_2\|'}\right\|'\\ &\le \frac{\|u_1-u_2\|'}{\|u_1\|'}+\frac{\|u_2\|'}{\|u_1\|'\,\|u_2\|'}\left|\|u_1\|'-\|u_2\|'\right|\\ &\le C\|u_1-u_2\|'+C\left|\|u_1\|'-\|u_2\|'\right|\\ &\le 4C^2(1-\cos\theta+\sin\theta).\end{aligned} \tag{6.29}$$

□

Proof of Theorem 6.22. We fix an N-dimensional chart (U,φ) and a countable dense set $D_X \subset X$. We will show that, for each $(\varepsilon,\sigma)\in(0,1)^2$, the measure $\mu \llcorner U$ admits a $(1,1+\varepsilon)$-biLipschitz Alberti representation with f-speed $\ge \sigma\,\mathrm{Lip} f$; the result about μ will then follow by applying the *gluing principle* Theorem 2.20.

We first consider the special case in which f is of the form $\langle v_0^\star,\varphi\rangle$ for some $v_0^\star \in \mathbb{R}^N\setminus\{0\}$. For each $\eta\in(0,1)$ we can use Egorov and Lusin Theorems to find disjoint compact sets $C_\alpha \in U$, and dual norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\alpha^\star$ on $\mathbb{R}^N$ such that:

$$\mu\left(U\setminus\bigcup_\alpha C_\alpha\right)=0; \tag{6.30}$$

$$\begin{aligned}&\frac{1}{1+\eta}\|\cdot\|_{TX}\le\|\cdot\|_\alpha\le(1+\eta)\|\cdot\|_{TX} \quad \text{(on the fibres of } TX\,|\,C_\alpha);\\ &\frac{1}{1+\eta}\|\cdot\|_{T^\star X}\le\|\cdot\|_\alpha^\star\le(1+\eta)\|\cdot\|_{T^\star X} \quad \text{(on the fibres of } T^\star X\,|\,C_\alpha).\end{aligned} \tag{6.31}$$

By Theorem 6.1 we can also assume that on the fibres of each $TX\,|\,C_\alpha$ one has:

$$\|\cdot\|_{TX} = \|\cdot\|_{D_X,d_X}. \tag{6.32}$$

Having fixed α, we will show that $\mu\llcorner C_\alpha$ admits a $(1,1+\varepsilon)$-biLipschitz Alberti representation with $\langle v_0^\star,\varphi\rangle$-speed $\ge\sigma\left\|v_0^\star\right\|_{T^\star X}$ for each $\sigma\in(0,1)$; the result about $\mu\llcorner U$ will follow again by using Theorem 2.20. As we can rescale $v_0^\star$, we can assume that $\left\|v_0^\star\right\|_\alpha^\star = 1$; we will denote by $v_0 \in S(\|\cdot\|_\alpha)$ a vector where $v_0^\star$ assumes the norm. We let M denote a constant such that:

$$\frac{1}{M}\|\cdot\|_{l^2}\le\|\cdot\|_\alpha\le M\|\cdot\|_{l^2}. \tag{6.33}$$

We fix $\varepsilon_0\in(0,1)$ and $\theta\in(0,\pi/2)$ and, using 2.28 and Theorem 2.18. we find a $(1,1+\varepsilon_0)$-biLipschitz Alberti representation $\mathcal{A}$ of $\mu\llcorner C_\alpha$ in the φ-direction of $\mathrm{Cone}\left(\frac{v_0}{\|v_0\|_{l^2}},\theta\right)$. Let $\mathcal{F}$ contain the components of φ and $\{f\}$, and let $\mathcal{C}$ contain χ_U. Using the Alberti representation $\mathcal{A}$ and (6.32) we conclude that for $\mu \llcorner C_\alpha$-a.e. p there is an $(\mathcal{F}, \mathcal{C})$-generic velocity vector $\gamma'(t) \in T_pX$ such that:

$$\begin{aligned} \operatorname{md}\gamma(t) = \left\|\gamma'(t)\right\|_{TX} &\in [1, 1+\varepsilon_0]; \\ (\varphi\circ\gamma)'(t) &\in \operatorname{Cone}\left(\frac{v_0}{\|v_0\|_{l^2}}, \theta\right). \end{aligned} \tag{6.34}$$

In particular, (6.34) and (6.31) imply that:

$$\left\|\gamma'(t)\right\|_\alpha \in \left[\frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0)\right]. \tag{6.35}$$

We now use Lemma 6.24 to get

$$\left\|\frac{(\varphi\circ\gamma)'(t)}{\left\|(\varphi\circ\gamma)'(t)\right\|_\alpha} - v_0\right\|_\alpha \le 4M^2(1-\cos\theta+\sin\theta); \tag{6.36}$$

as

$$\left|1 - \left\|(\varphi\circ\gamma)'(t)\right\|_\alpha\right| \le \max\left(1-\frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0)-1\right), \tag{6.37}$$

we obtain

$$\begin{aligned} \left\|(\varphi\circ\gamma)'(t) - v_0\right\|_\alpha &\le 4M^2(1-\cos\theta+\sin\theta) + \max\left(1-\frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0)-1\right) \\ &= a(\eta, \varepsilon_0, \theta), \end{aligned} \tag{6.38}$$

where $\lim_{\eta,\varepsilon_0,\theta\to 0} a(\eta, \varepsilon_0, \theta) = 0$. Recall that $t \in \operatorname{Dom}\gamma$ is a Lebesgue density point, and assume that $\langle v_0^\star, \varphi\rangle \circ \gamma$, which is $M\mathbf{L}(\varphi)(1+\varepsilon_0)$-Lipschitz because of (6.33), has been extended to a neighbourhood of t by using MacShane's Lemma:

$$\begin{aligned} \langle v_0^\star, \varphi\rangle\circ\gamma(t+h) - \langle v_0^\star, \varphi\rangle\circ\gamma(t) &= \int_t^{t+h} \left(\langle v_0^\star, \varphi\rangle\circ\gamma\right)'(s)\,ds \\ &\ge \int_{[t,t+h]\cap\operatorname{Dom}\gamma} \left(\langle v_0^\star, \varphi\rangle\circ\gamma\right)'(s) - \underbrace{M\mathbf{L}(\varphi)\,(1+\varepsilon_0)\mathcal{L}^1([t,t+h]\cap\operatorname{Dom}\gamma)}_{o(h)} \\ &= \int_{[t,t+h]\cap\operatorname{Dom}\gamma} \langle v_0^\star, v_0\rangle\,ds + \int_{[t,t+h]\cap\operatorname{Dom}\gamma} \langle v_0^\star, (\varphi\circ\gamma)'(s) - v_0\rangle\,ds + o(h) \\ &\ge \mathcal{L}^1([t,t+h]\cap\operatorname{Dom}\gamma) - h\,a(\eta,\varepsilon_0,\theta) + o(h), \end{aligned} \tag{6.39}$$

where in the last step we used the approximate continuity of $(\varphi\circ\gamma)'(s)$ at t. Now (6.39) implies that

$$(\langle v_0^\star, \varphi\rangle\circ\gamma)'(t) \ge \frac{1-a(\eta,\varepsilon_0,\theta)}{(1+\eta)^2(1+\varepsilon_0)} \operatorname{md}\gamma(t)\operatorname{Lip}\langle v_0^\star, \varphi\rangle(\gamma(t)), \tag{6.40}$$

and it suffices to choose $\eta, \varepsilon_0, \theta$ small enough to guarantee

$$\begin{aligned} \frac{1-a(\eta,\varepsilon_0,\theta)}{(1+\eta)^2(1+\varepsilon_0)} &\ge \sigma; \\ \varepsilon &\ge \varepsilon_0. \end{aligned} \tag{6.41}$$

We now consider the general case in which df is not constant. We let $V \subset U$ be a full-measure Borel subset where f is differentiable with respect to the chart functions φ. On the set where $df = 0$ we have $\operatorname{Lip} f = 0$, so we can assume that $df \neq 0$ on V. We fix $\eta > 0$ and use Lusin and Egorov Theorems to find disjoint compact sets $C_\alpha \subset V$ and $v_\alpha^\star \in \mathbb{R}^N \setminus \{0\}$ such that $\mu\left(V \setminus \bigcup_\alpha C_\alpha\right) = 0$ and:

$$\left\|df(x) - v_\alpha^\star\right\|_{T^*X} \le \eta\left\|df(x)\right\|_{T^*X} \quad (\forall x \in C_\alpha). \tag{6.42}$$

We fix $\sigma' \in (0,1)$ and, using the special case $f = \langle v^\star_\alpha, \varphi\rangle$, we obtain a $(1, 1+\varepsilon)$-biLipschitz Alberti representation $\mathcal{A}_\alpha = (P_\alpha, \nu_\alpha)$ of $\mu \llcorner C_\alpha$ with $\langle v^\star_\alpha, \varphi\rangle$-speed $\geq \sigma' \left\|v^\star_\alpha\right\|_{T^*X}$; then for P_α-a.e. γ and $\gamma^\star\nu_\gamma$-a.e. t we have:

$$\begin{aligned}(f\circ\gamma)'(t) &\geq (\langle v^\star_\alpha, \varphi\rangle \circ \gamma)'(t) - \eta \operatorname{md}\gamma(t) \|df\|_{T^*X}\\ &\geq \left(\sigma' \left\|v^\star_\alpha\right\|_{T^*X} - \eta \|df\|_{T^*X}\right) \operatorname{md}\gamma(t)\\ &\geq (\sigma' - (1+\sigma')\eta) \operatorname{Lip} f(\gamma(t)) \operatorname{md}\gamma(t),\end{aligned} \tag{6.43}$$

and it suffices to choose η small enough and σ' close to 1 to guaratee that $\sigma' - (1+\sigma')\eta \geq \sigma$.

The proof of (6.23) is now immediate. Let $\mathcal{F}$ contain the components of the chart functions and f, and let $\mathcal{C}$ contain the characteristic functions of the charts. Now, for each $\sigma \in (0,1)$, we conclude that for μ-a.e. $x \in X$ there is an $(\mathcal{F}, \mathcal{C})$-generic velocity vector $\gamma'(t) \in T_xX$ with

$$(f\circ\gamma)'(t) \geq \sigma \operatorname{Lip} f(\gamma(t)) \operatorname{md}\gamma(t); \tag{6.44}$$

observing that

$$\left|(f\circ\gamma)'(t)\right| \leq \operatorname{lip} f(\gamma(t)) \operatorname{md}\gamma(t), \tag{6.45}$$

we conclude that the Borel set

$$\left\{x \in X : \operatorname{lip} f(x) \geq \sigma \operatorname{Lip} f(x)\right\} \tag{6.46}$$

has full μ-measure, and then let $\sigma \nearrow 1$. □

6.3 A new proof of $g_f = \operatorname{Lip} f$ in PI-spaces

In this subsection we give a new proof of the characterization of the **minimal generalized upper gradient** g_f of a Lipschitz function f in a PI-space.

Before proceeding, we briefly recall some needed facts about PI spaces; see [14] for more detail.

We now assume that in addition to our standing assumptions from Subsection 2.1, our metric measure space (X, μ) is a PI space, i.e. μ is a doubling measure, and (X, μ) satisfies a Poincaré inequality in the sense of Heinonen-Koskela. Then:

(i) X is proper, i.e. closed balls are compact, since X is a complete and doubling metric space.
(ii) X is quasiconvex. This follows from the fact that X is complete, and a theorem of Semmes.
(iii) [14, Sec. 2] For every $1 < p < \infty$, and every Lipschitz function $f : X \to \mathbb{R}$, there is a canonical minimal generalized p-upper gradient $g_f \in L^p_{loc}(X, \mu)$. Here a generalized p-upper gradient for f is a function $g \in L^p_{loc}(X, \mu)$ such that there exist sequences $\{f_i\}, \{g_i\} \in L^p_{loc}(X, \mu)$, such that $f_i \xrightarrow{L^p_{loc}} f$, $g_i \xrightarrow{L^p_{loc}} g$, and g_i is an upper gradient for f_i for all i.
(iv) The Sobolev space $H^{1,p}(X, \mu)$ from [14, Sec. 2], is reflexive provided (X, μ) satisfies a $(1, p)$-Poincaré inequality.

We mention that [5] also gives a different proof that $g_f = \operatorname{Lip} f$ in PI spaces. In the same paper they show that the reflexivity of the Sobolev spaces, which plays an important role in our argument and in [14], is valid assuming only a doubling condition on the metric. However, their proof that $g_f = \operatorname{Lip} f$ requires both the doubling condition on the measure and the Poincaré inequality.

Our goal is to give a new proof of [14, Thm. 6.1]:

Theorem 6.47. *If (X, μ) is a PI-space with a $(1, p)$-Poincaré inequality, and if $f \in \operatorname{Lip}(X)$, then $\operatorname{Lip} f$ is a representative of the minimal generalized p-upper gradient of f (and hence the minimal generalized q-upper gradient of f is independent of q for all $q \geq p$).*

We first give some remarks on how the new proof differs from the original one. The original proof contained two steps:

(S1) Proof of Theorem 6.47 under the additional assumption that (X, μ) is a length space.
(S2) Removing the assumption that (X, μ) is a length space.

The argument for **(S1)** was motivated by the observation that, whenever (X, μ) is a length space and g is a *continuous* upper gradient of f, then $g \geq \operatorname{Lip} f$ holds at each point. Therefore the strategy in [14] was to prove an approximation result [14, Thm. 5.3] which states that for any $f \in \operatorname{Lip}(X) \cap H^{1,p}(X, \mu)$ there is a sequence $(f_k, h_k) \subset H^{1,p}(X, \mu) \times L^p(\mu)$ such that $f_k \to f$ in $H^{1,p}(X, \mu)$, the function h_k is a continuous upper gradient of f_k, and $h_k \to g_f$ in $L^p(\mu)$. This approximation result is probably the most technical part of Cheeger's original proof.

The first simplification of the new argument is that one does not need to handle first the case in which (X, μ) is a length space. The strategy of the proof is motivated by the observation (Lemma 6.48) that if g is a *bounded* upper gradient of f, then $g \geq \operatorname{Lip} f$ holds μ-a.e.: this is where Alberti representations are used. Had the minimal generalized upper gradient been defined by minimizing the p-energy on *bounded* upper gradients, then Theorem 6.47 would have followed directly from Lemma 6.48. However, as an upper gradient in $L^p(\mu)$ can be infinite on a null set, one needs, roughly speaking, to approximate f in $\operatorname{Lip}(X) \cap H^{1,p}(X, \mu)$ by functions which have bounded upper gradients. Here we use an instance of the argument "modulus equals capacity" [47] which appears also in the proof of [14, Thm. 5.3]: however, as we do not need to build approximations which use continuous upper gradients, there are fewer technical details to handle.

The following lemma relates bounded upper gradients and Alberti representations.

Lemma 6.48. *If (X, μ) is a PI-space, $u : X \to \mathbb{R}$ is Lipschitz, and g is a bounded upper gradient of u, then*

$$g \geq \operatorname{Lip} u \quad \mu\text{-a.e.} \tag{6.49}$$

Proof. For each $\varepsilon > 0$ we can find countably many disjoint compact sets $\{K_\alpha\}$ and nonnegative real numbers $\{\lambda_\alpha\}$ such that:

(1) For each $x \in K_\alpha$ one has $g(x) \in [\lambda_\alpha, \lambda_\alpha + \varepsilon)$;
(2) The $\{K_\alpha\}$ cover X in measure: $\mu\left(X \setminus \bigcup_\alpha K_\alpha\right) = 0$.

By Theorem 6.22 for $\mu \llcorner K_\alpha$-a.e. x there is a $(1, 1 + \varepsilon)$-biLipschitz fragment γ:

(1) The domain $\operatorname{Dom}\gamma$ is a compact subset of $[-1, \infty)$ and $\gamma(\operatorname{Dom}\gamma) \subset K_\alpha$;
(2) One has $\gamma(0) = x$ and:
$$\lim_{r \searrow 0} \frac{\mathcal{L}^1\left(\operatorname{Dom}\gamma \cap (-r, r)\right)}{2r} = 1; \tag{6.50}$$
(3) The point 0 is an approximate continuity point of $(u \circ \gamma)'$ and
$$(u \circ \gamma)'(0) \geq \frac{1}{1 + \varepsilon} \operatorname{Lip} u(x). \tag{6.51}$$

Let $[c, d]$ be the minimal interval containing $\operatorname{Dom}\gamma$ and let $\{(a_i, b_i)\}$ denote the set of components of $[c, d] \setminus \operatorname{Dom}\gamma$; we extend γ on each interval (a_i, b_i) by choosing a C-quasigeodesic joining $\gamma(a_i)$ to $\gamma(b_i)$: note that this is possible because a PI-space is C-quasiconvex for some C [14, Sec. 17][1]. Then:

$$\begin{aligned} \left| \int_0^r (u \circ \gamma)'(s)\, ds \right| = \left| u\left(\gamma(r)\right) - u(x) \right| &\leq \int_0^r g \circ \gamma \, \operatorname{md}\gamma(t)\, dt \\ &\leq (\lambda_\alpha + \varepsilon)(1 + \varepsilon) r + o(r); \end{aligned} \tag{6.52}$$

dividing by r and letting $r \searrow 0$ we get:

$$\operatorname{Lip} u(x) \leq (1 + \varepsilon)^2 \left(g(x) + \varepsilon\right), \tag{6.53}$$

and the result follows letting $\varepsilon \searrow 0$. □

1 This result is due to Semmes.

Remark 6.54. Note that in Lemma 6.48 we had to work with bounded upper gradients to establish (6.52); in fact, to apply the Fundamental Theorem of Calculus, one needs curves, and the K_α might only contain fragments, and thus, filling-in the fragments in K_α using that a PI-space is quasiconvex might produce curves where g is unbounded or infinite on a set of positive length. Note also that in a PI-space one can use curves instead of fragments in building Alberti representations; this follows from a general observation in [40] that if μ is a Radon measure on a quasiconvex metric space X, a Lipschitz Alberti representation of μ can be replaced by one which gives the same derivation and whose probability measure has support contained in the set of curves in X.

To prove Theorem 6.47 we can just consider, as in [14], upper gradients which are lower semicontinuous. In fact, the Vitali-Carathéodory Theorem [38, Thm. 2.25] states that for any $h \in L^1(\mu)$ and any $\varepsilon > 0$ there are functions u and v such that $u \le h \le v$, u is upper semicontinuous and bounded from above, v is lower semicontinuous and bounded from below, and $\|u - v\|_{L^1(\mu)} < \varepsilon$. Since we may assume μ is a finite measure by working in a ball, any upper gradient of f can be replaced, up to slightly increasing the $L^p(\mu)$-norm, by one which is lower semicontinous and bounded below by a small positive constant. We thus only need to prove:

Theorem 6.55. *Suppose (X, μ) is a PI-space, u is a real-valued Lipschitz function defined on X and g is a lower-semicontinuous upper gradient of u. Then:*

$$g \ge \operatorname{Lip} u \quad \mu\text{-a.e.} \tag{6.56}$$

To prove Theorem 6.55 we recall a consequence of the Poincaré inequality, which follows from the characterization of the Poincaré inequality in terms of the maximal function associated to an upper gradient [24, Lem. 5.15]. Suppose that g is an upper gradient for the function u and that $g \in L^p(\mu)$; consider for $N \in (0, \infty)$ the set:

$$A(g, N) = \left\{ x \in X : \sup_{r>0} -\!\!\!\!\!\!\int_{B(x,r)} g^p \, d\mu \le N^p \right\}; \tag{6.57}$$

then if $x, y \in A(g, N)$ are Lebesgue points of u, one has

$$\left| u(x) - u(y) \right| \le C N d(x, y), \tag{6.58}$$

where C is a universal constant that depends only on the PI-space (X, μ).

Proof. Let N, M be natural numbers and $S = A(g, N) \cap B(x, M)$; it suffices to show that (6.56) holds $\mu \llcorner S$-a.e. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$, and let:

$$u_n(x) = \inf \left\{ \int_\gamma (g \wedge n + \varepsilon) \, d\mathscr{H}^1_\gamma + u(y) : \ \gamma \text{ is a Lipschitz curve joining } x \text{ to } y \in S \right\}. \tag{6.59}$$

As (X, μ) is C-quasiconvex for some C, the function u_n is $C(n + \varepsilon)$-Lipschitz. Note also that $h_n = g \wedge n + \varepsilon$ is an upper gradient of u_n. We let S_m be a finite $\frac{1}{m}$-dense set in S, which exists because X is proper. Since the h_n are lower-semicontinuous and uniformly bounded away from zero, it follows that for each $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that, for $n \ge N_m$, one has (compare [47, 3.3, 3.4] and [14, Lem. 5.18]):

$$u_n(s) = u(s) \quad (\forall s \in S_m). \tag{6.60}$$

To see this, assume by contradiction that for some $s \in S_m$ and some subsequence $\{n_k\}$,

$$\sup_k u_{n_k}(s) < u(s) - \eta \,.$$

Then we could find a sequence of 1-Lipschitz curves $\{\gamma_k : [0, L_k] \to X\}$ with $\gamma_k(0) = s$ and $u(\gamma_k(L_k)) + \int_{\gamma_k} h_{n_k} \le u(s) - \eta$. But now we obtain $\sup L_k < \infty$ so by Arzela-Ascoli, after passing to a subsequence, we have $\gamma_k \xrightarrow{C^0} \gamma_\infty$

for some curve $\gamma_\infty : [0, L_\infty] \to X$, and by the lower-semicontinuity of g we get that $\int_{\gamma_\infty}(g+\varepsilon) < u(s)-u(\gamma(L_\infty))$, contradicting the assumption that g is an upper gradient of u.

Let v_n be obtained by truncating u_n so that

$$|v_n(x)| \le \sup_{y\in B(x,M)} |u_n(y)|; \tag{6.61}$$

thus, for $n \ge N_m$ one has:

$$v_n(s) = u(s) \quad (\forall s \in S_m). \tag{6.62}$$

Note that h_n is an upper gradient of v_n and that (6.58) implies that for every m, the function v_{N_m}, when restricted to the set S_1 of Lebesgue points of S, is $C(N + \varepsilon)$-Lipschitz; therefore, (6.60) implies that $v_{N_m} \to u$ uniformly on S_1 as $m \to \infty$. As the Banach space $H^{1,p}(\mu \llcorner B(x, M))$ is reflexive, by applying Mazur's Lemma we can find Lipschitz functions w_n and integers Q_n such that:

(1) The sequence $\{w_n\}$ converges to the function w in $H^{1,p}(\mu \llcorner B(x, M))$ and $w = u$ on S;
(2) Each function w_n is a convex combination of finitely many of the functions v_{N_m};
(3) The function $g \wedge Q_n + \varepsilon$ is an upper gradient for w_n.

We then recall that in a PI-space there is a constant C such that, for each Lipschitz function f, one has $Cg_f \ge \operatorname{Lip} f$ μ-a.e [14, Prop. 4.26]. As $w_n \to w$ in $H^{1,p}(\mu \llcorner B(x, M))$, one has that the generalized minimal upper gradients $\{g_{w_n-w}\}$ converge to 0 in $L^p(\mu \llcorner B(x, M))$; by the locality property of generalized minimal upper gradients [14, Cor. 2.25], as $u = w$ on S, we have that $\{g_{w_n-u}\}$ converges to 0 in $L^p(\mu \llcorner S)$; we thus conclude that $\operatorname{Lip}(w_n - u) \to 0$ in $L^p(\mu \llcorner S)$. As $|\operatorname{Lip} w_n - \operatorname{Lip} u| \le \operatorname{Lip}(w_n - u)$, we can then pass to a subquence such that $\operatorname{Lip} w_n \to \operatorname{Lip} u$ $\mu \llcorner S$-a.e. Now, by Lemma 6.48 we have that $g \wedge Q_n + \varepsilon \ge \operatorname{Lip} w_n$ holds $\mu \llcorner S$-a.e., and thus $g \ge \operatorname{Lip} u$ holds $\mu \llcorner S$-a.e. □

7 The geometry of blow-ups/tangent cones

In this section we show that, if (X, μ) is a differentiability space, blowing-up the measure μ at a generic point yields measures which possess Alberti representations concentrated on distinguished geodesic lines on which the blow-ups of the chart functions have constant derivatives, and are harmonic. This generalizes and strengthens the fact that in PI-spaces the blow-ups are generalized linear functions [14, Secs. 3, 10]. Weaker versions of the results presented here, where the blow-up of the measure is not discussed, have been obtained in [42], and [22]. The result in [42] is more general than [22] because it applies also in the context of Weaver derivations: we point out that the results in this section, under the assumption that μ is asymptotically doubling, have natural counterparts in that context. We first recall some notions of blow-ups of metric measure spaces and Lipschitz functions. Note that we use the terminology *blow-up* to avoid a conflict with the word *tangent* which is used for different objects in this paper; often, instances of what we call *blow-ups* are called *tangent cones / tangent spaces* in the literature.

7.1 Blow-ups of metric measure spaces and Lipschitz maps

Definition 7.1. A **blow-up of a metric space** X **at a point** p is a (complete) pointed metric space (Y, q) which is a pointed Gromov-Hausdorff limit of a sequence $(\frac{1}{r_n}X, p)$ where $r_n \searrow 0$: the notation $\frac{1}{r_n}X$ means that the metric on X is rescaled by $1/r_n$; the class of blow-ups of X at p is denoted by $\operatorname{Bw-up}(X, p)$.

Remark 7.2. In Subsection 8.3 we discuss blow-ups of metric spaces in a more general context which requires the notion of ultralimits: under suitable assumptions on X, a sequence $(\frac{1}{r_n}X, p)$ will always be precompact and the two notions will agree. This is the case, for example, if X is a doubling metric space. However, in the context of differentiability spaces we merely know (Theorem 2.21) that μ is asymptotically doubling, and that porous sets are μ-null. This implies that, for μ-a.e. $p \in X$, there is a compact set S_p such that: S_p is metrically doubling, and for each $\varepsilon > 0$, there is an $r_0 > 0$ such that, for each $r \le r_0$, the set $S_p \cap B(p, r)$ is εr-dense in $B(p, r)$. This allows essentially to reduce the existence of blow-ups to the case in which X is doubling.

Recall that if the sequence $(\frac{1}{r_n}X, p)$ converges to (Y, q) in the pointed Gromov-Hausdorff sense, there is a pointed metric space (Z, z) such that there are isometric embeddings $\iota_n : (\frac{1}{r_n}X, p) \to (Z, z)$ and $\iota : (Y, q) \to (Z, z)$, and, for each $R > 0$, one has:

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{y\in B(z,R)\cap\iota(Y)} \operatorname{dist}\left(\iota_n\left(\frac{1}{r_n}X\right), \{y\}\right) = 0,\\
&\lim_{n\to\infty} \sup_{y\in B(z,R)\cap\iota_n(\frac{1}{r_n}X)} \operatorname{dist}\left(\iota(Y), \{y\}\right) = 0.
\end{aligned} \tag{7.3}$$

In particular, each $q' \in Y$ can be *approximated* by a sequence $p'_n \in \frac{1}{r_n}X$ such that $\iota_n(p'_n) \to \iota(q')$ in Z. This notion can be made independent of the embedding in Z and one can **represent** each point $q' \in Y$ by some sequence $(p'_n) \subset X$ of points converging to p (compare the treatment with ultralimits in subsection 8.3). Moreover, if (p'_n) represents q', and if $(\tilde{p}'_n)$ represents $\tilde{q}'$, we have:

$$d_Y(q', \tilde{q}') = \lim_{n\to\infty} \frac{d_X(p'_n, \tilde{p}'_n)}{r_n}. \tag{7.4}$$

Definition 7.5. Let (X, μ) be a metric measure space; **a blow-up of** (X, μ) **at** p is a triple (Y, ν, q) such that one has $(\frac{1}{r_n}X, p) \to (Y, q) \in \text{Bw-up}(X, p)$, and, having chosen a pointed metric space (Z, z) and isometric embeddings $\iota_n : (\frac{1}{r_n}X, p) \to (Z, z)$ and $\iota : (Y, q) \to (Z, z)$ such that (7.3) holds, one has:

$$(\iota_n)_\sharp \frac{\mu}{\mu\left(B(p, r_n)\right)} \xrightarrow{\text{w}^\star} \iota_\sharp \nu. \quad \text{(convergence in the weak* topology)}. \tag{7.6}$$

The set of blow-ups of (X, μ) at p will be denoted by $\text{Bw-up}(X, \mu, p)$.

Remark 7.7. Note that if μ is asymptotically doubling and if porous sets are μ-null, then for μ-a.e. $p \in X$ one has $\text{Bw-up}(X, \mu, p) \neq \emptyset$. In fact, at a generic point p, for each sequence of scaling factors $r_n \searrow 0$, there is a subsequence r_{n_k} such that $(\frac{1}{r_{n_k}}X, p) \to (Y, q) \in \text{Bw-up}(X, p)$, and there is a doubling measure ν such that (7.6) holds.

We finally discuss blow-ups of Lipschitz mappings which take values into Euclidean spaces.

Definition 7.8. Let (X, μ) be a metric measure space and $\psi : X \to \mathbb{R}^N$ a Lipschitz map; then **a blow-up of** (X, μ, ψ) **at** p is a tuple (Y, ν, φ, q) such that one has that $(Y, \nu, q) \in \text{Bw-up}(X, \mu, p)$, where the blow-up is realized by considering scaling factors $r_n \searrow 0$, and where $\varphi : Y \to \mathbb{R}^N$ is a Lipschitz function such that, whenever $(p'_n) \subset X$ represents q', one has:

$$\varphi(q') = \lim_{n\to\infty} \frac{\psi(p'_n) - \psi(p)}{r_n}. \tag{7.9}$$

The set of blow-ups of (X, μ, ψ) at p will be denoted by $\text{Bw-up}(X, \mu, \psi, p)$.

Remark 7.10. If μ is asymptotically doubling and porous sets are μ-null, then for μ-a.e. $p \in X$ one has that $\text{Bw-up}(X, \mu, \psi, p) \neq \emptyset$ by an application of Ascoli-Arzelá.

Before proceeding to the main results about blowing-up Alberti representations, we first point out that the fact that Lip = lip a.e. (from [42] and Theorem 6.22 from Subsection 6.2) already implies a weak form of Theorem 1.12.

Suppose (U, ψ) is a chart for a differentiability space (X, μ), and that $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$ is realized by choosing scales $r_n \searrow 0$.

Let $f : X \to \mathbb{R}$ be a Lipschitz function that differentiable at p with respect to the $\{\psi^i\}_{i=1}^N$:

$$f(x) - f(p) = \sum_i a_i(\psi^i(x) - \psi^i(p)) \; + \; o(d(x, p)) \,.$$

Then it follows readily from the definitions that the maps

$$\frac{f - f(p)}{r_n} : \frac{1}{r_n} X \to \mathbb{R} \tag{7.11}$$

converge to the blow-up map $\hat{f} : Y \to \mathbb{R}$ given by

$$\hat{f}(y) = \sum_{i=1}^{N} a_i \varphi^i(y). \tag{7.12}$$

We now have:

Corollary 7.13. *Assume that for all $(a_1, \ldots, a_N) \in \mathbb{Q}^N$, the point p is an approximate continuity point of both* $\mathrm{Lip}(\sum_i a_i \psi^i)$ *and* $\mathrm{lip}(\sum_i a_i \psi^i)$ *. Then for any Lipschitz function $f : X \to \mathbb{R}$ which is differentiable w.r.t ψ at p, if $\hat{f} : Y \to \mathbb{R}$ denotes the function as in (7.12), then for all $y \in Y$, $r \in (0, \infty)$, the global Lipschitz constant* $\mathrm{LIP}(\hat{f})$ *satisfies:*

$$\mathrm{LIP}(\hat{f}) = \mathrm{Lip}(\hat{f})(y) = \mathrm{lip}(\hat{f})(y) = \frac{1}{r} \operatorname{var}(\hat{f}, y, r) . \tag{7.14}$$

Proof. This follows from the fact that Lip = lip almost everywhere ([42] and Theorem 6.22 from Section 6) and Lemmas 6.25, 6.26 from [27]. □

Under the assumptions of Corollary 7.13, the map $\sum_i a_i \psi^i = f \mapsto \hat{f}$ in (7.12) is injective by (7.14), so it yields a linear isomorphism $T_p^\star X \simeq \operatorname{span}\{\varphi^i\}$.

Definition 7.15. Under the assumptions of Corollary 7.13, if we identify $T_p^\star X$ with $\operatorname{span}\{\varphi^i\}$ using the isomophism above, the evaluation map yields a canonical map

$$E : Y \to (T_p X)^{\star\star} \simeq T_p X$$

where

$$E(y)(\sum_i a_i \varphi^i) = \sum_i a_i \varphi^i(y) . \tag{7.16}$$

A further consequence is:

Corollary 7.17. *The map*

$$E : Y \longrightarrow (T_p X, \|\cdot\|_x = \|\cdot\|^\star_{\mathrm{Lip}})$$

is 1*-Lipschitz.*

Proof. If $y_1, y_2 \in Y$, then by the definition of $\|\cdot\|^\star_{\mathrm{Lip}}$, there exist $(a_1, \ldots, a_N) \in \mathbb{R}^N$, such that if $f = \sum_i a_i \psi^i$, then $\mathrm{Lip}(f)(p) = \|df(p)\|_{\mathrm{Lip}} = \mathrm{LIP}(\hat{f}) = 1$, and $\|E(y_2) - E(y_1)\|^\star_{\mathrm{Lip}} = \hat{f}(E(y_2) - E(y_1))$, and so

$$\begin{aligned} \|E(y_2) - E(y_1)\|^\star_{\mathrm{Lip}} &= \hat{f}(E(y_2) - E(y_1)) = \hat{f}(y_2) - \hat{f}(y_1) \\ &\le \mathrm{LIP}(\hat{f}) d(y_1, y_2) = d(y_1, y_2) . \end{aligned}$$

□

7.2 Blowing up Alberti representations

We can now state the main result of this Section.

Theorem 7.18. *Let (U, ψ) be an N-dimensional differentiability chart for the differentiability space (X, μ); then for $\mu \llcorner U$-a.e. p, for each blow-up $(Y, \nu, \varphi, q) \in$* Bw-up(X, μ, ψ, p) *and for each unit vector $v_0 \in T_p X$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:*

(1) *Q is concentrated on the set* $\mathrm{Lines}(\varphi, v_0)$ *of unit speed geodesic lines in Y with* $(\varphi \circ \gamma)' = v_0$;
(2) *For each* $\gamma \in \mathrm{Lines}(\varphi, v_0)$ *the measure Φ_γ is given by:*

$$\Phi_\gamma = \mathscr{H}^1_\gamma. \tag{7.19}$$

Suppose that $X' \subset X$ and that the measures μ' and $\mu \llcorner X'$ are in the same measure class. Then an application of measure differentiation shows that for $\mu \llcorner X'$-a.e. p the sets $\mathrm{Bw\text{-}up}(X', \mu', p)$ and $\mathrm{Bw\text{-}up}(X, \mu, p)$ coincide. Given $(Y, \nu, \varphi, q) \in \mathrm{Bw\text{-}up}(X, \mu, \psi, p)$ we will then obtain the Alberti representations of ν by *blowing-up* Alberti representations of measures $\mu' \ll \mu$ which admit Alberti representations of a special form.

Definition 7.20 (Simplified Alberti representations)**.** We say that the Alberti representation $\mathcal{A} = (P, \Psi)$ of the measure μ' is **simplified** if there are $(C_0, D_0, \tau_0) \in (0, \infty)^3$ such that:

(1) The measure P is finite and is supported on the set of (C_0, D_0)-biLipschitz fragments whose domain is a subset of $[0, \tau_0]$;
(2) Denoting by $M(X)$ the set of finite Radon measures on X, Ψ is the Borel map:

$$\begin{aligned} \Psi : \mathrm{Frag}(X) &\to M(X) \\ \gamma &\mapsto \gamma_\sharp \left(\mathcal{L}^1 \llcorner \mathrm{Dom}\,\gamma\right). \end{aligned} \tag{7.21}$$

To prove Theorem 7.18 we will use the following technical result about blow-ups of a simplified Alberti representation $\mathcal{A}$.

Theorem 7.22. *Suppose that the simplified Alberti representation $\mathcal{A}$ of the finite measure $\mu' \ll \mu$ is in the ψ-direction of a cone $\mathcal{C}$ and that it has $\langle v_0, \psi\rangle$-speed $\geq \sigma_0 \|v_0\|_{T^*X}$. Then there is a Borel set U with full μ'-measure such that for each $p \in U$, for each* $(Y, \nu, \varphi, q) \in \mathrm{Bw\text{-}up}(X, \mu, \psi, p)$ *and each $R_0 > 0$ the measure $\nu \llcorner B(q, R_0)$ admits an Alberti representation $\mathcal{A}_{R_0} = (Q_{R_0}, \Phi)$ such that:*

(1) *The finite Radon measure Q_{R_0} has support contatined in a compact set $\mathcal{S}_{R_0} \subset \mathrm{Frag}(Y)$ of geodesic segments;*
(2) *The total mass of Q_{R_0} is bounded by* $\frac{D_0}{2R_0}\left(\mathrm{As}(\mu, p)\right)^{\log_2 R_0 + 1}$, *where* $\mathrm{As}(\mu, p)$ *denotes the asymptotic doubling constant of μ at p, i.e.:*

$$\mathrm{As}(\mu, p) = \limsup_{r \searrow 0} \frac{\mu\left(B(p, 2r)\right)}{\mu\left(B(p, r)\right)}; \tag{7.23}$$

(3) *The set $\mathcal{S}_{R_0}$ consists of those geodesic segments γ which have domain contained in* $\left[0, \frac{4R_0}{C_0}\right]$, *image contained in $\bar{B}(q, 2R_0)$, which have both endpoints lying outside of $B(q, \frac{3}{2}R_0)$, which intersect $\bar{B}(q, R_0)$, which have constant speed $\theta_\gamma \in [C_0, D_0]$, which satisfy:*

$$\mathrm{sgn}(s_2 - s_1)\left\langle v_0, \varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1)\right\rangle \geq \sigma_0 \theta_\gamma (s_2 - s_1)\,\mathrm{Lip}\left(\langle v_0, \psi\rangle\right)(p) \quad (\forall s_1, s_2 \in \mathrm{Dom}\,\gamma), \tag{7.24}$$

and such that there is a $w_\gamma \in \bar{\mathcal{C}}$ for which the following holds:

$$\varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1) = (s_2 - s_1) w_\gamma \quad (\forall s_1, s_2 \in \mathrm{Dom}\,\gamma); \tag{7.25}$$

(4) *For each $\gamma \in \mathcal{S}_{R_0}$ the measure Φ_γ is given by:*

$$\Phi_\gamma = \frac{1}{\theta_\gamma} \mathscr{H}^1_\gamma \llcorner B(q, r_0). \tag{7.26}$$

We now introduce a bit of terminology to split the measure on fragments in a *good* and a *bad* part.

Definition 7.27. Let $\varepsilon > 0$ and $S > 0$; with parameters as in Theorem 7.22, we denote by $\widetilde{\mathrm{Reg}}(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S)$ the set of pairs $(\gamma, p) \in \mathrm{Frag}(X) \times X$ such that:

(Reg1) The fragment γ is $[C_0, D_0]$-biLipschitz with domain contained in $[0, \tau_0]$;
(Reg2) There is a $t \in \mathrm{Dom}\,\gamma$ with $p = \gamma(t)$ and for each $r_1, r_2 \leq S$ one has:

$$\mathcal{L}^1\left(\mathrm{Dom}\,\gamma \cap [t - r_1, t + r_2]\right) \geq (1 - \varepsilon)(r_1 + r_2); \tag{7.28}$$

(Reg3) There are a $\theta \in [C_0, D_0]$ and $w \in \mathcal{C}$ such that if $r \leq S$ and $s_1, s_2 \in [t-r, t+r] \cap \operatorname{Dom}\gamma$ one has:

$$\begin{aligned} \left|d(\gamma(s_1), \gamma(s_2)) - \theta|s_1 - s_2|\right| &\leq \varepsilon|s_1 - s_2| \\ \left|\psi \circ \gamma(s_1) - \psi \circ \gamma(s_2) - w(s_1 - s_2)\right| &\leq \varepsilon|s_1 - s_2|; \end{aligned} \tag{7.29}$$

(Reg4) If $r \leq S$ and $s_1, s_2 \in [t-r, t+r]$ with $s_1 \leq s_2$ then:

$$\left\langle v_0, \psi \circ \gamma(s_2) - \psi \circ \gamma(s_1)\right\rangle \geq (\sigma_0 - \varepsilon)\theta \operatorname{Lip}(\langle v_0, \varphi\rangle)(p)(s_2 - s_1). \tag{7.30}$$

In the following we will usually fix a choice of $(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0)$ and vary $(\varepsilon, S) \in (0, \infty)^2$; we thus introduce the shorter notation $\mathrm{PAR}(\varepsilon, S)$ for $(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S)$. We denote by $\mathrm{Reg}(\mathrm{PAR}(\varepsilon, S))$ the subset of those $(\gamma, p) \in \widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S))$ such that:

(Reg5) For all $r_1, r_2 \leq S$ one has:

$$\mathcal{L}^1\left(\gamma^{-1}\left(\widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S))\right)_\gamma \cap [t - r_1, t + r_2]\right) \geq (1-\varepsilon)(r_1 + r_2), \tag{7.31}$$

where then notation $\left(\widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S))\right)_\gamma$ denotes the γ-section of the set $\widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S))$.

Lemma 7.32. *The set* $\mathrm{Reg}(\mathrm{PAR}(\varepsilon, S))$ *is a Borel subset of* $\mathrm{Frag}(X) \times X$.

Proof. We first show that the set $\widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S))$ is Borel. The set of fragments satisfying **(Reg1)** is closed in $\mathrm{Frag}(X)$. Now consider the set

$$\mathrm{IMG} = \left\{(\gamma, p) \in \mathrm{Frag}(X) \times X : p \in \gamma(\operatorname{Dom}\gamma)\right\}, \tag{7.33}$$

which is closed in $\mathrm{Frag}(X) \times X$; let $\mathrm{IMG}(C_0, D_0, \tau_0)$ denote the closed subset of those $(\gamma, p) \in \mathrm{IMG}$ such that γ satisfies **(Reg1)**; then the map:

$$\begin{aligned} \mathrm{Inv} : \mathrm{IMG}(C_0, D_0, \tau_0) &\to \mathbb{R} \\ (\gamma, p) &\mapsto \gamma^{-1}(p) \end{aligned} \tag{7.34}$$

is continuous. Using an argument similar to that used to prove that the map defined at (3.15) is Borel, we see that, for fixed $r_1, r_2 > 0$, the map:

$$\begin{aligned} \psi_{r_1, r_2} : \mathrm{Frag}(X) \times \mathbb{R} &\to \mathbb{R} \\ (\gamma, t) &\mapsto \mathcal{L}^1\left(\operatorname{Dom}\gamma \cap [t - r_1, t + r_2]\right) \end{aligned} \tag{7.35}$$

is Borel; then the set of pairs (γ, p) satisfying **(Reg1)–(Reg2)** is Borel since it can be written as:

$$\bigcap_{r_1, r_2 \in [0, S] \cap \mathbb{Q}} \left\{(\gamma, p) \in \mathrm{IMG}(C_0, D_0, \tau_0) : \psi_{r_1, r_2}\left(\gamma, \gamma^{-1}(p)\right) \geq (1-\varepsilon)(r_1 + r_2)\right\}. \tag{7.36}$$

That the set of pairs satisfying **(Reg3)–(Reg–4)** is Borel follows by arguments similar to those used in the proof of Lemma 3.4, compare (3.8), (3.10).

Consider the set:

$$\mathrm{TRIP} = \left\{(\gamma, p, t) \in \mathrm{Frag}(X) \times X \times \mathbb{R} : t \in \operatorname{Dom}\gamma, \gamma(t) = p\right\}, \tag{7.37}$$

which is closed in $\mathrm{Frag}(X) \times X \times \mathbb{R}$. We now fix $r_1, r_2 \geq 0$ and define the Borel set:

$$A_{r_1, r_2} = \left\{(\gamma, p, t) \in \widetilde{\mathrm{Reg}}(\mathrm{PAR}(\varepsilon, S)) \times \mathbb{R} \cap \mathrm{TRIP} : \gamma^{-1}(p) \in [t - r_1, t + r_2]\right\}; \tag{7.38}$$

using [25, Thm. 7.25] we get that the map:

$$\begin{aligned} \Omega_{r_1, r_2} : \mathrm{Frag}(X) \times \mathbb{R} \times M(X) &\to \mathbb{R} \\ (\gamma, t, \mu) &\mapsto \mu\left((A_{r_1, r_2})_{(\gamma, t)}\right) \end{aligned} \tag{7.39}$$

is Borel. The proof that $\mathrm{Reg}(\mathrm{PAR}(\varepsilon, S))$ is Borel is completed by observing that **(Reg5)** can be expressed as:

$$\Omega_{r_1, r_2}\left(\gamma, \gamma^{-1}(p), \Psi(\gamma)\right) \geq (1-\varepsilon)(r_1 + r_2) \quad (\forall r_1, r_2 \in [0, S] \cap \mathbb{Q}). \tag{7.40}$$

□

Consider the map Ψ defined in (7.21); we can decompose the measures $\Psi(\gamma)$ as follows:

$$\begin{aligned}\Psi_{\mathrm{PAR}(\varepsilon,S)}(\gamma) &= \Psi(\gamma) \llcorner \left(\mathrm{Reg}(\mathrm{PAR}(\varepsilon,S))\right)_{\gamma};\\ \Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}(\gamma) &= \Psi(\gamma) \llcorner \left(\mathrm{Reg}(\mathrm{PAR}(\varepsilon,S))\right)^{c}_{\gamma}.\end{aligned} \tag{7.41}$$

Lemma 7.42. *The maps $\Psi_{\mathrm{PAR}(\varepsilon,S)}$ and $\Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}$ are Borel. Thus, given an Alberti representation $\mathcal{A}$ of the finite measure μ' satisfying the assumptions of Theorem 7.22, we can define the finite Radon measures:*

$$\begin{aligned}\mu'_{\mathrm{PAR}(\varepsilon,S)} &= \int_{\mathrm{Frag}(X)} \Psi_{\mathrm{PAR}(\varepsilon,S)}(\gamma)\, dP(\gamma)\\ \mu'^{c}_{\mathrm{PAR}(\varepsilon,S)} &= \int_{\mathrm{Frag}(X)} \Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}(\gamma)\, dP(\gamma),\end{aligned} \tag{7.43}$$

which satisfy:

$$\mu'_{\mathrm{PAR}(\varepsilon,S)} + \mu'^{c}_{\mathrm{PAR}(\varepsilon,S)} = \mu', \tag{7.44}$$

and:

$$\lim_{S \searrow 0} \left\|\mu'^{c}_{\mathrm{PAR}(\varepsilon,S)}\right\| = 0. \tag{7.45}$$

Proof. By [25, Thm. 7.25] the map:

$$\begin{aligned}\Omega : \mathrm{Frag}(X) \times M(X) &\to \mathbb{R}\\ (\gamma,\mu) &\mapsto \mu \llcorner \left(\mathrm{Reg}(\mathrm{PAR}(\varepsilon,S))\right)_{\gamma}\end{aligned} \tag{7.46}$$

is Borel, and thus $\Psi_{\mathrm{PAR}(\varepsilon,S)}$ is Borel as $\Psi_{\mathrm{PAR}(\varepsilon,S)}(\gamma)$ can be written as $\Omega\left(\gamma, \Psi(\gamma)\right)$; the proof for $\Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}$ is similar.

Now note that $\left\|\Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}(\gamma)\right\| \le \tau_0$ and that, for each γ, one has

$$\lim_{S \searrow 0} \left\|\Psi^{c}_{\mathrm{PAR}(\varepsilon,S)}(\gamma)\right\| = 0, \tag{7.47}$$

as for $\mathcal{L}^1$-a.e. $t \in \mathrm{Dom}\,\gamma$ there is an $S(t)$ such that, for $s \le S(t)$, one has $(\gamma, \gamma(t)) \in \mathrm{Reg}(\mathrm{PAR}(\varepsilon, S))$. Then (7.45) follows by the Dominated Convergence Theorem. □

The next lemma follows from (7.45) and a standard argument in measure differentiation.

Lemma 7.48. *Let $\{\varepsilon_m\} \subset (0,\infty)$ be a sequence with $\sum_m \varepsilon_m < \infty$; then there are a Borel $U \subset X$ and a sequence of pairs $\{(s_m, S_m)\}_m \subset (0,\infty)^2$ such that:*

(1) One has $\mu(X \setminus U) \le \sum_m \varepsilon_m$ and, for each m, one also has $s_m \le S_m$;
(2) For each $x \in U$ and for each $r \le s_m$, one has:

$$\mu'^{c}_{\mathrm{PAR}(\varepsilon_m,S_m)}\left(B(x,r)\right) \le \varepsilon_m \mu'\left(B(x,r)\right). \tag{7.49}$$

Proof of Theorem 7.22. We fix a sequence ε_m such that $\sum_m \varepsilon_m < \infty$: the set U is the intersection of the set provided by Lemma 7.48 and the set of points p where the limit:

$$\lim_{r \searrow 0} \frac{\mu'\left(B(x,r_n)\right)}{\mu\left(B(x,r_n)\right)} \tag{7.50}$$

exists and is finite. Having fixed a point $p \in U$, we let r_n be a sequence converging to 0 such that the rescalings

$$\left(\frac{1}{r_n}X, \frac{\mu'}{\mu'\left(B(p,r_n)\right)}, \frac{\psi - \psi(p)}{r_n}, p\right) \tag{7.51}$$

converge to (Y, ν, φ, q) in the measured Gromov-Hausdorff sense. We let $X_n = \frac{1}{r_n}X$. As in the following we consider simultaneously different metric spaces, we will use subscripts to denote objects which "live" in a given metric space, e.g. $B_{X_n}(p, R_0)$ denotes the ball of radius R_0 and center p in the metric space X_n.

By the theory of measured Gromov-Hausdorff convergence we can find a compact metric space Z, which is a convex compact subset of some Banach space (e.g. ℓ^∞), which satisfies the following properties:

(Z1) There are isometric embeddings:

$$\begin{aligned} J_n &: \left(\bar{B}_{X_n}(p, 4R_0), p\right) \to (Z, q_Z) \\ J_\infty &: \left(\bar{B}_Y(q, 4R_0), q\right) \to (Z, q_Z); \end{aligned} \tag{7.52}$$

in the following we will often implicitly identify balls like $B_{X_n}(p, r)$ and $B_Y(q, r)$ with their images in Z;

(Z2) There are compact sets $K_n, \tilde{K}_n \subset Z$ and a sequence $\eta_n \searrow 0$ such that:

$$\begin{aligned} \bar{B}_{X_n}(p, R_0) \subset K_n &\subset \bar{B}_{X_n}(p, R_0 + \eta_n) \\ \bar{B}_{X_n}(p, 2R_0) \subset \tilde{K}_n &\subset \bar{B}_{X_n}(p, 2R_0 + \eta_n) \\ d_{Z,\mathrm{H}}(K_n, \bar{B}_Y(q, R_0)) &\le \eta_n \\ d_{Z,\mathrm{H}}(\tilde{K}_n, \bar{B}_Y(q, 2R_0)) &\le \eta_n, \end{aligned} \tag{7.53}$$

where $d_{Z,\mathrm{H}}(\cdot, \cdot)$ denotes the Hausdorff distance between subsets of Z;

(Z3) There is an $\mathbf{L}(\psi)$-Lipschitz function $\psi_Z : Z \to \mathbb{R}^N$ such that, denoting by ψ_{X_n} the restriction $\psi_Z|\bar{B}_{X_n}(p, 2R_0)$ and by ψ_Y the restriction $\psi_Z|\bar{B}_Y(q, 2R_0)$, one has:

$$\begin{aligned} \psi_{X_n} \circ J_n &= \frac{\psi - \psi(p)}{r_n} \\ \psi_Y \circ J_\infty &= \varphi; \end{aligned} \tag{7.54}$$

(Z4) Letting μ_n and μ_∞ denote, respectively, the measures

$$\begin{aligned} &J_{n\sharp} \frac{\mu' \llcorner B_{X_n}(p, R_0)}{\mu'\left(B_X(p, r_n)\right)} \\ &J_{\infty\sharp} \nu \llcorner B_Y(q, R_0), \end{aligned} \tag{7.55}$$

one has $\mu_n \xrightarrow{\mathrm{w}^\star} \mu_\infty$.

We chose Z convex to "fill-in" fragments to Lipschitz curves; specifically, let Curves(Z) denote the set of Lipschitz maps $\gamma : K \to Z$, where $K \subset \mathbb{R}$ is a (possibly degenerate) compact interval; we topologize Curves(Z) with the Vietoris topology. Let

$$\mathrm{Fill} : \mathrm{Frag}(Z) \to \mathrm{Curves}(Z) \tag{7.56}$$

be the map which extends a fragment γ to a Lispchitz curve, with domain the minimal compact interval $I(\gamma)$ containing $\mathrm{Dom}\,\gamma$, by extending γ linearly on each component of $I(\gamma) \setminus \mathrm{Dom}\,\gamma$. The map Fill is continuous.

Let $\Gamma_{X_n} \subset \mathrm{Frag}(X)$ denote the set of those $[C_0, D_0]$-biLipschitz fragments which intersect $\bar{B}_X(p, 2R_0 r_n)$; note that Γ_{X_n} is closed. We define maps:

$$\mathrm{Rep}_n : \Gamma_{X_n} \to \mathrm{Frag}(Z) \tag{7.57}$$

by composing $J_n \circ \left(\gamma|\gamma^{-1}(\bar{B}_X(p, 2R_0))\right)$, where we naturally identify γ with a fragment in X_n, with the unique affine map $A_\gamma : \mathbb{R} \to \mathbb{R}$ which has dilating factor $\frac{1}{r_n}$ and which maps the point:

$$\min\left\{t : t \in \gamma^{-1}(\bar{B}_X(p, 2R_0))\right\} \tag{7.58}$$

to 0. Note that Rep_n is continuous.

We will now refer back to the map Ψ defined in (7.21), adding subscripts regarding the metric space. From the definition of Rep_n we see that:

$$r_n \Psi_Z\left(\mathrm{Rep}_n(\gamma)\right) = J_{n\sharp} \Psi_{X_n}(\gamma) \llcorner \bar{B}_{X_n}(p, 2R_0) \tag{7.59}$$

Let $g \in C_c(Z)$ so that:

$$\lim_{n\to\infty} \int g\, d\mu_n = \int g\, d\mu_\infty; \tag{7.60}$$

then

$$\begin{aligned}\int g\, d\mu_n &= \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{B_{X_n}(p,R_0)} g \circ J_n\, d\mu' \\ &= \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{B_{X_n}(p,R_0)} g \circ J_n\, d(\mu'_{\mathrm{PAR}(\varepsilon_m,S_m)} + \mu'^{c}_{\mathrm{PAR}(\varepsilon_m,S_m)});\end{aligned} \tag{7.61}$$

Note that

$$\left| \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{B_{X_n}(p,R_0)} g \circ J_n\, d\mu'^{c}_{\mathrm{PAR}(\varepsilon_m,S_m)} \right| \le \|g\|_\infty \frac{\mu'^{c}_{\mathrm{PAR}(\varepsilon_m,S_m)}\left(B_X(p, r_nR_0)\right)}{\mu'\left(B_X(p, r_n)\right)}; \tag{7.62}$$

for n sufficiently large $r_nR_0 \le s_m$ so that by (7.49) and using that μ' is asymptotically doubling we conclude that:

$$\lim_{n\to\infty} \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{B_{X_n}(p,R_0)} g \circ J_n\, d\mu'^{c}_{\mathrm{PAR}(\varepsilon_m,S_m)} = 0. \tag{7.63}$$

We also introduce some notation to deal with regularity in Z and X; so we let:

$$\begin{aligned}\mathrm{PAR}_X(\varepsilon, S) &= \left(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S\right) \\ \mathrm{PAR}_Z(\varepsilon, S) &= \left(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \frac{4R_0}{C_0}, \varepsilon, S\right);\end{aligned} \tag{7.64}$$

in particular, inspection of conditions **(Reg1)**–**(Reg5)** shows that:

$$\begin{aligned}&\frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{B_{X_n}(p,R_0)} g \circ J_n\, d\mu'_{\mathrm{PAR}_X(\varepsilon_m,S_m)} \\ &\qquad = \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int\limits_{\Gamma_{X_n}} dP(\gamma) \int\limits_{Z} g\, r_n \chi_{B_{X_n}(p,R_0)}\, d\Psi_{\mathrm{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\mathrm{Rep}_n(\gamma)\right).\end{aligned} \tag{7.65}$$

Let $\tilde\Gamma_{X_n}$ be the Borel subset of those $\gamma \in \Gamma_{X_n}$ such that:

$$\chi_{B_{X_n}(p,R_0)} \Psi_{\mathrm{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\mathrm{Rep}_n(\gamma)\right) \neq 0; \tag{7.66}$$

then (7.66) implies that there is a $p_\gamma = \gamma(t) \in \left(\mathrm{Reg}(\mathrm{PAR}_X(\varepsilon_m, S_m))\right)_\gamma \cap B_X(p, r_nR_0)$. Note that the set $\mathcal{B}_{\gamma,n} = \gamma^{-1}\left(\bar B_X(p, 2r_nR_0)\right)$ has diameter at most $\frac{4R_0r_n}{C_0}$; let a_γ, b_γ be minimal such that the interval $[t_\gamma - a_\gamma, t_\gamma + b_\gamma]$ contains $\gamma^{-1}(\mathcal{B}_{\gamma,n})$. For n-sufficiently large one has $a_\gamma, b_\gamma \le S_m$ so that by **(Reg2)** the $\varepsilon_m(a_\gamma + b_\gamma)$-neighbhourhood of $\mathcal{B}_{\gamma,n}$ contains $[t_\gamma - a_\gamma, t_\gamma + b_\gamma]$. A similar conclusion holds for the smallest interval containing $\gamma^{-1}\left(B_{X_n}(p, R_0)\right)$ from which we get:

$$\begin{aligned}r_n \left\| \Psi_{\mathrm{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\mathrm{Rep}_n(\gamma)\right) \llcorner B_{X_n}(p, R_0) - \Psi_Z\left(\mathrm{Fill} \circ \mathrm{Rep}_n(\gamma)\right) \llcorner B_Z(q_Z, R_0) \right\| \\ \le \varepsilon_m r_n \frac{2R_0}{C_0}.\end{aligned} \tag{7.67}$$

Note also that:

$$\gamma\left(\mathrm{Dom}\,\gamma \cap [t_\gamma - \frac{r_nR_0}{D_0}, t_\gamma + \frac{r_nR_0}{D_0}]\right) \subset B_X(p, 2r_nR_0); \tag{7.68}$$

as for n sufficiently large one has $\frac{r_nR_0}{D_0} \le S_m$, we have:

$$r_n \Psi_{\mathrm{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\mathrm{Fill} \circ \mathrm{Rep}_n(\gamma)\right)\left(B_{X_n}(p, 2r_nR_0)\right) \ge 2(1 - \varepsilon_m)\frac{r_nR_0}{D_0}. \tag{7.69}$$

For n sufficiently large we also have $\frac{3r_nR_0}{C_0} \le S_m$ which implies:

$$\begin{aligned} \mathcal{L}^1\left(\operatorname{Dom}\gamma \cap \left[t_\gamma, t_\gamma + \frac{3r_nR_0}{C_0}\right]\right) &\ge (1-\varepsilon_m)\frac{3R_0}{C_0}r_n \\ \mathcal{L}^1\left(\operatorname{Dom}\gamma \cap \left[t_\gamma - \frac{3r_nR_0}{C_0}, t_\gamma\right]\right) &\ge (1-\varepsilon_m)\frac{3R_0}{C_0}r_n; \end{aligned} \tag{7.70}$$

so we can find $s_{1,\gamma} \le t_\gamma \le s_{2,\gamma}$ with:

$$\begin{aligned} \left|t_\gamma - s_{i,\gamma}\right| &\ge (1-\varepsilon_m)\frac{3r_nR_0}{C_0} \quad (\text{for } i = 1,2) \\ d_X(p, \gamma(s_{i,\gamma})) &\ge \left((1-\varepsilon_m)\frac{3R_0}{C_0} - R_0\right) r_n \quad (\text{for } i = 1,2); \end{aligned} \tag{7.71}$$

in particular, for m sufficiently large (7.71) implies that the maximum and minimum point in $\mathcal{B}_{\gamma,n}$ are mapped by γ outside of $B_X(p, \frac{3}{2}R_0r_n)$. Thus, the endpoints of $\text{Fill}\circ\text{Rep}_n(\gamma)$ lie out of $B_Z(q_Z, \frac{3}{2}R_0)$.

We now obtain an upper estimate for $P(\tilde{\Gamma}_{X_n})$ (note that we assume that n is sufficiently large depending on m):

$$\begin{aligned} 2(1-\varepsilon_m)\frac{r_nR_0}{D_0}P(\tilde{\Gamma}_{X_n}) &\le r_n \int_{\text{Frag}(X)} \Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}\left(\text{Rep}_n(\gamma)\right)\left(B_X(p, 2r_nR_0)\right)\, dP(\gamma) \\ &\le \mu'\left(B_X(p, 2R_0r_n)\right); \end{aligned} \tag{7.72}$$

in particular, using (7.67),

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\mu'\left(B_X(p,r_n)\right)}\left|\int_{\tilde{\Gamma}_{X_n}} dP(\gamma)\int g\, r_n\chi_{B_{X_n}(p,R_0)}d\Psi_{\text{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\text{Rep}_n(\gamma)\right)\right. \\ &\qquad\left. - \int_{\tilde{\Gamma}_{X_n}} dP(\gamma)\int g\, r_n\chi_{B_Z(q_Z,R_0)}\, d\Psi_{\text{PAR}_Z(\varepsilon_m,S_m/r_n)}\left(\text{Fill}\circ\text{Rep}_n(\gamma)\right)\right| \\ &\le \limsup_{n\to\infty}\frac{1}{\mu'\left(B_X(p,r_n)\right)}\|g\|_\infty P(\tilde{\Gamma}_{X_n})\, r_n\varepsilon_m\frac{2R_0}{C_0} \\ &\le \limsup_{n\to\infty}\|g\|_\infty\frac{\varepsilon_m}{1-\varepsilon_m}\frac{D_0}{C_0}\frac{\mu'\left(B_X(p,2r_nR_0)\right)}{\mu'\left(B_X(p,r_n)\right)} \\ &= O(\varepsilon_m), \end{aligned} \tag{7.73}$$

where in the last step we used that μ' is doubling. As $n\to\infty$ we can send $m\to\infty$ so that the left hand side of (7.73) converges to 0.

Let $\Omega_m \subset \text{Curves}(Z)$ denote the set of D_0-Lipschitz curves such that there exists a $\theta_\gamma \in [C_0, D_0]$ and a $w \in \bar{\mathcal{C}}$ such that (note the constant C_2 will be specified later):

(Ω1) For all $s_1, s_2 \in \operatorname{Dom}\gamma$ one has:

$$\left|d_Z(\gamma(s_1),\gamma(s_2)) - \theta_\gamma|s_1 - s_2|\right| \le C_2\varepsilon_m; \tag{7.74}$$

(Ω2) The domain of γ is a subset of $\left[0, \frac{4R_0}{C_0}\right]$;

(Ω3) The image of γ is contained in the $C_2\varepsilon_m$-neighbourhood of $\bar{B}_{X_n}(p, 2R_0)$;

(Ω4) For all $s_1, s_2 \in \operatorname{Dom}\gamma$ one has:

$$\left|\psi_Z\circ\gamma(s_1) - \psi_Z\circ\gamma(s_2) - w|s_1 - s_2|\right| \le C_2\varepsilon_m; \tag{7.75}$$

(Ω5) For all $s_1, s_2 \in \operatorname{Dom}\gamma$ with $s_2 \ge s_1$ one has:

$$\left\langle v_0, \psi_Z\circ\gamma(s_2) - \psi_Z\circ\gamma(s_1)\right\rangle \ge (\sigma_0 - \varepsilon_m)\theta_\gamma \operatorname{Lip}\left(\langle v_0, \varphi\rangle\right)(p)(s_2 - s_1) - C_2\varepsilon_m. \tag{7.76}$$

Note that the set Ω_m is compact. We also define Ω_∞ by requiring in **(Ω3)** that γ lies in $\bar{B}_Y(p, 2R_0)$ and that the error term ε_m is replaced by 0. In view of **(Reg1)**–**(Reg5)**, for an appropriate choice of C_2 one has $\mathrm{Fill}\circ\mathrm{Rep}_n(\tilde{\Gamma}_{X_n}) \subset \Omega_m$ for $n \geq N(m)$. If we let P_n denote the Radon measure on Curves(Z):

$$P_n = \frac{1}{\mu'\left(B_X(p, r_n)\right)} r_n \,\mathrm{Fill}\circ\mathrm{Rep}_n(\gamma)_\sharp P \llcorner \tilde{\Gamma}_{X_n}; \tag{7.77}$$

we have that P_n has support contained in Ω_m for $n \geq N(m)$. By (7.72) the total mass of P_n is bounded by:

$$\frac{D_0}{2(1-\varepsilon_m)R_0} \frac{\mu'\left(B_X(p, 2R_0 r_n)\right)}{\mu'\left(B_X(p, r_n)\right)}. \tag{7.78}$$

Moreover, an application of Ascoli-Arzelá shows that the set $\Omega = \bigcup_m \Omega_m \cup \Omega_\infty$ is compact; we can thus find a subsequence $n_m \geq N(m)$ such that $P_{n_m} \xrightarrow{\text{w}^\star} Q_{R_0}$. The previous discussion on the properties of the fragments in $\tilde{\Gamma}_{X_n}$ implies that the support spt Q_{R_0} of Q_{R_0} is a subset of $\mathcal{S}_{R_0} \subset \Omega_\infty$ and that point (2) in the statement of this Theorem follows from (7.78). We now observe that:

$$\begin{aligned} \frac{1}{\mu'\left(B_X(p, r_n)\right)} \int_{\tilde{\Gamma}_{X_{n_m}}} dP(\gamma) \int g r_n \chi_{B_Z(q_Z, R_0)}\, d\Psi_Z\left(\mathrm{Fill}\circ\mathrm{Rep}_{n_m}(\gamma)\right) \\ = \int_\Omega dP_{n_m}(\gamma) \int g\chi_{B_Z(q_Z, R_0)}\, d\Psi_Z(\gamma); \end{aligned} \tag{7.79}$$

fix a $\xi \in (0, 1)$, and let ψ_ξ be a continuous function, which takes values in $[0, 1]$ and which equals 1 on $\bar{B}_Z(q_Z, R_0 + \xi)$ and which vanishes out of $B_Z(q_Z, R_0 + 2\xi)$; then:

$$\left| \int_\Omega dP_{n_m}(\gamma) \int g\left(\chi_{B_Z(q_Z, R_0)} - \psi_\xi\right) d\Psi_Z(\gamma) \right| \leq P_{n_m}(\Omega) \|g\|_\infty \frac{2\xi}{C_0}; \tag{7.80}$$

as the map $\gamma \mapsto \int g\psi_\xi \, d\Psi_Z(\gamma)$ is continuous:

$$\lim_{m\to\infty} \int_\Omega dP_{n_m}(\gamma) \int g\psi_\xi \, d\Psi_Z(\gamma) = \int_\Omega dQ_{R_0}(\gamma) \int g\psi_\xi \, d\Psi_Z(\gamma). \tag{7.81}$$

Also,

$$\left| \int_\Omega dQ_{R_0}(\gamma) \int g\psi_\xi \, d\Psi_Z(\gamma) - \int_\Omega dQ_{R_0}(\gamma) \int g \, d\Psi_Z(\gamma) \llcorner B_Y(q, R_0) \right| \leq Q_{R_0}(\Omega) \|g\|_\infty \frac{\xi}{C_0}; \tag{7.82}$$

so we conclude that:

$$\lim_{m\to\infty} \int g \, d\mu_{n_m} = \int_{\mathcal{S}_{R_0}} dQ_{R_0} \int g \, d\Psi_Z(\gamma) \llcorner B_Y(q, R_0); \tag{7.83}$$

in particular, if we let:

$$\Phi_\gamma = \Psi_Z(\gamma) \llcorner B_Y(q, R_0) = \frac{1}{\theta_\gamma} \mathscr{H}^1_\gamma, \tag{7.84}$$

we get that (Q_{R_0}, Φ) gives an Alberti representation of $\nu \llcorner B_Y(q, R_0)$. It might be worth noting that in (7.84) we used that γ is a geodesic with constant speed θ_γ and that the function $\gamma \mapsto \theta_\gamma$ is continuous. □

Lemma 7.85. *Under the hypotheses of Theorem 7.18, there is a Borel $U \subset X$ with full μ-measure such that, for each $p \in U$, for each $(Y, \nu, \varphi, q) \in \mathrm{Bw\text{-}up}(X, \mu, \psi, p)$, for each $R_0 > 0$ and each $v_0 \in S(\|\cdot\|_{p,\mathrm{Lip}^*})$, the measure $\nu \llcorner B(q, R_0)$ admits an Alberti representation (Q_{R_0}, Φ) which satisfies the following conditions:*

(1) The measure Q_{R_0} is a finite Radon measure with total mass at most

$$\frac{1}{2R_0}\left(\mathrm{As}(\mu, p)\right)^{\log_2 R_0 + 1};$$

(2) *The support of Q_{R_0} is contatined in a compact set $\mathcal{S}_{R_0} \subset \mathrm{Frag}(Y)$ which consists of the unit-speed geodesic segments γ whose domain lies in $[0, 4R_0]$, whose image lies in $\bar{B}(q, 2R_0)$, which have both endpoints lying outside of $B(q, \frac{3}{2}R_0)$, which intersect $\bar{B}(q, R_0)$, and which satisfy:*

$$\varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1) = (s_2 - s_1)v_0 \quad (\forall s_1, s_2 \in \mathrm{Dom}\,\gamma); \tag{7.86}$$

(3) *For each $\gamma \in \mathcal{S}_{R_0}$ the measure Φ_γ is given by:*

$$\Phi_\gamma = \mathscr{H}^1_\gamma \llcorner B(q, R_0). \tag{7.87}$$

Proof. By Theorem 6.22 we can choose Borel maps $v_n : X \to TX$ [2] with $1 \le \|v_n\|_{TX} \le 1 + \frac{1}{n}$ and such that:

(1) For each $x \in X$ the closure of the set $\{v_n(x)\}_n$ contains $S(\|\cdot\|_{p,\mathrm{Lip}^*})$;
(2) For each n there is a measure μ'_n in the same measure class of μ and there are countably many disjoint compact sets $\{K_{n,\alpha}\}$ whose union has μ-negligible complement and such that the function v_n is constant on each $K_{n,\alpha}$;
(3) The measure $\mu'_n \llcorner K_{n,\alpha}$ admits a simplified and $(1, 1 + \frac{1}{n})$-biLipschitz Alberti representation $\mathcal{A}_{n,\alpha}$ in the ψ-direction of the cone $\mathcal{C}\left(v_n \llcorner K_{n,\alpha}/\|v_n \llcorner K_{n,\alpha}\|_2, \pi/2n\right)$ with $\langle v_n \llcorner K_{n,\alpha}, \psi\rangle$-speed $\ge (1 - 1/n)$.

Let U_n be a Borel subset of $\bigcup_\alpha K_{n,\alpha}$ with full μ-measure and such that, for each $p \in U_n \cap K_{n,\alpha}$, the conclusion of Theorem 7.22 holds taking $\mathcal{A} = \mathcal{A}_{n,\alpha}$. Let $U = \bigcap U_n$ and fix $p \in U$ and $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$. Choose a sequence n_m such that $v_{n_m}(p) \to v_0$ and let Q_{R_0,n_m}, Φ_{n_m} and $\mathcal{S}_{R_0,n_m}$ be the measures and sets of geodesics provided by Theorem 7.22. By Ascoli-Arzelá the set $\Omega = \mathcal{S}_{R_0} \bigcup_m \mathcal{S}_{R_0,n_m}$ is a compact subset of $\mathrm{Frag}(Y)$; as the measures Q_{R_0,n_m} are uniformly bounded and supported in Ω, we can pass to a subsequence such that $Q_{R_0,n_m} \xrightarrow{\mathrm{w}^\star} Q_{R_0}$. Note also that Q_{R_0} is supported in $\mathcal{S}_{R_0}$. For $g \in C_b(Y)$ one proves that:

$$\lim_{m\to\infty} \int_\Omega dQ_{R_0,n_m}(\gamma) \int g\, d(\Phi_{n_m})_\gamma = \int_{\mathcal{S}_{R_0}} dQ_{R_0}(\gamma) \int g\, d\Phi_\gamma \tag{7.88}$$

by using an argument similar to the one used to derive the estimates (7.80) and (7.82). Thus the pair (Q_{R_0}, Φ) provides the desired Alberti representation. □

To prove Theorem 7.18 we need to introduce a bit more of terminology. We can regard parametrized Lipschitz curves in Y, whose domain is a possibly infinite interval of $\mathbb{R}$, as elements of $F_c(\mathbb{R} \times Y)$ by identifying them with their graph. We denote by $\mathrm{Geo}(Y)$ the set of unit speed geodesic segments, half-lines or lines in Y; note that $\mathrm{Geo}(Y)$ is a K_σ. Moreover, if we let:

$$\begin{aligned} \Phi : \mathrm{Geo}(Y) &\to \mathrm{Rad} \\ \gamma &\mapsto \mathscr{H}^1_\gamma, \end{aligned} \tag{7.89}$$

then, for each $g \in C_c(Y)$, the map:

$$\begin{aligned} \Phi_g : \mathrm{Geo}(Y) &\to \mathbb{R} \\ \gamma &\to \int g d\Phi_\gamma = \int_{\mathbb{R}} g \circ \gamma(t)\, dt \end{aligned} \tag{7.90}$$

is continuous.

Proof of Theorem 7.18. Let U be the μ-full measure subset provided by Lemma 7.85 and consider $p \in U$ and $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$. Fix a diverging sequence of radii $\{R_n\}$ with $R_n > 2R_{n-1}$ and let Q_{R_n} and $\mathcal{S}_{R_n}$ be the corresponding measures and sets provided by Lemma 7.85. Note that $\mathcal{S}_{R_n}$ can also be regarded as a compact subset of $F_c(\mathbb{R} \times Y)$; for $i \le n$ we define the sets:

$$\mathcal{S}_{n,i} = \left\{\gamma \in \mathcal{S}_{R_n} : \mathrm{dist}(\gamma, q) \in (R_{i-1}, R_i]\right\}, \tag{7.91}$$

2 A choice of the representative of TX is implied.

where we take $R_0 = 0$, and observe that the sets $\mathcal{S}_{n,i}$ are Borel. We also consider the following Borel subsets of $F_c(\mathbb{R} \times Y) \times \mathbb{R}$:

$$\tilde{\mathcal{S}}_{i,n} = \left\{(\gamma, t) : t \in \operatorname{Dom}\gamma, \gamma \in \mathcal{S}_{i,n}, d\left(\gamma(t), q\right) \in (R_{i-1}, R_i]\right\}; \tag{7.92}$$

note that the sets $\tilde{\mathcal{S}}_{i,n}$ have compact sections, i.e. for $\gamma \in F_c$, each section $(\tilde{\mathcal{S}}_{i,n})_\gamma$ is compact. By the Lusin-Novikov Uniformization Theorem [25, Thm. 18.18], we can find Borel maps $\tau_{i,n} : \mathcal{S}_{i,n} \to \mathbb{R}$ such that $(\gamma, \tau_{i,n}(\gamma)) \in \tilde{\mathcal{S}}_{i,n}$. In particular, we can define a Borel map $\operatorname{Tran}_n : \operatorname{Geo}(Y) \to \operatorname{Geo}(Y)$ by requiring that for $\gamma \in \tilde{\mathcal{S}}_{i,n}$ the geodesic segment $\operatorname{Tran}_n(\gamma)$ is the composition of γ with the translation by $\tau_{n,i}(\gamma)$. Note that if $\gamma \in \tilde{\mathcal{S}}_{i,n}$ the extremes of $\operatorname{Dom}\gamma$ are at distance at least $\frac{3}{2}R_n - R_i$ from $\tau_{n,i}(\gamma)$ and so

$$\left[-\frac{R_n}{2}, \frac{R_n}{2}\right] \subset \operatorname{Dom}\operatorname{Tran}_n(\gamma) \subset [-3R_n, 3R_n]. \tag{7.93}$$

Let $Q_n = \operatorname{Tran}_{n\sharp} Q_{R_n}$ and denote by $K(m, i)$ the set of geodesic segments γ whose domain is contained in $[-3R_m, 3R_m]$, and which intersect $\bar{B}(q, R_i)$ in a point $p_\gamma = \gamma(t_\gamma)$ where t_γ is at distance at most $2R_i$ from 0. The set $K(\infty, i)$ is defined similarly by requiring γ to be a geodesic line. Note that the sets:

$$K(i) = K(\infty, i) \cup \bigcup_m K(m, i) \tag{7.94}$$

are compact and that Q_n is concentrated on the set $\bigcup_{i \le n} K(n, i)$. We now obtain an upper bound on $Q_n\left(K(i)\right)$:

$$\nu\left(B(q, 2R_i)\right) = \int_{\operatorname{Geo}(Y)} \mathscr{H}^1_\gamma\left(B(q, 2R_i)\right)\, dQ_n(\gamma) \ge \int_{K(i)} \mathscr{H}^1_\gamma\left(B(q, 2R_i)\right)\, dQ_n(\gamma); \tag{7.95}$$

if $\gamma \in \operatorname{Tran}_n\left(\mathcal{S}_{n,l}\right)$ and if $l \le i$ and $n \ge i$, one has $\mathscr{H}^1_\gamma\left(B(q, 2R_i)\right) \ge \frac{R_i}{2}$ so from (7.95) we obtain:

$$Q_n\left(K(i)\right) \le 2\frac{\nu\left(B(q, 2R_i)\right)}{R_i}. \tag{7.96}$$

In particular, we can pass to a subsequence and find a Radon measure Q on $\operatorname{Geo}(Y)$ such that for each i one has $Q_n \llcorner K(i) \xrightarrow{w^\star} Q \llcorner K(i)$; in particular $Q_n \xrightarrow{w^\star} Q$. Moreover, as Q_n is concentrated on $\bigcup_i K(n, i)$, the measure Q has support contained in $\operatorname{Lines}(\varphi, v_0)$. To show that (Q, Φ) gives an Alberti representation of ν we take $g \in C_c(Y)$ and choose i sufficiently large so that $\operatorname{spt} g \subset B(q, R_i)$:

$$\int_Y g\, d\nu = \int_{K(i)} dQ_n(\gamma) \int g\, d\mathscr{H}^1_\gamma = \int_{K(i)} \Phi_g(\gamma)\, dQ_n(\gamma), \tag{7.97}$$

and

$$\lim_{n\to\infty} \int_{K(i)} \Phi_g(\gamma)\, dQ_n(\gamma) = \int_{K(i)} \Phi_g(\gamma)\, dQ(\gamma) = \int_{\operatorname{Geo}(Y)} dQ(\gamma) \int g\, d\mathscr{H}^1_\gamma. \tag{7.98}$$

□

We now discuss consequences of Theorem 7.18 in terms of the canonical maps from blow-ups of X to the fibres of TX.

Retaining the assumptions of Theorem 7.18, suppose $(Y, \nu, \varphi, q) \in (X, \mu, \psi, p)$ is realized by choosing scales $r_n \searrow 0$.

Corollary 7.99 (cf. Theorem 1.12). *Let $p \in U$ be a point where the conclusion of Theorem 7.18 holds. Then:*

(1) *The canonical map $E : Y \to T_pX$ of Definition 7.15 is a metric submersion.*
(2) *For each $\tilde{q} \in Y$, and $v_0 \in T_pX$ there is a line $\gamma \in \operatorname{Lines}(\varphi, v_0)$ passing through it, and there is a $c_\gamma \in \mathbb{R}$ such that:*

$$E\left(\gamma(t)\right) = v_0(t - c_\gamma) \quad (\forall t \in \mathbb{R}). \tag{7.100}$$

Proof. (2). This follows immediately from Theorem 7.18 since the measure ν has full support in Y, and we have a blow-up Alberti representation of ν that is supported on lines in the direction v_0.

(1). The map E is 1-Lipschitz by Corollary 7.17. Given $y_1 \in Y$, $z_2 \in T_pX \setminus E(y_1)$, let $\Delta t = \|z_2 - E(y_1)\|_p$, and $v_0 = \frac{1}{\Delta t}(z_2 - E(y_1))$. By (2) there is a unit speed geodesic $\gamma : \mathbb{R} \to Y$ such that $E \circ \gamma$ has velocity v_0, and $\gamma(t_1) = y_1$ for some $t_1 \in \mathbb{R}$. Then putting $y_2 = \gamma(t_1 + \Delta t)$, we have

$$d(y_2, y_1) = d(\gamma(t_2), \gamma(t_1)) = \Delta t = \|z_2 - E(y_1)\|_p$$

and

$$E(y_2) = E(\gamma(t_1 + \Delta t)) = E(\gamma(t_1)) + \Delta t v_0 = z_2 \,.$$

□

Corollary 7.99 generalizes [14, Sec. 13] where the surjectivity of the map E was proven for the case in which (X, μ) is a PI-space. The surjectivity of the map E in the case in which (X, μ) is a differentiability space has already been proven in [22, 42].

7.3 Harmonicity of blow-up functions

In this subsection we prove Corollary 1.16.

Proof of Corollary 1.16. Let $u : X \to \mathbb{R}$ be a Lipschitz function, and suppose that $x \in X$ a point of differentiability of u where x is as in the statement of Theorem 1.12.

Note that (2) of Corollary 1.16 follows from Corollary 7.13.

Choose a unit vector $\xi \in (T_xX, \|\cdot\|_{TX}(x))$ supporting $du(x) \in T_x^\star X$, i.e.

$$du(x)(\xi) = \left\|du(x)\right\|_{T^*X} = \left\|du(x)\right\|_{T^*X} \cdot \|\xi\|_{TX} \,. \tag{7.101}$$

Since x is a point of differentiability, the blow-up $\hat{u}$ of u will be of the form $\hat{u} = \alpha \circ \hat{\varphi}_i$ for some $\alpha \in T_x^\star X$. Now consider an Alberti representation for $\hat{\mu}$ as in Theorem 1.12 (2), which is supported on unit speed geodesics γ with $(\hat{\varphi}_i \circ \gamma)' \equiv \xi$. Fix such a unit speed geodesic $\gamma : \mathbb{R} \to \hat{X}$. Note that for all $t \in \mathbb{R}$

$$(\hat{u} \circ \gamma)'(t) = \alpha\left((\hat{\varphi}_i \circ \gamma)'(t)\right) = \alpha(\xi) = \|du(x)\|_{T^*X} \,,$$

and by part (2) of the corollary we have

$$\|du(x)\|_{T^*X} \equiv \mathrm{Lip}(\hat{u})(\gamma(t)) = (\mathrm{Lip}(\hat{u} \circ \gamma))(t) \,.$$

If $v : \hat{X} \to \mathbb{R}$ is Lipschitz and agrees with $\hat{u}$ outside a compact subset $K \subset \hat{X}$, then for all $t \in \mathbb{R}$ we have $\mathrm{Lip}(v \circ \gamma)(t) \le \mathrm{Lip}(v)(\gamma(t))$, and for $\mathcal{L}$-a.e. $t \in \mathbb{R} \setminus \gamma^{-1}(K)$ we have

$$\mathrm{Lip}(v \circ \gamma) = |(v \circ \gamma)'(t)| = |(\hat{u} \circ \gamma)'(t)| = \mathrm{Lip}(\hat{u} \circ \gamma)(t) = \mathrm{Lip}(u)(x) \,.$$

Therefore if $\gamma^{-1}(K) \subset [a, b]$, then

$$\begin{aligned}
\int_{\gamma^{-1}(K)} [\mathrm{Lip}(v)(\gamma(t))]^p\,dt - \int_{\gamma^{-1}(K)} \left[\mathrm{Lip}(\hat{u})(\gamma(t))\right]^p\,dt &\ge \int_{\gamma^{-1}(K)} \left[\mathrm{Lip}(v \circ \gamma)(t))\right]^p\,dt - \int_{\gamma^{-1}(K)} \left[\mathrm{Lip}(\hat{u} \circ \gamma)(t))\right]^p\,dt \\
&= \int_{[a,b]} \left[\mathrm{Lip}(v \circ \gamma)(t))\right]^p\,dt - \int_{[a,b]} \left[\mathrm{Lip}(\hat{u} \circ \gamma)(t))\right]^p\,dt \\
&\ge \int_{[a,b]} |(v \circ \gamma)'(t)|^p\,dt - \int_{[a,b]} |(\hat{u} \circ \gamma)'(t)|^p\,dt \\
&\ge 0
\end{aligned}$$

by Jensen's inequality. Integrating this with respect to the measure on curves coming from the Alberti representation, we get that

$$\int_K \left[\mathrm{Lip}(v)\right]^p\,d\hat{\mu} \ge \int_K \left[\mathrm{Lip}(\hat{u})\right]^p\,d\hat{\mu} \,. \tag{7.102}$$

□

8 Lipschitz mappings $f : X \to Z$ and metric differentiation

8.1 The canonical subbundle determined by a pseudodistance

In this subsection we associate a canonical subbundle $\mathcal{W}_\varrho$ of $T^\star X$ to a Lipschitz compatible pseudometric ϱ; we denote by C_ϱ the Lipschitz constant of ϱ, that is, $\varrho \le C_\varrho d_X$.

Definition 8.1. Let Φ be a countable set of ϱ-Lipschitz functions and let V be a Φ-differentiability set; we define a subbundle $\mathcal{W}_\Phi$ of $T^\star X$ by letting, for $x \in V$, the fibre $\mathcal{W}_\Phi(x)$ equal the linear span of $\{df(x) : f \in \Phi\}$.

The collection $\mathrm{Sub}(\varrho)$ of subbundles associated to countable sets of ϱ-Lipschitz functions has a partial order $\preceq$: we say that $\mathcal{W}_\Phi \preceq \mathcal{W}_{\Phi'}$ if for μ-a.e. $x \in X$ one has $\mathcal{W}_\Phi(x) \subseteq \mathcal{W}_{\Phi'}(x)$.

Lemma 8.2. *The poset* $(\mathrm{Sub}(\varrho), \preceq)$ *contains a maximal element* $\mathcal{W}_\varrho$ *which we call* **the canonical subbundle associated to** ϱ.

Proof. As the constructions depend only on the measure class of μ, we can assume that μ is a probability measure. We basically follow the argument used in the proof of Lemma 2.33: to each $\mathcal{W}_\Phi \in \mathrm{Sub}(\varrho)$ we associate a "size", which is the expectation of the random variable $\dim \mathcal{W}_\Phi$:

$$\|\mathcal{W}_\Phi\| = \int \dim \mathcal{W}_\Phi(x)\, d\mu(x); \tag{8.3}$$

note that the finite dimensionality of $T^\star X$ implies that

$$S = \sup_{\mathcal{W}_\Phi \in \mathrm{Sub}(\varrho)} \|\mathcal{W}_\Phi\| < \infty. \tag{8.4}$$

Let $\mathcal{W}_{\Phi_n}$ be a maximizing sequence and let $\Phi_\infty = \bigcup_n \Phi_n$; then $\|\mathcal{W}_{\Phi_\infty}\| = S$. Suppose, by contradiction, that for some $\mathcal{W}_\Phi \in \mathrm{Sub}(\varrho)$ one has $\mathcal{W}_\Phi \not\preceq \mathcal{W}_{\Phi_\infty}$; then there is a positive measure set V such that, if $x \in V$, one has

$$\mathcal{W}_{\Phi_\infty}(x) \subsetneq \mathrm{span}\left(\mathcal{W}_\Phi(x) \cup \mathcal{W}_{\Phi_\infty}(x)\right); \tag{8.5}$$

but then we obtain the contradiction $\|\mathcal{W}_{\Phi \cup \Phi_\infty}\| > S$. □

Let $D_X \subset X$ be a countable dense set and $\Phi_{D_X,\varrho} = \{\varrho_x : x \in D_X\}$; we let $\mathcal{W}_{D_X,\varrho} = \mathcal{W}_{\Phi_{D_X,\varrho}}$. We now show that $\mathcal{W}_{D_X,\varrho}$ equals $\mathcal{W}_\varrho$: this is a stronger result than [28, Thm. 2.7] because it applies to subbundles associated to Lipschitz compatible pseudometrics, and does not require a Poincaré inequality.

Theorem 8.6. *For any countable dense set* $D_X \subset X$ *we have* $\mathcal{W}_{D_X,\varrho} = \mathcal{W}_\varrho$.

We offer two conceptually different proofs of Theorem 8.6.

Proof of Theorem 8.6 via a measurable Hahn-Banach. As $\mathcal{W}_{D_X,\varrho} \preceq \mathcal{W}_\varrho$, assume by contradiction that there is a positive measure Borel set U such that, for each $x \in U$ one has:

$$\mathcal{W}_{D_X,\varrho}(x) \subsetneq \mathcal{W}_\varrho(x). \tag{8.7}$$

Without loss of generality we can assume that there are 1-Lipschitz functions $\{\varphi_i\}_{i=1}^N$ such that $(U, \{\varphi_i\}_{i=1}^N)$ is a differentiability chart. Let

$$\begin{aligned}\tilde{U} = \Bigg\{(x,a) \in U \times \mathbb{R}^N : &\sum_{i=1}^N a_i d\varphi_i(x) \in \mathcal{W}_\varrho(x) \cap S(\|\cdot\|_{T^*X}(x)), \quad \text{and}\\ &\mathrm{dist}_{\|\cdot\|_{T^*X}(x)}\left(\sum_{i=1}^N a_i d\varphi_i(x), \mathcal{W}_{D_X,\varrho}(x)\right) \ge \frac{1}{2}\Bigg\}.\end{aligned} \tag{8.8}$$

Note that the distances in the fibre $T_x^\star X$ are computed with respect to the norm $\|\cdot\|_{T^*X}(x)$. The set $\tilde U$ is Borel and by (8.7) for each $x \in U$ the section $\tilde U_x$ is nonempty (compare [39, Lem. 4.22]) and compact. By the Lusin-Novikov Uniformization Theorem [25, Thm. 18.10] we obtain a unit-norm Borel section ω of $\mathcal{W}_\varrho \mid U$ satisfying:

$$\operatorname{dist}_{\|\cdot\|_{T^*X}(x)}\left(\omega(x), \mathcal{W}_{D_X,\varrho}(x)\right) \geq \frac{1}{2} \quad (\forall x \in U). \tag{8.9}$$

Using Hahn-Banach in each fibre $T_x^\star X$ and an argument similar to the one above, we obtain a Borel section ξ of $TX \mid U$ such that:

$$\begin{aligned} \|\xi\|_{TX} &\leq 2; \\ \langle \omega(x), \xi(x)\rangle &= 1 \quad (\forall x \in U), \end{aligned} \tag{8.10}$$

and such that $\xi(x)$ is annihilated by the functionals in $\mathcal{W}_{D_X,\varrho}(x)$. Up to shrinking U we can assume that there are $\tilde N \leq N$, $(1, \varrho)$-Lispchitz functions $\{\psi_i\}_{i=1}^{\tilde N}$ and bounded Borel maps $s_i : U \to \mathbb{R}$ satisfying:

$$\begin{aligned} \|s_i\| &\leq C; \\ \omega &= \sum_{i=1}^{\tilde N} s_i d\psi_i. \end{aligned} \tag{8.11}$$

Let $\mathcal{F}$ contain the φ_i, the ψ_i and the components of the chart functions; let $\mathcal{C}$ contain χ_U, the s_i and the characteristic functions of the charts; let $\mathcal{S}$ contain d_X and ϱ; by Theorem 5.3 we obtain an μ-measurable subset $V \subset U$ of full μ-measure with $G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ containing a dense set of directions in T_xX for each $x \in V$. In particular, fix $\varepsilon > 0$ and let $\gamma'(t) \in T_xX$ be an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vector such that:

$$\left\|\xi(x) - \gamma'(t)\right\|_{TX} \leq \varepsilon; \tag{8.12}$$

then

$$\left|\left\langle d\varrho_x, \gamma'(t)\right\rangle\right| \leq C_\varrho\varepsilon + \left|\left\langle d\varrho_x, \xi(x)\right\rangle\right| = C_\varrho\varepsilon; \tag{8.13}$$

by Theorem 4.3 we conclude that:

$$\varrho\text{-md}\gamma(t) \leq C_\varrho\varepsilon. \tag{8.14}$$

However, $\left\|\gamma'(t)\right\|_{TX} \leq 2 + \varepsilon$ and so

$$\left\langle \omega, \gamma'(t)\right\rangle \geq \frac{1}{2} - \varepsilon(2 + \varepsilon); \tag{8.15}$$

note also that

$$\left|\left\langle \omega, \gamma'(t)\right\rangle\right| = \left|\sum_{i=1}^{\tilde N} s_i\left(\gamma(t)\right)\ (\psi_i \circ \gamma)'(t)\right| \leq \tilde N C \max_i \left|(\psi_i \circ \gamma)'(t)\right|; \tag{8.16}$$

now choose $s_n \searrow 0$ with $t + s_n \in \operatorname{Dom}\gamma$; we have:

$$\left|\psi_i \circ \gamma(t + s_n) - \psi_i \circ \gamma(t)\right| \leq \varrho(\gamma(t + s_n), \gamma(t)) \leq o(s_n) + \int_{[t,t+s_n]\cap \operatorname{Dom}\gamma} \varrho\text{-md}\gamma(\tau)\, d\tau; \tag{8.17}$$

dividing by s_n and letting $n \nearrow \infty$ we get:

$$\left|(\psi_i \circ \gamma)'(t)\right| \leq \varrho\text{-md}\gamma(t). \tag{8.18}$$

Combining (8.14), (8.15), (8.16), and (8.18) we conclude that:

$$\frac{1}{2} - \varepsilon(2 + \varepsilon) \leq \tilde N\, C\, C_\varrho\varepsilon \tag{8.19}$$

which yields a contradiction if ε is sufficiently small. □

Proof of Theorem 8.6 via Weaver derivations. We show that if $K \subset X$ is compact and if f is ϱ-Lipschitz, for μ-a.e. $x \in K$ one has $df(x) \in \mathcal{W}_{D_X,\varrho}(x)$. Fix $n \in \mathbb{N}$ and choose a finite susbset $\{x_k\}_{k \in I_n} \subset D_X$ such that each $x \in K$ lies within d_X-distance at most $\frac{1}{n}$ from some x_k. To fix the ideas, suppose that f is (C, ϱ)-Lipschitz and define $f_n : K \to \mathbb{R}$ by:

$$f_n(x) = \inf \left\{ f(x_k) + C\varrho(x, x_k) : k \in I_n \right\} . \tag{8.20}$$

The functions $\{f_n\}_n$ are uniformly (C, ϱ)-Lipschitz and hence uniformly $(C\, C_\varrho, d_X)$-Lipschitz. By [42, Thm. 4.1] the exterior derivative operator d associated to the diffentiable structure is weak* continuous. In particular, let $L^2(\mu \llcorner K, T^*X)$ denote the L^2-space of sections of $T^*X \mid K$. Note that the dual of $L^2(\mu \llcorner K, T^*X)$ is $L^2(\mu \llcorner K, TX)$ and that these spaces are both reflexive by finite dimensionality of T^*X. Then as the $f_n \to f$ pointwise in K, we have that $df_n \to df$ weakly in $L^2(\mu \llcorner K, T^*X)$, and Mazur's Lemma and a standard argument give tail convex combinations g_n of the functions f_n with $dg_n \to df$ $\mu \llcorner K$-a.e. So the proof is completed if we show that each dg_n is a section of $\mathcal{W}_{D_X,\varrho}$, which happens if each df_n is a section of $\mathcal{W}_{D_X,\varrho}$. But for each n there are closed subsets $\{C_i\}_{i \in I_n}$ of K, such that $f_n \mid C_i = f(x_i) + C\varrho_{x_i}$, which gives $df_n \mid C_i = Cd\varrho_{x_i}$. □

We now associate to $\mathcal{W}_\varrho$ two a priori different norms on TX. Roughly speaking, we maximize the seminorms induced by sections of $\mathcal{W}_\varrho$. Recall that if f is ϱ-Lipschitz we can define the "big Lip" with respect to ϱ:

$$\varrho\text{-Lip}f(x) = \limsup_{r \searrow 0} \sup \left\{ \frac{|f(x) - f(y)|}{r} : \varrho(x, y) \le r \right\} , \tag{8.21}$$

and that the map $x \mapsto \varrho\text{-Lip}f(x)$ is Borel.

Let $\mathrm{Sec}_1(\varrho)$ denote the set of those sections ω of $\mathcal{W}_\varrho$ which are locally the differential of a $(1, \varrho)$-Lipschitz function; i.e. $\omega \in \mathrm{Sec}_1(\varrho)$ if and only if there are countably many disjoint Borel sets $\{V_\beta\}_\beta$ and countably many $(1, \varrho)$-Lipschitz functions $\{f_\beta\}_\beta$ such that $\mu\left(X \setminus \bigcup_\beta V_\beta\right) = 0$ and $\omega \mid V_\beta = df_\beta \mid V_\beta$. To each $\omega \in \mathrm{Sec}_1(\varrho)$ we associate a seminorm p_ω on TX by letting:

$$p_\omega(v) = |\langle \omega, v \rangle| . \tag{8.22}$$

We observe that $p_\omega \le C_\varrho \|\cdot\|_{TX}$ and denote by $\|\cdot\|_{\varrho,\mathrm{LIP}}$ the essential supremum (Lemma 2.33) of the collection $\{\|\cdot\|_\omega\}_{\omega \in \mathrm{Sec}_1(\varrho)}$.

Another way of obtaining seminorms on TX is to use arbitrary sections of $\mathcal{W}_\varrho$ and rescale them by the local ϱ-Lipschitz constant; note, however, that if u, v are both ϱ-Lipschitz, one can have $du = dv$ and $\varrho\text{-Lip}u \ne \varrho\text{-Lip}v$ on a set of positive measure. We are thus led to use a slightly more complicated framework. Let $\mathrm{Sec}_\star(\varrho)$ denote the set of countable pairs $\tilde{\omega} = \left\{(V_\beta, f_\beta)\right\}$ where the V_β are disjoint Borel sets with $\mu\left(X \setminus \bigcup_\beta V_\beta\right) = 0$, and the f_β are ϱ-Lipschitz functions. To each $\tilde{\omega} \in \mathrm{Sec}_\star(\varrho)$ we associate a seminorm $p_{\tilde{\omega}}$ on TX by letting, for $x \in V_\beta$ and $v \in T_xX$:

$$p_{\tilde{\omega}}(v) = \begin{cases} 0 & \text{if } \varrho\text{-Lip}f_\beta(x) = 0 \\ \frac{|\langle df_\beta(x), v \rangle|}{\varrho\text{-Lip}f_\beta(x)} & \text{otherwise;} \end{cases} \tag{8.23}$$

we denote by $\|\cdot\|_{\varrho,\mathrm{Lip}}$ the essential supremum (Lemma 2.33) of the collection $\{\|\cdot\|_\omega\}_{\omega \in \mathrm{Sec}_\star(\varrho)}$.

Theorem 8.24. *Let $D_X \subset X$ be a countable dense set. Then one has:*

$$\|\cdot\|_{D_X,\varrho} = \|\cdot\|_{\varrho,\mathrm{LIP}} = \|\cdot\|_{\varrho,\mathrm{Lip}} ; \tag{8.25}$$

in particular, if $D'_X \subset X$ is another countable dense set:

$$\|\cdot\|_{D_X,\varrho} = \|\cdot\|_{\tilde{D}_X,\varrho} ; \tag{8.26}$$

in the sequel, we will denote the **canonical norm** *(8.25) by $\|\cdot\|_\varrho$.*

Proof. Each pseudodistance function ϱ_x gives rise to an element of $\mathrm{Sec}_1(\varrho)$ and so $\|\cdot\|_{D_X,\varrho} \le \|\cdot\|_{\varrho,\mathrm{LIP}}$; to each $\omega \in \mathrm{Sec}_1(\varrho)$ one can associate $\tilde{\omega} \in \mathrm{Sec}_\star(\varrho)$ with $p_\omega \le p_{\tilde{\omega}}$ and so $\|\cdot\|_{\varrho,\mathrm{LIP}} \le \|\cdot\|_{\varrho,\mathrm{Lip}}$. We thus just prove that:

$$\|\cdot\|_{\varrho,\mathrm{Lip}} \le \|\cdot\|_{D_X,\varrho} . \tag{8.27}$$

It suffices to show that for any $\tilde{\omega} = \{(V_\beta, f_\beta)\} \in \mathrm{Sec}_\star(\varrho)$ one has

$$p_{\tilde{\omega}} \le \|\cdot\|_{D_X,\varrho} . \tag{8.28}$$

Let $\mathcal{F}$ contain the components of the chart functions and the functions $\{f_\beta\}_\beta$; let $\mathcal{C}$ contain the characteristic functions of the charts and the characteristic functions $\{\chi_{V_\beta}\}_\beta$; let $\mathcal{S}$ contain d_X and ϱ. Let V be an $\{f_\beta\}_\beta$-differentiability set and fix β; let $V'_\beta = V \cap V_\beta$; by Theorem 5.3 there is a full μ-measure μ-measurable subset $W_\beta \subset V'_\beta$ such that, for each $x \in W_\beta$ the set of $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vectors contains a dense set of directions. In particular, for each $v \in T_xX$ and $\varepsilon > 0$ we can find an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$-generic velocity vector $\gamma'(t) \in T_xX$ with $\|v - \gamma'(t)\|_{TX} \le \varepsilon$. Assume that ϱ-Lip$f_\beta(x) > 0$; note that the derivative $(f_\beta \circ \gamma)'(t)$ exists and is approximately continuous at t. Without loss of generality assume that $(f_\beta \circ \gamma)'(t) /= 0$; then we can find $s_n \searrow 0$ such that $t + s_n \in \mathrm{Dom}\,\gamma$ and $\varrho(\gamma(t + s_n), \gamma(t)) = r_n > 0$. We now obtain the estimate:

$$\begin{aligned} \left|(f_\beta \circ \gamma)(t + s_n) - (f_\beta \circ \gamma)(t)\right| &\le \sup\left\{\frac{|f_\beta(y) - f_\beta(x)|}{r_n} : \varrho(y, x) \le r_n\right\} \varrho(\gamma(t + s_n), \gamma(t)) \\ &\le \left(\varrho\text{-Lip}f_\beta(x) + O(1/n)\right) \left(\int_{[t,t+s_n]\cap \mathrm{Dom}\,\gamma} \varrho\text{-md}\gamma(\tau)\, d\tau + o(s_n)\right); \end{aligned} \tag{8.29}$$

dividing by s_n and letting $n \nearrow \infty$ we get, by approximate continuity of ϱ-mdγ at t:

$$\left|(f_\beta \circ \gamma)'(t)\right| \le \varrho\text{-Lip}f_\beta(x)\, \varrho\text{-md}\gamma(t). \tag{8.30}$$

Now Theorem 4.3 implies that ϱ-md$\gamma(t) = \|\gamma'(t)\|_{D_X,\varrho}$ and so:

$$\begin{aligned} \left|\langle df_\beta, \gamma'(t)\rangle\right| &\le \varrho\text{-Lip}f_\beta(x)\, \|\gamma'(t)\|_{D_X,\varrho} \\ &\le \varrho\text{-Lip}f_\beta(x)\, \|v\|_{D_X,\varrho} + \varepsilon C_\varrho\, \varrho\text{-Lip}f_\beta(x); \end{aligned} \tag{8.31}$$

let L denote the global Lipschitz constant of f_β; then:

$$\left|\langle df_\beta, v\rangle\right| \le \varrho\text{-Lip}f_\beta(x)\, \|v\|_{D_X,\varrho} + \varepsilon C_\varrho\, \varrho\text{-Lip}f_\beta(x) + \varepsilon L\, \|v\|_{TX} ; \tag{8.32}$$

so (8.28) follows by letting $\varepsilon \searrow 0$. □

8.2 Metric Differentiation for Lipschitz maps

We now reformulate the results of the previous subsection for a Lipschitz map $F : X \to Z$; throughout this subsection ϱ will denote the pull-back pseudometric $F^\star d_Z$. Putting together Theorems 8.6 and 8.24 we obtain:

Theorem 8.33. *Associated to the map F there is a canonical subbundle $\mathcal{W}_F$ of $T^\star X$ such that:*

(1) For each $g \in F^\star(\mathrm{Lip}(Z))$ (i.e. $g = h \circ F$ for some $h \in \mathrm{Lip}(Z)$) the section dg lies in $\mathcal{W}_F$;
(2) For each countable dense set $D_X \subset X$ the subbundle $\mathcal{W}_F$ coincides with the subbundle spanned by the sections $\{d\varrho_x : x \in D_X\}$.

Suppose now that $\mathcal{F}$ contains the components of the chart functions of (X, μ), that $\mathcal{C}$ contains the characteristic functions of the charts, and suppose also that $\mathcal{S}$ contains the pseudometric ϱ. The subbundle $\mathcal{W}_F$ induces a canonical seminorm $\|\cdot\|_F = \|\cdot\|_\varrho$ on TX such that, for each $(\mathcal{F}, \mathcal{C}, \mathcal{S})$-generic velocity vector $\gamma'(t)$ one has:

$$\|\gamma'(t)\|_F = \lim_{s\to 0} \frac{d_Z(F \circ \gamma(t + s), F \circ \gamma(t))}{|s|}. \tag{8.34}$$

Remark 8.35. In practice, it does not matter whether metric differentiation is formulated in terms of pseudometrics or Lipschitz maps. In fact, consider a Lipschitz compatible pseudometric ϱ on X and associate to it the Lipschitz map:

$$\begin{aligned} F : X &\to l^\infty(D_X) \\ y &\mapsto \{\varrho_x(y)\}_{x\in D_X} ; \end{aligned} \tag{8.36}$$

then we get $\|\cdot\|_{\varrho} . = \|\cdot\|_F$ and $\mathcal{W}_\varrho = \mathcal{W}_F$.

We now specialize the discussion to the case in which (Z, ν) is a differentiability space; throughout the remainder of this subsection we will fix choices of countable dense sets $D_X \subset X$ and $D_Z \subset Z$. The case of interest is when the measure $F_\sharp\mu$ is absolutely continuous with respect to ν. Using the Radon-Nikodym Theorem we can find a Borel subset $V_0 \subset Z$ such that $F_\sharp\mu \llcorner V_0$ and $\nu \llcorner V_0$ are in the same measure class. The case of interest is when $\nu(V_0) > 0$, which we will assume throughout the remainder of this subsection.

Let $U_0 = F^{-1}(V_0)$ and suppose that $g \in \mathrm{Lip}(Z)$ is differentiable at z_0 with respect to the Lipschitz functions $\{\psi^i\}_{i=1}^M$; suppose now that $z_0 = F(x_0)$ and that the functions $\{\psi^i \circ F\}_{i=1}^M$ are differentiable at x_0 with respect to the functions $\{\varphi^j\}_{j=1}^N$. We then obtain the chain rule:

$$g \circ F(x) - g \circ F(x_0) = \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\partial g}{\partial \psi^i}(z_0)\,\frac{\partial(\psi^i \circ F)}{\partial \varphi^j}\,\left(\varphi^j(x) - \varphi^j(x_0)\right) + o\left(d_X(x, x_0)\right). \tag{8.37}$$

The following Corollary is a consequence of the chain rule (8.37):

Corollary 8.38. *Let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be an atlas for (X, μ) and $\{(V_\beta, \psi_\beta)\}_\beta$ an atlas for (Z, ν). Then the subbundle $\mathcal{W}_F \mid U_0$ is spanned by the sections $\left\{d(\psi^i_\beta \circ F)\right\}_{\beta,i}$.*

Definition 8.39. As the measures $F_\sharp\mu \llcorner V_0$ and $\nu \llcorner V_0$ are in the same measure class, we obtain a pull-back map:

$$F^\star : T^\star Z \mid V_0 \to T^\star X \mid U_0, \tag{8.40}$$

which maps each section dg of $T^\star Z \mid V_0$ to the section $F^\star dg = d(g \circ F)$ of $T^\star X \mid U_0$. We define the push-forward map:

$$F_\star : TX \mid U_0 \to TZ \mid V_0 \tag{8.41}$$

by duality; that is, for $x \in U_0$, $v \in T_xX$ and $g \in \mathrm{Lip}(Z)$ we let:

$$\left\langle F_\star(v), dg\,|_{F(x)}\right\rangle = \left\langle v, \left(F^\star dg\right)_x\right\rangle. \tag{8.42}$$

We conclude this subsection by proving:

Theorem 8.43. *Let $\mathcal{W}_F^\perp$ denote the annihilator of $\mathcal{W}_F$: i.e. the fibre $\mathcal{W}_F^\perp(x)$ consists of those vectors in T_xX which are annihilated by the functionals in $\mathcal{W}_F(x)$. The seminorm $\|\cdot\|_F$ induces a norm on the quotient bundle $TX/\mathcal{W}_F^\perp$ which we will still denote by $\|\cdot\|_F$. Then $F_\star$ induces an injective isometry:*

$$F_\star : \left(TX/\mathcal{W}_F^\perp \mid U_0, \|\cdot\|_F\right) \to (TZ \mid V_0, \|\cdot\|_{TZ}). \tag{8.44}$$

The proof of Theorem 8.43 uses the following generalization of Theorem 5.3, whose proof is omitted.

Lemma 8.45. *Suppose that $\mathcal{F}$ contains the components of the $\{\varphi_\alpha\}_\alpha$ and of the $\{\psi_\beta \circ F\}_\beta$; suppose that $\mathcal{C}$ contains the $\{\chi_{U_\alpha}\}_\alpha$, the $\{\chi_{V_\beta}\}_\beta$ and χ_{U_0}; suppose that $\mathcal{S}$ contains ϱ. Suppose also that $\mathcal{F}'$ contains the components of the $\{\psi_\beta\}_\beta$ and that $\mathcal{C}'$ contains the $\{\chi_{V_\beta}\}$ and χ_{V_0}. Let*

$$G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}; \mathcal{F}', \mathcal{S}') = \left\{v \in T_xX : v = \gamma'(t), \text{ where } \gamma'(t) \text{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S})\text{-generic and } (F \circ \gamma)'(t) \text{ is } (\mathcal{F}', \mathcal{C}')\text{-generic}\right\}; \tag{8.46}$$

then there is a full μ-measure μ-measurable subset $U_1 \subset U_0$ such that, for each $x \in U_1$, $G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}; \mathcal{F}', \mathcal{S}')$ contains a dense set of directions.

Proof of Theorem 8.43. We apply Lemma 8.45 and show that for each $x \in U_1$ and each $v \in T_xX$ one has:

$$\|v\|_F = \left\|F_\star(v)\right\|_{TZ}; \tag{8.47}$$

by density of directions, we just need to show (8.47) for $v = \gamma'(t)$ where $\gamma'(t)$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S})$-generic and $(F \circ \gamma)'(t)$ is $(\mathcal{F}', \mathcal{C}')$-generic. By Theorem 4.3 applied in X to the pseudometric ϱ we get:

$$\left\|\gamma'(t)\right\|_F = \varrho\text{-md}\gamma(t); \tag{8.48}$$

note that by the definition of the ϱ-metric differential we have:

$$\varrho\text{-md}\gamma(t) = \text{md}\, F \circ \gamma(t); \tag{8.49}$$

finally, applying again Theorem 4.3 in Z to the metric d_Z, we get:

$$\left\|F_\star\gamma'(t)\right\|_{TZ} = \left\|(F \circ \gamma)'(t)\right\|_{TZ} = \text{md}\, F \circ \gamma(t). \tag{8.50}$$

□

8.3 Metric differentiation and blow-ups

In this subsection we generalize the results of Section 7 in the case in which one considers either a Lipschitz compatible pseudometric ϱ on X or a Lipschitz map $F : X \to Z$.

Definition 8.51. Let ϱ be a Lipschitz compatible pseudometric on X and (U, ψ) be an N-dimensional differentiability chart. A **blow-up of** (X, μ, ψ, ϱ) **at** p along the scales $r_n \searrow 0$ is a tuple $(Y, \nu, \varphi, \tilde{\varrho}, q)$ such that:

(1) The tuple (Y, ν, φ, q) is a blow-up of (X, μ, ψ) at p, i.e. the tuples:

$$\left(\frac{1}{r_n}X, \frac{\mu}{\mu\left(B(p, r_n)\right)}, \frac{\psi - \psi(p)}{r_n}, p\right) \tag{8.52}$$

converge to (Y, ν, φ, q) in the measured Gromov-Hausdorff sense;

(2) $\tilde{\varrho}$ is a Lipschitz compatible pseudometric on Y and if the points $y, y' \in Y$ are represented, respectively, by the sequences $[x_n], [x'_n] \subset X$, then:

$$\tilde{\varrho}(y, y') = \lim_{n\to\infty} \frac{\varrho(x_n, x'_n)}{r_n}. \tag{8.53}$$

We denote by Bw-up$(X, \mu, \psi, \varrho, p)$ the set of blow-ups of (X, μ, ψ, ϱ) at p.

Theorem 8.54. *Let (U, ψ) be an N-dimensional differentiability chart for the differentiability space (X, μ), and let ϱ be a Lipschitz compatible pseudometric. Then for $\mu \llcorner U$-a.e. p, for each blow-up $(Y, \nu, \varphi, \tilde{\varrho}, q) \in$* Bw-up$(X, \mu, \psi, \varrho, p)$*, and for each unit vector $v_0 \in T_pX$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:*

(1) Q is concentrated on the set Lines$(\varphi, v_0, \tilde{\varrho})$ *of unit speed geodesic lines in Y with:*

$$\begin{aligned}(\varphi \circ \gamma)' &= v_0;\\ \tilde{\varrho}(\gamma(t), \gamma(s)) &= \|v_0\|_\varrho\, |t - s|;\end{aligned} \tag{8.55}$$

(2) For each $\gamma \in$ Lines$(\varphi, v_0, \tilde{\varrho})$ *the measure Φ_γ is given by:*

$$\Phi_\gamma = \mathscr{H}^1_\gamma. \tag{8.56}$$

Proof. The proof follows the method used to prove Theorem 7.18; we just:

(1) add in condition **(Reg3)** that:

$$\left|\varrho(\gamma(s_1), \gamma(s_2)) - \left\|\gamma'(t)\right\|_\varrho |s_1 - s_2|\right| \le \varepsilon|s_1 - s_2|; \tag{8.57}$$

(2) require in Lemma 7.48 that U consists of points at which the map $x \mapsto \|\cdot\|_{\varrho}(x)$ is approximately continuous.

□

We now discuss what happens in the case of a Lipschitz map $F : X \to Z$. When we defined blow-ups of the chart functions there was no issue with the target space because $\mathbb{R}^N$ possesses a group of dilations. For a general map $F : X \to Z$ we first need to use ultramits [32, Sec. 2.4] to blow-up Z; we recall here the relevant constructions.

Definition 8.58. Let (Z, z_0) denote a pointed metric space and let $r_n \searrow 0$; we define a **blow-up** (W, w_0) **of** (Z, z_0) along the scales $r_n \searrow 0$ as an ultralimit of the sequence of pointed metric spaces $\left(\frac{1}{r_n} Z, z_0\right)$. Specifically, we choose a nonprincipal ultrafilter ω and consider the set $\tilde{W}$ of those sequences $[z_n] \subset Z$ such that:

$$\limsup_{n\to\infty} \frac{d_Z(z_n, z_0)}{r_n} < \infty. \tag{8.59}$$

We define a pseudometric $d_{\tilde{W}}$ on $\tilde{W}$ by:

$$d_{\tilde{W}}([z_n], [z'_n]) = \lim_{\omega} \frac{d_Z(z_n, z'_n)}{r_n}. \tag{8.60}$$

On $\tilde{W}$ we consider the equivalence relation:

$$[z_n] \sim [z'_n] \iff d_{\tilde{W}}([z_n], [z'_n]) = 0; \tag{8.61}$$

then $d_{\tilde{W}}$ induces a metric d_W on the quotient space $W = \tilde{W}/\sim$, and the base point w_0 is the equivalence class of the constant sequence $[z_0]$. We denote the set of blow-ups of Z at z_0 by $\text{Bw-up}(Z, z_0)$.

Consider now the case of a Lipschitz map $F : X \to Z$; having fixed scales $r_n \searrow 0$, we construct blow-ups $(Y, q) \in \text{Bw-up}(X, p)$ and $(W, w_0) \in \text{Bw-up}(Z, F(p))$. We then obtain a Lipschitz map $G : (Y, q) \to (W, w_0)$ by blowing up the graph of F at $(p, F(p))$. Specifically, if $[x_n] \subset X$ represents the point $y \in Y$, we let $G(y)$ be the equivalence class of the sequence $[F(x_n)]$. In general, we say **that a tuple** $(Y, \nu, \varphi, q; G, W, w_0)$ **is a blow-up of** $(X, \mu, \psi; F, Z)$ **at** p if: $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$, $(W_0, w_0) \in \text{Bw-up}(Z, F(p))$, and G is obtained by blowing up $F : X \to Z$ at p. We denote the set of blow-ups of $(X, \mu, \psi; F, Z)$ at p by $\text{Bw-up}(X, \mu, \psi, p; F, Z)$.

Applying Theorem 8.54 to the pseudometric $F^\star d_Z$ we get:

Theorem 8.62. *Let (U, ψ) be an N-dimensional differentiability chart for the differentiability space (X, μ), and let $F : X \to Z$ be a Lipschitz map. Then for $\mu \llcorner U$-a.e. p, for each blow-up $(Y, \nu, \varphi, q; G, W, w_0) \in$* $\text{Bw-up}(X, \mu, \psi, p; F, Z)$, *and for each unit vector $v_0 \in T_pX$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:*

(1) Q is concentrated on the set $\text{Lines}(\varphi, v_0, G)$ *of unit speed geodesic lines in Y with:*

$$\begin{aligned} (\varphi \circ \gamma)' &= v_0; \\ d_W(G \circ \gamma(t), G \circ \gamma(s)) &= \|v_0\|_F\,|t - s|; \end{aligned} \tag{8.63}$$

(2) For each $\gamma \in$ $\text{Lines}(\varphi, v_0, G)$ *the measure Φ_γ is given by:*

$$\Phi_\gamma = \mathscr{H}^1_\gamma. \tag{8.64}$$

Remark 8.65. In [14, Sec. 10] it was shown that if (X, μ) is a PI-space and if f is a real-valued Lipschitz map defined on X, at μ-a.e. p, blowing-up f at p always produces a *generalized linear function* g; in particular, the corresponding space Y contains through each point a geodesic line γ on which the blow-up F is affine, and such that γ behaves as an integral curve of the gradient of F. Applying Theorem 8.62 to the case in which $F = f$, one gets, through each point of Y, many geodesic lines on which the blow-up G is affine, and these geodesic lines can be used to obtain a Fubini-like decomposition of the measure ν. Among these geodesic lines, those where the slope of G is maximal correspond to the vector v_0 which is the derivative of f at p with respect to the coordinate functions ψ.

9 Examples

In this section we provide examples that illustrate how metric differentiation can be used to constrain the infinitesimal geometry of a Lipschitz map $F : X \to Y$, where X is a differentiability space. We will use a family of examples of differentiability spaces introduced in [20]: inverse limits of admissible inverse systems of metric measure graphs. For these spaces we find that for some natural classes of target spaces, blow-ups of arbitrary Lipschitz maps are quite degenerate.

In this section we will say that a map $\eta : \mathbb{R} \to Z$ into a metric space Z is a **geodesic with speed** σ if $d(\eta(s), \eta(t)) = \sigma|s - t|$ for all $s, t \in \mathbb{R}$; here we allow geodesics with speed 0, i.e. constant maps.

Theorem 9.1. *Let $(X_\infty, d_\infty, \mu_\infty)$ be the inverse limit of an admissible inverse system (see below). Let $A \subset X_\infty$ be a measurable subset, and $F : A \to Z$ be a Lipschitz map.*

Consider the following conditions:

- *(a) Z is CBB space, i.e. an Alexandrov space with curvature bounded below.*
- *(b) For every $z \in Z$ and every blow-up W of Z at z (in the sense of ultralimits, see Definition 8.58), two constant speed geodesics $\gamma, \gamma' : \mathbb{R} \to W$ which coincide on a nonempty open interval $(a, b) \subset \mathbb{R}$ coincide everywhere.*
- *(c) Z is an equiregular sub-Riemannian manifold.*
- *(d) Z is an Alexandrov space with curvature bounded above, and X_∞ satisfies the monotone bigon condition (Definition 9.18).*

If Z satisfies one of the conditions (a)–(c), then for μ_∞-a.e. $p \in X_\infty$, if $G : Y_\infty \to W$ is a blow-up of F at p, then G factors as $G = \bar{G} \circ \varphi$ where $\varphi : Y_\infty \to \mathbb{R}$ is 1-Lipschitz and $\bar{G} : \mathbb{R} \to W$ is a constant speed geodesic.

If Z satisfies (d), then for μ_∞-a.e. $p \in X_\infty$, every blow-up $G : Y_\infty \to W$ of F at p factors as $\bar{G} \circ \varphi$ where $\varphi : Y_\infty \to Z'$ is 1-Lipschitz, $\bar{G} : Z' \to W$ an isometric embedding, and Z' is a metric cone over a finite set, i.e. the union of finitely many geodesics rays leaving a basepoint.

In fact the argument gives slightly more precise control, see (9.19) and (9.20) below. Also, the argument can be generalized somewhat further, see Remark 9.21.

Remark 9.2. Note that when the target Z is a Banach space with the Radon-Nikodym Property the same conclusion as in the cases (a)–(c) follows by differentiating F along the Alberti representations, compare the discussion on RNP-differentiability in [16].

Theorem 9.1 has the following consequence:

Corollary 9.3. *Under the assumptions of Theorem 9.1, if $F : A \to Z$ is a bi-Lipschitz embedding, then there is a 1-rectifiable subset $A_1 \subset A$ such that $\mu(A \setminus A_1) = 0$.*

The proof of the corollary is given after the proof of Theorem 9.1.

Definition 9.4 (Admissible inverse systems, [20]). We consider an inverse system of metric measure graphs:

$$\cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} X_{i+1} \xleftarrow{\pi_{i+1}} \cdots, \tag{9.5}$$

where the index i can range either over $\mathbb{Z}$ or over $\mathbb{N} \cup \{0\}$: in the former case we will say that the inverse system is **signed**, and in the latter case that it is **unsigned**. We denote the metric and measure on X_i by d_i and μ_i respectively. Having fixed an integer $m \geq 2$ and parameters $\Delta, C, \theta \in (0, \infty)$, we say that the inverse system $\{X_i, \pi_i\}$ is **admissible** if it satisfies the following axioms:

- **(Ad1)** Each metric space (X_i, d_i) is a nonempty connected graph with vertices of valence $\leq \Delta$ and such that each edge of X_i is isometric to an interval of length m^{-i} with respect to the path metric d_i;
- **(Ad2)** Let X_i' denote the graph obtained by subdividing each edge of X_i into m edges of length $m^{-(i+1)}$. Then π_i induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X_i', d_i)$ which is open, simplicial and an isometry on every edge;
- **(Ad3)** For each $x_i \in X_i'$ the inverse image $\pi_i^{-1}(x_i) \subset X_{i+1}$ has d_{i+1}-diameter at most $\theta\, m^{-(i+1)}$;

(Ad4) Each graph X_i is equipped with a measure μ_i which restricts to a multiple of arclength on each edge; if e_1, e_2 are two adjacent edges of X_i we have:

$$\frac{\mu_i(e_1)}{\mu_i(e_2)} \in [C^{-1}, C]; \tag{9.6}$$

(Ad5) The measures $\{\mu_i\}$ are compatible with the projections $\{\pi_i\}$: $\pi_{i\sharp}\mu_{i+1} = \mu_i$;

(Ad6) Let $\mathrm{St}(x, G)$ denote the star of a vertex x in a graph G, i.e. the union of all the edges containing x. Then, for each vertex $v_i' \in X_i'$ and each $v_{i+1} \in \pi_i^{-1}(v_i')$, the quantity:

$$\frac{\mu_{i+1}\left(\pi_i^{-1}(e_i') \cap \mathrm{St}(v_{i+1}, X_{i+1})\right)}{\mu_i(e_i')} \tag{9.7}$$

is the same for all edges $e_i' \in \mathrm{St}(v_i', X_i')$;

(Ad7) If the inverse system $\{X_i, \pi_i\}$ is unsigned we will assume that $X_0 \simeq [0, 1]$, $\mu_0 = \mathcal{L}^1 \llcorner [0, 1]$ and we will denote by φ_i the map:

$$\varphi_i = \pi_1 \circ \cdots \circ \pi_{i-1}. \tag{9.8}$$

If the inverse system $\{X_i, \pi_i\}$ is signed we require the existence of open surjective maps $\varphi_i : X_i \to \mathbb{R}$ which are, regarding $\mathbb{R}$ as a graph of edges $\left\{[km^{-i}, (k+1)m^{-i}]\right\}_{k\in\mathbb{Z}}$, simplicical and restrict to isometries on every edge. Moreover, we require that the $\{\varphi_i\}$ are compatible with the $\{\pi_i\}$:

$$\varphi_i \circ \pi_i = \varphi_{i+1} \quad (\forall i). \tag{9.9}$$

An immediate consequence of the axioms **(Ad1)–(Ad7)** is that the metric measure spaces (X_i, d_i, μ_i) converge in the measured Gromov-Hausdorff sense[3] to a metric measure space $(X_\infty, d_\infty, \mu_\infty)$ which is called **the inverse limit** of the admissible inverse system. If $\{X_i, \pi_i\}$ is unsigned, then (X_∞, d_∞) is compact geodesic and μ_∞ is a doubling probability measure; if $\{X_i, \pi_i\}$ is signed, (X_∞, d_∞) is proper geodesic and μ_∞ is a doubling measure. In both cases there are 1-Lipschitz maps $\pi_{\infty,k} : X_\infty \to X_k$ satisfying:

$$\begin{aligned} \pi_{k-1} \circ \pi_{\infty,k} &= \pi_{\infty,k-1} \\ \pi_{\infty,k\sharp}\mu_\infty &= \mu_k. \end{aligned} \tag{9.10}$$

For $j > k$ we will use the short-hand notation $\pi_{j,k}$ to denote the map $\pi_k \circ \cdots \circ \pi_{j-1}$. Moreover, the maps $\varphi_i : X_i \to \mathbb{R}$ or $[0, 1]$ pass to the limit giving a 1-Lipschitz map $\varphi_\infty : X_\infty \to \mathbb{R}$ or $[0, 1]$ satisfying:

$$\varphi_\infty(q) = \varphi_i(\pi_{\infty,i}(q)) \quad (\forall q \in X_\infty, \forall i \in \mathbb{Z} \text{ or } \mathbb{N} \cup \{0\}) \tag{9.11}$$

We now define a special class of paths in X_i or X_∞.

Definition 9.12. Let $I \subseteq \mathbb{R}$ be connected and $\gamma : I \to X_i$ continuous, where we allow $i = \infty$. We say that γ is a **monotone geodesic** if $\varphi_i \circ \gamma : I \to \mathbb{R}$ or $[0, 1]$ is either a strictly increasing or decreasing affine map. In particular, the axioms **(Ad1)–(Ad7)** imply that γ is a constant speed geodesic in (X_i, d_i). Moreover, by axioms **(Ad2)** and **(Ad7)**, if $j > i$ and if $\gamma_i : I \to X_i$ is a monotone geodesic, then for each $q_j \in \pi_{j,i}^{-1}\left(\gamma_i(I)\right)$, one can lift γ_i to obtain a monotone geodesic $\gamma_j : I \to X_j$ passing through q_j and satisfying $\pi_{j,i} \circ \gamma_j = \gamma_i$.

We now summarize some important consequences of the axioms **(Ad1)–(Ad7)**.

Theorem 9.13. *Let $\{X_i, \pi_i\}$ be an admissible inverse system and let X_∞ denote the inverse limit; then:*

(1) The metric measure space $(X_\infty, d_\infty, \mu_\infty)$ admits a $(1, 1)$-Poincaré inequality; in particular, it is a differentiability space with a single differentiability chart $(X_\infty, \varphi_\infty)$;

3 If $\{X_i, \pi_i\}$ is signed we consider the convergence in the pointed sense by choosing basepoints $\{q_i\}_{i\in\mathbb{Z}}$ satisfying $\pi_i(q_{i+1}) = q_i$.

(2) *The (X_i, μ_i)'s and (X_∞, μ_∞) admit distinguished Alberti representations whose support is precisely the set of all monotone geodesics $\gamma : I \to X_\infty$ where $\varphi_\infty \circ \gamma = \mathrm{id}_I$, and $I = [0, 1]$ if $\{X_i, \pi_i\}$ is unsigned, and $I = \mathbb{R}$ if it is signed. These Alberti representations are compatible with projection.*

The proof of Theorem 9.13 is contained in [20]; note, however, that in [20] only the case of what we call unsigned inverse systems is discussed: the modifications for the case of signed inverse systems are straightforward. Alberti representations are not explicitly mentioned in [20], but part (2) in Theorem 9.13 follows from the discussion in [20, Sec. 6].

Remark 9.14. In [41] it is shown that the collection of admissible inverse systems defined in [20] contains an uncountable family of the form $\{\{(X_i, d_i, \mu_{i,\alpha})\}_i\}_{\alpha\in\mathcal{A}}$, such that inverse limits $\{(X_\infty, d_\infty, \mu_{\infty,\alpha})\}_{\alpha\in\mathcal{A}}$ —which are all PI spaces [20]— realize an uncountable family $\{\mu_{\infty,\alpha}\}_{\alpha\in\mathcal{A}}$ of mutually singular measures on the same inverse limit metric space (X_∞, d_∞).

Theorem 9.15. *Let X_∞ be the inverse limit of an admissible inverse system $\{X_i, \pi_i\}$ and let $\psi = \varphi_\infty$; if the system is unsigned assume also that $p \notin \psi^{-1}(\{0, 1\})$. Then each element of* $\mathrm{Bw\text{-}up}(X_\infty, \mu_\infty, \psi, p)$ *is of the form $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q)$ where:*

(1) *The metric measure space $(Y_\infty, d_\infty, \nu_\infty)$ is the inverse limit of a signed admissible inverse system $\{Y_i, \pi_i\}$, and φ is the function φ_∞ corresponding to Y_∞.*
(2) *The parameters σ anc c satisfy:*

$$\begin{aligned} &\sigma \in [1, m] \\ &c = \frac{1}{\nu_\infty\left(B_{Y_\infty}(q, 1/\sigma)\right)} . \end{aligned} \tag{9.16}$$

(3) *The basepoint q satisfies $\varphi(q) = 0$.*
(4) *Furthermore, up to renormalization, the blow-up of the distinguished Alberti representation of (X_∞, μ_∞) is the distinguished Alberti reprsentation on (Y_∞, ν_∞).*

We may now apply Theorem 8.62 to obtain the following:

Theorem 9.17. *Let X_∞ be the inverse limit of an admissible inverse system $\{X_i, \pi_i\}$. Let $A \subset X_\infty$ be a measurable subset, and $F : X_\infty \supset A \to Z$ be Lipschitz.*

Then there is a full μ_∞-measure subset $S_F \subset A$ such that, for each $p \in S_F$ and each $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in$ $\mathrm{Bw\text{-}up}(X_\infty, \mu_\infty, \psi, p; F, Z)$, one has that G maps each unit-speed monotone geodesic line $\gamma : \mathbb{R} \to Y_\infty$ to a (possibly degenerate[4]) geodesic line in W with constant speed $\sigma^{-1} \left\|\partial_\psi \,|_p\right\|_F$.

Definition 9.18. An admissible inverse system $\{(X_i, \pi_i)\}$ satisfies the **monotone bigon condition** if there is a constant D such that for every i, if $y_1, y_2 \in X_\infty$ project under $\pi_i : X_\infty \to X_i$ to the same point, there are monotone geodesic segments $\gamma_1, \gamma_2 \subset X_\infty$ of length $< Dm^{-i}$ such that $y_i \in \gamma_i$, and γ_1, γ_2 have the same endpoints.

One may readily check that some standard examples of admissible systems, for instance Examples 1.2 and 1.4 from [19], satisfy the monotone bigon condition.

Proof of Theorem 9.1. We apply Theorem 9.17, and will show that under each of the assumptions (a)–(d), there is a full measure subset of points $p \in A$ such that if $G : Y_\infty \to W$ is as Theorem 9.17, then:

- If one of (a)–(c) holds then G factors as

$$G \xrightarrow{\varphi_\infty} \mathbb{R} \xrightarrow{\bar{G}} W \tag{9.19}$$

4 This happens iff $\left\|\partial_\psi \,|_p\right\|_F = 0$, i.e. when $G \circ \gamma$ is constant.

where $\bar{G}$ is a geodesic with constant speed $\sigma_0 = \sigma^{-1} \left\| \partial_\psi \, |_p \right\|_F$. Here we allow $\sigma_0 = 0$, in which case $\bar{G}$ and G are constant maps.

- If (d) holds, then G factors as $\bar{G} \circ \pi^{\infty}_{-\infty}$ where $\pi^{\infty}_{-\infty} : Y_\infty \to Y_{-\infty}$ is the projection map from the inverse limit to the direct limit, and

$$\bar{G} : Y_{-\infty} \to W \tag{9.20}$$

is a map whose restriction of any monotone geodesic in $Y_{-\infty}$ is a geodesic in W with constant speed $\sigma_0 = \sigma^{-1} \left\| \partial_\psi \, |_p \right\|_F$.

If $\sigma_0 = 0$, then the restriction of G to any monotone geodesic is constant, and since any two points in Y_∞ may be joined by a piecewise monotone geodesic path, this implies that G is constant. Then the above assertions are clear. Therefore we assume that $\sigma_0 > 0$.

We first assume that (b) holds. This also covers case (a) since (a) $\implies$ (b).

Let Γ be the set of monotone geodesics $\gamma : \mathbb{R} \to Y_\infty$ such that $\varphi_\infty \circ \gamma = \mathrm{id}_{\mathbb{R}}$. We define an equivalence relation on Γ by saying that $\gamma_1 \sim \gamma_2$ if there is a geodesic $\eta : \mathbb{R} \to W$ with constant speed σ_0 such that $G \circ \gamma_i = \eta \circ \varphi_\infty \circ \gamma_i$ for $i = \{1, 2\}$.

Note that if the images of $\gamma_1, \gamma_2 \in \Gamma$ intersect in an interval, then they are equivalent by assumption (b). By concatenating rays to form monotone geodesics, it follows that if the images of γ_1 and γ_2 intersect even in a single point, they are equivalent. Now suppose the images of $\gamma_1, \gamma_2 \in \Gamma$ intersect even in a single point, i.e. for some $t_1, t_2 \in \mathbb{R}$, we have $\gamma_1(t_1) = \gamma_2(t_2)$. Then $t_1 = \varphi_\infty \circ \gamma_1(t_1) = \varphi_\infty \circ \gamma_2(t_2) = t_2$. Hence we may define a monotone geodesic $\gamma_3 \in \Gamma$ by letting $\gamma_3(t) = \gamma_1(t)$ if $t \le t_1$, and $\gamma_3(t) = \gamma_2(t)$ otherwise. Now γ_3 is equivalent to both γ_1 and γ_2, since its image shares an interval with each of their images, due to the fact that $\sigma_0 > 0$. Now define an equivalence relation on Y_∞ by saying that $y_1, y_2 \in Y_\infty$ are equivalent if some (or equivalently every) $\gamma_1 \in \Gamma$ passing through y_1 is equivalent to some (or every) geodesic $\gamma_2 \in \Gamma$ passing through y_2. The cosets of this relation on Y_∞ are closed, and any path γ that is a concatenation of finitely many segments from monotone geodesics lies in a single coset. Therefore there is only one coset and G factors as claimed.

Suppose (c) holds. Since the conclusion is local, we may assume without loss of generality that there is a smooth map $\Psi : Z \to \mathbb{R}^k$, where k is the dimension of the horizontal space, and the derivative $D\Psi$ restricts to an isomorphism on every horizontal space. Hence by [36], for every $z \in Z$, the blow-up of Z at z is a Carnot group W, and the blow-up $\hat{\Psi} : W \to \mathbb{R}^k$ of Ψ at z yields the horizontal coordinate for W.

We now proceed as before, except that we take p to be a point of differentiability of the composition $\Psi \circ F : A \to \mathbb{R}^k$. Passing to the blow-up $G : Y_\infty \to W$, from differentiability we get that $\hat{\Psi} \circ G = \alpha \circ \varphi_\infty$ where $\alpha : \mathbb{R} \to \mathbb{R}^k$ is an affine map. It follows that for every monotone geodesic $\gamma \in \Gamma$, the composition $G \circ \gamma$ projects under the horizontal coordinate $\hat{\Psi} : W \to \mathbb{R}^k$ to the same constant speed geodesic $\alpha \circ \varphi_\infty \circ \gamma$. This implies that for every $\gamma \in \Gamma$, the composition $G \circ \gamma$ is an integral curve of the left invariant vector field determined by α. In particular, if two such geodesics agree at a point, then they coincide.

Now consider the equivalence relation on Γ defined as before. If $\gamma_1, \gamma_2 \in \Gamma$ agree at some $t \in \mathbb{R}$, then by the above discussion $G \circ \gamma_1 = G \circ \gamma_2$, i.e. $\gamma_1 \sim \gamma_2$. The rest of the argument is the same.

Now assume (d) holds. Again consider the map $G : Y_\infty \to W$ obtained by applying metric differentiation at a point $p \in X_\infty$. As above, for every $\gamma \in \Gamma$, the composition $G \circ \gamma$ is a geodesic of constant speed σ_0. One checks readily that Y_∞ inherits the monotone bigon condition. Pick $y_1, y_2 \in Y_\infty$, and suppose that $\pi_j(y_1) = \pi_j(y_2)$. By the monotone bigon condition, there exist $\gamma_i \in \Gamma$ such that γ_i passes through y_i and the maps $\gamma_1, \gamma_2 : \mathbb{R} \to Y_\infty$ agree outside a a compact subset of $\mathbb{R}$. Then $G \circ \gamma_1$ and $G \circ \gamma_2$ are constant speed geodesics in a $CAT(0)$ space, and they agree outside a compact subset of $\mathbb{R}$. It follows that $\gamma_1 = \gamma_2$, so in particular $G(y_1) = G(y_2)$. We have thus shown that G factors through a map $G_j : Y_j \to W$ which has the property that its restriction to any monotone geodesic in Y_j is a geodesic in W with speed σ_0. This implies that G factors as $\bar{G} \circ \pi^{\infty}_{-\infty}$, where $\pi^{\infty}_{-\infty} : Y_\infty \to Y_{-\infty}$ is the projection map to the direct limit $Y_{-\infty}$ of the signed inverse system $\{(Y_i, \pi_i)\}$, and $\bar{G} : Y_{-\infty} \to W$ is a map whose restriction to any monotone geodesic in $Y_{-\infty}$ is a geodesic in W with constant speed σ_0. Finally note that as there is a uniform bound on the cardinality of the links of the vertices of Y_j as $j \to -\infty$, then $Y_{-\infty}$ is a finite union of geodesic rays issuing from a basepoint. □

Remark 9.21. Note that the previous argument for points (a)–(c) can be run more generally under the assumptions that G maps monotone geodesics to constant speed geodesics in W which cannot branch, i.e. two such geodesics must coincide if they agree on an open interval. Note that it suffices to verify this non-branching property for the geodesics that arise from blow-up; one may restrict these geodesics by exploiting differentiability of auxiliary Lipschitz functions, such as the function Ψ appearing in case (c).

Proof of Corollary 9.3. Let $F : A \to Z$ be an L-bilipschitz embedding.

We first assume that we are in one of the cases (a)-(c). Therefore we know that there is a measurable subset $A_1 \subset A$ with $\mu(A \setminus A_1) = 0$ such that for all $x \in A_1$, for every blow-up $G : Y_\infty \to W$ as in (the proof of) Theorem 9.1, we have that $G = \eta \circ \varphi_\infty$ where $\eta : \mathbb{R} \to W$ is a geodesic with constant speed lying in the interval $[L^{-1}, L]$. Since G is also L-bilipschitz, we conclude that for every $y_1, y_2 \in Y_\infty$, we have

$$d(\varphi_\infty(y_1), \varphi_\infty(y_2)) \geq L^{-2} d(y_1, y_2)\,. \tag{9.22}$$

This implies that for all $x \in A_1$ there is an $r(x) > 0$ such that

$$d(\varphi_\infty(x'), \varphi_\infty(x)) > \frac{1}{2} L^{-2} d(x', x) \tag{9.23}$$

for all $x' \in B(x, r(x))$; otherwise there would be a sequence $x'_k \to x$ violating (9.23), and by rescaling and passing to a limit, we get a blow-up contradicting (9.22). Now put

$$S_j = \{x \in A_1 \mid r(x) > j^{-1}\}\,.$$

Then $\varphi_\infty\big|_{S_j} : S_j \to \mathbb{R}$ is $2L^2$-bilipschitz on $\frac{r(x)}{2}$-balls, and $A_1 = \cup_j S_j$. This shows that A_1 is rectifiable.

Now suppose we are in case (d).

Let

$$A_0 = \{x \in A \mid \pi_j(x) \text{ is not a vertex for any } j\} \subset A\,.$$

Note that $A \setminus A_0$ is μ-null. Then there is a full measure subset $A_1 \subset A_0$ such that for every $x \in A_1$, every blow-up $G : Y_\infty \to W$ of F at x has a factorization $G = \bar{G} \circ \pi^{\infty}_{-\infty}$ as in the proof of Theorem 9.1, and reasoning as above we get that

$$d(\pi^{\infty}_{-\infty}(y_1), \pi^{\infty}_{-\infty}(y_2)) \geq L^{-2} d(y_1, y_2) \tag{9.24}$$

for all $y_1, y_2 \in Y_\infty$.

Now for $x \in A_1$, using (9.24) and a contradiction argument, we get that for every C there is a j_x such that for all $j \geq j_x$, the projection $\pi_j : X_{j+1} \to X_j$ maps the ball $B(\pi_{j+1}(x), Cm^{-j}) \subset X_{j+1}$ bijectively onto $B(\pi_j(x), Cm^{-j}) \subset X_j$. Iterating this, it follows that for all $k \geq j \geq j_x$, the projection $\pi^k_j : X_k \to X_j$ maps $B(\pi_k(x), Cm^{-k})$ bijectively onto $B(\pi_j(x), Cm^{-k})$. Since $\pi_j(x)$ is not a vertex, for large enough k we conclude that $B(\pi_k(x), Cm^{-k})$ has no branch points, i.e. it is isometric to an interval. As C is arbitrary, this together with (9.24) implies that for every blow-up $G : Y_\infty \to W$ of F at x, the direct limit $Y_{-\infty}$ has no branch points. Hence we are in the same situation as cases (a)-(c), and we may complete the proof as before.

□

Acknowledgement: J.C. was supported by a collaboration grant from the Simons foundation, and NSF grant DMS-1406407. B.K. was supported by a Simons Fellowship, a Simons collaboration grant, and NSF grants DMS-1105656, and DMS-1405899. A.S. was supported by the by the ETH Zurich Postdoctoral Fellowship Program and the Marie Curie Actions for People COFUND Program.

We would like to thank David Bate, Guy David, and Sean Li for drawing our attention to errors in an earlier version of this paper. We also thank the referees for a number of helpful comments.

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