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The geometry of rank drop in a class of face-splitting matrix products: Part II

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Abstract: Given $k \leq 9$ points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, we characterize rank deficiency of the $k \times 9$ matrix Z_k with rows $x_i^\top \otimes y_i^\top$ in terms of the geometry of the point configurations $\{x_i\}$ and $\{y_i\}$. In [3] we presented results for the cases $k \leq 6$. In this paper we deal with the remaining cases $k = 7, 8$ and 9. The results involve the interplay of quadric surfaces, cubic curves and Cremona transformations.

Keywords: Computer vision, quadric surface, cubic curve, Cremona transformation.

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1 Introduction

We are interested in solving the following problem, where $\otimes$ denotes the Kronecker product:

Problem 1.1. *Given $k \leq 9$ points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, consider the $k \times 9$ matrix Z_k whose rows are $x_i^\top \otimes y_i^\top$ for $i = 1, \dots, k$, i.e.,*

$$Z_k = \begin{bmatrix} x_1^\top \otimes y_1^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{bmatrix}.$$

Delineate the geometry of point configurations $\{x_i\}$ and $\{y_i\}$ for which $\operatorname{rank}(Z_k) < k$.

Note that Problem 1.1 can be rephrased geometrically and generalized to any algebraic variety.

Problem 1.2. *Given $k \leq 9$ points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, delineate the geometry of the point configurations $\{x_i\}$ and $\{y_i\}$ for which the subspace spanned by the images of these points under the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ has dimension less than $k - 1$.*

Problem 1.1 arises in the study of reconstruction problems in 3D computer vision. For background on the problem and related work we direct the reader to Part I of this work [3] where Problem 1.1 was solved for $k \leq 6$. The results relied on the classical invariant theory of points in $\mathbb{P}^2$ and the theory of cubic surfaces. In this paper we complete the characterization for the remaining cases $k = 7, 8, 9$. Once again, the results can be phrased in terms of classical algebraic geometry and invariants.

Semi-genericity

Throughout this paper, we will concern ourselves with point configurations that are *semi-generic*; a configuration of k point pairs (x_i, y_i) is *semi-generic* if every subset of $k - 1$ point pairs is fully generic. That is, we say that a property holds for a semi-generic choice of $(x_i, y_i) \in (\mathbb{P}^2 \times \mathbb{P}^2)^k$ if there is a nonempty Zariski open set $\mathcal{U} \subseteq (\mathbb{P}^2 \times \mathbb{P}^2)^{k-1}$ so that the property holds whenever $\{(x_i, y_i) : i \neq j\}$ lies in $\mathcal{U}$ for all $j = 1, \dots, k$. Despite the

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name, semi-genericity is actually a stronger notion than usual genericity. We use this name because often the property of interest for points in $(\mathbb{P}^2 \times \mathbb{P}^2)^k$ is that two algebraic conditions coincide, whereas generic points satisfy neither algebraic condition. As a small example of this usage, let us instead consider a semi-generic pair of points x_1, x_2 in the line $\mathbb{R}$. Consider $f(x_1, x_2) = x_1(x_2 - 1)(x_1 - x_2)$. Then $f(x_1, x_2) = 0$ if and only if $x_1 = 0$, $x_2 = 1$, or $x_1 = x_2$. For generic (x_1, x_2), $f(x_1, x_2) \neq 0$. Semi-genericity only allows us to exclude algebraic conditions on x_1 and x_2 individually. In this example, a semi-generic pair of points (x_1, x_2) satisfies $f(x_1, x_2) = 0$ if and only if $x_1 = x_2$. This holds whenever $x_1, x_2 \in \mathcal{U} = \mathbb{R}\backslash\{0, 1\}$.

Summary of results and organization of the paper

In [3] we studied Problem 1.1 algebraically by decomposing the ideal generated by the maximal minors of Z_k into its prime components and examining only those components that did not correspond to rank drop conditions for a submatrix of Z_k with at most $k - 1$ rows, called *inherited conditions*, for the rank deficiency of Z_k. Through this we obtained both algebraic conditions that completely characterized rank drop, and geometric conditions that characterized rank drop under mild genericity assumptions. This method cannot be applied to the cases $k = 7, 8, 9$ due to computational limitations. Additionally, in these cases, the novel component of rank drop has a greater dimension than all the components of inherited conditions. Previously, for $k \leq 5$ the novel component had a strictly lower dimension than the variety of inherited conditions, and for $k = 6$ the novel component had dimension equal to that of the inherited conditions variety. For this reason, we largely concern ourselves only with the geometric characterization of rank drop for semi-generic configurations with $k = 7, 8, 9$, rather than an algebraic characterization beyond the vanishing of the maximal minors of Z_k.

In Section 2 we establish a number of facts about Cremona transformations, cubic curves, and projective reconstructions that we will use throughout the paper. In Section 3 we study the problem for $k = 8$ and prove that Z_k is rank deficient exactly when there is a quadratic Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $f(x_i) = y_i$ for all i (Theorem 3.1). To do so, we establish a correspondence between three sets: lines in the nullspace of Z_k, quadrics passing through a projective reconstruction of the input point pairs, and Cremona transformations sending $x_i \mapsto y_i$ (Theorem 3.16 which depends on Theorem 3.2). We refer to this as the *trinity correspondence* and it is the foundation for all of our results in this paper. In Section 4 we study the problem for $k = 7$ and prove that Z_k is rank deficient exactly when there are cubic curves in each copy of $\mathbb{P}^2$, passing through all seven points, and an isomorphism between these curves that sends $x_i \mapsto y_i$ (Theorem 4.2). We further prove that this occurs exactly when seven particular cubic curves in each copy of $\mathbb{P}^2$ are coincident and we provide an algebraic characterization when this occurs (Theorem 4.11). In Section 5 we answer Problem 1.1 for $k = 9$, which is largely straight-forward (Theorem 5.1). We summarize our results in Section 6 and state a geometric consequence about reconstructions of semi-generic point pairs of size six, seven and eight.

2 Background and tools

2.1 Quadratic Cremona transformations and cubic curves

Definition 2.1. A *quadratic Cremona transformation* of $\mathbb{P}^2$ is a birational automorphism $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined as $f(x) = (f_1(x) : f_2(x) : f_3(x))$ where f_1, f_2, f_3 are homogeneous quadratic polynomials in $x = (x_1, x_2, x_3)$.

We drop the word “quadratic” from now on as all the Cremona transformations we consider will be quadratic. Each Cremona transformation can be obtained by blowing up three points a_1, a_2, a_3 in the domain (called *base points*) at which the transformation is not defined, and collapsing three lines $\gamma_1, \gamma_2, \gamma_3$ (called *exceptional lines*) which contain pairs of base points: for distinct i, j, k, the line γ_i contains a_j, a_k. Generically, the base points and exceptional lines of a Cremona transformation will all be distinct; when they are not all distinct, the transformation is said to be degenerate. In this paper we will consider only non-degenerate Cremona transformations.

The inverse of a Cremona transformation f is also a Cremona transformation with base points b_1, b_2, b_3 and exceptional lines τ_1, τ_2, τ_3 in the codomain of f. The map f sends $\gamma_i \mapsto b_i$ while f^{-1} sends $\tau_i \mapsto a_i$. For simplicity we will often refer to both the base points in the domain and the base points in the codomain (i.e. the base points of f^{-1}) as the base points of f. The standard Cremona transformation is

$$f(x_1, x_2, x_3) = (x_2x_3 : x_1x_3 : x_1x_2) \tag{1}$$

which has base points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and exceptional lines $x_i = 0$ for $i = 1, 2, 3$. This transformation is an involution since it is its own inverse, and the base points and exceptional lines of f^{-1} are again $(1:0:0)$, $(0:1:0$, $(0:0:1)$ and $x_i = 0$ for $i = 1, 2, 3$. All Cremona transformations differ from the standard one only by projective transformations as stated below.

Lemma 2.2. *Let g be a Cremona transformation and let f be the standard Cremona involution. Then there are projective transformations H_1, H_2 such that $g = H_1 \circ f \circ H_2$.*

Proof. Let $a_1, a_2, a_3 \in \mathbb{P}^2$ denote the base points of g. The coordinates (g_1, g_2, g_3) of g form a basis for the three-dimensional vector space of quadratics vanishing on the points a_1, a_2, a_3. Another basis is $h = (\ell_2\ell_3, \ell_1\ell_3, \ell_1\ell_2)$ where $\ell_i \in \mathbb{C}[x, y, z]_1$ defines the line joining a_j and a_k for every labeling $\{i, j, k\} = \{1, 2, 3\}$. Therefore there is some invertible linear transformation H_1 for which $g = H_1h$. Similarly, (ℓ_1, ℓ_2, ℓ_3) is a basis for $\mathbb{C}[x, y, z]_1$ and so there is a linear transformation H_2 for which $H_2(x, y, z) = (\ell_1, \ell_2, \ell_3)$. The map h is given by $f \circ H_2$ and so $g = H_1 \circ f \circ H_2$. □

Throughout this paper we are interested in $\mathbb{P}^2 \times \mathbb{P}^2$ and we typically denote points in the first $\mathbb{P}^2$ by x and those in the second $\mathbb{P}^2$ by y. The notation $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ will help to keep this correspondence clear.

Lemma 2.3. *Let $f : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ be a Cremona transformation. If f and f^{-1} have base points $e^x_1 = e^y_1 = (1:0:0)$, $e^x_2 = e^y_2 = (0:1:0)$, $e^x_3 = e^y_3 = (0:0:1)$ in the domain and codomain, then f has the form*

$$f(x_1, x_2, x_3) = (ax_2x_3 : bx_1x_3 : cx_1x_2) \tag{2}$$

where $a, b, c \in \mathbb{C}\backslash\{0\}$.

Proof. Suppose that $f = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are quadratic polynomials. Since f is undefined at the three base points in the domain, it follows that f_1, f_2, f_3 contain only the monomials x_1x_2, x_1x_3, x_2x_3. Moreover, we know that $f(x_1, x_2, 0) = (0:0:1)$. It follows that f_1, f_2 do not contain the monomial x_1x_2. In examining the other two exceptional lines, we find that f_1, f_2, f_3 contain only one monomial each and that f has the desired form. □

We note that the choice of (a, b, c) is equivalent to specifying a single point correspondence $p \mapsto q$, where neither p nor q lie on an exceptional line. It follows that a Cremona transformation has 14 degrees of freedom: six from the base points in the domain, six from the base points in the codomain, and two from the choice of a single point correspondence.

Next we prove some facts about Cremona transformations and isomorphisms of cubic curves.

Definition 2.4. Let f be a Cremona transformation with base points $B(f)$. For a curve $C \subset \mathbb{P}^2$, define $f(C) := \overline{f(C\backslash B(f))}$, and for a given point p, let $\nu_p(C)$ be the multiplicity of the curve C at the point p.

Lemma 2.5 (See [4]). *Let $C \subset \mathbb{P}^2$ be a plane curve of degree n and let f be a Cremona transformation. Then*

$$\deg(f(C)) = 2n - \sum_{p \in B(f)} \nu_p(C). \tag{3}$$

In particular, if C is a smooth cubic curve then $f(C)$ is also a cubic curve if and only if the base points of f lie on C. In this case, $f^{-1}(f(C)) = C$ implies that the base points of f^{-1} lie on $f(C)$.

Using this, we can prove the following result.

Lemma 2.6. *Let C be a smooth cubic curve and let f be a Cremona transformation with base points $a_1, a_2, a_3 \in C$ in the domain and b_1, b_2, b_3 in the co-domain. Then $f(C)$ is a smooth cubic curve and $\bar{f} : C \to f(C)$, defined by taking the closure of $f|_{C\backslash B(f)}$, is an isomorphism.*

Proof. By Lemma 2.5, $f(C)$ is a cubic curve. Moreover, since $f^{-1}(f(C)) = C$ is a cubic curve, it also follows that $b_1, b_2, b_3 \in f(C)$. The fact that $\bar{f}$ is an isomorphism follows from the corollary after [8, § 1.6, Theorem 2] which says that a birational map between nonsingular projective plane curves is regular at every point, and is a one-to-one correspondence. □

Given a smooth cubic curve C, any automorphism $g : C \to C$ is of the form $u \mapsto au + b$ with $a = \pm 1$, $b \in C$, where addition is defined via the group law on C. Theorem 1.3 in [4] states that given a smooth cubic curve C and an automorphism $g : C \to C$ defined by some multiplier $a = \pm 1$ and some translation $b \in C$, the automorphism g is induced by a Cremona transformation with base points a_1, a_2, a_3 if and only if $a(a_1 + a_2 + a_3) = 3b$, where again, addition is with respect to the group law on C. In particular, every automorphism of C is induced by a two-parameter family of Cremona transformations, which we obtain by picking the first two base points arbitrarily and then letting the third base point be determined by the equation $a_3 = a(3b - a_1 - a_2)$.

We can use this to prove a converse to Lemma 2.6.

Lemma 2.7. *Let $f : C \to C'$ be an isomorphism of smooth cubic plane curves. Then there is a two-parameter family of Cremona transformations $f'_\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $f'_\sigma|_C = f$. The base points of these Cremona transformations will lie on the cubic curves.*

Proof. Since C and C' are isomorphic, they have the same Weierstraß form C_0. There are therefore homographies $H_1, H_2 \in \mathrm{PGL}(3)$ such that $H_1(C) = C_0 = H_2(C')$ and therefore $H_1^{-1}H_2(C') = C$. Then $H_1^{-1}H_2 \circ f : C \to C$ is an automorphism of C and it follows by [4, Theorem 1.3] that this is induced by some two-parameter family of Cremona transformations g_σ; the members of this family are obtained by picking the first two base points arbitrarily on C and then letting the third base point be determined by the equation $a_3 = a(3b - a_1 - a_2)$. Then $f'_\sigma := H_2^{-1}H_1 \circ g_\sigma$ is the desired family of Cremona transformations. By Lemma 2.5 the base points of each of these Cremona transformations lie on the cubic curves. □

2.2 Fundamental matrices and projective reconstruction

In this paper we will be concerned with pairs of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2. In the context of computer vision, these arise as *projective cameras* which are linear projections from $\mathbb{P}^3(\mathbb{R}) \dashrightarrow \mathbb{P}^2(\mathbb{R})$, represented by (unique) matrices $A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3\times4})$ of rank three, such that $\pi_i(p) \sim A_i p$ for all *world points* $p \in \mathbb{P}^3(\mathbb{R})$. The notation $\sim$ indicates equality in projective space. The centers c_i are the unique points in $\mathbb{P}^3(\mathbb{R})$ such that $A_i c_i = 0$ for $i = 1, 2$. The projections we consider in this paper are slightly more general in that they work over $\mathbb{C}$; they are represented by rank three matrices $A_i \in \mathbb{P}(\mathbb{C}^{3\times4})$ and send $p \in \mathbb{P}^3$ to $A_i p \in \mathbb{P}^2$.

In the vision setting, the image formation equations $A_i p = \lambda_i \pi_i(p)$ with $i = 1, 2$ and some $\lambda_i \in \mathbb{R}$ imply that for all $p \in \mathbb{P}^3(\mathbb{R})$ one has

$$0 = \det \begin{bmatrix} A_1 & \pi_1(p) & 0 \\ A_2 & 0 & \pi_2(p) \end{bmatrix} = \pi_2(p)^\top F \pi_1(p) \tag{4}$$

for a unique matrix $F \in \mathbb{P}(\mathbb{R}^{3\times3})$ of rank two, determined by (A_1, A_2); see [6, Chapter 9.2]. This matrix F is called the *fundamental matrix* of the cameras/projections (A_1, A_2) / (π_1, π_2). It defines the bilinear form $B_F(x, y) = y^\top F x$ such that $B_F(\pi_1(p), \pi_2(p)) = \pi_2(p)^\top F \pi_1(p) = 0$ for all $p \in \mathbb{P}^3(\mathbb{R})$. The entries of F are certain 4×4 minors of the 6×4 matrix obtained by stacking A_1 on top of A_2. The points $e^x := \pi_1(c_2)$ and $e^y := \pi_2(c_1)$ are called the *epipoles* of F. It is well-known, see [6, Chapter 9.2], that e^x and e^y are the unique points in $\mathbb{P}^2$ such that $Fe^x = 0 = (e^y)^\top F$. Conversely, for every rank-two matrix $F \in \mathbb{P}(\mathbb{R}^{3\times3})$ there exists, up to projective transformation, a unique pair of cameras (A_1, A_2) / linear projections $\pi_1, \pi_2 : \mathbb{P}^3(\mathbb{R}) \dashrightarrow \mathbb{P}^2(\mathbb{R})$ with fundamental matrix F, see [6, Theorem 9.10]. All of these facts extend verbatim over $\mathbb{C}$, and we call a rank two matrix $F \in \mathbb{P}(\mathbb{C}^{3\times3})$ a *fundamental matrix* of (π_1, π_2) if it satisfies (4).

Equation (4) is a constraint on the images of a world point in two cameras. Going the other way, given k point pairs $(x_i, y_i) \in \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$, one can ask if they admit a *projective reconstruction*, namely a pair of real cameras A_1, A_2 and real world points $p_1, \ldots, p_k$ such that $A_1 p_i \sim x_i$ and $A_2 p_i \sim y_i$ for $i = 1, \ldots, k$. A necessary condition for a reconstruction is the existence of a rank-two matrix $F \in \mathbb{P}(\mathbb{R}^{3\times3})$ such that $y_i^\top F x_i = 0$ for $i = 1, \ldots, k$, called a *fundamental matrix* of the point pairs $(x_i, y_i)_{i=1}^k$. Note that vec(F) lies in the nullspace of $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$. The necessary and sufficient conditions for the existence of a projective reconstruction of $(x_i, y_i)_{i=1}^k$ are (1) the existence of a fundamental matrix F and (2) for each i, either $Fx_i = 0$ and $y_i^\top F = 0$, or neither x_i nor y_i lie in the right and left nullspaces of F; see [7]. In this paper, we extend the above definition to $\mathbb{C}$ and call any rank-two matrix $F \in \mathbb{P}(\mathbb{C}^{3\times 3})$ that lies in the nullspace of Z_k a *fundamental matrix* of the point pairs $(x_i, y_i)_{i=1}^k$.

3 The case $k = 8$

In this section we characterize the rank deficiency of $Z = Z_8 = (x_i^\top \otimes y_i^\top)_{i=1}^8$ when the point pairs (x_i, y_i) are semi-generic. When k is fixed we often write Z instead of Z_k.

Theorem 3.1. *For eight semi-generic point pairs $(x_i, y_i)_{i=1}^8$, the matrix Z drops rank if and only if there exists a Cremona transformation $f : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $f(x_i) = y_i$ for all i.*

Proof of the if-direction. Suppose that we have a Cremona transformation $f : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $f(x_i) = y_i$ for $i = 1, \ldots, 8$. After homographies we can assume that f is the basic quadratic involution mapping (x_1, x_2, x_3) to (x_2x_3, x_1x_3, x_1x_2). Then

$$Z = \begin{bmatrix}
x_{11}x_{12}x_{13} & x_{11}^2x_{13} & x_{11}^2x_{12} & x_{12}^2x_{13} & x_{11}x_{12}x_{13} & x_{11}x_{12}^2 & x_{12}x_{13}^2 & x_{11}x_{13}^2 & x_{11}x_{12}x_{13} \\
x_{21}x_{22}x_{23} & x_{21}^2x_{23} & x_{21}^2x_{22} & x_{22}^2x_{23} & x_{21}x_{22}x_{23} & x_{21}x_{22}^2 & x_{22}x_{23}^2 & x_{21}x_{23}^2 & x_{21}x_{22}x_{23} \\
x_{31}x_{32}x_{33} & x_{31}^2x_{33} & x_{31}^2x_{32} & x_{32}^2x_{33} & x_{31}x_{32}x_{33} & x_{31}x_{32}^2 & x_{32}x_{33}^2 & x_{31}x_{33}^2 & x_{31}x_{32}x_{33} \\
x_{41}x_{42}x_{43} & x_{41}^2x_{43} & x_{41}^2x_{42} & x_{42}^2x_{43} & x_{41}x_{42}x_{43} & x_{41}x_{42}^2 & x_{42}x_{43}^2 & x_{41}x_{43}^2 & x_{41}x_{42}x_{43} \\
x_{51}x_{52}x_{53} & x_{51}^2x_{53} & x_{51}^2x_{52} & x_{52}^2x_{53} & x_{51}x_{52}x_{53} & x_{51}x_{52}^2 & x_{52}x_{53}^2 & x_{51}x_{53}^2 & x_{51}x_{52}x_{53} \\
x_{61}x_{62}x_{63} & x_{61}^2x_{63} & x_{61}^2x_{62} & x_{62}^2x_{63} & x_{61}x_{62}x_{63} & x_{61}x_{62}^2 & x_{62}x_{63}^2 & x_{61}x_{63}^2 & x_{61}x_{62}x_{63} \\
x_{71}x_{72}x_{73} & x_{71}^2x_{73} & x_{71}^2x_{72} & x_{72}^2x_{73} & x_{71}x_{72}x_{73} & x_{71}x_{72}^2 & x_{72}x_{73}^2 & x_{71}x_{73}^2 & x_{71}x_{72}x_{73} \\
x_{81}x_{82}x_{83} & x_{81}^2x_{83} & x_{81}^2x_{82} & x_{82}^2x_{83} & x_{81}x_{82}x_{83} & x_{81}x_{82}^2 & x_{82}x_{83}^2 & x_{81}x_{83}^2 & x_{81}x_{82}x_{83}
\end{bmatrix} \tag{5}$$

which one can see is rank deficient because its first, fifth and ninth columns are the same. □

In order to prove the only-if direction of Theorem 3.1, we develop a number of tools in § 3.1. The proof of Theorem 3.1 will then be completed in Subsection 3.2.

3.1 The trinity of lines, quadrics and Cremona transformations

In order to establish the *trinity correspondence*, we need to introduce some genericity conditions for our main objects of interest. We say that a line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ is *generic* if it contains exactly three rank-two matrices. These lines are generic in the usual sense, since almost all lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$ intersect the degree-three determinantal variety $\mathcal{D} := \{X \in \mathbb{P}(\mathbb{C}^{3\times 3}) : \det(X) = 0\}$ in three distinct points. Furthermore, given a pair of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with distinct centers c_1, c_2 we say that a smooth quadric Q through c_1, c_2 is *permissible* if it does not contain the line $\overline{c_1c_2}$ connecting the two centers.

Theorem 3.2 (Trinity correspondence). *Consider the following three sets:*

(1) $\mathcal{L}$*: the set of all generic lines ℓ in $\mathbb{P}(\mathbb{C}^{3\times 3})$,*
(2) $\mathcal{Q}$*: the set (up to projective equivalence) of pairs of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2, along with a permissible quadric $Q \subset \mathbb{P}^3$ through c_1, c_2,*
(3) $\mathcal{C}$*: the set of (non-degenerate) Cremona transformations from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.*

Then there is a $1:1$ *correspondence between* $\mathcal{L}$ *and* $\mathcal{C}$*, a* $1:3$ *correspondence between* $\mathcal{L}$ *and* $\mathcal{Q}$*, and a* $3:1$ *correspondence between* $\mathcal{Q}$ *and* $\mathcal{C}$*, such that the diagram* (6) *commutes:*

$$\begin{tikzcd} & \mathcal{Q} \arrow[dr, "3:1"] & \\ \mathcal{L} \arrow[ur, "1:3"] \arrow[rr, leftrightarrow] & & \mathcal{C} \end{tikzcd} \tag{6}$$

A similar theorem holds for lines which pass through exactly two rank-two matrices; however, we do not prove it here.

We first show that for fixed linear projections π_1, π_2 with centers $c_1 \neq c_2 \in \mathbb{P}^3$, there is a bijection between the quadrics that contain c_1, c_2 and lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$ through the fundamental matrix F of (π_1, π_2). This result is well-known in the context of computer vision (see [1], [5]), but we write an independent proof below.

Lemma 3.3. *Fix a pair of linear projections* $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *with non-coincident centers* c_1, c_2 *and let* F *be its fundamental matrix. There is a* $1:1$ *correspondence between the quadrics* $Q \subset \mathbb{P}^3$ *through* c_1, c_2 *and the lines* $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ *through* F*.*

Proof. Applying projective transformations, we can assume that $c_1 = (1:0:0:0)$, $c_2 = (0:1:0:0)$, $\pi_1(u_1:u_2:u_3:u_4) = (u_2:u_3:u_4)$ and $\pi_2(u_1:u_2:u_3:u_4) = (u_1:u_3:u_4)$. If $F = (F_{ij})$ is the fundamental matrix of (π_1, π_2), then for all $u \in \mathbb{P}^3$ we have

$$0 = \pi_2(u)^\top F \pi_1(u) = \langle F, \pi_2(u)\pi_1(u)^\top \rangle = \left\langle \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \begin{pmatrix} u_1u_2 & u_1u_3 & u_1u_4 \\ u_2u_3 & u_3^2 & u_3u_4 \\ u_2u_4 & u_3u_4 & u_4^2 \end{pmatrix} \right\rangle. \tag{7}$$

Since the entries in position (2, 3) and (3, 2) of $\pi_2(u)\pi_1(u)^\top$ are the same, F is a scalar multiple of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and $B_F(x, y) = x_3y_2 - x_2y_3$. In particular, there exists some $p \in \mathbb{P}^3$ with $\pi_1(p) = x$ and $\pi_2(p) = y$ if and only if $x_3y_2 = x_2y_3$.

Consider the image of $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}(\mathbb{C}^{3\times 3})$ where $\varphi(u) = \pi_2(u)\pi_1(u)^\top$. By (7), $\varphi(\mathbb{P}^3)$ is contained in the hyperplane $F^\perp \subset \mathbb{P}(\mathbb{C}^{3\times 3})$. Any matrix in $\mathbb{P}(\mathbb{C}^{3\times 3})$ can be written as $sF + M$ for some scalar s and $M \in F^\perp$. Therefore,

$$\langle sF + M, \pi_2(u)\pi_1(u)^\top \rangle = \pi_2(u)^\top M \pi_1(u) \tag{8}$$

since $\pi_2(u)^\top F \pi_1(u) = 0$, and any linear function on the image of φ can be identified with its image in $F^\perp$. On the other hand, a line ℓ in $\mathbb{P}(\mathbb{C}^{3\times 3})$ through F is of the form $\{sF + tM : (s:t) \in \mathbb{P}^1\}$, where $M \in F^\perp$. Therefore, lines through F are in bijection with linear functions on $\varphi(\mathbb{P}^3)$, up to scaling.

The monomials $u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3^2, u_3u_4, u_4^2$ form a basis for the 7-dimensional vector space of homogeneous quadratic polynomials that vanish on c_1, c_2. Thus any quadratic polynomial in $\mathbb{C}[u_1, u_2, u_3, u_4]_2$ vanishing at c_1 and c_2 can be written as $\langle M, \pi_2(u)\pi_1(u)^\top \rangle$ for a unique matrix $M \in F^\perp$. This gives a linear isomorphism between linear functions on the image of φ, up to global scaling (which have been identified with lines through F), and quadrics passing though c_1 and c_2. □

Corollary 3.4. *Let* $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *be two linear projections with centers* $c_1 \neq c_2$ *and fundamental matrix* F*. Let* ℓ_F *be a line in* $\mathbb{P}(\mathbb{C}^{3\times 3})$ *through* F*. The correspondence* $\ell_F \mapsto Q$*, where* $Q \subset \mathbb{P}^3$ *is a quadric passing through* c_1, c_2*, is as follows. Let* $M \in \ell_F$ *be any* $M \neq F$*. Then* Q *is cut out by the bilinear form*

$$B_M(\pi_1(p), \pi_2(p)) = \pi_2(p)^\top M \pi_1(p) = 0. \tag{9}$$

The following result is well-known and can be proven by writing a comprehensive list of the equivalence classes, under projective transformation, of quadrics through a pair of distinct points and then testing an example from each class.

Lemma 3.5 ([1], [5], [6, Result 22.11]). *Under the* $1:1$ *correspondence in Lemma* 3.3*, the line* ℓ *corresponds to a permissible quadric* Q *through* c_1, c_2 *if and only if* ℓ *is a generic line.*

Next we prove that permissible quadrics through c_1, c_2 give rise to quadratic Cremona transformations from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Recall that all Cremona transformations we consider are assumed to be non-degenerate.

Lemma 3.6. *Fix* $\pi_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *to be linear projections with non-coincident centers* c_i *for* $i = 1, 2$*. A permissible quadric* Q *through* c_1, c_2 *defines a Cremona transformation* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *such that* $f(\pi_1(p)) = \pi_2(p)$ *for any point* $p \in Q$*. The base points of* f *are* $\pi_1(c_2)$ *and the image under* π_1 *of the two lines contained in* Q *passing through* c_1*. Similarly, the base points of* f^{-1} *are* $\pi_2(c_1)$ *and the image under* π_2 *of the two lines contained in* Q *passing through* c_2*.*

Proof. Since $c_1, c_2 \in Q$, the restriction of π_1 (and π_2) to Q is generically $1:1$. Therefore, $\pi_1(Q)$ and $\pi_2(Q)$ are each birational to a $\mathbb{P}^2$. The map f will be $\pi_2 \circ (\pi_1|_Q)^{-1}$. Let us check that this is a quadratic Cremona transformation.

As before, we can take $\pi_1(u) = (u_2 : u_3 : u_4)$ and $\pi_2(u) = (u_1 : u_3 : u_4)$. Then $c_1 = (1:0:0:0)$ is the kernel of π_1, and we are given that it lies on Q. As we saw already, these assumptions imply that Q is defined by the vanishing of a polynomial of the form $q(u) = \alpha u_1 u_2 + \beta u_1 + \gamma u_2 + \delta$ where $\alpha \in \mathbb{C}$ is a scalar, $\beta, \gamma \in \mathbb{C}[u_3, u_4]$ are of degree 1, and $\delta \in \mathbb{C}[u_3, u_4]$ is of degree 2. We can then write q as

$$q(u) = a u_1 + b \tag{10}$$

where $a = (\alpha u_2 + \beta)$, $b = (\gamma u_2 + \delta) \in \mathbb{C}[u_2, u_3, u_4]$ with $\deg(a) = 1$, $\deg(b) = 2$. The map $(\pi_1|_Q)^{-1}$ is then given by

$$x \mapsto (-b(x) : x_1 a(x) : x_2 a(x) : x_3 a(x)) =: (u_1 : u_2 : u_3 : u_4). \tag{11}$$

To verify this, first check that $\pi_1(u) = a(x) \cdot x$ where $\cdot$ denotes scalar multiplication. To see that $u \in Q$ we compute

$$\begin{aligned} q(u) &= u_1 \cdot a(u_2, u_3, u_4) + b(u_2, u_3, u_4) \\ &= u_1 \cdot a(\pi_1(u)) + b(\pi_1(u)) \\ &= -b(x) \cdot a(a(x) \cdot x) + b(a(x) \cdot x) \\ &= -b(x)a(x)a(x) + a(x)^2 b(x) = 0 \end{aligned} \tag{12}$$

where the last equality comes from the homogeneity of a, b with $\deg(a) = 1$, $\deg(b) = 2$.

Composing with π_2 we have

$$\pi_2 \circ (\pi_1|_Q)^{-1}(x) = (-b(x) : x_2 a(x) : x_3 a(x)), \tag{13}$$

whose coordinates are indeed quadratic. Since $f = \pi_2 \circ (\pi_1|_Q)^{-1}$ is defined by quadratics and generically $1:1$, it is a quadratic Cremona transformation.

To show that this transformation is non-degenerate, we must demonstrate that it has three unique base points. To understand the base points of f, recall that on a smooth quadric surface there are two distinct (possibly complex) lines passing through each point. The images of the two lines passing through c_1 under the projection π_1 will each be a single point. Therefore f is not well-defined on these image points in $\mathbb{P}^2$. Similarly, f is undefined on $\pi_1(c_2)$ since $\pi_2(\pi_1^{-1}(\pi_1(c_2))) = \pi_2(c_2) = 0$. Therefore these three points are exactly the base points of f in the domain. Finally, because $\overline{c_1 c_2} \not\subset Q$, these base points are all distinct. The base points in the codomain can be found symmetrically. □

Thus far we have shown that if we fix linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with centers $c_1 \neq c_2$ in $\mathbb{P}^3$, then there is a bijection between permissible quadrics through c_1, c_2 and generic lines through the fundamental matrix F of (π_1, π_2). Furthermore, there is a map sending each generic line through F (permissible quadric through c_1, c_2) to the Cremona transformation from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\pi_2 \circ (\pi_1|_Q)^{-1}$. These correspondences are summarized in (14), where $\mathcal{L}_F$ is the set of all generic lines through F and $\mathcal{Q}_F$ is the set of all permissible quadrics through c_1, c_2.

$$\begin{tikzcd} & \mathcal{Q}_F \arrow[dl] \arrow[dr] & \\ \mathcal{L}_F \arrow[ur] \arrow[rr, dashed] & & \mathcal{C} \end{tikzcd} \tag{14}$$

We can make the correspondence between generic lines through F and Cremona transformations even more explicit.

Lemma 3.7. *Given a generic line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$, the set of points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ satisfying $y^T Mx = 0$ for all $M \in \ell$ coincides with the closure of the graph $\{(x, f(x)) : x \in \mathbb{P}^2 \setminus B(f)\}$ of a unique Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. This gives a $1:1$ correspondence between generic lines $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ and Cremona transformations $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Moreover, when $F \in \ell$ has rank two, this Cremona transformation agrees with that induced by the maps $\mathcal{L}_F \to \mathcal{Q}_F \to \mathcal{C}$.*

Proof. Since ℓ is generic, we may assume without loss of generality that $\ell = \operatorname{span}\{F, M\}$ where F has rank two. This gives a pair of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2 with fundamental matrix F. In the 1:1 correspondence $\mathcal{L}_F \leftrightarrow \mathcal{Q}_F$ given in Corollary 3.4, the line ℓ corresponds to the permissible quadric Q given by the zero set of $q(u) = \pi_2(u)^\top M\pi_1(u)$. By Lemma 3.6, the Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ corresponding to $q(u)$ in the correspondence $\mathcal{Q}_F \to \mathcal{C}$ satisfies $f(\pi_1(p)) = \pi_2(p)$ for all $p \in Q \setminus \{c_1, c_2\}$. Since $\pi_1(Q)$ is dense in $\mathbb{P}^2$, the graph of f and the set $\{(\pi_1(p), \pi_2(p)) : p \in Q\setminus\{c_1, c_2\}\} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are both two-dimensional, as is their intersection. Each is the image of an irreducible variety under a rational map and so the Zariski-closures of these two sets are equal. By construction, this is contained in the zero sets of $y^T Fx$ and $y^T Mx$, as $\pi_2(p)^T F\pi_1(p) = 0$ for all $p \in \mathbb{P}^3$ and $\pi_2(p)^T M\pi_1(p) = 0$ for all $p \in Q$. Since F, M are linearly independent, the variety $\{(x, y) : y^T Fx = y^T Mx = 0\}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is two-dimensional. It therefore coincides with the Zariski-closure of the graph of f.

Conversely, suppose that $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation. We claim that $\{f(x)x^\top : x \in \mathbb{C}^3\}$ spans a 7-dimensional linear space $V \subset \mathbb{C}^{3\times 3}$. Up to projective transformations on $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$, we can take f to be the standard Cremona involution, giving

$$f(x)x^\top = \begin{pmatrix} x_1x_2x_3 & x_1^2x_3 & x_1^2x_2 \\ x_2^2x_3 & x_1x_2x_3 & x_1x_2^2 \\ x_2x_3^2 & x_1x_3^2 & x_1x_2x_3 \end{pmatrix}. \tag{15}$$

One can check explicitly that seven distinct monomials appear in this matrix and so the span of all such matrices is 7-dimensional. Projectively, the orthogonal complement gives a line $\ell = V^\perp$ in $\mathbb{P}(\mathbb{C}^{3\times 3})$. By definition, ℓ is exactly the set of all matrices M such that $y^\top Mx = 0$ for all (x, y) in the graph of f. Under the assumption that f is the standard Cremona transformation, ℓ is the span of the diagonal matrices $F_1 = \operatorname{diag}(1, -1, 0)$ and $F_2 = \operatorname{diag}(0, 1, -1)$; in general ℓ will be projectively equivalent to this line. We can verify that this line contains exactly the three rank-two matrices $F_1, F_2, F_1 + F_2$, and is therefore generic. □

Remark 3.8. Given $\ell = \operatorname{span}\{F, M\}$ we can solve for the coordinates of the corresponding Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given $x \in \mathbb{P}^2$, the corresponding point $y = f(x)$ will be the left kernel of the 3×2 matrix $\begin{pmatrix} Fx & Mx \end{pmatrix}$. The coordinates of y can be written explicitly in terms of the 2×2 minors of this matrix, which are quadratic in x. Note that, up to scaling, this formula for y is independent of the choice of basis $\{F, M\}$ for ℓ. Any point $x \in \mathbb{P}^2$ for which $\begin{pmatrix} Fx & Mx \end{pmatrix}$ has rank at most 1 will be a base point of this Cremona transformation. In particular, if $Fx = 0$, then x is a base point of f. As we will see below, there are three such points when ranging over all rank-two matrices in ℓ.

The next two results finish off the proof of the trinity correspondence (6) and the proof of Theorem 3.2.

Lemma 3.9. *Let ℓ be a generic line in $\mathbb{P}(\mathbb{C}^{3\times 3})$, i.e., ℓ contains three rank-two matrices F_1, F_2, F_3.*

(1) *Then ℓ gives rise to three permissible quadrics $Q_1, Q_2, Q_3 \subset \mathbb{P}^3$, each containing the centers of a pair of linear projections with fundamental matrices F_1, F_2, F_3 respectively.*
(2) *The quadrics Q_1, Q_2, Q_3, in conjunction with their distinguished linear projections, all induce the same Cremona transformation f. The base points of f are e_1^x, e_2^x, e_3^x in the domain and e_1^y, e_2^y, e_3^y in the codomain, where e_i^x and e_i^y generate the right and left nullspaces of F_i respectively.*

Proof. A generic line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ intersects the determinantal variety $\mathcal{D}$ cut out by $\det X = 0$ in three rank-two matrices F_1, F_2, F_3. Each F_i is the fundamental matrix of a pair of linear projections $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers, and by Lemma 3.3 and Lemma 3.5 there is a unique permissible quadric Q_i through these centers corresponding to the line ℓ. By Lemma 3.7, each of these quadrics induces the same Cremona transformation $f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

To conclude, we show that the base points of f and f^{-1} are e_1^x, e_2^x, e_3^x and e_1^y, e_2^y, e_3^y, respectively. We show that e_1^x, e_2^x, e_3^x are the base points of f and the argument for the base points of f^{-1} follows symmetrically. First, note that each e_i^x is a base point of f. This follows from Remark 3.8, since each $F_i \in \ell$ has rank two. Since the Cremona transformation f has three base points, it only remains to show that these points are distinct. If $e_1^x = e_2^x$, then by linearity $Fe_1^x = 0$ for all $F \in \ell = \operatorname{span}\{F_1, F_2\}$. This would imply that $\operatorname{rank}(F) \leq 2$ for all $F \in \ell$, contradicting the genericity of the line ℓ. □

Corollary 3.10. *The correspondence* $\mathcal{Q} \to \mathcal{C}$ *is* $3:1$.

Proof. Let $\mathbf{Q} = (Q, \pi_1, \pi_2) \in \mathcal{Q}$ be a permissible quadric along with a pair of linear projections that correspond to $f \in \mathcal{C}$. If F is the fundamental matrix associated to (π_1, π_2), then there exists a unique generic line ℓ through F corresponding to Q by Lemma 3.3 and Lemma 3.5. With the full trinity correspondence, this line ℓ contains three fundamental matrices F_1, F_2, F_3 corresponding to $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \in \mathcal{Q}$ that each produce the Cremona transformation f. Moreover, by Lemma 3.7 this line ℓ is the unique line in $\mathbb{P}(\mathbb{C}^{3\times 3})$ corresponding to f. Therefore if $\mathbf{Q}' \in \mathcal{Q}$ is such that $\mathbf{Q}' \mapsto f$ it follows that π_1', π_2' have one of F_1, F_2, F_3 as their fundamental matrix and that the quadric Q' is produced by the line ℓ. We conclude that $\mathbf{Q}'$ is, up to projective equivalence, one of $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$. □

This completes the proof of Theorem 3.2. A consequence of Theorem 6 is the following generalization of Problem 1.2.

Theorem 3.11. *Given a generic codimension-two subspace* $V \subset \mathbb{P}(\mathbb{C}^{3\times 3})$*, the intersection of* V *with* R_1*, the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$*, is a del Pezzo surface of degree six, and can be described explicitly via the trinity correspondence. Specifically, if* $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *is the Cremona transformation corresponding to the line* $V^\perp$*, then*

$$V \cap R_1 = \{g(x)x^\top : x \in \mathbb{P}^2\} \cup \{xg^{-1}(x)^\top : x \in \mathbb{P}^2\}.$$

Proof. For convenience, we denote

$$V_1 := \{g(x)x^\top : x \in \mathbb{P}^2\} \cup \{xg^{-1}(x)^\top : x \in \mathbb{P}^2\}.$$

To see that this is a degree-six del Pezzo surface, we show that V_1 can be obtained as the blowup of $\mathbb{P}^2$ in three non-collinear points, specifically, at the base points of g: e_1^x, e_2^x, e_3^x. Let $\pi_x : V_1 \dashrightarrow \mathbb{P}^2$ be the morphism defined by $\pi_x(vu^\top) = u$. Let the ℓ_i^y be the exceptional lines of g such that $g^{-1}(\ell_i^y) = e_i^x$. Then π_x is $1:1$ except on three mutually skew lines $\{y(e_i^x)^\top : y \in \ell_i^y\}$ which are taken to the points $\{e_i^x\}$. Therefore V_1 is the blowup of $\mathbb{P}^2$ in three non-collinear points and is a del Pezzo surface of degree six.

In particular, V_1 must be Zariski closed and it follows by Lemma 3.7 that $V \cap R_1 = V_1$. □

3.2 Back to the proof of Theorem 3.1

Before we can adapt the trinity correspondence to the reconstruction of point pairs, we need to address a certain kind of degeneracy. Given a configuration of point pairs $P = (x_i, y_i)_{i=1}^k$ consider the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ and its right nullspace $\mathcal{N}_Z$.

Lemma 3.12. *Suppose that* $P = (x_i, y_i)_{i=1}^k$ *admits a generic line* $\ell \subseteq \mathcal{N}_Z$ *(passing through three rank-two matrices* F_1, F_2, F_3*). Then for all* $j = 1, 2, 3$ *there is no* i *such that* $y_i^\top F_j = 0 = F_j x_i$.

Proof. Suppose, without loss of generality, $y_1^\top F_1 = 0 = F_1 x_1$. From the matrix F_1 and the line ℓ through it we obtain a pair of projections π_1, π_2 with centers c_1, c_2 and a smooth permissible quadric Q passing through them. Then $\pi_2(c_1)$ and $\pi_1(c_2)$ are the left and right epipoles of F_1, but since $y_1^\top F_1 = 0 = F_1 x_1$, it must be that $y_1 \sim \pi_2(c_1)$ and $x_1 \sim \pi_1(c_2)$. On the other hand, for any point p on the line connecting c_1, c_2, we have

$$\pi_2(p)^\top F_2 \pi_1(p) = \pi_2(c_1) F_2 \pi_1(c_2) = y_1^\top F_2 x_1 = 0$$

since $F_2 \in \mathcal{N}_Z$. Therefore, by Corollary 3.4, $p \in Q$ and thus $\overline{c_1 c_2} \subset Q$, which is a contradiction since Q is permissible. □

Even though a rank-two matrix F on a generic line in $\mathcal{N}_Z$ cannot have $y_i^\top F = 0 = Fx_i$, it might be that one of the equations hold. We name this type of degeneracy in the following definition.

Definition 3.13. A generic line $\ell \subseteq \mathcal{N}_Z$ is *P-degenerate* if there exists a rank-two matrix $F \in \ell$ such that either $Fx_i = 0$ or $y_i^\top F = 0$ for some i. We call a generic line that is not P-degenerate a *P-generic* line.

Any rank-two matrix F in a P-generic line will give a reconstruction $c_1, c_2, p_1, \ldots, p_k$ of the point pairs P. That is, there will be linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with centers c_1, c_2 so that $\pi_2^\top(p) F \pi_1(p) = 0$ for all $p \in \mathbb{P}^3$ and $(x_i, y_i) = (\pi_1(p_i), \pi_2(p_i))$ for all $i = 1, \ldots, k$. A smooth quadric Q will contain two lines through any of its points.

Definition 3.14. A quadric $Q \subset \mathbb{P}^3$ passes *degenerately* through a reconstruction $c_1, c_2, \{p_i\}_{i=1}^k$ of P if it passes through these $k + 2$ points and contains the line through a center point c_i and a reconstructed point p_j.

Definition 3.15. A Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ maps $x_i \mapsto y_i$ *degenerately* if x_i is a base point of f and y_i lies on the corresponding exceptional line, or symmetrically, y_i is a base point of f^{-1} and x_i lies on the corresponding exceptional line.

Generically, the trinity correspondence specializes to the reconstruction of point pairs in an intuitive way.

Theorem 3.16. *Given a configuration of point pairs $P = (x_i, y_i)_{i=1}^k$ and the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$, define the following subsets of $\mathcal{L}, \mathcal{Q}, \mathcal{C}$:*

(1) *$\mathcal{L}_P$: the set of all P-generic lines $\ell \subseteq \mathcal{N}_Z :=$ nullspace(Z),*

(2) *$\mathcal{Q}_P$: the set (up to projective equivalence) of all permissible quadrics passing non-degenerately through some reconstruction $c_1, c_2, p_1, \ldots, p_k$ of P,*

(3) *$\mathcal{C}_P$: the set of all Cremona transformations $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ mapping $x_i \mapsto y_i$ non-degenerately for all $i = 1, \ldots, k$.*

Then there is a $1:1$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{C}_P$, a $1:3$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{Q}_P$, and a $3:1$ correspondence between the elements of $\mathcal{Q}_P$ and $\mathcal{C}_P$ as in the diagram

$$\begin{tikzcd} & \mathcal{Q}_P \arrow[dr, "3:1"] & \\ \mathcal{L}_P \arrow[ur, "1:3"] \arrow[rr, leftrightarrow] & & \mathcal{C}_P \end{tikzcd} \tag{16}$$

Proof. We need to show that the trinity correspondence (6) can be restricted to the sets $\mathcal{L}_P, \mathcal{Q}_P, \mathcal{C}_P$. We will therefore examine each leg of this diagram.

($\mathcal{L}_P \to \mathcal{Q}_P$) We begin by considering a P-generic line $\ell = \text{span}\{F, M\} \subseteq \mathcal{N}_Z$. Without loss of generality, we can take F to be one of the three fundamental matrices in ℓ with corresponding projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2 that give reconstructions $p_1, \ldots, p_k \in \mathbb{P}^3$ of the point pairs P. By Lemma 3.3, the line ℓ corresponds to a smooth permissible quadric Q defined by the vanishing of $q(u) = \pi_2(u)^T M \pi_1(u)$. For any point p_i in the reconstruction, we have

$$q(p_i) = \pi_2(p_i)^\top M \pi_1(p_i) = y_i^\top M x_i = 0 \tag{17}$$

since $M \in \ell \subset \mathcal{N}_Z$. Therefore Q passes through the reconstruction $c_1, c_2, p_1, \ldots, p_k$. It remains to show that it does so non-degenerately. By Lemmas 3.6 and 3.9, a reconstructed point p_i lies on one of the lines through c_1 (or symmetrically through c_2) if and only if there exists $M \in \ell$ such that $Mx_i = 0$ (or symmetrically $y_i^\top M = 0$). Since ℓ is P-generic there is no such M, implying that the quadric passes through the reconstruction non-degenerately.

($\mathcal{Q}_P \to \mathcal{C}_P$) Consider a permissible quadric Q passing through a reconstruction $c_1, c_2, p_1, \ldots, p_k$ of P with linear projections π_1, π_2. As in Theorem 3.2, the tuple (Q, π_1, π_2) induces a Cremona transformation $f := \pi_2 \circ (\pi_1|_Q)^{-1}$. By Lemma 3.6, the base points of f are the images of the point c_2 and each of the lines in Q passing through c_1. Since $p_i \neq c_2$ and does not belong to these lines, the point $x_i = \pi_1(p_i)$ is not a base point of f. Similarly, the base points of f^{-1} are the images of the point c_1 and the lines in Q passing through c_2 under π_2, so a symmetric argument shows that $y_i = \pi_2(p_i)$ is not a base point of f^{-1}. Therefore f maps $x_i = \pi_1(p_i)$ to $y_i = \pi_2(p_i)$ non-degenerately.

($\mathcal{C}_P \to \mathcal{L}_P$) Consider a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $x_i \mapsto y_i$ non-degenerately for all i. As in Lemma 3.7, f corresponds to a unique line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ defined by the property that $f(x)^\top Mx = 0$ for all $M \in \ell$ and $x \in \mathbb{P}^2$. In particular, $y_i^\top M x_i = 0$ for all $M \in \ell$ and $i = 1, \ldots, k$, implying that $\ell \subseteq \mathcal{N}_Z$. By assumption, no point x_i is a base point of f and no point y_i is a base point of f^{-1}. By Lemma 3.9, it then follows that $Mx_i \neq 0$ and $y_i^\top M \neq 0$ for all $M \in \ell$. Therefore ℓ is not P-degenerate. □

Remark 3.17. The assumptions of non-degeneracy can be removed from the 1:1 correspondence between generic lines in $\mathcal{N}_Z$ and Cremona transformations mapping $x_i \mapsto y_i$. Extending this to quadrics is more subtle, as some rank-two matrices $F \in \ell \subset \mathcal{N}_Z$ may not give full reconstructions of the point pairs P.

Proof of the only-if direction of Theorem 3.1. For 8 semi-generic point pairs, the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^8$ is rank deficient exactly when $\mathcal{N}_Z =: \ell$ is a line. This line ℓ is generic because it is also the nullspace of any submatrix of Z of size 7×9 and the corresponding seven point pairs are generic. Pick a subset of seven point pairs, say $(x_i, y_i)_{i=1}^7$, from the original eight pairs. Since these seven point pairs are generic, and ℓ is also generic, we can assume that $Fx_i \neq 0$ and $y_i^\top F \neq 0$ for any rank-two matrix $F \in \ell$ and all $i = 1, \ldots, 7$. On the other hand, if we pick a different set of seven point pairs, say $(x_i, y_i)_{i=2}^8$, then ℓ is also the nullspace of the corresponding Z_7 and by the same argument as before, $Fx_i \neq 0$ and $y_i^\top F \neq 0$ for any rank-two matrix $F \in \ell$ and all $i = 2, \ldots, 8$. Therefore, ℓ is P-generic.

Since ℓ is P-generic, by Theorem 3.16, ℓ gives rise to a Cremona transformation $f : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $f(x_i) = y_i$ for $i = 1, \ldots, 8$. This finishes the proof of Theorem 3.1. □

We end this section by demonstrating the trinity correspondence for an example, beginning with a single quadric through a reconstruction.

Example 3.18. Consider the quadric $Q \subset \mathbb{P}^3$ defined by the equation $x^2 + y^2 - z^2 - w^2 = 0$ and the following 10 points $p_1, \ldots, p_8, c_1, c_2 \in Q$:

$$
\begin{aligned}
c_1 &= (1:0:0:1) & c_2 &= (0:1:0:1)\\
p_1 &= (5:12:13:0) & p_2 &= (13:0:5:12)\\
p_3 &= (12:5:13:0) & p_4 &= (3:4:5:0)\\
p_5 &= (4:3:5:0) & p_6 &= (3:4:0:5)\\
p_7 &= (4:3:0:5) & p_8 &= (5:0:4:3).
\end{aligned}
$$

The two projections (cameras) with centers c_1, c_2 have matrices

$$
A_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

and we can calculate the image points and epipoles:

$$
\begin{aligned}
e_x &= (-1:1:0) & e_y &= (1:-1:0)\\
x_1 &= (5:12:13) & y_1 &= (5:12:13)\\
x_2 &= (1:0:5) & y_2 &= (13:-12:5)\\
x_3 &= (12:5:13) & y_3 &= (12:5:13)\\
x_4 &= (3:4:5) & y_4 &= (3:4:5)\\
x_5 &= (4:3:5) & y_5 &= (4:3:5)\\
x_6 &= (-2:4:0) & y_6 &= (3:-1:0)\\
x_7 &= (-1:3:0) & y_7 &= (4:-2:0)\\
x_8 &= (2:0:4) & y_8 &= (5:-3:4).
\end{aligned}
$$

The point pairs (x_i, y_i) give us the matrix

$$
Z_8 = \begin{bmatrix}
25 & 60 & 65 & 60 & 144 & 156 & 65 & 156 & 169\\
13 & -12 & 5 & 0 & 0 & 0 & 65 & -60 & 25\\
144 & 60 & 156 & 60 & 25 & 65 & 156 & 65 & 169\\
9 & 12 & 15 & 12 & 16 & 20 & 15 & 20 & 25\\
16 & 12 & 20 & 12 & 9 & 15 & 20 & 15 & 25\\
-6 & 2 & 0 & 12 & -4 & 0 & 0 & 0 & 0\\
-4 & 2 & 0 & 12 & -6 & 0 & 0 & 0 & 0\\
10 & -6 & 8 & 0 & 0 & 0 & 20 & -12 & 16
\end{bmatrix}
$$

which we can check is rank deficient and has nullspace spanned by the vectors

$$m_1 = (-1, 1, 0, -1, -1, 0, 0, 0, 1), \qquad m_2 = (0, 0, -1, 0, 0, -1, 1, 1, 0).$$

The reconstruction we started with has fundamental matrix

$$F = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 1\\ -1 & -1 & 0 \end{bmatrix}$$

and if we take a different matrix

$$M = \begin{bmatrix} -1 & -1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$

in the nullspace of Z_8 we can verify that $A_2^\top M A_1$ yields the original quadric Q:

$$(x, y, z, w)A_2^\top M A_1(x, y, z, w)^\top = (x, y, z, w)\begin{bmatrix} -1 & -1 & 0 & 1\\ 1 & -1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 1 & 1 & 0 & -1 \end{bmatrix}(x, y, z, w)^\top = -x^2 - y^2 + z^2 + w^2.$$

The other two possible choices for fundamental matrices in the nullspace of Z_8 are

$$F_2 = \begin{bmatrix} -1 & -1 & 1\\ 1 & -1 & 1\\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad F_3 = \begin{bmatrix} -1 & -1 & -1\\ 1 & -1 & -1\\ 1 & 1 & 1 \end{bmatrix},$$

which have epipoles $e_x^2 = (0 : 1 : 1)$, $e_y^2 = (-1 : 0 : 1)$, $e_x^3 = (0 : -1 : 1)$ and $e_y^3 = (1 : 0 : 1)$. Moreover, we can verify that there is a unique Cremona transformation

$$f(x_1, x_2, x_3) = (x_1^2 - x_2^2 + x_3^2, x_1^2 + 2x_1x_2 + x_2^2 - x_3^2, 2x_1x_3)$$

such that $f(x_i) = y_i$ for all i. This Cremona transformation has base points exactly matching the epipoles. Finally, we can check that each camera center lies on two real lines on the quadric Q, parameterized by $(a : b) \in \mathbb{P}^1$ as

$$\ell_x^2 = (a : b : b : a), \quad \ell_x^3 = (a : -b : b : a), \quad \ell_y^2 = (-b : a : b : a), \quad \text{and} \quad \ell_y^3 = (b : a : b : a)$$

whose images are exactly the other two possible pairs of epipoles/base points (e_x^2, e_y^2) and (e_x^3, e_y^3).

4 The case $k = 7$

We now come to the case of $k = 7$ point pairs. In order to understand the case of seven point pairs, we first need to understand six generic point pairs $(x_i, y_i)_{i=1}^6$. In this case, the nullspace $\mathcal{N}_Z$ of the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^6$ is projectively a plane and $\mathcal{N}_Z \cap \mathcal{D} =: C$ is a cubic curve in $\mathbb{P}(\mathbb{C}^{3\times 3})$ lying in the plane $\mathcal{N}_Z$. By our genericity assumption, C misses all rank-one matrices in $\mathcal{D}$ and hence every point on C is a fundamental matrix of $(x_i, y_i)_{i=1}^6$. Let κ_x and κ_y denote the quadratic maps that take a rank-two matrix $M \in \mathbb{P}(\mathbb{C}^{3\times 3})$ to its right and left nullvectors respectively. As a consequence of the classical theory of blowups and cubic surfaces as discussed in [3], the maps $C \to \kappa_x(C) =: C_x \subset \mathbb{P}_x^2$ and $C \to \kappa_y(C) =: C_y \subset \mathbb{P}_y^2$ are isomorphisms when $(x_i, y_i)_{i=1}^6$ is generic; we will go into more detail on the nature of these isomorphism in Subsection 4.2.1.

$$\begin{tikzcd} & C \arrow[dl, "\kappa_x"'] \arrow[dr, "\kappa_y"] & \\ \mathbb{P}_x^2 \supset C_x & & C_y \subset \mathbb{P}_y^2 \end{tikzcd} \tag{18}$$

By the composition $\kappa_y \circ \kappa_x^{-1}$, we get that C_x and C_y are isomorphic cubic curves. However, this isomorphism is not particularly useful; for instance, it does not take $x_i \mapsto y_i$. By construction, the curves C_x and C_y consist exactly of all possible epipoles of the fundamental matrices of $(x_i, y_i)_{i=1}^6$ in $\mathbb{P}_x^2$ and $\mathbb{P}_y^2$. We therefore call C_x and C_y the right and left *epipolar curves* of $(x_i, y_i)_{i=1}^6$. We will see that these cubic curves are closely tied to both rank drop and the trinity relationship established in Theorem 3.16.

Example 4.1. Consider the following six point pairs:

$$\begin{aligned} x_1 &= (0:0:1) \quad y_1 = (0:0:1) & x_2 &= (1:0:1) \quad y_2 = (1:0:1) \\ x_3 &= (0:1:1) \quad y_3 = (0:1:1) & x_4 &= (1:1:1) \quad y_4 = (1:1:1) \\ x_5 &= (3:5:1) \quad y_5 = (7:-2:1) & x_6 &= (-7:11:1) \quad y_6 = (3:13:1) \end{aligned}$$

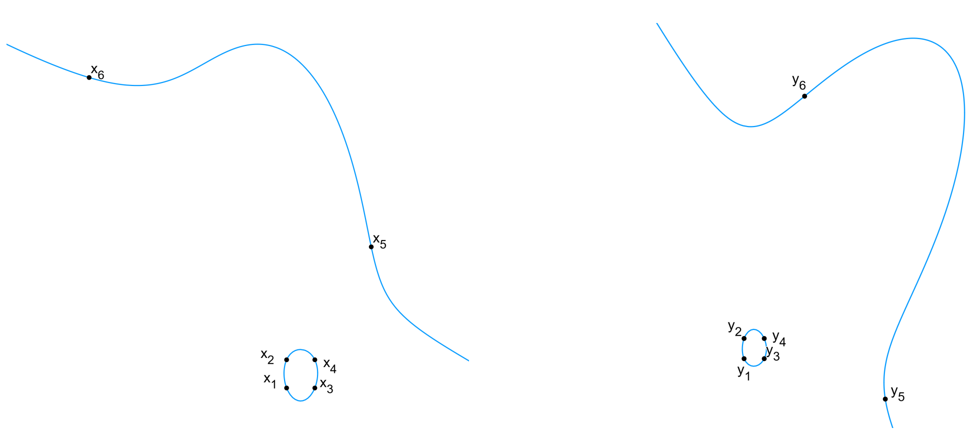

Figure 1: The cubic curves C_x and C_y from Example 4.1, with x_i and y_i labeled.

Figure 1 shows the curves C_x and C_y. Observe that $x_i \in C_x$ and $y_i \in C_y$ for all $i = 1, \dots, 6$, a fact we will prove in Section 4.1. The curves C_x and C_y are cut out by $g_x(u) = 0$ and $g_y(v) = 0$ in $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ where

$$g_x(u) = 447u_1^3 + 775u_1^2u_2 + 113u_1u_2^2 + 118u_2^3 - 4083u_1^2u_3 - 888u_1u_2u_3 - 1521u_2^2u_3 + 3636u_1u_3^2 + 1403u_2u_3^2,$$
$$g_y(v) = 447v_1^3 - 136v_1^2v_2 - 12v_1v_2^2 + 118v_2^3 - 3608v_1^2v_3 + 148v_1v_2v_3 - 1478v_2^2v_3 + 3161v_1v_3^2 + 1360v_2v_3^2.$$

In Section 4.2 we use classical invariant theory to derive the polynomials g_x and g_y.

Given seven point pairs $(x_i, y_i)_{i=1}^7$, denote the epipolar curves obtained by excluding the ith point pair as $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$. In the event that these curves are equal for all choices of i, we denote $C_x := C_x^{\hat{1}} = \cdots = C_x^{\hat{7}}$ and $C_y := C_y^{\hat{1}} = \cdots = C_y^{\hat{7}}$. We will see that this equality is necessary (Theorem 4.2) and sufficient (Theorem 4.11) for $Z_7 = (x_i^\top \otimes y_i^\top)_{i=1}^7$ to be rank deficient.

The maps κ_x, κ_y are not the only way to derive the epipolar curves C_x, C_y; it is also possible to obtain them via the trinity correspondence (16). This will be the subject of Subsection 4.1 and will allow us to prove the following result:

Theorem 4.2. *For 7 semi-generic point pairs $(x_i, y_i)_{i=1}^7$, the matrix Z_7 is rank deficient if and only if there exist cubic curves C_1 through $x_1, \dots, x_7$ and C_2 through $y_1, \dots, y_7$ as well as an isomorphism $f : C_1 \to C_2$ such that $x_i \mapsto y_i$. Moreover, if this holds then $C_1 = C_x$ and $C_2 = C_y$.*

This is the first of the two main results in this section and it is the more geometric theorem, to be proved at the end of Subsection 4.1. In Subsection 4.2.1 we use the theory of cubic surfaces as in [3] to obtain explicit equations for the epipolar curves. In Subsection 4.2.2 we use these explicit equations to characterize rank deficiency of Z_7 using 14 algebraic equations and to prove our second main result, Theorem 4.11, which is the more algebraic theorem. Finally, in Section 4.3 we collect some further results outside the assumption of semi-genericity.

4.1 Rank drop and cubic curves

Before addressing the cases of six generic point pairs and seven semi-generic point pairs, we establish an analogue of Lemma 3.7 to show how general projective planes in $\mathbb{P}(\mathbb{C}^{3\times3})$ give rise to Cremona transformations of cubic curves.

Lemma 4.3. *Let $\mathcal{P} \subset \mathbb{P}(\mathbb{C}^{3\times3})$ be a projective plane not containing any rank-one matrix. The set of points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ satisfying $y^TMx = 0$ for all $M \in \mathcal{P}$ coincides with the closure of the graph $\{(x, f(x)) : x \in C_x^{\mathcal{P}}\}$ of the restriction of a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to a cubic curve $C_x^{\mathcal{P}}$. Moreover, there is a two-dimensional family of Cremona transformations $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, indexed by generic lines $\ell \subset \mathcal{P}$ as in Lemma* 3.7*, with the same restriction to $C_x^{\mathcal{P}}$.*

Proof. The curve $C_x^{\mathcal{P}}$ consists of the set of points $x \in \mathbb{P}^2$ for which there exists an $M \in \mathcal{P}$ with $Mx = 0$. When $\mathcal{P} = \mathcal{N}_Z$, this is the epipolar curve C_x described above. By choosing a basis $\{M_1, M_2, M_3\}$ for $\mathcal{P}$ we can write any $M \in \mathcal{P}$ as $aM_1 + bM_2 + cM_3$. Given $x \in \mathbb{P}^2$ there exists $(a : b : c) \in \mathbb{P}^2$ with $(aM_1 + bM_2 + cM_3)x = 0$ if and only if $\det\begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix} = 0$. Therefore $C_x^{\mathcal{P}}$ is defined by the vanishing of this determinant, which is a cubic form in x_1, x_2, x_3. Symmetrically the cubic curve $C_y^{\mathcal{P}}$ defined by the vanishing of the determinant of the matrix with rows $y^\top M_j$ coincides with C_y when $\mathcal{P} = \mathcal{N}_Z$.

Let $\ell = \operatorname{span}\{M_1, M_2\} \subset \mathcal{P} \subset \mathbb{P}(\mathbb{C}^{3\times3})$ be a generic line. By Lemma 3.7, there is a Cremona transformation $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose graph is the set of points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ satisfying $y^TMx = 0$ for all $M \in \ell$. As in Remark 3.8, the map f_ℓ transforms x into $\ker\begin{pmatrix} M_1x & M_2x \end{pmatrix}$. For $x \in C_x^{\mathcal{P}}$ except the three base points of f_ℓ, the left kernel of $\begin{pmatrix} M_1x & M_2x \end{pmatrix}$ is also the left kernel of the rank-two 3×3 matrix $\begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix}$, which is independent of the choice of $\ell = \operatorname{span}\{M_1, M_2\} \subset \mathcal{P}$.

Note that the graph $\{(x, f_\ell(x)) : x \in C_x^{\mathcal{P}}\}$ and the set of points $(x, y) \in \mathbb{P}^2_x \times \mathbb{P}^2_y$ satisfying $y^TMx = 0$ for all $M \in \mathcal{P}$ have the same projection onto $\mathbb{P}^2_x$, namely $C_x^{\mathcal{P}}$. For any $x \in C_x^{\mathcal{P}}$, the corresponding point y is given by $f_\ell(x) = \ker\begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix}$. □

4.1.1 Six point pairs. Let $(x_i, y_i)_{i=1}^6$ be a set of six generic point pairs, $Z = (x_i, y_i)_{i=1}^6$ and let F be any choice of fundamental matrix (i.e., a rank-two matrix on the projective plane $\mathcal{N}_Z$). Genericity guarantees a reconstruction $p_1, \ldots, p_6, c_1, c_2 \in \mathbb{P}^3$, of $(x_i, y_i)_{i=1}^6$ from F. Recall that c_1, c_2 are the centers of camera projections π_1, π_2 and $p_1, \ldots, p_6$ are world points such that $\pi_1(p_j) = x_j$ and $\pi_2(p_j) = y_j$ for all $j = 1, \ldots, 6$.

Since $\mathcal{N}_Z$ is a two-dimensional plane, it contains a pencil of lines through F, see (14) and (16), which corresponds to a pencil of quadrics Q_λ, each passing through the reconstruction. The intersection of these quadrics, also obtainable as the intersection of any two distinct quadrics in the pencil, is a quartic space curve $W \subset \mathbb{P}^3$ that must also pass through the reconstruction. Since c_1, c_2 are on W, $\pi_1(W) \subset \mathbb{P}_x^2$ and $\pi_2(W) \subset \mathbb{P}_y^2$ are cubic curves. We will see that these cubic curves are independent of the choice of F, and that they are exactly the epipolar curves C_x and C_y. We will use this derivation to study their special properties arising from the trinity relationship. The following lemma assumes the setup just described.

Lemma 4.4. *For six generic point pairs $(x_i, y_i)_{i=1}^6$ we have the following.*

(1) *The cubic curves $\pi_1(W)$ and $\pi_2(W)$ are the right and left epipolar curves C_x, C_y, respectively; in particular, they are independent of the choice of F.*
(2) *The points x_i lie on C_x and the points y_i lie on C_y for $i = 1, \ldots, 6$.*
(3) *There exists a two-parameter family of Cremona transformations $f_\ell : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$, indexed by lines ℓ in the projective plane $\mathcal{N}_Z$, such that the following holds:*
   - *$f_\ell(x_i) = y_i$ for $i = 1, \ldots, 6$,*
   - *the restriction of f_ℓ to a map $C_x \to C_y$ is independent of ℓ, and*
   - *the base points of all the Cremona transformations f_ℓ lie in C_x, C_y.*

Proof. Let F be a fundamental matrix in $\mathcal{N}_Z$. Since $(x_i, y_i)_{i=1}^6$ is generic, F can be any element of the cubic curve $C = \mathcal{N}_Z \cap \mathcal{D}$, and we can use F to obtain a reconstruction consisting of world points $p_1, \ldots, p_6$ and cameras corresponding to linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$.

The quartic space curve W is defined by quadrics of the form $q(u) = \pi_2(u)^\top M \pi_1(u)$ where $M \in \mathcal{P} \cap F^\perp$. Therefore $\pi_1(W)$ contains the cubic plane curve C_x defined by $\{x \in \mathbb{P}^2 : \exists M \in \mathcal{N}_Z \text{ such that } Mx = 0\}$. Since $c_1 \in W$, $\pi_1(W)$ is a cubic plane curve and so these must be equal. A symmetric argument shows that $\pi_2(W) = C_y$. Since W contains each point p_i, this also implies that $x_i = \pi_1(p_i)$ belongs to C_x and $y_i = \pi_2(p_i)$ belongs to C_y for $i = 1, \ldots, 6$.

By Lemma 4.3, for any generic line $\ell \subset \mathcal{N}_Z$ the restriction of the Cremona transformation $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to the cubic C_x is independent of the choice of ℓ. By Theorem 3.16 we have $f_\ell(x_i) = y_i$ for all i. As in Lemma 3.9, the base points of f_ℓ are the right kernels of the three rank-two matrices $F_1, F_2, F_3 \in \ell$ and therefore belong to C_x. Similarly, the base points of f_ℓ^{-1} are the left kernels of these matrices and so belong to C_y. □

Remark 4.5. Given a rank two matrix $F \in \mathcal{N}_Z$, it may be the case that $Fx_i = 0$ (or $y_i^\top F = 0$) for some i. However, even in this case we can still apply the trinity (6) to obtain a pencil of quadrics (and a pencil of Cremona transformations), and from them the cubic curves C_x, C_y with the isomorphism between them. Therefore, even if ℓ is such that x_i is a base point of f_ℓ, the restriction of f_ℓ to a map $C_x \to C_y$, as in Lemma 2.6, would still satisfy $x_i \mapsto y_i$.

4.1.2 From six points to seven. The trinity correspondence has allowed us to prove a number of properties of the epipolar curves corresponding to six generic point pairs. In particular, we know that there is an isomorphism $f : C_x \to C_y$ that sends $x_i \mapsto y_i$ for $i = 1, \ldots, 6$ which is induced by a two-parameter family of Cremona transformations $\mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$. For seven generic point pairs, the following corollary holds.

Lemma 4.6. *Let $(x_i, y_i)_{i=1}^7$ be seven semi-generic point pairs. Then the rank of $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ drops if and only if there exist cubic curves C_1, C_2 through $x_1, \ldots, x_7$ and $y_1, \ldots, y_7$ respectively, as well as a two-parameter family of Cremona transformations $f_\ell : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $f(x_i) = y_i$ for all i and the family is well-defined on the restriction $C_1 \to C_2$. Furthermore, if this holds then $C_1 = C_x$ and $C_2 = C_y$.*

Proof. ($\Rightarrow$) Under semi-genericity, Z is rank deficient if and only if the nullspace of Z and the nullspaces of each of its 6×9 submatrices are identical. In particular, if P_i is the subset of 6 point pairs obtained by excluding the

ith, then, using the notation from Theorem 3.16, $\mathcal{L}_{P_1} = \cdots = \mathcal{L}_{P_7}$. Applying Lemma 4.4, we find that the pairs of curves $C_x^{\hat{i}}, C_y^{\hat{i}}$ are identical for all i. Accordingly, we omit the superscripts and identify them as C_x and C_y respectively. Similarly, the family of Cremona transformations satisfies $\mathcal{C}_{P_1} = \cdots = \mathcal{C}_{P_7}$, and, as in Lemma 4.4, restricting this family to the map $C_x \to C_y$ yields a well-defined isomorphism with the property $x_i \mapsto y_i$ for all i.

($\Leftarrow$) For this direction, we use Theorem 3.16. In particular, the existence of such a family of Cremona transformations implies that $\dim(\mathcal{L}_P) = \dim(\mathcal{C}_P) = 2$ as illustrated in (16). Since there is a two-dimensional family of lines ℓ in the projective nullspace of Z, we must have $\operatorname{rank}(Z) < 7$. We now need to verify that $C_1 = C_x$ and $C_2 = C_y$. It follows by Lemma 2.5 that the curves C_1, C_2 contain all possible base points of the Cremona transformations f_ℓ. Furthermore, by Lemma 3.9 the sets of all such base points in the domain and codomain is exactly the set of all possible right and left epipoles. It follows that $C_x \subset C_1$ and $C_y \subset C_2$ and therefore the curves are equal. □

Proof of Theorem 4.2. ($\Rightarrow$) This direction follows from Lemma 4.6. In particular, the isomorphism is exactly that obtained by restricting the family of Cremona transformations to the map $C_x \to C_y$.

($\Leftarrow$) Assume that such curves C_1, C_2 exist, as well as the desired isomorphism $C_1 \to C_2$. By Lemma 2.7 there is a two-parameter family of Cremona transformations $\mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ whose restriction $C_1 \to C_2$ yields this isomorphism. It follows from Lemma 4.6 that Z is rank deficient and that $C_1 = C_x$ and $C_2 = C_y$. □

4.2 The Cremona hexahedral form of C_x and C_y

In this subsection we return to the original characterization of the cubic curves C_x and C_y as the images under the quadratic maps κ_x and κ_y of the curve C as in (18). We will see that it is possible to derive explicit equations for these curves using the classical theory of cubic surfaces and a special invariant-theoretic representation of them called the Cremona hexahedral form. These ideas intersect substantially with the characterization of rank drop of Z_6 in [3]; in particular, we draw on the connection between six generic points pairs $(x_i, y_i)_{i=1}^6$ and cubic surfaces. We begin by explicitly characterizing the curve $C = \mathcal{N}_Z \cap \mathcal{D}$ as the planar section of a cubic surface; we will then use this characterization in conjunction with material from [3] to find explicit equations for the curves C_x and C_y.

4.2.1 Six generic point pairs again. Suppose we have six generic point pairs $(x_i, y_i)_{i=1}^6$; in particular, $Z = (x_i^\top \otimes y_i^\top)_{i=1}^6$ has full rank. Let $Z_{\hat{j}}$ denote the 5×9 matrix obtained by deleting the jth row of Z. Then $\mathcal{N}_{Z_{\hat{j}}} \cong \mathbb{P}^3$ and $S_{\hat{j}} := \mathcal{N}_{Z_{\hat{j}}} \cap \mathcal{D}$ is a smooth cubic surface in $\mathcal{N}_{Z_{\hat{j}}}$ by the genericity assumption, and hence all points on it have rank two. It was shown in [3] that $S_{\hat{j}}$ is the blowup of $\mathbb{P}_x^2$ at $(\{x_i\}_{i=1}^6 \setminus \{x_j\}) \cup \{\bar{x}_j\}$ where $\bar{x}_j$ is a new point that arises from $\{x_i\}_{i=1}^6 \setminus \{x_j\}$, see Lemma 6.1 of [3] for its derivation and formula. Symmetrically, $S_{\hat{j}}$ is also the blowup of $(\{y_i\}_{i=1}^6 \setminus \{y_j\}) \cup \{\bar{y}_j\}$ in $\mathbb{P}_y^2$ where $\bar{y}_j$ is a new point determined by $\{y_i\}_{i=1}^6 \setminus \{y_j\}$. The quadratic maps $\kappa_x^{\hat{j}} : S_{\hat{j}} \to \mathbb{P}_x^2$ and $\kappa_y^{\hat{j}} : S_{\hat{j}} \to \mathbb{P}_y^2$ are $1:1$ except on the exceptional lines of the blowup. The curve C is given by

$$C = \mathcal{N}_Z \cap \mathcal{D} = \mathcal{N}_{Z_{\hat{j}}} \cap \mathcal{D} \cap (x_j^\top \otimes y_j^\top)^\perp = S_{\hat{j}} \cap (x_j^\top \otimes y_j^\top)^\perp.$$

Therefore, C cuts each of the exceptional lines of the blowup in one point, and therefore the restrictions of κ_x, κ_y to C are isomorphisms.

For a set of six points $u_1, \ldots, u_6 \in \mathbb{P}^2$, set $[ijk] := \det[u_i\, u_j\, u_k]$ and define

$$[(ij)(kl)(rs)] := [ijr][kls] - [ijs][klr]. \tag{19}$$

This is a classical invariant of $u_1, \ldots, u_6$ under the action of PGL(3) whose vanishing expresses that the lines $\overline{u_i u_j}$, $\overline{u_k u_l}$ and $\overline{u_r u_s}$ meet in a point; compare [2, pp. 169]. Using these invariants, Coble [2, page 170] defines the following six scalars:

$$\begin{aligned}
\bar{a} &= [(25)(13)(46)] + [(51)(42)(36)] + [(14)(35)(26)] + [(43)(21)(56)] + [(32)(54)(16)]\\
\bar{b} &= [(53)(12)(46)] + [(14)(23)(56)] + [(25)(34)(16)] + [(31)(45)(26)] + [(42)(51)(36)]\\
\bar{c} &= [(53)(41)(26)] + [(34)(25)(16)] + [(42)(13)(56)] + [(21)(54)(36)] + [(15)(32)(46)]\\
\bar{d} &= [(45)(31)(26)] + [(53)(24)(16)] + [(41)(25)(36)] + [(32)(15)(46)] + [(21)(43)(56)]\\
\bar{e} &= [(31)(24)(56)] + [(12)(53)(46)] + [(25)(41)(36)] + [(54)(32)(16)] + [(43)(15)(26)]\\
\bar{f} &= [(42)(35)(16)] + [(23)(14)(56)] + [(31)(52)(46)] + [(15)(43)(26)] + [(54)(21)(36)]
\end{aligned} \tag{20}$$

Coble also defines the following six cubic polynomials that vanish on $u_1, \dots, u_6$:

$$\begin{aligned}
a(u) &= [25u][13u][46u] + [51u][42u][36u] + [14u][35u][26u] + [43u][21u][56u] + [32u][54u][16u]\\
b(u) &= [53u][12u][46u] + [14u][23u][56u] + [25u][34u][16u] + [31u][45u][26u] + [42u][51u][36u]\\
c(u) &= [53u][41u][26u] + [34u][25u][16u] + [42u][13u][56u] + [21u][54u][36u] + [15u][32u][46u]\\
d(u) &= [45u][31u][26u] + [53u][24u][16u] + [41u][25u][36u] + [32u][15u][46u] + [21u][43u][56u]\\
e(u) &= [31u][24u][56u] + [12u][53u][46u] + [25u][41u][36u] + [54u][32u][16u] + [43u][15u][26u]\\
f(u) &= [42u][35u][16u] + [23u][14u][56u] + [31u][52u][46u] + [15u][43u][26u] + [54u][21u][36u]
\end{aligned} \tag{21}$$

These cubic polynomials are *covariants* of $u_1, \dots, u_6$ under the action of PGL(3).

It is a well-known result in algebraic geometry that every smooth cubic surface is the blowup of six points in $\mathbb{P}^2$. The blowup procedure furnishes an algorithm to find a determinantal representation of the surface. However, these representations do not directly reflect the six points that were blown up. The *Cremona hexahedral form* of a smooth cubic surface provides explicit equations for the surface in terms of the points being blown up. It consists of the following polynomials:

$$\begin{aligned}
z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 &= 0\\
z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0\\
\bar{a}z_1 + \bar{b}z_2 + \bar{c}z_3 + \bar{d}z_4 + \bar{e}z_5 + \bar{f}z_6 &= 0.
\end{aligned} \tag{22}$$

Furthermore, the cubic surface can also be parameterized by

$$\overline{\{(a(u) : b(u) : c(u) : d(u) : e(u) : f(u)) \ : \ u \in \mathbb{P}^2\}}. \tag{23}$$

We will now use the above facts to obtain explicit equations (that depend on $(x_i, y_i)_{i=1}^6$) of the epipolar curves C_x and C_y. In what follows, we index $\bar{a}, \dots, \bar{f}$ and $a(u), \dots, f(u)$ with x (respectively y) when $u_i = x_i$ (respectively $u_i = y_i$).

Definition 4.7. Given six point pairs $(x_i, y_i)_{i=1}^6$ we define the following cubic polynomials:

$$\begin{aligned}
g_x(u) &:= \bar{a}_y a_x(u) + \bar{b}_y b_x(u) + \bar{c}_y c_x(u) + \bar{d}_y d_x(u) + \bar{e}_y e_x(u) + \bar{f}_y f_x(u),\\
g_y(v) &:= \bar{a}_x a_y(v) + \bar{b}_x b_y(v) + \bar{c}_x c_y(v) + \bar{d}_x d_y(v) + \bar{e}_x e_y(v) + \bar{f}_x f_y(v).
\end{aligned} \tag{24}$$

Given seven point pairs $(x_i, y_i)_{i=1}^7$, let $g_x^{\hat{i}}$ and $g_y^{\hat{i}}$ denote the above cubic polynomials obtained from the point pairs $(x_j, y_j)_{j \neq i}$.

The polynomials g_x, g_y played a prominent role in the rank drop of Z_6 in [3].

Lemma 4.8. *Given generic point pairs $(x_i, y_i)_{i=1}^6$, let $C = \mathcal{N}_Z \cap \mathcal{D}$, $C_x = \kappa_x(C) \subset \mathbb{P}_x^2$ and $C_y = \kappa_y(C) \subset \mathbb{P}_y^2$. Also let S_x be the blowup of $\mathbb{P}_x^2$ at $x_1, \dots, x_6$ and let S_y be the blowup of $\mathbb{P}_y^2$ at $y_1, \dots, y_6$, each expressed in Cremona hexahedral form. Then the following hold true:*

(1) *The plane cubic curves C_x and C_y have defining equations $g_x(u) = 0$ and $g_y(v) = 0$ respectively.*

(2) *The cubic curve $C \cong S_x \cap S_y$ which has equations*

$$\begin{aligned} z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 &= 0 \\ z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0 \\ \bar{a}_x z_1 + \bar{b}_x z_2 + \bar{c}_x z_3 + \bar{d}_x z_4 + \bar{e}_x z_5 + \bar{f}_x z_6 &= 0 \\ \bar{a}_y z_1 + \bar{b}_y z_2 + \bar{c}_y z_3 + \bar{d}_y z_4 + \bar{e}_y z_5 + \bar{f}_y z_6 &= 0. \end{aligned} \tag{25}$$

(3) *The cubic curve $S_x \cap S_y$ is the image of C_x under the blowup of $\mathbb{P}_x^2$ at $x_1, \dots, x_6$ and also the image of C_y under the blowup of $\mathbb{P}_y^2$ at $y_1, \dots, y_6$.*

Proof. We begin with the first item. By Lemma 4.4, $x_i \in C_x$ for all i and by Definition 4.7, $g_x(x_i) = 0$ for all i since the cubic polynomials in (21) vanish on the x_i. For fixed $i = 1, \dots, 6$, consider the 5 point pairs left after excluding (x_i, y_i) and let (u_i, v_i) be the unique new point pair (cf. Lemma 6.1 in [3]) such that the configuration

$$\{(x_1, y_1), \dots, (x_6, y_6), (u_i, v_i)\} \setminus \{(x_i, y_i)\} \tag{26}$$

is rank deficient. For convenience, we assume without loss of generality that $i = 6$. In other words, if $Z_{\hat{6}} = (x_i \otimes y_i)_{i=1}^5$ then (u_6, v_6) is the unique point pair such that $S_{\hat{6}} = \mathcal{N}_{Z_{\hat{6}}} \cap \mathcal{D}$ can be obtained both by blowing up $\mathbb{P}_x^2$ in the points $x_1, \dots, x_5, u_6$ and by blowing up $\mathbb{P}_y^2$ in the points $y_1, \dots, y_5, v_6$. It follows that the curve $C \subset S_{\hat{6}}$ cuts the exceptional lines corresponding to u_6, v_6 exactly once each and therefore $u_6 \in C_x$ and $v_6 \in C_y$; it follows symmetrically that $u_i \in C_x$ and $v_i \in C_y$ for all $i = 1, \dots, 6$. One can check using a computer algebra package that $g_x(u_6) = 0$ and $g_y(v_6) = 0$ after fixing points as in Lemma 6.1 in [3]; it follows symmetrically that $g_x(u_i) = 0$ and $g_y(v_i) = 0$ for all i. Finally, since C_x and the curve cut out by g_x share 12 distinct points, they must be the same cubic curve; similarly we can conclude that C_y is cut out by g_y. This finishes the proof of the first claim.

To prove the second and third claims, recall that $\kappa_x : C \to C_x$ is an isomorphism. Let $\kappa'_x : S_x \to \mathbb{P}_x^2$ and $\kappa'_y : S_y \to \mathbb{P}_y^2$ be the blow down morphisms. The Cremona hexahedral forms of S_x and S_y give

$$S_x \cap S_y = \{z \in S_x \ : \ \bar{a}_y z_1 + \dots + \bar{f}_y z_6 = 0\}. \tag{27}$$

By (23),

$$S_x = \overline{\{(a_x(u) : b_x(u) : c_x(u) : d_x(u) : e_x(u) : f_x(u)) \ : \ u \in \mathbb{P}^2\}} \tag{28}$$

and since C_x is cut out by $g_x(u) = 0$, we get that

$$S_x \cap S_y = \overline{\{(a_x(u) : \dots : f_x(u)) : \bar{a}_y a_x(u) + \dots + \bar{f}_y f_x(u) = 0, u \in \mathbb{P}_x^2\}} = \overline{\{(a_x(u) : \dots : f_x(u)) : u \in C_x\}}. \tag{29}$$

Therefore, $S_x \cap S_y$ is exactly the image of C_x under the blowup of $\mathbb{P}_x^2$ at $x_1, \dots, x_6$. Restricting κ_x to $\kappa'_x|_{S_x \cap S_y}$: $S_x \cap S_y \to C_x$ we obtain an isomorphism, and we have $S_x \cap S_y \cong C_x \cong C$, which proves the second claim. Finally, we note that by a symmetric argument, $S_x \cap S_y$ is also exactly the image of C_y under the blowup of $\mathbb{P}_y^2$ at $y_1, \dots, y_6$ proving the third claim as well. □

Example 4.9 (Example 4.1, continued). One can verify that the polynomials (24) define the same cubic curves as those in Example 4.1. We then pick a specific point $x_7 = (0 : 1403 : 118) \in C_x$. Using a computer algebra package, one can compute the unique point $y_7 = (1802855 : 1562942 : 171287)$ such that $Z = (x_i, y_i)_{i=1}^7$ is rank deficient. It is straight-forward to verify that $y_7 \in C_y$. Moreover, there is a two-parameter family of Cremona transformations f_ℓ such that $x_i \mapsto y_i$ for $i = 1, \dots, 6$ and for all members of this family $f_\ell(x_7) = (1802855 : 1562942 : 171287)$, which lines up with Lemma 4.6. These points can be seen on the cubic curve in Figure 2.

4.2.2 Algebraic conditions for the rank deficiency of Z_7. We are now ready to present our main algebraic result for rank drop given $k = 7$ point pairs. We begin with a basic lemma that will connect all of our results in the main theorem.

Lemma 4.10. *Let $(x_i, y_i)_{i=1}^7$ be seven semi-generic points. Then $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient if and only if $C^{\hat{1}} = \dots = C^{\hat{7}}$ where $C^{\hat{i}}$ is the cubic curve $\mathcal{N}_{Z_{\hat{i}}} \cap \mathcal{D}$.*

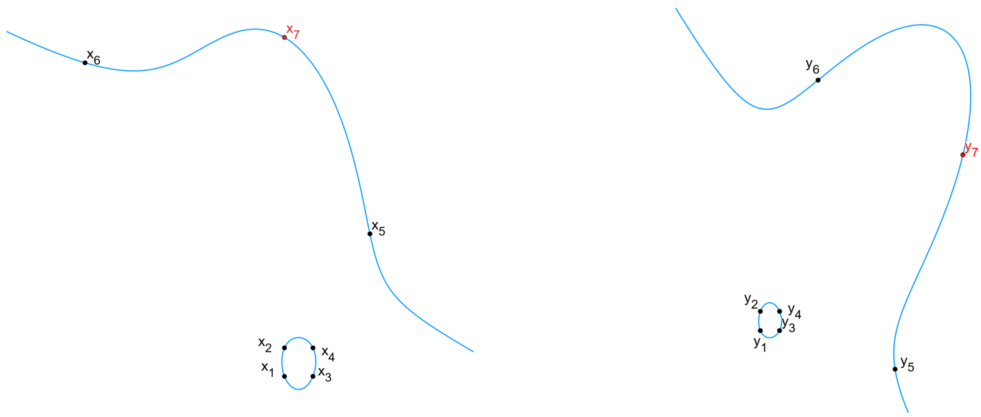

Figure 2: The cubic curves C_x and C_y, with x_7 and y_7 highlighted.

Proof. By semi-genericity, Z is rank deficient if and only if $\mathcal{N}_Z = \mathcal{N}_{Z_{\hat{1}}} = \cdots = \mathcal{N}_{Z_{\hat{7}}}$ for each 6×9 submatrix $Z_{\hat{i}}$ of Z. Since $C^{\hat{i}} = \mathcal{N}_{Z_{\hat{i}}} \cap \mathcal{D}$, the matrix Z is rank deficient if and only if $C = C^{\hat{1}} = \cdots = C^{\hat{7}}$. □

The following theorem, which is the main result of this subsection, allows us to check for rank drop without computing Cremona transformations.

Theorem 4.11. *For seven semi-generic point pairs $(x_i, y_i)_{i=1}^7$, the following are equivalent:*

(1) *$Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient.*

(2) *We have $x_i \in C_x^{\hat{i}}$ and $y_i \in C_y^{\hat{i}}$ for all $i = 1, \ldots, 7$.*

(3) *We have $g_x^{\hat{i}}(x_i) = 0$ and $g_y^{\hat{i}}(y_i) = 0$ for all $i = 1, \ldots, 7$.*

(4) *All seven cubic curves in $\mathbb{P}_x^2$ are equal: $C_x^{\hat{7}} = \cdots = C_x^{\hat{1}}$.*

(5) *All seven cubic curves in $\mathbb{P}_y^2$ are equal: $C_y^{\hat{7}} = \cdots = C_y^{\hat{1}}$.*

Proof. By Lemma 4.8, (2) is equivalent to (3). We next prove that (1) implies (4) and (5). If Z is rank deficient, then $C^{\hat{1}} = \cdots = C^{\hat{7}}$ by Lemma 4.10. Applying the quadratic maps κ_x and κ_y we obtain (4) and (5). To prove the reverse direction we will show (4) implies (1); the proof that (5) implies (1) is symmetric. In particular, we will show that $C_x^{\hat{i}} = C_x^{\hat{j}}$ if and only if $C^{\hat{i}} = C^{\hat{j}}$. For ease of notation, we assume $i = 6$ and $j = 7$. Consider the five point pairs $(x_i, y_i)_{i=1}^5$ and the matrix $Z_5 = (x_i^\top \otimes y_i^\top)_{i=1}^5$. Then $S = \mathcal{N}_{Z_5} \cap \mathcal{D}$ is a cubic surface and $\kappa_x : S \to \mathbb{P}_x^2$ and $\kappa_y : S \to \mathbb{P}_y^2$ are $1:1$ except on the six exceptional lines in each case. Moreover, we can obtain the cubic curves $C^{\hat{6}}$ and $C^{\hat{7}}$ by intersecting this surface with a plane. We can conclude that $\kappa_x(C^{\hat{6}}) = \kappa_x(C^{\hat{7}})$ only if $C^{\hat{6}} = C^{\hat{7}}$. It then follows that (4) implies (1), and symmetrically, (5) implies (1).

We now prove that (1) implies (2). Fix $i \in \{1, \ldots, 7\}$. Then $x_j \in C_x^{\hat{i}}$ for all $j \neq i$ by Lemma 4.4. Moreover, since $C_x^{\hat{i}} = C_x^{\hat{j}}$ by hypothesis it follows that $x_i \in C_x^{\hat{i}}$. The other equalities follow symmetrically.

Finally, we prove that (2) implies (1). Since $x_j \in C_x^{\hat{i}}$ and $y_j \in C_y^{\hat{i}}$ for $j \neq i$ by construction, the additional hypothesis (2) gives that $x_1, \ldots, x_7 \in \bigcap_{i=1}^7 C_x^{\hat{i}}$ and $y_1, \ldots, y_7 \in \bigcap_{i=1}^7 C_y^{\hat{i}}$. We fix the first five point pairs $(x_i, y_i)_{i=1}^5$ and consider the 5×9 matrix $Z_5 = (x_i^\top \otimes y_i^\top)_{i=1}^5$. Consider the cubic surface $S = \mathcal{N}_{Z_5} \cap \mathcal{D}$ paired with the maps κ_x and κ_y. The cubic curves $C^{\hat{6}}$ and $C^{\hat{7}}$ are obtained by intersecting S with a plane. By genericity, the four matrices $\kappa_x^{-1}(x_6)$, $\kappa_x^{-1}(x_7)$, $\kappa_y^{-1}(y_6)$, $\kappa_y^{-1}(y_7)$ are all distinct. Moreover, they are all contained in

$$C^{\hat{6}} \cap C^{\hat{7}} = (\mathcal{N}_{Z^{\hat{6}}} \cap \mathcal{D}) \cap (\mathcal{N}_{Z^{\hat{7}}} \cap \mathcal{D}) = \mathcal{N}_Z \cap \mathcal{D} \tag{30}$$

which can also be realized as the intersection of the cubic surface S with two planes. If $\mathcal{N}_Z$ were one-dimensional, it would intersect $\mathcal{D}$ in at most three points. Since we have found $4 > 3$ distinct points in $\mathcal{N}_Z \cap \mathcal{D}$, $\mathcal{N}_Z$ must have projective dimension ≥ 2, implying (1). □

4.3 Beyond semi-genericity

Given seven semi-generic point pairs $(x_i, y_i)_{i=1}^7$, we have now fully characterized the conditions under which the matrix Z_7 will be rank deficient. This characterization was given geometrically (Theorem 4.2) and then algebraized using 14 polynomials (Theorem 4.11). We now move away from the assumptions of semi-genericity. We will first examine how Z_7 becomes rank deficient without these assumptions and, to some extent, generalize our algebraic condition (Theorem 4.11) to this case. We will also consider configurations where $(x_i, y_i)_{i=1}^7$ are fully generic, and therefore Z_7 must have full rank; in this case, we can use the cubic curves $C_x^{\hat{i}}$, $C_y^{\hat{i}}$ and their associated polynomials to characterize the epipoles of the possible fundamental matrices in terms of classical invariants.

We begin by presenting two relatively simple, but highly degenerate, conditions for the rank deficiency of Z_7. One of these conditions is that Z_7 will be rank deficient if $\{x_i\}$ and $\{y_i\}$ are equal up to a change of coordinates.

Lemma 4.12. *Suppose we have point pairs $(x_i, y_i)_{i=1}^7$ and an invertible projective transformation H such that $Hx_i = y_i$ for all i. Then $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient.*

Proof. Since rank drop is a projective invariant, we can assume $x_i = y_i$ for all i. Then the equations $y_i^\top F x_i = x_i^\top F x_i = 0$, $i = 1, \dots, 7$ hold for all 3×3 skew-symmetric matrices $F \in \mathrm{Skew}_3$. Since Skew_3 is a three-dimensional vector space, $\dim(\mathcal{N}_Z) \geq 3$ and $\mathrm{rank}(Z) \leq 9 - 3 = 6$. □

The second simple condition is that the rank of Z will drop if the points in either $\mathbb{P}^2$ lie in a line.

Lemma 4.13. *Suppose $(x_i, y_i)_{i=1}^7$ is such that either $\{x_i\}$ or $\{y_i\}$ are on a line. Then $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient.*

Proof. Suppose the y_i's are on a line. Then we may assume that $y_i = (m_i, 0, 1)$ after a change of coordinates. Then simple column operations on Z show that it is rank deficient. □

Remark 4.14. We note that the existence of such configurations does not necessarily imply that the rank drop variety is reducible. We suspect that these configurations are in the Zariski closure of the generic rank drop component.

It is simple to check that in both of the above cases we have $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for $i = 1, \dots, 7$, suggesting a possible generalization of Theorem 4.11(3). This is possible to some extent. In particular, even without any genericity assumptions, if Z_7 is rank deficient then these 14 polynomial equations hold.

Lemma 4.15. *If $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient, then $g_x^{\hat{i}}(x_i) = 0$ and $g_y^{\hat{i}}(y_i) = 0$ for all i.*

Proof. Let I be the ideal generated by the 14 polynomials $g_x^{\hat{i}}(x_i)$ and $g_y^{\hat{i}}(y_i)$ for $i = 1, \dots, 7$ in the polynomial ring $\mathbb{C}[x_{ij}, y_{ij} : i = 1, \dots, 7, j = 1, 2, 3]$, treating $(x_i, y_i)_{i=1}^7$ as symbolic. If Z is the appropriate symbolic 7×9 matrix then it can be verified using Macaulay2 that I is contained in the ideal generated by the maximal minors of Z. □

However, the converse does not hold in general. We present two examples of highly degenerate configurations where the 14 equations hold, but Z_7 is not rank deficient.

Example 4.16. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 0 & 1 & 3 & 4 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 1 & 4 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{31}$$

where x_1, x_2, x_3, x_4, x_7 are on a line and $x_5 = x_6$. Similarly, y_1, y_2, y_3, y_5, y_6 are on a line and $y_4 = y_7$. We can verify that $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for $i = 1, \dots, 7$ and that the matrix Z is not rank deficient. In particular, $\mathcal{N}_Z$ is spanned by the two singular matrices

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the latter of which has rank one.

Example 4.17. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 1 & 2 & 5 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 1 & 5 & 1 & 0 & 2 & 6 & 8 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{32}$$

where $\{x_i\}_{i=1}^4, \{y_i\}_{i=1}^4$ and $\{x_i\}_{i=5}^7, \{y_i\}_{i=5}^7$ are on distinct lines in each image. We can verify that $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for $i = 1, \dots, 7$ and that the matrix Z is not rank deficient. In particular, $\mathcal{N}_Z$ is spanned by the two rank one matrices

$$\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

While the focus of this paper has been on the conditions under which Z drops rank, the tools we have developed have applications beyond rank drop. In particular, for a fully generic configuration of seven point pairs we can use the cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$ to find the possible epipoles of fundamental matrices. While this has minimal practical application, it is significant in that the characterization is entirely in terms of classical projective invariants.

Lemma 4.18. *Let $(x_i, y_i)_{i=1}^7$ be generic point pairs. In particular, we assume that $\mathcal{N}_Z$ is one-dimensional and contains three rank-two matrices F_1, F_2, F_3, two of which may be complex. Then the epipoles of these fundamental matrices e_1^x, e_2^x, e_3^x and e_1^y, e_2^y, e_3^y can be obtained as the unique three points in the intersections $\bigcap_{i=1}^7 C_x^{\hat{i}} \subset \mathbb{P}_x^2$ and $\bigcap_{i=1}^7 C_y^{\hat{i}} \subset \mathbb{P}_y^2$.*

Proof. Consider the two cubic curves $C_x^{\hat{7}}$ and $C_x^{\hat{6}}$. The intersection $A_{6,7} = C_x^{\hat{7}} \cap C_x^{\hat{6}}$ will contain exactly nine points. We know that $x_1, \dots, x_5 \in A_{6,7}$. Additionally, let (u_6, v_6) be the pair of rank drop points, as in Lemma 5.1 of [3], associated to $(x_i, y_i)_{i=1}^5$. Then, by Lemma 4.15 we have $u_6 \in A_{6,7}$ as well. There should be three more points in the intersection. Let f be the unique Cremona transformation $f: \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $x_i \mapsto y_i$ for $i = 1, \dots, 7$. This f is contained in the two-parameter family of Cremona transformations $\mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $x_i \mapsto y_i$ for $i = 1, \dots, 6$. By Lemma 4.4 the base points of f are contained in $C_x^{\hat{7}}$. By a symmetric argument these base points are also contained in $C_x^{\hat{6}}$ and we can conclude that these three base points are the last three points in the intersection. By Lemma 3.9 these base points are exactly the epipoles of the fundamental matrices, and it follows by symmetry that $e_x^1, e_x^2, e_x^3 \in \bigcap_{i=1}^7 C_x^{\hat{i}}$. Clearly the points $x_1, \dots, x_5, u_6$ are not in $\bigcap_{i=1}^7 C_x^{\hat{i}}$ generically, and thus these three base points are the unique points in the intersection of all seven cubic curves. Symmetrically, e_1^y, e_2^y, e_3^y are the unique points in $\bigcap_{i=1}^7 C_y^{\hat{i}}$. □

Example 4.19. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 3 & 2 & 5 & 0 & 4 & -20 & -4 \\ 0 & 7 & 3 & 3 & 2 & 25 & 7 \\ 1 & 1 & 2 & 1 & 5 & 12 & 2 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -49 & -15 & -3 & -5 & 5 & 7 \\ -1 & 14 & 25 & 0 & 10 & 4 & 4 \\ 1 & 9 & 4 & 1 & 6 & 2 & 1 \end{bmatrix} \tag{33}$$

We can then construct the seven cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$ in each $\mathbb{P}^2$. See Figure 3. Each set of seven cubic curves has three common intersection points. If we compute $\mathcal{N}_Z$ we find that there are exactly three possible real fundamental matrices. These matrices have epipoles

$$\begin{aligned} e_x^1 &= (0:0:1) & e_y^1 &= (0:0:1) \\ e_x^2 &= (-2:3:1) & e_y^2 &= (-3:4:1) \\ e_x^3 &= (4:3:4) & e_y^3 &= (3:2:2) \end{aligned} \tag{34}$$

and we can see that these are exactly the three common intersection points.

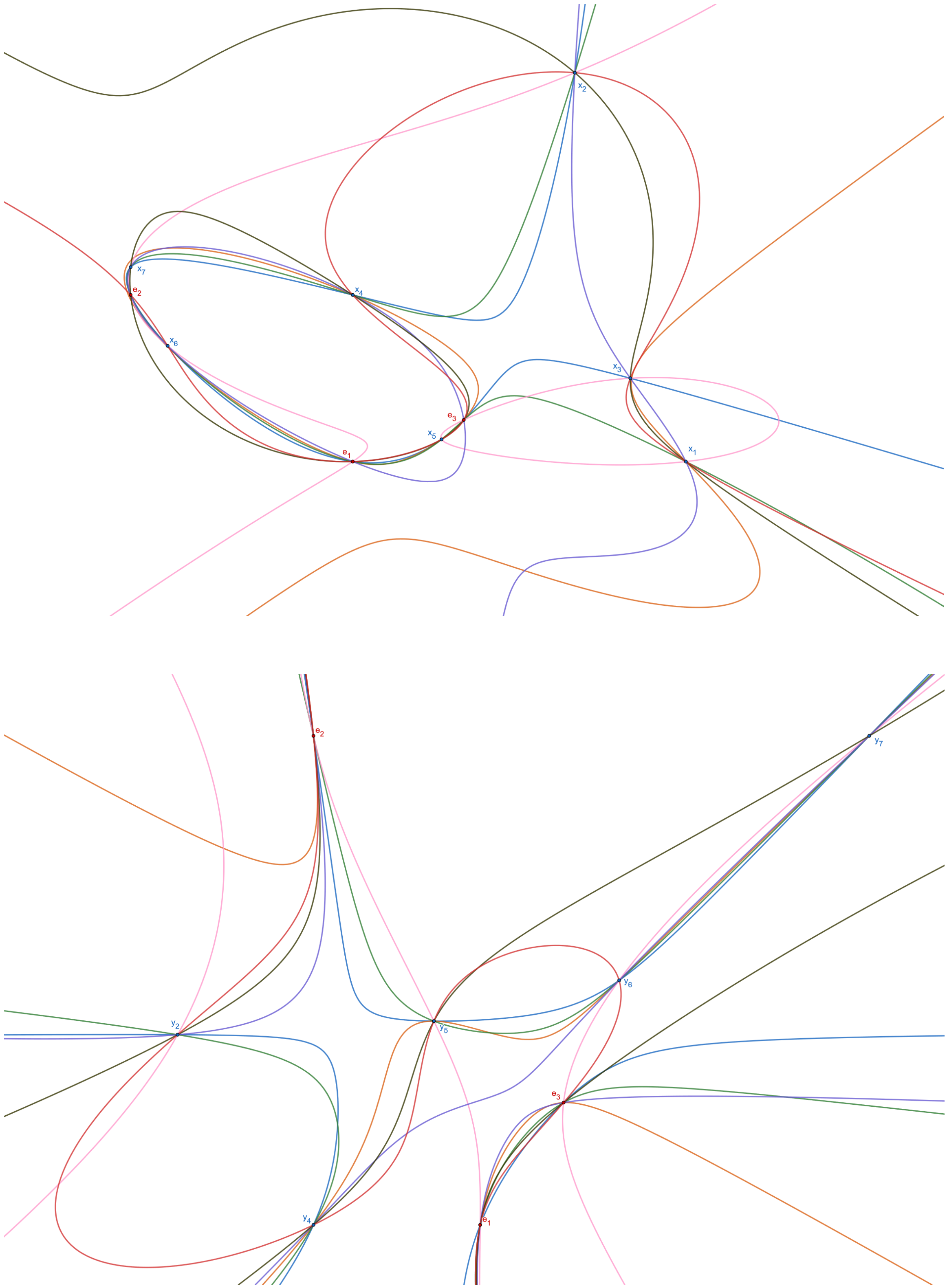

Figure 3: The cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$. The intersection points are exactly the three possible epipoles associated to the fundamental matrices.

5 The case $k = 9$

We finish by characterizing the rank deficiency of $Z = (x_i^\top \otimes y_i^\top)_{i=1}^9$, and this time we make no assumptions on the point pairs $(x_i, y_i)_{i=1}^9$. A simple algebraic characterization of rank drop in this case is that $\det(Z) = 0$. This is a single polynomial equation but as mentioned already, typically this equation does not shed much light on the geometry of the points $\{x_i\}$ and $\{y_i\}$ that makes Z rank deficient. By the methods of invariant theory, it is possible to write $\det(Z)$ as a polynomial in the brackets $[ijk]_x$ and $[ijk]_y$ constructed from $\{x_i\}$ and $\{y_i\}$ which may or may not offer geometric insight. Below we provide a geometric characterization of rank drop in terms of the two point sets in $\mathbb{P}_x^2$ and $\mathbb{P}_y^2$. The result is straight-forward.

Recall that if a, b are distinct points in $\mathbb{P}^2$, then $a \times b \in \mathbb{P}^2$ is the normal of the line containing a and b, i.e., $u \in \operatorname{Span}\{a, b\}$ if and only if $u^\top(a \times b) = 0$. In what follows we let ℓ_{ab} denote the line spanned by a, b. Its normal $a \times b = [a]_\times b$ where $[a]_\times$ is the 3×3 skew symmetric matrix that expresses cross products with a as a matrix-vector multiplication.

Theorem 5.1. *The matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^9$ is rank deficient if and only if there is a projective transformation $T : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $y_i^\top(Tx_i) = 0$ for $i = 1, \dots, 9$, or equivalently, y_i lies on the line with normal vector Tx_i for $i = 1, \dots, 9$. This manifests in three possible ways depending on the rank of T:*

(1) *There exists a line $\ell \subset \mathbb{P}_x^2$ and a line $\ell' \subset \mathbb{P}_y^2$ such that for each i, we have $x_i \in \ell$ or $y_i \in \ell'$ (both may happen for a given i).*

(2) *There are two points $e \in \mathbb{P}_x^2$ and $e' \in \mathbb{P}_y^2$ and a $\mathbb{P}^1$-homography sending the pencil of lines through e to the pencil of lines through e' such that $\ell_{ex_i} \mapsto \ell_{e'y_i}$ for each i.*

(3) *There is some $T \in \mathrm{PGL}(3)$ such that y_i lies on the line with normal vector Tx_i for each i.*

Proof. The first statement is trivial. The matrix Z is rank deficient if and only if $\mathcal{N}_Z \subset \mathbb{P}^8$ contains at least one point. Representing such a point by $T \in \mathbb{P}(\mathbb{C}^{3\times3})$ we have $(x_i^\top \otimes y_i^\top)\operatorname{vec}(T) = y_i^\top(Tx_i) = 0$ for $i = 1, \dots, 9$.

(1) If $\operatorname{rank}(T) = 1$, then $T = uv^\top$ for some $u, v \in \mathbb{C}^3$. Therefore, $(y_i^\top u)(v^\top x_i) = 0$ for $i = 1, \dots, 9$ which is equivalent to saying that for each i, at most one of $u^\top y_i$ or $v^\top x_i$ can be non-zero. Therefore there exist lines ℓ (with normal v) and ℓ' (with normal u) such that for each i, we have $x_i \in \ell$ or $y_i \in \ell'$.

(2) Suppose that $\operatorname{rank}(T) = 2$. Let $e \in \mathbb{P}_x^2$ be the unique point in the right nullspace of T and let $e' \in \mathbb{P}_y^2$ be the unique point in the left nullspace of T. The pencil of all lines through e (respectively e') can be identified with $\mathbb{P}^1$.
Pick any line ℓ not passing through e and suppose its normal is n. Then the projective transformation $T[n]_\times$ is a $\mathbb{P}^1$-homography that takes $\ell_{ex_i} \to \ell_{e'y_i}$; see [6, Result 9.5]. Indeed, suppose the intersection of ℓ and ℓ_{ex_i} is u_i. Since u_i is orthogonal to both n and $e \times x_i$, we have $u_i \sim n \times (e \times x_i) = [n]_\times(e \times x_i)$. Since u_i lies on ℓ_{ex_i}, we have $u_i = \lambda e + \mu x_i$ for some scalars λ, μ, and since ℓ does not contain e, we obtain $u_i \neq e$ which implies that $\mu \neq 0$. Therefore

$$T[n]_\times(e \times x_i) = Tu_i = \lambda Te + \mu Tx_i = 0 + \mu Tx_i \sim Tx_i$$

which says that the normal of ℓ_{ex_i} is mapped to Tx_i by $T[n]_\times$. We just need to argue that Tx_i is the normal of $\ell_{e'y_i}$ to finish the proof. For this check that $(e')^\top Tx_i = 0$ since $(e')^\top T = 0$ and $y_i^\top Tx_i = 0$ by assumption. Therefore the line spanned by e' and y_i has normal Tx_i.

(3) If $\operatorname{rank}(T) = 3$ then T is a homography (an invertible projective transformation). Then $y_i^\top Tx_i = 0$ for $i = 1, \dots, 9$ implies that y_i lies on the line with normal Tx_i for each i. □

Remark 5.2. In the proof of (2), if $x_i = e$ for some i then $[e]_\times e = 0$ and similarly, if $y_j = e'$ for some j then $[e']_\times y_j = 0$. Therefore, the $\mathbb{P}^1$-homography will not work for the indices i, j where $x_i = e$ or $y_j = e'$.

Remark 5.3. As we saw, if seven of the nine points on either side are on a line then the rank of Z_9 will drop. Condition (1) allows for the situations where s points with $3 \le s \le 6$ on one side are on a line and the $9 - s$ complementary y points are on a line.

Example 5.4. (1) Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 3 \end{bmatrix}. \tag{35}$$

One can check that all 8×9 submatrices of Z have rank 8. If the coordinates of $\mathbb{P}^2$ are u_1, u_2, u_3 then $x_1, \dots, x_4$ lie on the line $u_1 = 0$ and $y_5, \dots, y_9$ lie on the line $u_2 = 0$ and Z must drop rank by Condition (1). Indeed, the unique element in the nullspace of Z is the rank-one matrix

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{36}$$

(2) Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 3 \end{bmatrix}. \tag{37}$$

Again, Z and all its 8×9 submatrices have rank 8. The unique element in $\mathcal{N}_Z$ is the rank-two matrix

$$T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}. \tag{38}$$

The points $e = e' = (1, 1, 0)^\top$ are generators of the right and left nullspaces of T. Note that $x_5 = e$ and $y_7 = e'$. Pick $\bar{\ell} = (1, 2, 3)^\top$. Then $e^\top \bar{\ell} \neq 0$. Now check that $[e']_\times Y = (T[\bar{\ell}]_\times)[e]_\times X$. Indeed,

$$[e]_\times = [e']_\times = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \qquad [\bar{\ell}]_\times = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix},$$

$$\text{and} \qquad [e']_\times Y = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -3 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 0 & 0 & 3 & 3 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & -3 & -3 & 0 & -3 & -3 & -3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 3 & -3 & -3 & 3 \end{bmatrix} = (T[\bar{\ell}]_\times)[e]_\times X \tag{39}$$

except in the columns of X and Y where $x_i = e$ and $y_j = e'$.

Here is another example where the epipoles do not appear among the x_i's or y_j's. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & -1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 & 2 & -1 & -1 \end{bmatrix}. \tag{40}$$

The unique element in $\mathcal{N}_Z$ is the rank-two matrix

$$T = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}. \tag{41}$$

The points $e = (-1, 1, -2)^\top$ and $e' = (1, 2, -1)^\top$ generate the right and left nullspaces of T. Pick $\bar{\ell} = e$. Then $e^\top e \neq 0$. Now check that $[e']_\times Y = (T[e]_\times)[e]_\times X$. Indeed,

$$[e']_\times Y = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & -1 & -4 \\ -1 & -1 & -2 & -2 & -3 & -2 & -3 & 0 & 0 \\ -2 & -1 & -2 & -1 & -2 & 0 & -1 & -1 & -4 \end{bmatrix}$$
$$\sim \begin{bmatrix} 0 & -12 & -6 & -18 & -24 & -12 & -30 & 6 & 18 \\ 6 & 12 & 6 & 12 & 18 & 6 & 18 & 0 & 0 \\ 12 & 12 & 6 & 6 & 12 & 0 & 6 & 6 & 18 \end{bmatrix} = (T[e]_\times)[e]_\times X. \tag{42}$$

(3) Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y with

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 15 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -5 \end{bmatrix}. \tag{43}$$

The unique element in $\mathcal{N}_Z$ is the rank-three matrix

$$T = \begin{bmatrix} 0 & 1 & -4 \\ 1 & 0 & 3 \\ -4 & 3 & 0 \end{bmatrix}. \tag{44}$$

By construction, $y_i^\top T x_i = 0$ for $i = 1, \dots, 9$.

6 Conclusion

In combination with [3], we now have a complete characterization of how rank deficiency of the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ occurs for all values of $k = 2, \dots, 9$. We have also demonstrated a strong correspondence between lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$, quadric surfaces in $\mathbb{P}^3$, and quadratic Cremona transformations of $\mathbb{P}^2$ under appropriate genericity assumptions, which we have named the *trinity correspondence*. We conclude with a simple corollary of our work that highlights the geometry of reconstructions of semi-generic point pairs of sizes six, seven and eight.

Corollary 6.1. *Let $(x_i, y_i)_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$ be semi-generic. Then we get the following:*

- *When $k = 6$, Z_6 is rank deficient exactly when a reconstruction $p_1, \dots, p_6, c_1, c_2$ is a Cayley octad (eight points in the intersection of three generic quadrics).*
- *When $k = 7$, Z_7 is rank deficient exactly when the points $p_1, \dots, p_7, c_1, c_2$ of any reconstruction lie on a quartic curve that arises as the intersection of two quadrics.*
- *When $k = 8$, Z_8 is rank deficient exactly when the points $p_1, \dots, p_8, c_1, c_2$ of any reconstruction lie on a quadric.*

Proof. When $k = 8$, the matrix Z_8 is rank deficient exactly when $\mathcal{N}_{Z_8}$ is a line. By the semi-genericity of the point pairs, this line is P-generic and does not contain any rank-one matrices. Any reconstruction of the point pairs corresponds to a fundamental matrix F on this line, and by Lemma 3.3 the reconstruction lies on a quadric. Similarly, if the point pairs have a reconstruction, given by some fundamental matrix F which lies on a quadric, then there is a corresponding line through F in $\mathcal{N}_{Z_8}$ and Z_8 is rank deficient.

When $k = 7$, Z_7 is rank deficient exactly when $\mathcal{N}_{Z_7}$ is a plane. Given any reconstruction $p_1, \dots, p_7, c_1, c_2$ of the point pairs, let F be the corresponding fundamental matrix. By semi-genericity of the point pairs, $\mathcal{N}_{Z_7}$ is a generic plane that intersects $\mathcal{D}$ in a curve C of rank-two matrices. If we take any two lines through F in $\mathcal{N}_{Z_7}$ then as in Lemma 3.3 we obtain two quadrics Q_1, Q_2 whose intersection is a quartic curve through the reconstruction. Similarly, if any reconstruction corresponding to a fundamental matrix F' lies on two distinct quadrics then there are two distinct lines through F' in $\mathcal{N}_{Z_7}$ and Z_7 is rank deficient.

For $k = 6$, Z_6 is rank deficient if and only if $\mathcal{N}_{Z_6}$ is a 3-dimensional plane. Equivalently, every rank-two matrix $F \in \mathcal{N}_{Z_6}$ lies on a net of lines in $\mathcal{N}_{Z_6}$, which corresponds to a net of quadrics containing the reconstruction corresponding to F. It follows that if the reconstruction lies on a Cayley octad $Q_1 \cap Q_2 \cap Q_3$ then Z_6 is rank deficient. For the other direction, suppose that Z_6 is rank deficient. Then the reconstruction lies on a net of quadrics $Q_1 \cap Q_2 \cap Q_3$ and we need to show that this intersection contains exactly the 8 points $\{p_i\}_{i=1}^6, c_1, c_2$. If $p' \in Q_1 \cap Q_2 \cap Q_3$ is any point distinct from c_1, c_2, then $\pi_2(p')^\top M \pi_1(p') = 0$ for all $M \in \mathcal{N}_{Z_6}$. Due to semi-genericity, the hypothesis of [3, Lemma 6.1] holds for any subset of 5 point pairs, and it follows that $(\pi_1(p'), \pi_2(p')) = (x_i, y_i)$ for some i. We can conclude that $p' = p_i$ and the intersection is indeed a Cayley octad. □

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References

[1] M. Bråtelund, Critical configurations for two projective views, a new approach. *J. Symbolic Comput.* **120** (2024), Paper No. 102226, 22 pages. MR4583113 Zbl 07725351

[2] A. B. Coble, Point sets and allied Cremona groups. I. *Trans. Amer. Math. Soc.* **16** (1915), 155–198. MR1501008 Zbl 02618070

[3] E. Connelly, S. Agarwal, A. Ergur, R. Thomas, The geometry of rank drop in a class of face-splitting matrix products: Part I. *Adv. Geom.* **24** (2024), 369–394.

[4] J. Diller, Cremona transformations, surface automorphisms, and plane cubics. *Michigan Math. J.* **60** (2011), 409–440. MR2825269 Zbl 1244.14012

[5] R. Hartley, F. Kahl, Critical configurations for projective reconstruction from multiple views. *International Journal of Computer Vision* **71** (2007), 5–47. Zbl 1477.68366

[6] R. Hartley, A. Zisserman, *Multiple view geometry in computer vision*. Cambridge Univ. Press 2003. MR2059248 Zbl 1072.68104

[7] H. L. Lee, On the existence of a projective reconstruction. Preprint 2020, arXiv:1608.05518

[8] I. R. Shafarevich, *Basic algebraic geometry* 1. Springer 2013. MR3100243 Zbl 1273.14004