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Inequalities for f^*-vectors of lattice polytopes

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Abstract: The Ehrhart polynomial $\mathrm{ehr}_P(n)$ of a lattice polytope P counts the number of integer points in the n-th dilate of P. The f^*-vector of P, introduced by Felix Breuer in 2012, is the vector of coefficients of $\mathrm{ehr}_P(n)$ with respect to the binomial coefficient basis $\{\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{d}\}$, where $d = \dim P$. Similarly to h/h^*-vectors, the f^*-vector of P coincides with the f-vector of its unimodular triangulations (if they exist). We present several inequalities that hold among the coefficients of f^*-vectors of lattice polytopes. These inequalities resemble striking similarities with existing inequalities for the coefficients of f-vectors of simplicial polytopes; e.g., the first half of the f^*-coefficients increases and the last quarter decreases. Even though f^*-vectors of polytopes are not always unimodal, there are several families of polytopes that carry the unimodality property. We also show that for any polytope with a given Ehrhart h^*-vector, there is a polytope with the same h^*-vector whose f^*-vector is unimodal.

Keywords: Lattice polytope, Ehrhart polynomial, Gorenstein polytope, f^*-vector, h^*-vector, unimodality.

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1 Introduction

For a d-dimensional *lattice polytope* $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Z}^d$) and a positive integer n, let $\mathrm{ehr}_P(n)$ denote the number of integer lattice points in nP. Ehrhart's famous theorem [11] says that $\mathrm{ehr}_P(n)$ evaluates to a polynomial in n. Similar to the situations with other combinatorial polynomials, it is useful to express $\mathrm{ehr}_P(n)$ in different bases; here we consider two such bases consisting of binomial coefficients:

$$\mathrm{ehr}_P(n) = \sum_{k=0}^{d} h_k^* \binom{n+d-k}{d} = \sum_{k=0}^{d} f_k^* \binom{n-1}{k}. \tag{1}$$

We call $(f_0^*, f_1^*, \ldots, f_d^*)$ the f^*-*vector* and $(h_0^*, h_1^*, \ldots, h_d^*)$ the h^*-*vector* of P. Stanley [15] proved that the h^*-vector of any lattice polytope is nonnegative (whereas the coefficients of $\mathrm{ehr}_P(n)$ written in the standard monomial basis can be negative). Breuer [9] proved that the f^*-vector of any lattice polytopal complex is nonnegative (whereas the h^*-vector of a complex can have negative coefficients); his motivation was that various combinatorially-defined polynomials can be realized as Ehrhart polynomials of complexes and so the nonnegativity of the f^*-vector yields a strong constraint for these polynomials. The f^*- and h^*-vector can also be defined through the *Ehrhart series* of P:

$$\mathrm{Ehr}_P(z) := 1 + \sum_{n\geq 1} \mathrm{ehr}_P(n) z^n = \frac{\sum_{k=0}^{d} h_k^* z^k}{(1-z)^{d+1}} = 1 + \sum_{k=0}^{d} f_k^* \left(\frac{z}{1-z}\right)^{k+1}.$$

It is thus sometimes useful to add the definition $f_{-1}^* := 1$. The polynomial $\sum_{k=0}^{d} h_k^* z^k$ is the h^*-*polynomial* of P, and its degree is the *degree* of P. For general background on Ehrhart theory, see, e.g., [3]. The f^*- and h^*-vectors

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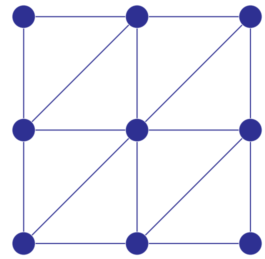

Figure 1: A (regular) unimodular triangulation of the square $[-1, 1]^2$.

share the same relation as f- and h-vectors of polytopes/polyhedral complexes, namely

$$\sum_{k=0}^{d} h_k^* z^k = \sum_{k=0}^{d+1} f_{k-1}^* z^k (1-z)^{d-k+1} \tag{2}$$

$$h_k^* = \sum_{j=-1}^{k-1} (-1)^{k-j-1} \binom{d-j}{k-j-1} f_j^* \tag{3}$$

$$f_k^* = \sum_{j=0}^{k+1} \binom{d-j+1}{k-j+1} h_j^* . \tag{4}$$

The (very special) case that P admits a unimodular triangulation yields the strongest connection between f^*/h^*-vectors and f/h-vectors: in this case the f^*/h^*-vector of P equals the f/h-vector of the triangulation, respectively.

Example 1. Let P be the square $[-1, 1]^2$. The unimodular triangulation of P shown in Figure 1 has f-vector $(f_0, f_1, f_2) = (9, 16, 8)$, where f_i counts its i-dimensional faces. Equivalently, $f^*(P) = (9, 16, 8)$, and one easily checks that (1) yields the familiar Ehrhart polynomial $\mathrm{ehr}_P(n) = (2n + 1)^2$.

Example 2. The f^*-vector of a d-dimensional unimodular simplex Δ equals $[\binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d+1}]$, coinciding with the f-vector of Δ considered as a simplicial complex. If we include $f_{-1}^* = 1$, it gives the only instance of a *symmetric* f^*-vector of a lattice polytope P, since the equality $f_{-1}^* = f_d^*$ combined with (4) implies that $h_i^* = 0$ for $1 \le i \le d$.

There has been much research on (typically linear) constraints for the h^*-vector of a given lattice polytope; see, e.g., [16; 17]. On the other hand, f^*-vectors seem to be much less studied, and our goal is to rectify that situation. Our motivating question is how close the f^*-vector of a given lattice polytope is to being *unimodal*, i.e., the f^*-coefficients increase up to some point and then decrease. Our main results are as follows.

Theorem 3. *Let $d \ge 2$ and let P be a d-dimensional lattice polytope. Then*

(a) $f_0^* < f_1^* < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \le f^*_{\lfloor \frac{d}{2} \rfloor}$;

(b) $f^*_{\lfloor \frac{3d}{4} \rfloor} > f^*_{\lfloor \frac{3d}{4} \rfloor + 1} > \cdots > f_d^*$;

(c) $f_k^* \le f_{d-1-k}^*$ *for integer* $0 \le k \le \frac{d-2}{2}$.

Examples 1 and 2 yield cases of polytopes for which the inequalities $f^*_{\lfloor \frac{3d}{4} \rfloor - 1} < f^*_{\lfloor \frac{3d}{4} \rfloor}$ and $f^*_{\lfloor \frac{d}{2} \rfloor} > f^*_{\lfloor \frac{d}{2} \rfloor + 1}$ hold, respectively, and thus the strings of inequalities in (a) and (b) can, in general, not be extended further. We record the following immediate consequence of Theorem 3.

Corollary 4. *Let P be a d-dimensional lattice polytope. Then for $0 \le k \le d$,*

$$f_k^* \ge \min\{f_0^*, f_d^*\}.$$

We remark that one can prove that if P is of degree $d \ge 2$, equal to its dimension, then $f_0^* \le f_d^*$, except when the h^*-vector of P satisfies $h_2^* = \cdots = h_d^* = 1$.

Theorem 5. *The f^*-vector of a d-dimensional lattice polytope, where $1 \le d \le 13$, is unimodal. On the other hand, there exists a 15-dimensional lattice simplex with nonunimodal f^*-vector.*

Even though f^*-vectors are quite different from f-vectors of polytopes, the above results resemble striking similarities with existing theorems on f-vectors. Namely, Björner [5; 6; 7] proved that the f-vector of a simplicial d-polytope satisfies the inequalities in Theorem 3 (with the *s removed, the last coordinate dropped, and $\frac{d-2}{2}$ replaced by $\frac{d-3}{2}$ in (c)). In fact, Björner also showed that in the f-analogue of Theorem 3(b) the decrease starts from $\lfloor \frac{3(d-1)}{4} \rfloor$ instead of $\lfloor \frac{3d}{4} \rfloor$, and that the inequalities in Theorem 3(a) and (b) cannot be further extended, by constructing a simplicial polytope with f-vector that peaks at f_j, for any j with $\lfloor \frac{d}{2} \rfloor \le j \le \lfloor \frac{3(d-1)}{4} \rfloor$.

Corollary 4 compares the entries of the f^*-vector with the minimum between the first and the last entry. Note that a similar relation for f-vectors of polytopes was recently proven by Hinman [14], answering a question of Bárány from the 1990s. (Hinman also proved a stronger result, namely certain lower bounds for the ratios f_k/f_0 and f_k/f_{d-1}.)

The f-analogue of Theorem 5 for simplicial polytopes is again older: Björner [5] showed that the f-vector of any simplicial d-polytope is unimodal for $d \le 15$ (later improved to $d \le 19$ by Eckhoff [10]), and he and Lee [4] produced examples of 20-dimensional simplicial polytopes with nonunimodal f-vectors.

For a special class of polytopes we can increase the range in Theorem 3(b). A lattice polytope P is *Gorenstein of index g* if

- nP contains no interior lattice points for $1 \le n < g$,
- gP contains a unique interior lattice point, and
- $\mathrm{ehr}_P(n-g)$ equals the number of interior lattice points in nP, for $n > g$.

This is equivalent to P having degree $d+1-g$ and a symmetric h^*-vector (with respect to its degree).

Theorem 6. *Let P be a d-dimensional Gorenstein polytope of index g. Then*

$$f^*_{k-1} > f^*_k \quad \text{for} \quad \tfrac{1}{2}\left(d+1+\left\lfloor \tfrac{d+1-g}{2} \right\rfloor\right) \le k \le d.$$

Going even further, for a certain class of polytopes we can prove unimodality of the f^*-vector, a consequence of the following refinement of Theorem 3(b) for polytopes with degree $< \frac{d}{2}$.

Theorem 7. *Let P be a d-dimensional lattice polytope with positive degree $\le s$. Then*

$$f^*_{k-1} > f^*_k \quad \text{for} \quad \lceil \tfrac{d+s}{2} \rceil \le k \le d,$$

unless the degree of P is 0, i.e., P is a unimodular simplex with f^*-vector as in Example 2.

This theorem implies that lattice d-polytopes of degree s satisfying $s^2 - s - 1 \le \frac{d}{2}$ have a unimodal f^*-vector (see Proposition 9 below for details). One family with asymptotically small degree, compared to the dimension, is given by taking iterated pyramids. Given a polytope $P \subset \mathbb{R}^d$, we denote by $\mathrm{Pyr}(P) \subset \mathbb{R}^{d+1}$ the convex hull of P and the $(d+1)$st unit vector. It is well known that P and $\mathrm{Pyr}(P)$ have the same h^*-vector (ignoring an extra 0), and so we conclude:

Corollary 8. *If P is any lattice polytope then $\mathrm{Pyr}^n(P)$ has unimodal f^*-vector for sufficiently large n.*

2 Proofs

We start with a few warm-up proofs which only use the fact that h^*-vectors are nonnegative.

Proof of Theorem 3(a). It follows by (4) and the nonnegativity of $h^*(P)$ that, for $1 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$,

$$f^*_k - f^*_{k-1} = \sum_{j=0}^{k+1} \left(\binom{d+1-j}{k+1-j} - \binom{d+1-j}{k-j} \right) h^*_j \ge 0.$$

In fact, $f^*_k - f^*_{k-1}$ is bounded below by $\left(\binom{d+1}{k+1} - \binom{d+1}{k}\right) h^*_0 > 0$, for $1 \le k < \left\lfloor \frac{d}{2} \right\rfloor$, since $h^*_0 = 1$. □

Proof of Theorem 3(c). For $0 \le k \le \frac{d-2}{2}$, it follows by (4) that

$$f^*_{d-1-k} - f^*_k = \sum_{j=0}^{d-k} \left(\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j$$
$$= \sum_{j=0}^{d-1-2k} \left(\binom{d+1-j}{k+1} - \binom{d+1-j}{k+1-j} \right) h^*_j + \sum_{j=d-2k}^{d-k} \left(\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j .$$

We have $\binom{d+1-j}{k+1} - \binom{d+1-j}{k+1-j} \ge 0$ since $k+1-j \le k+1 \le \frac{d+1-j}{2}$ holds for $0 \le j \le d-1-2k$. Similarly, $\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \ge 0$ holds because $k+1-j \le d-k-j \le \frac{d+1-j}{2}$ for $d-2k \le j$. Therefore, it follows by the nonnegativity of the h^*-vector that $f^*_{d-1-k} - f^*_k \ge 0$. □

Proof of Theorem 7. Since $h^*_j = 0$ for $j \ge s+1$, equation (4) gives

$$f^*_{k-1} - f^*_k = \sum_{j=0}^{s} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j = \sum_{j=0}^{s} \frac{2k-d-j}{k+1-j} \binom{d+1-j}{k-j} h^*_j .$$

For $\frac{d+s}{2} \le k \le d$, we have $k+1-j > 0$ and $2k-d-j > 0$ for all $j = 0, \ldots, s-1$, and $k+1-j > 0$, $2k-d-j \ge 0$ for $j = s$. Therefore, the claim follows by the nonnegativity of the h^*-vector and the positivity of h^*_0. □

Proposition 9. *Let P be a d-dimensional lattice polytope that has degree at most s, with $s \ge 1$. If $d \ge 2s^2 - 2s - 2$ then the f^*-vector of P is unimodal with a (not necessarily "sharp") peak at f^*_p, where $\lfloor \frac{d}{2} \rfloor \le p \le \lceil \frac{d+s}{2} \rceil - 1$.*

Proof. By Theorems 3(a) and 7, it suffices to show that $f^*_{\lfloor \frac{d}{2} \rfloor + i} \ge f^*_{\lfloor \frac{d}{2} \rfloor + i+1}$ implies $f^*_{\lfloor \frac{d}{2} \rfloor + i+1} \ge f^*_{\lfloor \frac{d}{2} \rfloor + i+2}$, or that $2f^*_{\lfloor \frac{d}{2} \rfloor + 1+i} - f^*_{\lfloor \frac{d}{2} \rfloor + 2+i} - f^*_{\lfloor \frac{d}{2} \rfloor + i} \ge 0$ for $0 \le i \le \frac{s}{2} - 2$.

As $h^*_j = 0$ for $j \ge s+1$, by equation (4) we can express $2f^*_{\lfloor \frac{d}{2} \rfloor + 1+i} - f^*_{\lfloor \frac{d}{2} \rfloor + 2+i} - f^*_{\lfloor \frac{d}{2} \rfloor + i}$ as the sum

$$\sum_{j=0}^{s} \left(2\binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 2 - j + i} - \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 3 - j + i} - \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 1 - j + i} \right) h^*_j$$
$$= \sum_{j=0}^{s} \left(\frac{2(\lceil \frac{d}{2} \rceil - i)}{\lfloor \frac{d}{2} \rfloor + 2 - j + i} - \frac{(\lceil \frac{d}{2} \rceil - i)(\lceil \frac{d}{2} \rceil - 1 - i)}{(\lfloor \frac{d}{2} \rfloor + 2 - j + i)(\lfloor \frac{d}{2} \rfloor + 3 - j + i)} - 1 \right) \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 1 - j + i} h^*_j .$$

Since $d \ge \max\{2s^2 - 2s - 2, 0\}$ we have that $(\lfloor \frac{d}{2} \rfloor + 3 - j + i)(\lfloor \frac{d}{2} \rfloor + 2 - j + i)$ is positive for $j = 0, \ldots, s$ and since h^* is nonnegative, it remains to show that

$$\begin{aligned} &2(\lceil \tfrac{d}{2} \rceil - i)(\lfloor \tfrac{d}{2} \rfloor + 3 - j + i) - (\lceil \tfrac{d}{2} \rceil - i)(\lceil \tfrac{d}{2} \rceil - 1 - i) - (\lfloor \tfrac{d}{2} \rfloor + 2 - j + i)(\lfloor \tfrac{d}{2} \rfloor + 3 - j + i) \\ &= d - (2j - 5)(\lceil \tfrac{d}{2} \rceil - \lfloor \tfrac{d}{2} \rfloor) + 4i(\lceil \tfrac{d}{2} \rceil - \lfloor \tfrac{d}{2} \rfloor) - 12i + 4ij - 4i^2 - 6 + 5j - j^2 \\ &= \begin{cases} d - 4i^2 + 4ij - 12i - j^2 + 5j - 6 & \text{if } d \text{ is even}, \\ d - 4i^2 + 4ij - 8i - j^2 + 3j - 1 & \text{if } d \text{ is odd}, \end{cases} \end{aligned} \tag{5}$$

is nonnegative for $0 \le j \le s$. Indeed, the conditions $j \le s$ and $0 \le i \le \frac{s}{2} - 2$ imply that (5) is bounded below by

$$d - 4i^2 - 12i - j^2 - 6 \ge d - 4(\tfrac{s}{2} - 2)^2 - 12(\tfrac{s}{2} - 2) - s^2 - 6 = d - 2s^2 + 2s + 2,$$

which is nonnegative by assumption. □

The next proofs use more than just the nonnegativity of the h^*-vector. The first result needs the following elementary lemma on binomial coefficients.

Lemma 10. *Let j, k, n be positive integers such that $k \le n + 1 - j$. Then, for $n \ne 2k - 1$,*

$$\left| \binom{n}{k} - \binom{n}{k-1} \right| \ge \left| \binom{n-j}{k} - \binom{n-j}{k-1} \right| .$$

Proof. It suffices to prove the statement for the cases i) $j = 1$ and the quantities $\binom{n}{k} - \binom{n}{k-1}$ and $\binom{n-1}{k} - \binom{n-1}{k-1}$ having the same sign, and ii) the point when the signs change, i.e., $n = 2k$ and $j = 2$. To show case i), we simplify

$$\left|\binom{n}{k} - \binom{n}{k-1}\right| = \frac{(n-1)!}{k!(n-k)!}\frac{n}{n-k+1}\,|n-2k+1|$$

and

$$\left|\binom{n-1}{k} - \binom{n-1}{k-1}\right| = \frac{(n-1)!}{k!(n-k)!}\,|n-2k|\,.$$

If $n \geq 2k$ then the inequalities

$$\frac{n}{n-(k-1)}(n-2k+1) \geq n-2k+1 > n-2k$$

imply that

$$\left|\binom{n}{k} - \binom{n}{k-1}\right| > \left|\binom{n-1}{k} - \binom{n-1}{k-1}\right|. \tag{6}$$

If $n \leq 2k-2$, we have $k(-2k+2+n) \leq 0$ which is equivalent to

$$\frac{n}{n-(k-1)}(2k-n-1) \geq 2k-n$$

and so again (6) holds as a weak inequality.

To show case ii), we compute

$$\left|\binom{2k}{k} - \binom{2k}{k-1}\right| = \frac{(2k)!}{k!(k+1)!} = \frac{(2k-2)!}{k!(k-1)!}\frac{2k(2k-1)}{k(k+1)}$$

and

$$\left|\binom{2k-2}{k} - \binom{2k-2}{k-1}\right| = \frac{(2k-2)!}{k!(k-1)!}.$$

Since $2(2k-1) \geq (k+1)$ for any positive k, we conclude that

$$\left|\binom{2k}{k} - \binom{2k}{k-1}\right| \geq \left|\binom{2k-2}{k} - \binom{2k-2}{k-1}\right|. \qquad \square$$

We are now prepared to prove Theorem 3(b).

Proof of Theorem 3(b). The inequality $f^*_{d-1} > f^*_d$ holds by Theorem 7. Now, let $\lfloor \frac{3d}{4} \rfloor + 1 \leq k < d$. By equation (4),

$$f^*_{k-1} - f^*_k = \sum_{j=0}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j. \tag{7}$$

The difference $\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j}$ is nonnegative whenever $k - j \geq \lfloor \frac{d+1-j}{2} \rfloor$ and negative otherwise. Hence, the difference is nonnegative whenever $j \leq 2k-d$ and negative whenever $j > 2k-d$. Since $2d-2k < 2k+1-d$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k$, from (7) we obtain

$$f^*_{k-1} - f^*_k \geq \sum_{j=0}^{2d-2k} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{8}$$

$$+ \sum_{j=2k+1-d}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j, \tag{9}$$

where the differences appearing in (8) are nonnegative and the ones in (9) are negative. Our aim is to compare the sums in (8) and (9) so as to conclude that $f^*_{k-1} - f^*_k$ is positive.

Using standard identities for binomial coefficients, the right hand-side of (8) equals

$$\sum_{j=0}^{2d-2k}\left(\sum_{l=j}^{2d-2k-1}\left(\binom{d-l}{k-l}-\binom{d-l}{k+1-l}\right)+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\right)h_j^*$$
$$=\sum_{l=0}^{2d-2k-1}\left(\left(\binom{d-l}{k-l}-\binom{d-l}{k+1-l}\right)\sum_{j=0}^{2d-2k-1-l}h_j^*\right)+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h_j^*,$$

whence we conclude that the right hand-side of (8) is bounded below by

$$\left(\binom{d}{k}-\binom{d}{k+1}\right)h_0^*+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h_j^*>\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h_j^* \tag{10}$$

since $\binom{d}{k}-\binom{d}{k+1}>0$ for $\lfloor\frac{3d}{4}\rfloor+1\le k<d$, and $h_0^*=1$, $h_j^*\ge 0$ for $j=1,\dots,2d-2k-1$.

On the other hand, for the differences appearing in (9), using that $2d-2k<j$ and $j\le k+1$, it follows by Lemma 10 that

$$\left|\binom{d+1-(2d-2k)}{d+1-k}-\binom{d+1-(2d-2k)}{d-k}\right|\ge\left|\binom{d+1-j}{d+1-k}-\binom{d+1-j}{d-k}\right|,$$

i.e.,

$$\left|\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right|\ge\left|\binom{d+1-j}{k-j}-\binom{d+1-j}{k+1-j}\right|.$$

Hence for $j\ge 2k+1-d$,

$$-\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\le\binom{d+1-j}{k-j}-\binom{d+1-j}{k+1-j}.$$

Since both $-\binom{d+1-j}{k-j}+\binom{d+1-j}{k+1-j}$ and h_j^* are nonnegative for $j\ge 2k+1-d$, the sum in (9) is bounded below by

$$-\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=2k+1-d}^{d}h_j^*. \tag{11}$$

Now (10) and (11) yield

$$f_{k-1}^*-f_k^*>\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\left(\sum_{j=0}^{2d-2k}h_j^*-\sum_{j=2k+1-d}^{d}h_j^*\right).$$

Hibi [12] showed that the inequality

$$\sum_{j=0}^{m+1}h_j^*\ge\sum_{j=d-m}^{d}h_j^* \tag{12}$$

holds for $m=0,\dots,\lfloor\frac{d}{2}\rfloor-1$. Since $2d-2k-1\le\lfloor\frac{d}{2}\rfloor-1$ for $\lfloor\frac{3d}{4}\rfloor+1\le k$, we can use (12) to finally obtain

$$f_{k-1}^*-f_k^*>0.$$ □

Proof of Theorem 5. If $d=1$ or 2, there is nothing to prove. If $3\le d\le 6$, then by Theorem 3, either

$$f_0^*\le\cdots\le f_{\lfloor\frac{d}{2}\rfloor}^*\ge f_{\lfloor\frac{3d}{4}\rfloor}^*\ge\cdots\ge f_d^*$$

or

$$f_0^*\le\cdots\le f_{\lfloor\frac{d}{2}\rfloor}^*\le f_{\lfloor\frac{3d}{4}\rfloor}^*\ge\cdots\ge f_d^*.$$

For $7\le d\le 13$, we will show that if $f_i^*\ge f_{i+1}^*$, then $f_{i+1}^*\ge f_{i+2}^*$ for $\lfloor\frac{d}{2}\rfloor\le i\le\lfloor\frac{3d}{4}\rfloor-2$. By Theorem 3, this will imply the unimodality of $(f_0^*,f_1^*,\dots,f_d^*)$. Below, we examine each value of d separately, and make use of the inequality

$$\sum_{j=1}^{m+1}(h_j^*-h_{d+1-j}^*)>0, \tag{13}$$

which holds for $m = 0, \dots, \lfloor \frac{d}{2} \rfloor - 1$ by [16, Remark 1.2], as well as the nonnegativity of h^*-vectors to deduce that $f^*_{i+1} - f^*_{i+2} \ge c(f^*_i - f^*_{i+1})$ for some nonnegative real c in each case.

Suppose that $d = 7$ and $f^*_3 \ge f^*_4$. Then, by (4) (and $h^*_0 = 1$), we compute

$$2f^*_4 - f^*_3 - f^*_5 = 14h^*_0 + 14h^*_1 + 10h^*_2 + 5h^*_3 + h^*_4 - h^*_5 - h^*_6 > \sum_{j=1}^{3}(h^*_j - h^*_{8-j}),$$

hence $f^*_4 - f^*_5 \ge f^*_3 - f^*_4$. Likewise, for $d = 8$ we have

$$2f^*_5 - f^*_4 - f^*_6 = 6h^*_0 + 14h^*_1 + 14h^*_2 + 10h^*_3 + 5h^*_4 + h^*_5 - h^*_6 - h^*_7 > \sum_{j=1}^{3}\left(h^*_j - h^*_{9-j}\right),$$

and similarly for $d = 9$,

$$f^*_5 - f^*_6 - 2(f^*_4 - f^*_5) > \sum_{j=1}^{4}\left(h^*_j - h^*_{10-j}\right),$$

and for $d = 10$,

$$2f^*_6 - f^*_5 - f^*_7 > 2\sum_{j=1}^{5}\left(h^*_j - h^*_{11-j}\right).$$

For $d = 11$, we need to consider two values: $i = 5$ and $i = 6$. The claim follows since

$$f^*_6 - f^*_7 - 2(f^*_5 - f^*_6) > 2\sum_{j=1}^{5}\left(h^*_j - h^*_{12-j}\right),$$

and

$$f^*_7 - f^*_8 - \frac{4}{5}(f^*_6 - f^*_7) > 3\sum_{j=1}^{5}\left(h^*_j - h^*_{12-j}\right).$$

For $d = 12$, i also takes two values: $i = 6$ and $i = 7$. Now,

$$f^*_7 - f^*_8 - \frac{5}{4}(f^*_6 - f^*_7) > 3\sum_{j=1}^{6}\left(h^*_j - h^*_{13-j}\right),$$

and also

$$f^*_8 - f^*_9 - \frac{1}{2}(f^*_7 - f^*_8) > 3\sum_{j=1}^{6}\left(h^*_j - h^*_{13-j}\right).$$

Finally, for $d = 13$, the desired inequality holds for both values $i = 6$ and $i = 7$ as weak inequality:

$$f^*_7 - f^*_8 - \frac{7}{3}(f^*_6 - f^*_7) \ge 3\sum_{j=1}^{6}\left(h^*_j - h^*_{14-j}\right)$$

and

$$2f^*_8 - f^*_7 - f^*_9 \ge 4\sum_{j=1}^{6}\left(h^*_j - h^*_{14-j}\right).$$

To construct a polytope with nonunimodal f^*-vector, we employ a family of simplices introduced by Higashitani [13]. Concretely, denote the jth unit vector by e_j and let

$$\Delta_w := \operatorname{conv}\{0, e_1, e_2, \dots, e_{14}, w\}$$

where

$$w := (\underbrace{1, 1, \dots, 1}_{7}, \underbrace{131, 131, \dots, 131}_{7}, 132).$$

It has h^*-vector

$$(1, \underbrace{0, 0, \dots, 0}_{7}, 131, \underbrace{0, 0, \dots, 0}_{7})$$

and, via (4), f^*-vector

$$(16, 120, 560, 1820, 4368, 8008, 11440, 13001, \mathbf{12488}, \mathbf{11676}, \mathbf{11704}, 10990, 7896, 3788, 1064, 132). \qquad \square$$

We record the following consequence of Theorem 5, which follows by the nonnegativity of h^*-vectors.

Corollary 11. *Every lattice polytope of degree at most* 5 *has unimodal f^*-vector.*

Proof. Let P be a d-dimensional lattice polytope of degree at most 5. We know from Theorem 5 that f^* is unimodal when $d \leq 13$.

Suppose that $d \geq 14$. The proof is similar to the proof of Proposition 9, but we need to be a bit more precise with bounds. By Theorems 3(a) and 7, it suffices to show that $f^*_{\lfloor \frac{d}{2} \rfloor + i} \geq f^*_{\lfloor \frac{d}{2} \rfloor + i + 1}$ implies $f^*_{\lfloor \frac{d}{2} \rfloor + i + 1} \geq f^*_{\lfloor \frac{d}{2} \rfloor + i + 2}$, for $i = 0, \ldots, \lceil \frac{d+5}{2} \rceil - \lfloor \frac{d}{2} \rfloor - 3$. Note that $\lceil \frac{d+5}{2} \rceil = \lfloor \frac{d}{2} \rfloor + \lceil \frac{5}{2} \rceil$, hence $i = 0$. Arguing as in the proof of Proposition 9, we can reduce the proof to showing that the expression in (5) in Proposition 9 is nonnegative for $0 \leq j \leq 5$ and $i = 0$, i.e., that

$$d - (2j - 5)\left(\left\lceil \frac{d}{2} \right\rceil - \left\lfloor \frac{d}{2} \right\rfloor\right) - 6 + j(5 - j) \geq 0. \tag{14}$$

The inequality in (14) holds for $0 \leq j \leq 5$ since

$$d - (2j - 5)\left(\left\lceil \frac{d}{2} \right\rceil - \left\lfloor \frac{d}{2} \right\rfloor\right) - 6 + j(5 - j) \geq d - 11.$$ □

Proof of Theorem 6. Let $s := d + 1 - g$ (and $\frac{1}{2}\left(d + 1 + \left\lfloor \frac{d+1-g}{2} \right\rfloor\right) \leq k \leq d$). We first consider the case that s is odd; the case s even will be similar. Since $h^*_j = 0$ for $j > s$ and $h^*_j = h^*_{s-j}$,

$$\begin{aligned} f^*_{k-1} - f^*_k &= \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j + \sum_{j=\lfloor \frac{s}{2} \rfloor + 1}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h^*_j . \end{aligned}$$

Because we assume $k \geq \frac{1}{2}(d + 1 + \lfloor \frac{s}{2} \rfloor)$,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} > 0$$

for $0 \leq j \leq \lfloor \frac{s}{2} \rfloor$. The inequality

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} > 0$$

follows directly if $\binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \geq 0$ or $k - s + j + 1 < 0$. Otherwise, Lemma 10 implies that, for the same range of j,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \geq 0.$$

In fact, the last inequality is strict for $k \geq \frac{1}{2}(d + 1 + \lfloor \frac{s}{2} \rfloor)$; the proof is analogous to that of Lemma 10. Finally we use that $h^*_j \geq 0$ (for $0 \leq j \leq \lfloor \frac{s}{2} \rfloor$) and $h^*_0 = 1$ to deduce that $f^*_{k-1} - f^*_k > 0$.

The computations in the case s even is very similar. Now we write

$$\begin{aligned} f^*_{k-1} - f^*_k &= \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\ &= \sum_{j=0}^{\frac{s}{2}-1} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h^*_j + \left(\binom{d-\frac{s}{2}+1}{k-\frac{s}{2}} - \binom{d-\frac{s}{2}+1}{k-\frac{s}{2}+1} \right) h^*_{\frac{s}{2}} \end{aligned}$$

and use the same argumentation as in the case s odd. □

3 Concluding remarks

There are many avenues to explore f^*-vectors, e.g., along analogous studies of h^*-vectors, and we hope the above results form an enticing starting point. The techniques in our proof of Theorem 5 do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidate f^*-vectors with corresponding h^*-vectors that satisfy all inequalities discussed in [16]. It is unknown, though, if such polytopes exist.

Higashitani [13, Theorem 1.1] provided examples of d-dimensional polytopes with nonunimodal h^*-vector for all $d \geq 3$. Therefore, by Theorem 5 we have examples of polytopes that have such an h^*-vector but their f^*-vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal h^*-vector and nonunimodal f^*-vector. By Corollary 11, such polytopes would need to have degree at least 6.

Whenever one detects that a given polynomial is unimodal, it is natural to ask about the stronger property that the polynomial is log concave or, even stronger, real rooted. Our methods do not yield these properties but it would be interesting if one could extend, e.g., Corollary 8 or Proposition 9 along these lines.

Finally, starting with Stapledon's work [16], there has been much recent attention to symmetric decompositions of h- and h^*-polynomials; see, e.g., [1; 2] and, in particular, [8] where analogous decompositions for f-vectors are discussed. We believe this line of research is worthy of attention with regards to understanding f^*-vectors and the inequalities that hold among their coefficients.

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References

[1] C. A. Athanasiadis, Triangulations of simplicial complexes and theta polynomials. Preprint 2022, arXiv:2209.01674
[2] E. Bajo, M. Beck, Boundary h^*-polynomials of rational polytopes. *SIAM J. Discrete Math.* **37** (2023), 1952–1969. MR4633746 Zbl 1521.52005
[3] M. Beck, S. Robins, *Computing the continuous discretely*. Springer 2015. MR3410115 Zbl 1339.52002
[4] L. J. Billera, C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes. *J. Combin. Theory Ser. A* **31** (1981), 237–255. MR635368 Zbl 0479.52006
[5] A. Björner, The unimodality conjecture for convex polytopes. *Bull. Amer. Math. Soc.* (*N.S.*) **4** (1981), 187–188. MR598684 Zbl 0458.52004
[6] A. Björner, Face numbers of complexes and polytopes. In: *Proceedings of the International Congress of Mathematicians, Vol.* 1, 2 (*Berkeley, Calif.,* 1986), 1408–1418, Amer. Math. Soc. 1987. MR934345 Zbl 0657.00005
[7] A. Björner, Partial unimodality for f-vectors of simplicial polytopes and spheres. In: *Jerusalem combinatorics* '93, volume 178 of *Contemp. Math.*, 45–54, Amer. Math. Soc. 1994. MR1310573 Zbl 0815.52011
[8] P. Brändén, L. Solus, Symmetric decompositions and real-rootedness. *Int. Math. Res. Not. IMRN* (2021), no. **10**, 7764–7798. MR4259159 Zbl 1473.05329
[9] F. Breuer, Ehrhart f^*-coefficients of polytopal complexes are non-negative integers. *Electron. J. Combin.* **19** (2012), Paper 16, 22 pages. MR3001653 Zbl 1270.52020
[10] J. Eckhoff, Combinatorial properties of f-vectors of convex polytopes. *Normat* **54** (2006), 146–159. MR2288936
[11] E. Ehrhart, Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris* **254** (1962), 616–618. MR130860 Zbl 0100.27601
[12] T. Hibi, *Algebraic combinatorics on convex polytopes*. Carslaw Publications, Glebe 1992. MR3183743 Zbl 0772.52008
[13] A. Higashitani, Counterexamples of the conjecture on roots of Ehrhart polynomials. *Discrete Comput. Geom.* **47** (2012), 618–623. MR2891252 Zbl 1239.52011
[14] J. Hinman, A positive answer to Bárány's question on face numbers of polytopes. *Combinatorica* **43** (2023), 953–962. MR4648588 Zbl 07745908

[15] R. P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen–Macaulay rings. *Duke Math. J.* **43** (1976), 511–531. MR444514 Zbl 0335.05010

[16] A. Stapledon, Inequalities and Ehrhart δ-vectors. *Trans. Amer. Math. Soc.* **361** (2009), 5615–5626. MR2515826 Zbl 1181.52024

[17] A. Stapledon, Additive number theory and inequalities in Ehrhart theory. *Int. Math. Res. Not. IMRN* (2016), no. **5**, 1497–1540. MR3509934 Zbl 1342.52017