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Variations on the Weak Bounded Negativity Conjecture

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Abstract: We present two applications of Hao's proof of the *Weak Bounded Negativity Conjecture*. First, we address the so-called *Weighted Bounded Negativity Conjecture* and we prove that all but finitely many reduced and irreducible curves C on the blow-up of $\mathbb{P}^2$ at n points satisfy the inequality $C^2 \geq \min\{-\frac{1}{12}n(C.L+27), -2\}$, where L is the pull-back of a line. Next, we turn to the widely open conjecture that the canonical degree $C.K_X$ of an integral curve on a smooth projective surface X is bounded from above by an expression of the form $A(g-1)+B$, where g is the geometric genus of C and A, B are constants depending only on X. We prove that this conjecture holds with $A = -1$ under the assumptions $h^0(X, -K_X) = 0$ and $h^0(X, 2K_X + C) = 0$.

Keywords: Smooth projective surface, irreducible curve, self-intersection, bounded negativity conjecture.

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1 Introduction

The celebrated *Bounded Negativity Conjecture*, going back (at least) to Federigo Enriques (see for instance [3], Conjecture 1.1 and the historical remarks following its statement), predicts that on every smooth surface X the self-intersection C^2 of any reduced and irreducible curve C on X is bounded below by a constant $-b_X$ depending only on X, thus $C^2 \geq -b_X$. The extreme difficulty of such a conjecture, which is still open even for the blow-up of the projective plane $\mathbb{P}^2$ at $n \geq 10$ general points, motivated the formulation of the so-called *Weak Bounded Negativity Conjecture* (see [2], Conjecture 3.3.4), where the lower bound on C^2 is allowed to depend on the geometric genus of C. A complete proof of this weaker version has been provided by Hao in [5].

Here we present some applications of Hao's result to two still open conjectures in the same circle of ideas.

The first conjecture we address is a bounded negativity statement known as *Weighted Bounded Negativity Conjecture* (see [2], Conjecture 3.7.1), where the lower bound on C^2 is allowed to depend on the degree of C with respect to any nef and big divisor on X. The recent paper [6] collects several partial results towards this direction. In particular, in [6], Theorem 3.1, by using Orevkov–Sakai–Zaidenberg's inequality, it is proven that if Y is the blow-up of $\mathbb{P}^2$ at n distinct points and C is a reduced and irreducible curve on Y, then $C^2 \geq -2nC.L$, where L is the pull-back of a line.[1] In the same paper (see p. 370), as a consequence of the Plücker–Teissier formula, the previous bound is improved to $C^2 \geq -nC.L$. By applying only elementary tools in algebraic surface theory, we obtain the following partial improvement:

Theorem 1. *Let Y be the blow-up of $\mathbb{P}^2$ at n arbitrary (proper or infinitely near) points and let L be the pull-back of a (general) line in $\mathbb{P}^2$. Then all but finitely many reduced and irreducible curves C on Y satisfy the inequality*

$$C^2 \geq \min\Big\{-\frac{1}{6}n(C.L+3), -2\Big\}.$$

1 We guess that the above statement implicitly assumes $C.L > 0$, in order to exclude the exceptional divisors.

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As a consequence of [5], Corollary 1.8, we sharpen the previous bound as follows:

Theorem 2. *Let Y be the blow-up of $\mathbb{P}^2$ at n arbitrary (proper or infinitely near) points and let L be the pull-back of a (general) line in $\mathbb{P}^2$. Then all but finitely many reduced and irreducible curves C on Y satisfy the inequality*

$$C^2 \geq \min\Big\{-\frac{1}{12}n(C.L+27), -2\Big\}.$$

Next, we turn to another widely open conjecture (see [1], Conjecture 1, and [4], Conjecture 5.1) related to bounded negativity. Let C be an integral curve on a smooth complex projective surface X. We denote by $g = g(C)$ its geometric genus and by by $k_C := C.K_X$ its *canonical degree*.

Conjecture 3 (Vojta). *Let X be a smooth projective surface. There exist constants A, B such that for any integral curve C we have $k_C \leq A(g-1)+B$.*

Note that if $h^0(X,-K_X) > 0$ then $k_C \leq 0$ for all but finitely many curves C on X, hence we can assume that $h^0(X,-K_X) = 0$.

Moreover, if C is smooth then $k_C = 2(g-2) - C^2$, so in this case the Bounded Negativity Conjecture would imply $k_C \leq 2(g-1) + b_X - 2$. In the same vein, Hao's proof of the Weak Bounded Negativity Conjecture yields the following result:

Theorem 4. *Let X be a smooth projective surface with $h^0(X,-K_X) = 0$ and let C be a smooth irreducible curve of genus g on X.*

(a) *If $h^0(X, 2K_X + C) \neq 0$ then $k_C \leq 4(g-1) + 3c_2(X) - K_X^2$.*
(b) *If $h^0(X, 2K_X + C) = 0$ then $k_C \leq -(g-1) - K_X^2 - \chi(\mathcal{O}_X)$.*

Item (a) is precisely [5], Corollary 1.8 (just note that $h^0(X, 2K_X + C) \neq 0$ implies $h^0(X, 2(K_X + C)) \neq 0$), while item (b) follows from the proof of [5], Lemma 1.3, which works under the weaker assumption $h^0(X, 2K_X+C) = 0$ and in the smooth case gives the better bound $C^2 \geq 3g + K_X^2 + \chi(\mathcal{O}_X) - 3$.

The extension of item (a) to singular curves is known to be a highly nontrivial problem. A partial result towards this direction is [7], Theorem 1, item (4): if C is integral and some multiple of $K_X + C$ is effective then $k_C \leq 4(g-1) + 3c_2(X) - K_X^2 + n$, where n is the number of ordinary nodes and ordinary triple points of C. As pointed out by Miyaoka in [8], Remark D, this yields *a similar bound for $C.K_X$ provided C contains neither ordinary double points nor ordinary triple points. Strangely, complicated singularities of high multiplicity do no harm to estimating canonical degrees. Curves with many ordinary double points are technically the most difficult to deal with.*

We focus on item (b) and we obtain the following complete generalization to the singular case:

Theorem 5. *Let X be a smooth projective surface with $h^0(X,-K_X) = 0$ and let C be an integral curve of geometric genus g on X. If $h^0(X, 2K_X + C) = 0$ then*

$$k_C \leq -(g-1) - K_X^2 - \chi(\mathcal{O}_X).$$

Our argument is inspired by [5], proof of Theorem 1.9, which in turn closely follows [2], proof of Proposition 3.5.3.

We work over the complex field $\mathbb{C}$.

2 The proofs

Lemma 6. *Let X be a smooth projective surface and let C be a reduced and irreducible curve on X. Then for every integer $m \neq 1$ we have*

$$\begin{aligned} C^2 = {} & \frac{1}{m-1}\chi(\mathcal{O}_X) + \frac{1}{2}mK_X^2 + 2p_a(C) + \frac{1}{m-1}p_a(C) - 2 - \frac{1}{m-1} \\ & - \frac{1}{m-1}h^0(mK_X + C) + \frac{1}{m-1}h^1(mK_X + C) - \frac{1}{m-1}h^0(-(m-1)K_X - C). \end{aligned}$$

Proof. Just apply the Riemann–Roch theorem to $mK_X + C$, Serre duality to $h^2(mK_X + C)$ and the adjunction formula to C:

$$\begin{aligned} h^0(mK_X + C) - h^1(mK_X + C) + h^0(-(m-1)K_X - C) = \\ &= h^0(mK_X + C) - h^1(mK_X + C) + h^2(mK_X + C) = \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}(mK_X + C)((m-1)K_X + C) = \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}(m(m-1)K_X^2 + (2m-1)K_X.C + C^2) = \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}(m(m-1)K_X^2 + (2m-1)(2p_a(C) - 2 - C^2) + C^2) = \\ &= \chi(\mathcal{O}_X) + \frac{1}{2}m(m-1)K_X^2 + (m-1)(2p_a(C) - 2) + p_a(C) - 1 - (m-1)C^2. \end{aligned}$$ □

Lemma 7. *Notation as in Lemma 6. If $h^0(-mK_X) \neq 0$ for some $m \geq 1$, then all but finitely many reduced and irreducible curves C on X satisfy the inequality $C^2 \geq -2$.*

Proof. Let E be an effective divisor in $|-mK_X|$. If C is not one of the finitely many curves in the support of E then $-K_X.C \geq 0$, hence $C^2 = -K_X.C + 2p_a(C) - 2 \geq -2$. □

Lemma 8. *Notation as in Theorem 1. Let m_0 be the integer such that*

$$\frac{C.L + 1}{3} \leq m_0 \leq \frac{C.L + 3}{3}.$$

Then $h^0(m_0K_Y + C) = 0$.

Proof. We have $(m_0K_Y + C).L = -3m_0 + C.L \leq -1 < 0$ and L is nef, hence the divisor $m_0K_Y + C$ cannot be effective. □

Proof of Theorem 1. We apply Lemma 6 to $X = Y$, in particular we have $\chi(\mathcal{O}_Y) = 1$ and $K_Y^2 = 9 - n$. By Lemma 7 we may assume that $h^0(-(m-1)K_Y - C) = h^0(-(m-1)K_Y) = 0$. By setting $m = m_0$ as in Lemma 8 we obtain

$$C^2 \geq -\frac{1}{2}m_0 n \geq -\frac{1}{6}n(C.L + 3).$$ □

Proof of Theorem 2. If $h^0(2K_Y + C) = 0$, then by arguing as in the proof of Theorem 1 with $m = 2$ we obtain $C^2 \geq \min\{-n, -2\}$. Assume now $h^0(2K_Y + C) \neq 0$, so that in particular $h^0(2(K_Y + C)) \neq 0$. By Lemma 7 we may also assume $h^0(-K_Y) = 0$. Hence we are in the position to apply [5], Corollary 1.8, and deduce

$$C^2 \geq K_Y^2 - 3c_2(Y) + 2 - 2p_a(C) = 9 - n - 3(3 + n) + 2 - 2p_a(C) = -4n + 2 - 2p_a(C).$$

On the other hand, by arguing as in the proof of Theorem 1, we obtain

$$C^2 \geq -\frac{1}{6}n(C.L + 3) + 2p_a(C) - 2.$$

It follows that

$$2C^2 \geq -4n + -\frac{1}{6}n(C.L + 3)$$

and we conclude

$$C^2 \geq -\frac{1}{12}n(C.L + 27).$$ □

Proof of Theorem 5. The idea is to blow up X resolving step by step the singularities of C. If the assumptions hold at each step provided they hold at the previous one and the conclusion holds at each step provided it holds at the next one, then the statement follows recursively from the smooth case, namely from item (b) of Theorem 4.

Let X be a smooth projective surface and let C be an integral curve of geometric genus g on X. Let $p \in C$ be a point with multiplicity $\mathrm{mult}_p(C) = m \geq 2$ and let $\pi : \tilde{X} \to X$ be the blow-up of X at p. Let E be the exceptional divisor of the blow-up and let $\tilde{C} = \pi^*(C) - mE$ be the strict transform of C.

We claim the following:

(i) If $h^0(X, -K_X) = 0$ then $h^0(\tilde{X}, -K_{\tilde{X}}) = 0$.

(ii) If $h^0(X, 2K_X + C) = 0$ then $h^0(\tilde{X}, 2K_{\tilde{X}} + \tilde{C}) = 0$.

(iii) If $k_{\tilde{C}} \leq -(g-1) - K_{\tilde{X}}^2 - \chi(\mathcal{O}_{\tilde{X}})$ then $k_C \leq -(g-1) - K_X^2 - \chi(\mathcal{O}_X)$.

Indeed, (i) follows from $K_{\tilde{X}} = \pi^*(K_X) + E$ (for details see [5], Lemma 1.10).

Next, for (ii) we have $h^0(\tilde{X}, 2K_{\tilde{X}} + \tilde{C}) = h^0(\tilde{X}, 2(\pi^*(K_X) + E) + \pi^*(C) - mE)) \leq h^0(\tilde{X}, 2\pi^*(K_X) + \pi^*(C)) = h^0(\tilde{X}, \pi^*(2K_X + C)) = h^0(X, 2K_X + C) = 0$.

Finally, we have $k_C = K_X.C = K_{\tilde{X}}.\tilde{C} - m = k_{\tilde{C}} - m \leq -(g-1) - K_{\tilde{X}}^2 - \chi(\mathcal{O}_{\tilde{X}}) - m = -(g-1) - (K_X^2 - 1) - \chi(\mathcal{O}_X) - m < -(g-1) - K_X^2 - \chi(\mathcal{O}_X)$. □

Remark 9. If the curve $C = \sum_i^n C_i$ is reducible (but still reduced), the argument above works verbatim by setting $g := \sum_i^n g_i - (n-1)$, where g_i is the geometric genus of the irreducible component C_i. Indeed, after finitely many blow-ups we obtain a curve $\tilde{C}$ which is the disjoint union of the normalizations of the curves C_i. The arithmetic genus of $\tilde{C}$ is $p_a(\tilde{C}) = \sum_i^n g_i - (n-1) = g$ and the proof of [5], Lemma 1.3, implies $\tilde{C}^2 \geq 3p_a(\tilde{C}) + K_{\tilde{X}}^2 + \chi(\mathcal{O}_{\tilde{X}}) - 3$, hence we have $K_{\tilde{X}}.\tilde{C} \leq -(p_a(\tilde{C}) - 1) - K_{\tilde{X}}^2 - \chi(\mathcal{O}_{\tilde{X}}) = -(g-1) - K_{\tilde{X}}^2 - \chi(\mathcal{O}_{\tilde{X}})$, exactly as in the integral case.

Remark 10. Under the same assumptions, the argument above yields the following stronger inequality:

$$K_X.C \leq -(g-1) - K_X^2 - \chi(\mathcal{O}_X) - \sum_{p \in C} (\mathrm{mult}_p(C) - 1) \leq -(g-1) - K_X^2 - \chi(\mathcal{O}_X) - \#\mathrm{Sing}(C).$$

Remark 11. The argument above does not apply to the extension to singular curves of item (a) of Theorem 4 because the analogue of item (iii) does not work. Indeed, if $k_{\tilde{C}} \leq 4(g-1) + 3c_2(\tilde{X}) - K_{\tilde{X}}^2$ then $k_C = k_{\tilde{C}} - m \leq 4(g-1) + 3(c_2(X) + 1) - (K_X^2 - 1) - m = 4(g-1) + 3c_2(X) - K_X^2 + (4 - m)$, with $4 - m > 0$ if $2 \leq m \leq 3$.

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