DE GRUYTER Adv. Calc. Var. 2025; 18(3): 979–1010

Research Article

Bogi Kim, Jehan Oh* and Abhrojyoti Sen

Parabolic Lipschitz truncation for multi-phase problems: The degenerate case

https://doi.org/10.1515/acv-2025-0003
Received January 6, 2025; accepted March 28, 2025

Abstract: This article is devoted to exploring the Lipschitz truncation method for parabolic multi-phase problems. The method is based on Whitney decomposition and covering lemmas with a delicate comparison scheme of appropriate alternatives to distinguish phases, as introduced by the first and second authors in [B. Kim and J. Oh, Higher integrability for weak solutions to parabolic multi-phase equations, *J. Differential Equations* **409** (2024), 223–298].

Keywords: Lipschitz truncation, parabolic multi-phase problems, parabolic systems, degenerate case, energy estimates

MSC 2020: Primary 35B65; secondary 35K65, 35K55, 35D30, 35A01

Communicated by: Juha Kinnunen

1 Introduction

In this paper, we prove energy estimates for weak solutions to parabolic multi-phase problems of type

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u + a(z)|\nabla u|^{q-2}\nabla u + b(z)|\nabla u|^{s-2}\nabla u) = -\operatorname{div}(|F|^{p-2}F + a(z)|F|^{q-2}F + b(z)|F|^{s-2}F) \quad \text{in } \Omega_T,$$

where $\Omega_T := \Omega \times (0, T)$ represents a space-time cylinder with a bounded open set $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, $2 \leq p < q < s < \infty$ and the modulating coefficients $a(\cdot)$ and $b(\cdot)$ are nonnegative and Hölder continuous.

Energy estimates are very important for proving the existence and regularity results for partial differential equations. We prove the energy estimates using the Lipschitz truncation method for parabolic multi-phase problems. Here, the Lipschitz truncation is a method of redefining a given function so that it keeps its values on a specific "good set", while redefining it in "bad sets" using a partition of unity related to a Whitney covering argument. Acerbi and Fusco [1, 2] have introduced the Lipschitz truncation for elliptic problems. Furthermore, Kinnunen and Lewis [30] and Kim, Kinnunen and Särkiö [28] have studied related methods for the parabolic p-Laplace system and for the parabolic double phase systems, respectively.

In this paper, we deal with a parabolic multi-phase system

$$u_t - \operatorname{div} \mathcal{A}(z, u, \nabla u) = -\operatorname{div} \mathcal{B}(z, F) \quad \text{in } \Omega_T, \tag{1.1}$$

where Carathéodory vector fields $\mathcal{A} : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ and $\mathcal{B} : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ satisfy the following growth conditions: for any $z \in \Omega_T$, $v \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{Nn}$, there exist two constants $0 < \nu \leq L < \infty$ such that

$$\nu H(z, |\xi|) \leq \mathcal{A}(z, v, \xi) \cdot \xi, \quad |\mathcal{A}(z, v, \xi)||\xi| \leq LH(z, |\xi|) \tag{1.2}$$

***Corresponding author: Jehan Oh,** Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea, e-mail: jehan.oh@knu.ac.kr. https://orcid.org/0000-0001-9602-5111
Bogi Kim, Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea, e-mail: rlaqhrl4@knu.ac.kr
Abhrojyoti Sen, Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60629 Frankfurt, Germany, e-mail: sen@math.uni-frankfurt.de

and

$$|\mathcal{B}(z,\xi)||\xi| \le LH(z,|\xi|), \tag{1.3}$$

where the function $H : \Omega_T \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$H(z,\kappa) = \kappa^p + a(z)\kappa^q + b(z)\kappa^s$$

for $z \in \Omega_T$ and $\kappa \in \mathbb{R}^+$. Furthermore, the source term $F : \Omega_T \to \mathbb{R}^{Nn}$ satisfies

$$\iint_{\Omega_T} H(z,|F|)\,dz < +\infty. \tag{1.4}$$

We assume that the modulating coefficients $a : \Omega_T \to \mathbb{R}^+$ and $b : \Omega_T \to \mathbb{R}^+$ satisfy

$$q \le p + \frac{2\alpha}{n+2}, \quad 0 \le a \in C^{\alpha,\frac{\alpha}{2}}(\Omega_T) \quad \text{for some } \alpha \in (0,1], \tag{1.5}$$

and

$$s \le p + \frac{2\beta}{n+2}, \quad 0 \le b \in C^{\beta,\frac{\beta}{2}}(\Omega_T) \quad \text{for some } \beta \in (0,1]. \tag{1.6}$$

Here, $a \in C^{\alpha,\frac{\alpha}{2}}(\Omega_T)$ means that $a \in L^\infty(\Omega_T)$ and there exists a Hölder constant $[a]_\alpha := [a]_{\alpha,\frac{\alpha}{2};\Omega_T} > 0$ such that

$$|a(x_1,t_1) - a(x_2,t_2)| \le [a]_\alpha\big(|x_1 - x_2| + \sqrt{|t_1 - t_2|}\big)^\alpha$$

for all $x_1, x_2 \in \Omega$ and $t_1, t_2 \in (0,T)$.

Definition 1.1. A function $u : \Omega \times (0,T) \to \mathbb{R}^N$ satisfying

$$u \in C(0,T; L^2(\Omega,\mathbb{R}^N)) \cap L^1(0,T; W^{1,1}(\Omega,\mathbb{R}^N))$$

and

$$\iint_{\Omega_T} [H(z,|u|) + H(z,|\nabla u|)]\,dz < \infty$$

is a weak solution to (1.1) if

$$\iint_{\Omega_T} [-u \cdot \varphi_t + \mathcal{A}(z,u,\nabla u) \cdot \nabla\varphi]\,dz = \iint_{\Omega_T} [\mathcal{B}(z,F) \cdot \nabla\varphi]\,dz$$

for every $\varphi \in C_0^\infty(\Omega_T,\mathbb{R}^N)$.

The parabolic multi-phase problems derive from elliptic double phase problems. The elliptic double phase problems of type

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = -\operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F)$$

was first introduced in [35–38]. These problems originate from the Lavrentiev phenomenon and the homogenization of strongly anisotropic materials. According to [13, 21], the conditions

$$a(\cdot) \in C^\alpha(\Omega), \quad \alpha \in (0,1] \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{n} \tag{1.7}$$

and

$$u \in L^\infty(\Omega), \quad a(\cdot) \in C^\alpha(\Omega), \quad \alpha \in (0,1] \quad \text{and} \quad q \le p + \alpha \tag{1.8}$$

are sharp for obtaining regularity results of weak solutions. In fact, when (1.7) or (1.8) holds, the gradient of a weak solution u is Hölder continuous, see [6, 13, 14, 19]. Moreover, the Calderón–Zygmund estimates have been discussed in [3, 15, 17]. Also, Baroni, Colombo and Mingione [5] have investigated the Harnack's inequality. In addition, other regularity results for elliptic double phase problems have been discussed in [8–11, 22, 23, 25, 31, 32]. The regularity results for elliptic multi-phase problems given by

$$-\operatorname{div}\bigg(|\nabla u|^{p-2}\nabla u + \sum_{i=1}^m a_i(x)|\nabla u|^{p_i-2}\nabla u\bigg) = -\operatorname{div}\bigg(|F|^{p-2}F + \sum_{i=1}^m a_i(x)|F|^{p_i-2}F\bigg) \quad \text{in } \Omega,$$

with $1 < p < p_1 \le \cdots \le p_m$ and $0 \le a_i(\cdot) \in C^{0,\alpha_i}(\bar{\Omega})$, $\alpha_i \in (0,1]$, have also been discussed in [4, 16, 18, 20].

On the other hand, the regularity for parabolic double phase problems

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u + a(z)|\nabla u|^{q-2}\nabla u) = -\operatorname{div}(|F|^{p-2}F + a(z)|F|^{q-2}F) \quad \text{in } \Omega_T$$

with $2 \le p < q$ has been studied recently. If (1.5) holds, the existence of weak solutions has been discussed in [12], see also [28, 34]. The gradient higher integrability and the Calderón–Zygmund-type estimate have been studied in [26, 27]. Moreover, the gradient higher integrability results for degenerate and singular parabolic multi-phase problems have also been studied in [24] and [33], respectively.

In this paper, our goal is to prove the energy estimates of the weak solution to (1.1). For this, we denote parabolic cylinders $U_{r,\tau}(z_0)$ and $Q_r(z_0)$ by

$$U_{r,\tau}(z_0) := B_r(x_0) \times \ell_\tau(t_0) \quad \text{and} \quad Q_r(z_0) := B_r(x_0) \times I_r(t_0) \tag{1.9}$$

with

$$\ell_\tau(t_0) := (t_0 - \tau, t_0 + \tau) \quad \text{and} \quad I_r(t_0) := (t_0 - r^2, t_0 + r^2). \tag{1.10}$$

Our main theorem of this paper is as follows.

Theorem 1.2. *Let u be a weak solution to* (1.1). *Then, for $U_{R_2,S_2}(z_0) \subset \Omega_T$, $R_1 \in [\frac{R_2}{2}, R_2)$ and $S_1 \in [\frac{S_2}{2^2}, S_2)$, there exists a constant c depending on n, p, q, s, ν and L such that the following inequality holds:*

$$\begin{aligned} &\sup_{t \in (t_0 - S_1, t_0 + S_1)} \fint_{B_{R_1}(x_0)} \frac{|u - (u)_{U_{R_1,S_1}(z_0)}|^2}{S_1}\,dx + \iint\!\!\!\!\!\!-\,_{U_{R_1,S_1}(z_0)} H(z, |\nabla u|)\,dz \\ &\le c \iint\!\!\!\!\!\!-\,_{U_{R_2,S_2}(z_0)} H\Big(z, \frac{|u - (u)_{U_{R_2,S_2}(z_0)}|}{R_2 - R_1}\Big)\,dz + c \iint\!\!\!\!\!\!-\,_{U_{R_2,S_2}(z_0)} \frac{|u - (u)_{U_{R_2,S_2}(z_0)}|^2}{S_2 - S_1}\,dz + c \iint\!\!\!\!\!\!-\,_{U_{R_2,S_2}(z_0)} H(z, |F|)\,dz. \end{aligned}$$

Remark 1.3. In [24], the energy estimate presented in their Lemma 3.1 is derived under the assumption that $|\nabla u| \in L^s(\Omega_T)$. Consequently, the gradient higher integrability result [24, Theorem 1.2] also relies on this assumption. However, in light of the energy estimate above, [24, Theorem 1.2] can be established under the weaker assumption specified in Definition 1.1.

Remark 1.4. We would like to mention that the existence and uniqueness results for (1.1) with Dirichlet boundary condition can be obtained by closely following the proofs of [28, Theorem 2.6 and Theorem 2.7]. Therefore, we choose not to pursue such aspects in this paper.

To prove Theorem 1.2, we will construct the Whitney decomposition by dividing into p-, (p, q)-, (p, s)- and (p, q, s)-phases in Section 2. Furthermore, in this section, we prove the Vitali covering argument by dividing it into a total of sixteen cases as in [24] and establish the related properties, which we summarize in Lemma 2.18. In Section 3, we define the Lipschitz truncation and establish the related properties. We then prove the energy estimates in Section 4.

2 Whitney decomposition and covering lemmas

In this section, we construct a family of Whitney decomposition and show that the family covers the bad set $E(\Lambda)^c$ via a Vitali covering argument. The construction of such a decomposition is heavily used in the subsequent sections.

2.1 Auxiliary definitions

Let $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $\varrho > 0$. Parabolic cylinders with quadratic scaling in time are denoted as

$$Q_\varrho(z_0) = B_\varrho \times I_\varrho(t_0),$$

where $B_\varrho = B_\varrho(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < \varrho\}$ and $I_\varrho(t_0) = (t_0 - \varrho^2, t_0 + \varrho^2)$.

Let us define the strong maximal function for $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ as

$$Mf(z) = \sup_{z\in Q} \fint\!\!\fint_Q |f|\,dw, \tag{2.1}$$

where the supremum is taken over cubes $Q \subset \mathbb{R}^{n+1}$. As in [28], using the Hardy–Littlewood–Wiener maximal function theorem with respect to space and time, we obtain the following lemma.

Lemma 2.1. *For $1 < \sigma < \infty$ and $f \in L^\sigma(\mathbb{R}^{n+1})$, there exists a constant $c = c(n, \sigma)$ such that*

$$\iint_{\mathbb{R}^{n+1}} |Mf|^\sigma\,dz \le c \iint_{\mathbb{R}^{n+1}} |f|^\sigma\,dz.$$

Without loss of generality we can assume that the modulating coefficient functions $a(\cdot)$ and $b(\cdot)$ are defined in $\mathbb{R}^{n+1}$ satisfying $a(\cdot) \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^{n+1})$ and $b(\cdot) \in C^{\beta,\frac{\beta}{2}}(\mathbb{R}^{n+1})$, see for instance [28] and [29, Theorem 2.7]. Let $f \in L^p(\mathbb{R}^{n+1})$, $f \ge 0$, be such that

$$\iint_{\mathbb{R}^{n+1}} (f^p + af^q + bf^s)\,dz < \infty,$$

and let $d = \frac{s-1+p}{2}$. By Lemma 2.1, there exists $\Lambda_0 > 1 + \|a\|_{L^\infty(\mathbb{R}^{n+1})} + \|b\|_{L^\infty(\mathbb{R}^{n+1})}$ such that

$$\iint_{\mathbb{R}^{n+1}} \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})(z)\Big)^{\frac{p}{d}}\,dz \le c(n,p,s) \iint_{\mathbb{R}^{n+1}} (f^p + af^q + bf^s)\,dz \le \Lambda_0.$$

Let $\Lambda > \Lambda_0$ and we define the set

$$E(\Lambda) = \Big\{z \in \mathbb{R}^{n+1} : M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(z) \le \Lambda^{\frac{d}{p}}\Big\}. \tag{2.2}$$

Chebyshev's inequality implies that

$$\lim_{\Lambda\to\infty} \Lambda|E(\Lambda)^c| \le \lim_{\Lambda\to\infty} \iint_{E(\Lambda)^c} \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})(z)\Big)^{\frac{p}{d}}\,dz = 0. \tag{2.3}$$

Now, we write $K \ge 2$ as

$$\begin{aligned} K &:= 2 + 800[a]_\alpha\bigg(\frac{1}{|B_1|}\iint_{\mathbb{R}^{n+1}} \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})\Big)^{\frac{p}{d}}(z)\,dz\bigg)^{\frac{\alpha}{n+2}} \\ &\quad + 800[b]_\beta\bigg(\frac{1}{|B_1|}\iint_{\mathbb{R}^{n+1}} \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})\Big)^{\frac{p}{d}}(z)\,dz\bigg)^{\frac{\beta}{n+2}}. \end{aligned} \tag{2.4}$$

Note that for each $z \in E(\Lambda)^c$, there exists a unique $\lambda_z > 1$ such that $\Lambda = \lambda_z^p + a(z)\lambda_z^q + b(z)\lambda_z^s$. We then consider a family of metrics $\{d_z(\cdot,\cdot)\}_{z\in E(\Lambda)^c}$ given by

$$d_z(z_1,z_2) = \begin{cases} \max\Big\{|x_1 - x_2|, \sqrt{\lambda_z^{p-2}|t_1 - t_2|}\Big\} & \text{if } K^2\lambda_z^p \ge a(z)\lambda_z^q \text{ and } K^2\lambda_z^p \ge b(z)\lambda_z^s, \\ \max\Big\{|x_1 - x_2|, \sqrt{g_q(z,\lambda_z)\lambda_z^{-2}|t_1 - t_2|}\Big\} & \text{if } K^2\lambda_z^p < a(z)\lambda_z^q \text{ and } K^2\lambda_z^p \ge b(z)\lambda_z^s, \\ \max\Big\{|x_1 - x_2|, \sqrt{g_s(z,\lambda_z)\lambda_z^{-2}|t_1 - t_2|}\Big\} & \text{if } K^2\lambda_z^p \ge a(z)\lambda_z^q \text{ and } K^2\lambda_z^p < b(z)\lambda_z^s, \\ \max\Big\{|x_1 - x_2|, \sqrt{g_{q,s}(z,\lambda_z)\lambda_z^{-2}|t_1 - t_2|}\Big\} & \text{if } K^2\lambda_z^p < a(z)\lambda_z^q \text{ and } K^2\lambda_z^p < b(z)\lambda_z^s \end{cases}$$

for $z_1 = (x_1,t_1)$, $z_2 = (x_2,t_2) \in \mathbb{R}^{n+1}$, where the functions g_q, g_s, and $g_{q,s}$, which are q, s, and (q,s)-growth functions, respectively, defined as

$$g_q(z,\kappa) := \kappa^p + a(z)\kappa^q, \tag{2.5}$$

$$g_s(z,\kappa) := \kappa^p + b(z)\kappa^s, \tag{2.6}$$

$$g_{q,s}(z,\kappa) := \kappa^p + a(z)\kappa^q + b(z)\kappa^s \tag{2.7}$$

for $z \in \Omega_T$ and $\kappa \in \mathbb{R}^+$. For every $z \in E(\Lambda)^c$, we define a distance function z to $E(\Lambda)$ as

$$4r_z = d_z(z, E(\Lambda)) = \inf_{w\in E(\Lambda)} d_z(z,w). \tag{2.8}$$

Using this distance, we construct a family of open subsets $\{U_z^\Lambda\}_{z\in E(\Lambda)^c}$ defined as

$$U_z^\Lambda = \begin{cases} Q_{r_z,\lambda_z} & \text{if } K^2\lambda_z^p \geq a(z)\lambda_z^q \text{ and } K^2\lambda_z^p \geq b(z)\lambda_z^s, \\ Q^q_{r_z,\lambda_z} & \text{if } K^2\lambda_z^p < a(z)\lambda_z^q \text{ and } K^2\lambda_z^p \geq b(z)\lambda_z^s, \\ Q^s_{r_z,\lambda_z} & \text{if } K^2\lambda_z^p \geq a(z)\lambda_z^q \text{ and } K^2\lambda_z^p < b(z)\lambda_z^s, \\ Q^{q,s}_{r_z,\lambda_z} & \text{if } K^2\lambda_z^p < a(z)\lambda_z^q \text{ and } K^2\lambda_z^p < b(z)\lambda_z^s. \end{cases}$$

Here, the intrinsic cylinders denoted as follows:

$$\begin{aligned} Q_{r_z,\lambda_z}(z) &:= B_{r_z}(x)\times I_{r_z,\lambda_z}(t), \\ Q^q_{r_z,\lambda_z}(z) &:= B_{r_z}(x)\times I^q_{r_z,\lambda_z}(t), \\ Q^s_{r_z,\lambda_z}(z) &:= B_{r_z}(x)\times I^s_{r_z,\lambda_z}(t), \\ Q^{q,s}_{r_z,\lambda_z}(z) &:= B_{r_z}(x)\times I^{q,s}_{r_z,\lambda_z}(t) \end{aligned}$$

for any $z=(x,t)\in\mathbb{R}^n\times(0,T)$, $r_z>0$, $\lambda_z\geq 1$, where

$$\begin{aligned} I_{r_z,\lambda_z}(t) &:= (t-\lambda_z^{2-p}r_z^2,\ t+\lambda_z^{2-p}r_z^2), \\ I^q_{r_z,\lambda_z}(t) &:= \Big(t-\frac{\lambda_z^2}{g_q(z,\lambda_z)}r_z^2,\ t+\frac{\lambda_z^2}{g_q(z,\lambda_z)}r_z^2\Big), \\ I^s_{r_z,\lambda_z}(t) &:= \Big(t-\frac{\lambda_z^2}{g_s(z,\lambda_z)}r_z^2,\ t+\frac{\lambda_z^2}{g_s(z,\lambda_z)}r_z^2\Big), \\ I^{q,s}_{r_z,\lambda_z}(t) &:= \Big(t-\frac{\lambda_z^2}{g_{q,s}(z,\lambda_z)}r_z^2,\ t+\frac{\lambda_z^2}{g_{q,s}(z,\lambda_z)}r_z^2\Big). \end{aligned}$$

Let $f\in L^1(\Omega_T)$ be a function, and let $Q\subset\Omega_T$ be a measurable set with finite positive measure. We define the integral average of f over Q by

$$(f)_Q := \fint\!\!\fint_Q f\,dz.$$

If Q is one of the four intrinsic cylinders defined above, we denote it as follows:

$$\begin{aligned} (f)_{z_0;\rho,\lambda} &:= \fint\!\!\fint_{Q_{\rho,\lambda}(z_0)} f\,dz, \\ (f)^{(q)}_{z_0;\rho,\lambda} &:= \fint\!\!\fint_{Q^q_{\rho,\lambda}(z_0)} f\,dz, \\ (f)^{(s)}_{z_0;\rho,\lambda} &:= \fint\!\!\fint_{Q^s_{\rho,\lambda}(z_0)} f\,dz, \\ (f)^{(q,s)}_{z_0;\rho,\lambda} &:= \fint\!\!\fint_{Q^{q,s}_{\rho,\lambda}(z_0)} f\,dz. \end{aligned}$$

The integral average on a ball $B_\rho(x_0)\subset\Omega$ is denoted by

$$(f)_{x_0;\rho}(t) := \fint_{B_\rho(x_0)} f(x,t)\,dx, \qquad t\in(0,T).$$

We intend to use a Vitali-type argument to find a countable collection of points $z_i\in E(\Lambda)^c$ such that the corresponding subfamily satisfies some properties. From now on, we denote

$$\begin{gathered} \lambda_i=\lambda_{z_i}, \quad d_i(\cdot,\cdot)=d_{z_i}(\cdot,\cdot) \\ Q_i=Q_{r_{z_i},\lambda_{z_i}}, \quad Q_i^q=Q^q_{r_{z_i},\lambda_{z_i}}, \quad Q_i^s=Q^s_{r_{z_i},\lambda_{z_i}}, \quad Q_i^{q,s}=Q^{q,s}_{r_{z_i},\lambda_{z_i}} \end{gathered}$$

and

$$U_i = B_i \times I_i = \begin{cases} Q_i & \text{if } K^2\lambda_i^p \geq a(z_i)\lambda_i^q \text{ and } K^2\lambda_i^p \geq b(z_i)\lambda_i^s, \\ Q_i^q & \text{if } K^2\lambda_i^p < a(z_i)\lambda_i^q \text{ and } K^2\lambda_i^p \geq b(z_i)\lambda_i^s, \\ Q_i^s & \text{if } K^2\lambda_i^p \geq a(z_i)\lambda_i^q \text{ and } K^2\lambda_i^p < b(z_i)\lambda_i^s, \\ Q_i^{q,s} & \text{if } K^2\lambda_i^p < a(z_i)\lambda_i^q \text{ and } K^2\lambda_i^p < b(z_i)\lambda_i^s. \end{cases} \tag{2.9}$$

Moreover, we denote

$$\begin{gathered} d_i(U_i, E(\Lambda)) = \inf_{z \in U_i, w \in E(\Lambda)} d_i(z, w), \quad \mathcal{I} = \Big\{ j \in \mathbb{N} : \frac{2}{K} U_i \cap \frac{2}{K} U_j \neq \emptyset \Big\}, \\ K_i = 200K^3 \quad \text{for } U_i = Q_i, Q_i^q \text{ or } Q_i^s \qquad \text{and} \qquad K_i = 200 \quad \text{for } U_i = Q_i^{q,s}. \end{gathered} \tag{2.10}$$

2.2 Some preliminary estimates

We start with some basic estimates.

Lemma 2.2. *Let $z \in E(\Lambda)^c$. Assume $K^2\lambda_z^p < a(z)\lambda_z^q$ and $K^2\lambda_z^p < b(z)\lambda_z^s$. Then $\frac{a(z)}{2} \leq a(\tilde{z}) \leq 2a(z)$ and $\frac{b(z)}{2} \leq b(\tilde{z}) \leq 2b(z)$ for any $\tilde{z} \in 200KQ_{r_z}(z)$.*

Proof. We claim that $2[a]_\alpha(200Kr_z)^\alpha < a(z)$ and $2[b]_\beta(200Kr_z)^\beta < b(z)$. We will only prove the second statement, and the proof of the first statement is similar. On the contrary, let us assume $b(z) \leq 2[b]_\beta(200Kr_z)^\beta$. Since $Q_{r_z,\lambda_z}^{q,s}(z) \subset E(\Lambda)^c$ and $a(z)\lambda_z^q + b(z)\lambda_z^s < \Lambda$, we get

$$a(z)\lambda_z^q + b(z)\lambda_z^s < \fint\!\!\fint_{Q_{r_z,\lambda_z}^{q,s}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw.$$

Now using $\Lambda = \lambda_z^p + a(z)\lambda_z^q + b(z)\lambda_z^s < 2[a(z)\lambda_z^q + b(z)\lambda_z^s]$, we obtain

$$\begin{aligned} a(z)\lambda_z^q + b(z)\lambda_z^s &< \fint\!\!\fint_{Q_{r_z,\lambda_z}^{q,s}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw \\ &= \frac{\Lambda\lambda_z^{-2}}{2|B_1|r_z^{n+2}} \iint_{Q_{r_z,\lambda_z}^{q,s}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw \\ &< \frac{(a(z)\lambda_z^q + b(z)\lambda_z^s)\lambda_z^{-2}}{|B_1|r_z^{n+2}} \iint_{Q_{r_z,\lambda_z}^{q,s}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw. \end{aligned}$$

Raising the power to $\frac{\beta}{n+2}$ in both of sides to the above expression, we have

$$r_z^\beta\lambda_z^{\frac{2\beta}{n+2}} < \bigg(\frac{1}{|B_1|} \iint_{Q_{r_z,\lambda_z}^{q,s}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw\bigg)^{\frac{\beta}{n+2}} \leq \frac{K}{800[b]_\beta}.$$

Now using $K^2\lambda_z^p < b(z)\lambda_z^s$, $s \leq p + \frac{2\beta}{n+2}$ and the counter-assumption, we get

$$K^2\lambda_z^p < b(z)\lambda_z^s \leq 2[b]_\beta(200Kr_z)^\beta\lambda_z^p\lambda_z^{\frac{2\beta}{n+2}} \leq \frac{1}{2}K^2\lambda_z^p,$$

which is a contradiction. This completes the proof of $2[b]_\beta(200Kr_z)^\beta < b(z)$. Now using the Hölder continuity of $b(z)$, we get

$$2[b]_\beta(200Kr_z)^\beta < b(z) \leq \inf_{\tilde{z} \in 200KQ_{r_z}(z)} b(\tilde{z}) + [b]_\beta(200Kr_z)^\beta,$$

and hence

$$[b]_\beta(200Kr_z)^\beta \leq \inf_{\tilde{z} \in 200KQ_{r_z}(z)} b(\tilde{z}).$$

It follows that

$$\sup_{\tilde{z} \in 200KQ_{r_z}(z)} b(\tilde{z}) \leq \inf_{\tilde{z} \in 200KQ_{r_z}(z)} b(\tilde{z}) + [b]_\beta(200Kr_z)^\beta < 2 \inf_{\tilde{z} \in 200KQ_{r_z}(z)} b(\tilde{z})$$

and this proves $\frac{b(z)}{2} \leq b(\tilde{z}) \leq 2b(z)$ for any $\tilde{z} \in 200KQ_{r_z}(z)$. The statement for $a(\cdot)$ can be proved similarly. □

Lemma 2.3. *Let $z \in E(\Lambda)^c$. Assume $K^2\lambda_z^p \geq a(z)\lambda_z^q$ and $K^2\lambda_z^p < b(z)\lambda_z^s$. Then we have $\frac{b(z)}{2} \leq b(\tilde{z}) \leq 2b(z)$ for any $\tilde{z} \in 200KQ_{r_z}(z)$. Moreover, we have*

$$[a]_\alpha(800Kr_z)^\alpha\lambda_z^q \leq (K^2-1)\lambda_z^p. \tag{2.11}$$

Proof. First, we note that $Q^s_{r_z,\lambda_z}(z) \subset E(\Lambda)^c$, and therefore

$$\begin{aligned} g_s(z,\lambda_z) &< \fint\!\!\fint_{Q^s_{r_z,\lambda_z}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw \\ &= \frac{g_s(z,\lambda_z)\lambda_z^{-2}}{2|B_1|r_z^{n+2}} \iint_{Q^s_{r_z,\lambda_z}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw. \end{aligned}$$

Raising power $\frac{\alpha}{n+2}$ in the above expression, we have

$$r_z^\alpha\lambda_z^{\frac{2\alpha}{n+2}} < \Bigg(\frac{1}{|B_1|}\iint_{Q^s_{r_z,\lambda_z}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw\Bigg)^{\frac{\alpha}{n+2}} \leq \frac{K-1}{800[a]_\alpha}.$$

Using $q \leq p + \frac{2\alpha}{n+2}$, we get

$$[a]_\alpha(800Kr_z)^\alpha\lambda_z^q \leq [a]_\alpha 800Kr_z^\alpha\lambda_z^p\lambda_z^{\frac{2\alpha}{n+2}} \leq [a]_\alpha 800K\lambda_z^p\frac{K-1}{800[a]_\alpha} \leq (K^2-1)\lambda_z^p.$$

Moreover, the proof of the first statement follows from the previous lemma. This completes the proof. □

Lemma 2.4. *Let $z \in E(\Lambda)^c$. Assume $K^2\lambda_z^p < a(z)\lambda_z^q$ and $K^2\lambda_z^p \geq b(z)\lambda_z^s$. Then we have $\frac{a(z)}{2} \leq a(\tilde{z}) \leq 2a(z)$ for any $\tilde{z} \in 200KQ_{r_z}(z)$. Furthermore, we have*

$$[b]_\beta(800Kr_z)^\beta\lambda_z^s \leq (K^2-1)\lambda_z^p. \tag{2.12}$$

Proof. Since $Q^q_{r_z,\lambda_z} \subset E(\Lambda)^c$, we have

$$\begin{aligned} g_q(z,\lambda_z) &< \fint\!\!\fint_{Q^s_{r_z,\lambda_z}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw \\ &= \frac{g_q(z,\lambda_z)\lambda_z^{-2}}{2|B_1|r_z^{n+2}} \iint_{Q^s_{r_z,\lambda_z}(z)} \Big(M\Big(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}}\Big)(w)\Big)^{\frac{p}{d}}\,dw. \end{aligned}$$

Moreover, following the previous lemma, we get

$$[b]_\beta(800Kr_z)^\beta\lambda_z^s \leq (K^2-1)\lambda_z^p.$$

Also, the first statement follows from Lemma 2.2. This completes the proof. □

Lemma 2.5. *Let $z,\tilde{z} \in E(\Lambda)^c$. Assume $K^2\lambda_z^p \geq a(z)\lambda_z^q$ and $K^2\lambda_z^p < b(z)\lambda_z^s$, and $\tilde{z} \in 200KQ_{r_z}(z)$. Then we have $K^{-\frac{2}{p}}\lambda_{\tilde{z}} \leq \lambda_z \leq K^{\frac{2}{p}}\lambda_{\tilde{z}}$.*

Proof. From Lemma 2.2, we get

$$\frac{b(z)}{2} \leq b(\tilde{z}) \leq 2b(z). \tag{2.13}$$

Now, note that it is enough to prove $K^{-\frac{2}{p}}\lambda_{\tilde{z}} \leq \lambda_z$. On contrary, let us assume $\lambda_z < K^{-\frac{2}{p}}\lambda_{\tilde{z}}$. Then using (2.11), (2.13) and the counter assumption, we obtain

$$\begin{aligned} \Lambda &= \lambda_z^p + a(z)\lambda_z^q + b(z)\lambda_z^s \\ &\leq \lambda_z^p + [a]_\alpha(200Kr_z)^\alpha\lambda_z^q + a(\tilde{z})\lambda_z^q + b(z)\lambda_z^s \\ &< \lambda_z^p + (K^2-1)\lambda_z^p + a(\tilde{z})K^{-\frac{2q}{p}}\lambda_{\tilde{z}}^q + 2b(\tilde{z})K^{-\frac{2s}{p}}\lambda_{\tilde{z}}^s \\ &\leq K^2\lambda_z^p + a(\tilde{z})K^{-\frac{2q}{p}}\lambda_{\tilde{z}}^q + b(\tilde{z})K^{1-\frac{2s}{p}}\lambda_{\tilde{z}}^s \\ &< \lambda_{\tilde{z}}^p + a(\tilde{z})\lambda_{\tilde{z}}^q + b(\tilde{z})\lambda_{\tilde{z}}^s = \Lambda, \end{aligned}$$

which is a contradiction. □

Lemma 2.6. *Let $z, \tilde{z} \in E(\Lambda)^c$. Assume $K^2\lambda_z^p < a(z)\lambda_z^q$ and $K^2\lambda_z^p \geq b(z)\lambda_z^s$, and $\tilde{z} \in 200KQ_{r_z}(z)$. Then we have $K^{-\frac{2}{p}}\lambda_{\tilde{z}} \leq \lambda_z \leq K^{\frac{2}{p}}\lambda_{\tilde{z}}$.*

Proof. From Lemma 2.4, we have

$$\frac{a(z)}{2} \leq a(\tilde{z}) \leq a(z) \tag{2.14}$$

for any $\tilde{z} \in 200KQ_{r_z}(z)$. We will show $K^{-\frac{2}{p}} \leq \lambda_{\tilde{z}}$. On contrary, let us assume $K^{-\frac{2}{p}} > \lambda_{\tilde{z}}$. Then using (2.12) and (2.14), we get

$$\begin{aligned}\Lambda &= \lambda_z^p + a(z)\lambda_z^q + b(z)\lambda_z^s \\ &\leq \lambda_z^p + a(z)\lambda_z^q + [b]_\beta(200Kr_z)^\beta\lambda_z^s + b(\tilde{z})\lambda_z^s \\ &< \lambda_z^p + (K^2-1)\lambda_z^p + 2a(\tilde{z})K^{-\frac{2q}{p}}\lambda_{\tilde{z}}^q + b(\tilde{z})K^{-\frac{2s}{p}}\lambda_{\tilde{z}}^s \\ &\leq K^2\lambda_z^p + a(\tilde{z})K^{1-\frac{2q}{p}}\lambda_{\tilde{z}}^q + b(\tilde{z})K^{-\frac{2s}{p}}\lambda_{\tilde{z}}^s \\ &\leq \lambda_{\tilde{z}}^p + a(\tilde{z})\lambda_{\tilde{z}}^q + b(\tilde{z})\lambda_{\tilde{z}}^s = \Lambda,\end{aligned}$$

which gives a contradiction. This completes the proof. □

Lemma 2.7. *Let $z, \tilde{z} \in E(\Lambda)^c$. Assume $\frac{a(z)}{2} \leq a(\tilde{z}) 2a(z)$ and $\frac{b(z)}{2} \leq b(\tilde{z}) \leq 2b(z)$. Then we have $2^{-\frac{1}{p}}\lambda_{\tilde{z}} \leq \lambda_z \leq 2^{\frac{1}{p}}\lambda_{\tilde{z}}$. Moreover, the above inequality holds provided that $K^2\lambda_z^p < a(z)\lambda_z^q$ and $K^2\lambda_z^p < b(z)\lambda_z^s$, and $\tilde{z} \in 200KQ_{r_z}(z)$.*

Proof. It is enough to prove the first statement. The second statement of the lemma follows from Lemma 2.2. We claim $\lambda_z \leq 2^{\frac{1}{p}}\lambda_{\tilde{z}}$. In contrast, assume $\lambda_z > 2^{\frac{1}{p}}\lambda_{\tilde{z}}$. Using the hypothesis and the counter assumption, we get

$$\begin{aligned}\Lambda = \lambda_z^p + a(z)\lambda_z^q + b(z)\lambda_z^s &> 2\lambda_{\tilde{z}}^p + 2^{\frac{q}{p}-1}a(\tilde{z})\lambda_{\tilde{z}}^q + 2^{\frac{s}{p}-1}b(\tilde{z})\lambda_{\tilde{z}}^s \\ &\geq \lambda_{\tilde{z}}^p + a(\tilde{z})\lambda_{\tilde{z}}^q + b(\tilde{z})\lambda_{\tilde{z}}^s = \Lambda,\end{aligned}$$

which is a contradiction. □

Lemma 2.8. *Let $z \in E(\Lambda)^c$. Assume that $K^2\lambda_z^p \geq a(z)\lambda_z^q$ and $K^2\lambda_z^p \geq b(z)\lambda_z^s$. Then $[a]_\alpha(50Kr_z)^\alpha\lambda_z^q \leq (K^2-1)\lambda_z^p$ and $[b]_\beta(50Kr_z)^\beta\lambda_z^s \leq (K^2-1)\lambda_z^p$.*

Proof. The proof can be completed analogously to the argument presented in the proof of Lemma 2.2. □

Lemma 2.9. *Let $z \in E(\Lambda)^c$. Assume that $K^2\lambda_z^p \geq a(z)\lambda_z^q$ and $K^2\lambda_z^p \geq b(z)\lambda_z^s$. If $\tilde{z} \in 50KQ_{r_z}(z)$, then $\lambda_z \leq K^{\frac{2}{p}}\lambda_{\tilde{z}}$.*

Proof. The proof can be completed from Lemma 2.5 and Lemma 2.6. □

2.3 Vitali covering and their properties

In this subsection, we want to choose a countable collection of intrinsic cylinders from $\{U_z^\Lambda\}_{z\in E(\Lambda)^c}$ such that

$$E(\Lambda)^c \subset \bigcup_{z\in E(\Lambda)^c} \frac{1}{K}U_z^\lambda \quad \text{and} \quad \frac{1}{6K^6}U_{z_1}^\Lambda \cap \frac{1}{6K^6}U_{z_2}^\Lambda = \emptyset \quad \text{for any } z_1,\ z_2 \in E(\Lambda)^c.$$

First, we claim that $\{r_z : z \in E(\Lambda)^c\}$ is uniformly bounded. Indeed, since, by (2.3), $0 < |E(\Lambda)^c| < \infty$ and $\Lambda > 1 + \|a\|_{L^\infty(\mathbb{R}^{n+1})} + \|b\|_{L^\infty(\mathbb{R}^{n+1})}$, there exist $\lambda > 1$ and $R > 0$, which are in particular independent of $z \in E(\Lambda)^c$, such that

$$\lambda^p + \|a\|_{L^\infty(\mathbb{R}^{n+1})}\lambda^q + \|b\|_{L^\infty(\mathbb{R}^{n+1})}\lambda^s = \Lambda \tag{2.15}$$

and $|B_R \times (-\lambda^2\Lambda^{-1}R^2, \lambda^2\Lambda^{-1}R^2)| = |E(\Lambda)^c|$. It clear that $\lambda \leq \lambda_z \leq \Lambda$ for any $z \in E(\Lambda)^c$. Hence, if $r_z > R$ for some $z \in E(\Lambda)^c$, then

$$|E(\Lambda)^c| = |B_R \times (-\lambda^2\Lambda^{-1}R^2, \lambda^2\Lambda^{-1}R^2)| < |U_z^\Lambda| \leq |E(\Lambda)^c|.$$

This is a contradiction, and we conclude $r_z \leq R$ for any $z \in E(\Lambda)^c$.

Let $\mathcal{F} = \{\frac{1}{6K^6}U_z^\Lambda\}_{z\in E(\Lambda)^c}$ and, for each $j \in \mathbb{N}$,

$$\mathcal{F}_j = \left\{\frac{1}{6K^6}U_z^\Lambda \in \mathcal{F} : \frac{R}{2^j} < r_z \leq \frac{R}{2^{j-1}}\right\}.$$

Since $r_z \leq R$ for any $z \in E(\Lambda)^c$, we obtain

$$\mathcal{F} = \bigcup_{j\in\mathbb{N}} \mathcal{F}_j.$$

Then, in the same way as [28, Subsection 3.3], we have a countable subcollection $\mathcal{G}$ of pairwise disjoint cylinders in $\mathcal{F}$. From now on, we show that the $6K^5$-times the cylinders in $\mathcal{G}$ covers $E(\Lambda)^c$. Fix $\frac{1}{6K^6}U_{\tilde{z}}^{\Lambda} \in \mathcal{F}$. Then there exists $i \in \mathbb{N}$ such that $\frac{1}{6K^6}U_{\tilde{z}}^{\Lambda} \in \mathcal{F}_i$ and by the construction of $\mathcal{G}$ in [28] there exists a cylinder $\frac{1}{6K^6}U_{z}^{\Lambda} \in \bigcup_{j=1}^{i} \mathcal{G}_j$ such that

$$\frac{1}{6K^6}U_{\tilde{z}}^{\Lambda} \cap \frac{1}{6K^6}U_{z}^{\Lambda} \neq \emptyset. \tag{2.16}$$

Since $\frac{1}{6K^6}U_{\tilde{z}}^{\Lambda} \in \mathcal{F}_i$ and $\frac{1}{6K^6}U_{z}^{\Lambda} \in \mathcal{F}_j$ for some $j \in \{1,\dots,i\}$, we obtain

$$r_{\tilde{z}} \leq 2r_z. \tag{2.17}$$

We claim that

$$\frac{1}{6K^6}U_{\tilde{z}}^{\Lambda} \subset \frac{1}{K}U_{z}^{\Lambda}. \tag{2.18}$$

Let $z = (x,t)$ and $\tilde{z} = (\tilde{x},\tilde{t})$. By (2.16), (2.17) and the proof of the standard Vitali covering lemma, we obtain

$$\frac{1}{6K^6}B_{r_{\tilde{z}}}(y) \subset \frac{1}{K}B_{r_z}(x).$$

Moreover, by (2.17) and the standard Vitali covering argument, we obtain that $Q_{r_{\tilde{z}}}(\tilde{z}) \subset 5Q_{r_z}(z)$ and hence, we conclude

$$\tilde{z} \in 5Q_{r_z}(z) \subset 200KQ_{r_z}(z). \tag{2.19}$$

Now, we prove the inclusion in (2.18) in the time direction, by considering the sixteen cases depicted in Table 1.

$U_{\tilde{z}}^{\Lambda}$ U_{z}^{Λ}	$Q_{r_z,\lambda_z}(z)$	$Q^q_{r_z,\lambda_z}(z)$	$Q^s_{r_z,\lambda_z}(z)$	$Q^{q,s}_{r_z,\lambda_z}(z)$
$Q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$	(1-1)	(1-2)	(1-3)	(1-4)
$Q^q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$	(2-1)	(2-2)	(2-3)	(2-4)
$Q^s_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$	(3-1)	(3-2)	(3-3)	(3-4)
$Q^{q,s}_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$	(4-1)	(4-2)	(4-3)	(4-4)

Table 1: The combinations of $U_{\tilde{z}}^{\Lambda}$ and U_{z}^{Λ}.

Case (1-1). By (2.16) and (2.17), for $\tau \in \frac{1}{6K^6}I_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{t})$, we obtain

$$\begin{aligned}|\tau - t| &\leq |\tau - \tilde{t}| + |\tilde{t} - t| \\ &\leq 2\lambda_{\tilde{z}}^{2-p}\Big(\frac{r_{\tilde{z}}}{6K^6}\Big)^2 + \lambda_z^{2-p}\Big(\frac{r_z}{6K^6}\Big)^2 \\ &\leq (8\lambda_{\tilde{z}}^{2-p} + \lambda_z^{2-p})\Big(\frac{r_z}{6K^6}\Big)^2.\end{aligned}$$

Lemma 2.9 and (2.19) imply

$$\begin{aligned}|\tau - t| &\leq (8K^{\frac{2(p-2)}{p}} + 1)\lambda_z^{2-p}\Big(\frac{r_z}{6K^6}\Big)^2 \\ &\leq (9K^2)\lambda_z^{2-p}\Big(\frac{r_z}{6K^6}\Big)^2 \\ &\leq \lambda_z^{2-p}\Big(\frac{r_z}{K}\Big)^2,\end{aligned}$$

and hence $\tau \in \frac{1}{K}I_{r_z,\lambda_z}(t)$.

Cases (2-1), (3-1) and (4-1). Since $Q^q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z}) \subset Q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$, $Q^s_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z}) \subset Q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$ and $Q^{q,s}_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z}) \subset Q_{r_{\tilde{z}},\lambda_{\tilde{z}}}(\tilde{z})$, by the previous argument, we obtain the conclusion.

Case (1-2). By (2.16) and (2.17), for $\tau \in \frac{1}{6K^6} I_{r_{\tilde z},\lambda_{\tilde z}}(\tilde t)$, we obtain

$$\begin{aligned}|\tau - t| &\le |\tau - \tilde t| + |\tilde t - t|\\ &\le 2\lambda_{\tilde z}^{2-p}\Big(\frac{r_{\tilde z}}{6K^6}\Big)^2 + \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{6K^6}\Big)^2\\ &\le \Big(8\lambda_{\tilde z}^{2-p} + \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big)\Big(\frac{r_z}{6K^6}\Big)^2.\end{aligned}$$

From $a(\tilde z)\lambda_{\tilde z}^q \le K^2\lambda_{\tilde z}^p$, we have

$$\lambda_{\tilde z}^{2-p} = 2\frac{\lambda_{\tilde z}^2}{\lambda_{\tilde z}^p + \lambda_{\tilde z}^p} \le 2K^2\frac{\lambda_{\tilde z}^2}{g_q(\tilde z,\lambda_{\tilde z})}.$$

Since (1.5) implies $\frac{q}{p} \le 2$, we obtain from Lemma 2.4, Lemma 2.6, (1.5) and (2.19) that

$$\frac{\lambda_{\tilde z}^2}{g_q(\tilde z,\lambda_{\tilde z})} \le \frac{K^6\lambda_z^2}{\lambda_z^p + a(\tilde z)\lambda_z^q} \le 2K^6\frac{\lambda_z^2}{g(z,\lambda_z)}. \tag{2.20}$$

Thus, we conclude

$$|\tau - t| \le (32K^8 + 1)\frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{6K^6}\Big)^2 \le \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{K}\Big)^2$$

and hence $\tau \in \frac{1}{K} I^q_{r_z,\lambda_z}(t)$.

Cases (1-3), (1-4), (2-3), (3-2), (2-4) and (3-4). By Lemmas 2.4, 2.3, 2.5, 2.6 and 2.7, we obtain from the above argument the conclusion.

Case (2-2). By (2.16) and (2.17), for $\tau \in \frac{1}{6K^6}$, we obtain

$$\begin{aligned}|\tau - t| &\le |\tau - \tilde t| + |\tilde t - t|\\ &\le 2\frac{\lambda_{\tilde z}^2}{g_q(\tilde z,\lambda_{\tilde z})}\Big(\frac{r_{\tilde z}}{6K^6}\Big)^2 + \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{6K^6}\Big)^2\\ &\le \Big(\frac{8\lambda_{\tilde z}^2}{g_q(\tilde z,\lambda_{\tilde z})} + \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big)\Big(\frac{r_z}{6K^6}\Big)^2.\end{aligned}$$

Then (2.20) implies

$$|\tau - t| \le (16K^8 + 1)\frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{6K^6}\Big)^2 \le \frac{\lambda_z^2}{g_q(z,\lambda_z)}\Big(\frac{r_z}{K}\Big)^2$$

and hence $\tau \in \frac{1}{K} I^q_{r_z,\lambda_z}(t)$.

Cases (3-3) and (4-4). In the same way as the above case, we easily obtain the conclusion.

Cases (4-2) and (4-3). We obtain the conclusion for Cases (2-2) and (2-3) in the same way for Cases (2-1), (3-1) and (4-1).

Thus, we have established a countable covering family $\{\frac{1}{K}U_i\}_{i\in\mathbb{N}}$ of intrinsic cylinders defined as in (2.9) and with pairwise disjoint $\frac{1}{6K^6}U_i$. Now, we prove some properties of the collection $\{U_i\}_{i\in\mathbb{N}}$ that will be summarized in Lemma 2.18 at the end.

Lemma 2.10. *We have* $3r_i \le d_i(U_i, E(\Lambda)) \le 4r_i$ *for every* $i \in \mathbb{N}$.

Proof. Since, by the definition of r_i in (2.8),

$$d_i(U_i, E(\Lambda)) \le d_i(z_i, E(\Lambda)) = 4r_i,$$

the second inequality is satisfied. Moreover, from the triangle inequality we obtain that, for any $z \in U_i$,

$$d_i(z, E(\Lambda)) \ge d_i(z_i, E(\Lambda)) - d_i(z, z_i) \ge 4r_i - r_i = 3r_i.$$

Since $z \in U_i$ is arbitrary, we have $d_i(U_i, E(\Lambda)) \ge 3r_i$. □

Next, we prove the property (v) in Lemma 2.18.

Lemma 2.11. *We have $(12K^2)^{-1}r_j \le r_i \le 12K^2 r_j$ for every $i \in \mathbb{N}$ and $j \in \mathcal{I}$. Moreover, if $U_j = Q_j^{q,s}$, then $r_i \le 12r_j$*

Proof. It is enough to prove that

$$r_i \le 12K^2 r_j$$

for $j \in \mathbb{N}$. If $r_i < r_j$, then clearly the conclusion is satisfied. Thus, we assume $r_j \le r_i$. Then

$$z_j \in 4Q_{r_i}(z_i). \tag{2.21}$$

Let $w \in \frac{2}{K}U_i \cap \frac{2}{K}U_j$. Since $d_i(z_i, w) \le \frac{2}{K}r_i \le 2r_i$ and $d_j(z_j, w) \le \frac{2}{K}r_j \le 2r_j$, using Lemma 2.10 and the triangle inequality as in [28, Lemma 3.8], we get

$$r_i \le d_i(w, E(\Lambda)) \quad \text{and} \quad d_j(w, E(\Lambda)) \le 6r_j. \tag{2.22}$$

To complete the proof, we consider 16 cases in Table 1 with $z = z_i$ and $\tilde{z} = z_j$.

Case (1-1), (2-1), (3-1), (4-1), (1-2), (1-3) and (1-4). Using Lemma 2.5, Lemma 2.6 and Lemma 2.7 and following the proof of Case 1, Case 2 and Case 3 in [28, Lemma 3.8], we have the conclusion.

Case (2-2). We obtain from Lemma 2.6 and $K^2\lambda_j^p \ge b(z_j)\lambda_j^s$ that

$$\begin{aligned} g_q(z_i, \lambda_i)\lambda_i^{-2} &\le g_{q,s}(z_i, \lambda_i)\lambda_i^{-2} \\ &\le K^{\frac{4}{p}} g_{q,s}(z_j, \lambda_j)\lambda_j^{-2} \\ &\le 2K^{2+\frac{4}{p}} g_q(z_j, \lambda_j)\lambda_j^{-2} \\ &\le 4K^4 g_q(z_j, \lambda_j)\lambda_j^{-2}, \end{aligned}$$

where the last inequality follows from $\frac{1}{p} \le \frac{1}{2}$. Therefore, we obtain

$$d_i(z, w) \le 2K^2 d_j(z, w) \quad \text{for any } z \in E(\Lambda),$$

and hence, by (2.22), we have the conclusion.

Case (2-3), (2-4), (3-3), (3-2) and (3-4). Proceed similarly to the proof above.

Case (4-4). Using Lemma 2.7 and following the proof of [28, Case 3 in Lemma 3.8], we have $r_i \le 12r_j$.

Case (4-2) and (4-3). Note that, by Lemma 2.7,

$$\begin{aligned} g_q(z_i, \lambda_i)\lambda_i^{-2} &\le g_{q,s}(z_i, \lambda_i)\lambda_i^{-2} \\ &= g_{q,s}(z_j, \lambda_j)\lambda_i^{-2} \\ &\le 2^{\frac{2}{p}} g_{q,s}(z_j, \lambda_j)\lambda_j^{-2} \end{aligned}$$

and, similarly,

$$g_s(z_i, \lambda_i)\lambda_i^{-2} \le 2^{\frac{2}{p}} g_{q,s}(z_i, \lambda_i)\lambda_j^{-2}.$$

Thus, we obtain $d_i(z, w) \le 2d_j(z, w)$ and, therefore, conclude $r_i \le 12r_j$. □

By this lemma, we get

$$\frac{2}{K}Q_{r_j}(z_j) \subset 200KQ_{r_i}(z_i) \tag{2.23}$$

for all $j \in \mathcal{I}$. Using this, we summarize Lemma 2.5, Lemma 2.6, Lemma 2.7 and Lemma 2.9.

Lemma 2.12. *For any $i \in \mathbb{N}$ and $j \in \mathcal{I}$, we have $K^{-\frac{2}{p}}\lambda_j \le \lambda_i \le K^{\frac{2}{p}}\lambda_j$. Moreover, if $U_i = Q_i^{q,s}$, then $2^{-\frac{1}{p}}\lambda_j \le \lambda_i \le 2^{\frac{1}{p}}\lambda_j$.*

We show from the previous two lemmas that the measures of the neighboring cylinders are comparable.

Lemma 2.13. *There exists c depending on n and K such that*

$$\sup_{\substack{i \in \mathbb{N} \\ j \in \mathcal{I}}} \frac{|U_i|}{|U_j|} \le c.$$

Proof. We divide it into sixteen cases as in Lemma 2.11.

Case (1-1), (1-2), (1-3) and (1-4). We obtain from $I_i \subset I_{r_i,\lambda_i}$, Lemma 2.11 and Lemma 2.12 that

$$\frac{|U_i|}{|U_j|} \le \frac{|Q_i|}{|Q_j|} = \frac{2|B_1| r_i^{n+2}\lambda_i^{2-p}}{2|B_1| r_j^{n+2}\lambda_j^{2-p}} \le \frac{(12K^2)^{n+2} r_j^{n+2} K^{\frac{2(p-2)}{p}} \lambda_j^{2-p}}{r_j^{n+2}\lambda_j^{2-p}} \le (12K^2)^{n+3}$$

for all $i \in \mathbb{N}$ and $j \in \mathcal{I}$.

Case (2-2), (2-4), (3-3), (3-4) and (4-4). We conclude in a similar way from Lemma 2.11 and Lemma 2.12.

Case (2-1). We get from $a(z_i)\lambda_i^q \le K^2\lambda_i^p$, $b(z_i)\lambda_i^s \le K^2\lambda_i^p$, Lemma 2.11 and Lemma 2.12 that

$$\begin{aligned}\frac{|U_i|}{|U_j|} = \frac{|Q_i|}{|Q_j^q|} &= \frac{2|B_1| r_i^{n+2}\lambda_i^2 g_q(z_j,\lambda_j)}{2|B_1| r_j^{n+2}\lambda_j^2\lambda_i^p} = \frac{r_i^{n+2}\lambda_i^2 g_q(z_j,\lambda_j)}{r_j^{n+2}\lambda_j^2\lambda_i^p}\\ &\le \frac{(12K^2)^{n+2}K^{\frac{4}{p}}\lambda_j^2 r_j^{n+2} g_q(z_j,\lambda_j)}{\lambda_j^2 r_j^{n+2}\lambda_i^p} = \frac{3(12K^2)^{n+3} g_q(z_j,\lambda_j)}{\lambda_i^p+\lambda_i^p+\lambda_i^p}\\ &\le \frac{2(12K^2)^{n+3}K^2\Lambda}{\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s} \le c(n,K)\end{aligned}$$

for all $i \in \mathbb{N}$ and $j \in \mathcal{I}$.

Case (3-1), (4-1), (3-2), (2-3), (4-2), (4-3). Similarly, we obtain the conclusion. □

Now, we establish the inclusion property of U_i and U_j for any $i \in \mathbb{N}$ and $j \in \mathcal{I}$.

Lemma 2.14. *Let $i \in \mathbb{N}$ be such that $U_i = Q_i$. We have $\frac{2}{K}U_j \subset 50K^2 Q_i$ for every $j \in \mathcal{I}$.*

Proof. Since $\frac{2}{K}B_i \cap \frac{2}{K}B_j \ne \emptyset$ and $r_j \le 12K^2 r_i$, we clearly obtain $\frac{2}{K}B_j \subset 50KB_i$. It remains to prove the inclusion in the time direction. As $I_j \subset I_{r_j,\lambda_j}(t_j)$, for $\tau \in \frac{2}{K}I_j$, we have from Lemma 2.11 and Lemma 2.12 that

$$\begin{aligned}|\tau - t_i| \le |\tau - t_j| + |t_j - t_i| &\le 2\lambda_j^{2-p}\Big(\frac{2}{K}r_j\Big)^2 + \lambda_i^{2-p}\Big(\frac{2}{K}r_i\Big)^2\\ &\le 2(K^{-\frac{2}{p}}\lambda_i)^{2-p}(24Kr_i)^2 + \lambda_i^{2-p}(2r_i)^2 \le \lambda_i^{2-p}(50K^2 r_i)^2\end{aligned}$$

and hence $\frac{2}{K}U_j \subset 50K^2 Q_i$. □

Lemma 2.15. *Let $i \in \mathbb{N}$ be such that either $U_i = Q_i^q$ or $U_i = Q_i^s$ holds. Then $\frac{2}{K}U_j \subset 100K^3 U_i$ for every $j \in \mathcal{I}$.*

Proof. We may assume that $U_i = Q_i^q$. Since $\frac{2}{K}B_i \cap \frac{2}{K}B_j \ne \emptyset$ and $r_j \le 12K^2 r_i$, clearly, we obtain $\frac{2}{K}B_j \subset 50KB_i$. It remains to prove the inclusion in the time direction. Since $I_j \subset I_{r_j,\lambda_j}(t_j)$, it is enough to check only when $U_j = Q_j$. For $\tau \in \frac{2}{K}I_{r_j,\lambda_j}(t_j)$, we have

$$|\tau - t_i| \le |\tau - t_j| + |t_j - t_i| \le 2\lambda_j^{2-p}\Big(\frac{2}{K}r_j\Big)^2 + \frac{\lambda_i^2}{g_q(z_i,\lambda_i)}\Big(\frac{2}{K}r_i\Big)^2.$$

Since $K^2\lambda_j^p \ge a(z_j)\lambda_j^q$, $K^2\lambda_j^p \ge b(z_j)\lambda_j^s$ and Lemma 2.11 and Lemma 2.12 give $r_j \le 12K^2 r_i$ and $\lambda_j \le K^{\frac{2}{p}}\lambda_i$, we obtain

$$\begin{aligned}\lambda_j^{2-p}\Big(\frac{2}{K}r_j\Big)^2 &= 3\frac{\lambda_j^2}{\lambda_j^p+\lambda_j^p+\lambda_j^p}\Big(\frac{2}{K}r_j\Big)^2 \le 12\frac{\lambda_j^2}{\lambda_j^p + a(z_j)\lambda_j^q + b(z_j)\lambda_j^s} r_j^2\\ &\le 12\frac{K^{\frac{4}{p}}\lambda_i^2}{\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s}(12K^2 r_i)^2 \le \frac{\lambda_i^2}{g_q(z_i,\lambda_i)}(48K^3 r_i)^2.\end{aligned}$$

Thus, we have

$$|\tau - t_i| \le 2\frac{\lambda_i^2}{g_q(z_i,\lambda_i)}(48K^3 r_i)^2 + \frac{\lambda_i^2}{g_q(z_i,\lambda_i)}(2r_i)^2 \le \frac{\lambda_i^2}{g_q(z_i,\lambda_i)}(100K^3 r_i)^2,$$

and hence $\frac{2}{K}U_j \subset 100K^3 U_i$. □

Lemma 2.16. *Let $i \in \mathbb{N}$ be such that $U_i = Q_i^{q,s}$. Then we have $\frac{2}{K}U_j \subset 200U_i$ for each $j \in \mathcal{I}$.*

Proof. The proof can be obtained by using $r_j \le 12r_i$ and $2^{-\frac{1}{p}}\lambda_j \le \lambda_i \le 2^{\frac{1}{p}}\lambda_j$ instead of $(12K^2)^{-1}r_j \le r_i \le 12K^2 r_j$ and $K^{-\frac{2}{p}}\lambda_j \le \lambda_i \le K^{\frac{2}{p}}\lambda_j$ in the proof of the above lemma. □

Through the above three lemmas, we obtain the condition

$$\frac{2}{K}U_j \subset K_i U_i \quad \text{for all } i \in \mathbb{N} \text{ and } j \in \mathcal{I}, \tag{2.24}$$

where K_i is defined in (2.10). Finally, we prove that the cardinality of $\mathcal{I}$ is uniformly bounded.

Lemma 2.17. *There exists a constant c depending only on n and K such that $|\mathcal{I}| \le c$ for every $i \in \mathbb{N}$.*

Proof. Since the cylinders $\{\frac{1}{6K^6}U_j\}_{j\in\mathbb{N}}$ are disjoint, we get (2.24) and Lemma 2.13 that

$$|200K^4 U_i| \ge \Big|\bigcup_{j\in\mathcal{I}} \frac{1}{6K^6}U_j\Big| = \sum_{j\in\mathcal{I}} \Big|\frac{1}{6K^6}U_j\Big| \ge \sum_{j\in\mathcal{I}} c(n,K)|U_i| = c(n,K)|\mathcal{I}||U_i|,$$

and hence $|\mathcal{I}| \le c(n,K)$. □

We summarize the above results below.

Lemma 2.18. *Let K be as in (2.4) and $E(\Lambda)$ as in (2.2). There exists a collection $\{\frac{1}{K}U_i\}_{i\in\mathbb{N}}$ of cylinders defined as in (2.9) satisfying the following properties:*

(i) $E(\Lambda)^c \subset \bigcup_{i\in\mathbb{N}} \frac{1}{K}U_i$.
(ii) $\frac{1}{6K^6}U_i \cap \frac{1}{6K^6}U_j = \emptyset$ *for every $i,j \in \mathbb{N}$ with $i \ne j$.*
(iii) $3r_i \le d_i(U_i, E(\Lambda)) \le 4r_i$ *for every $i \in \mathbb{N}$.*
(iv) $4U_i \subset E(\Lambda)^c$ *and* $5U_i \cap E(\Lambda) \ne \emptyset$ *for every $i \in \mathbb{N}$.*
(v) $(12K^2)^{-1}r_j \le r_i \le 12K^2 r_j$ *for every $i \in \mathbb{N}$, $j \in \mathcal{I}$.*
(vi) $K^{-\frac{2}{p}}\lambda_j \le \lambda_i \le K^{\frac{2}{p}}\lambda_j$ *for every $j \in \mathcal{I}$.*
(vii) *There exists a constant $c = c(n,K)$ such that $|U_i| \le c|U_j|$ for every $i \in \mathbb{N}$ and $j \in \mathcal{I}$.*
(viii) $\frac{2}{K}U_j \subset K_i U_i$ *for every $i \in \mathbb{N}$, $j \in \mathcal{I}$.*
(ix) *If $U_i = Q_i$, then there exists a constant $c = c([a]_\alpha, \alpha, K)$ such that $r_i^\alpha \lambda_i^q \le c\lambda_i^p$ and $r_i^\beta \lambda_i^s \le c\lambda_i^p$.*
(x) *If $U_i = Q_i^q$, then $\frac{a(z_i)}{2} \le a(z) \le 2a(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$. If $U_i = Q_i^s$, then $\frac{b(z_i)}{2} \le b(z) \le 2b(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$ and if $U_i = Q_i^{q,s}$, then $\frac{a(z_i)}{2} \le a(z) \le 2a(z_i)$ and $\frac{b(z_i)}{2} \le b(z) \le 2b(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$.*
(xi) *For any $i \in \mathbb{N}$, the cardinality of $\mathcal{I}$, denoted by $|\mathcal{I}|$, is finite. Moreover, there exists a constant $c = c(n,K)$ such that $|\mathcal{I}| \le c$.*

2.4 Partition of unity

The following lemma demonstrates the construction of a partition of unity subordinate to Whitney decomposition $\{\frac{2}{K}U_i\}_{i\in\mathbb{N}}$.

Lemma 2.19. *There exists a partition of unity $\{\omega_i\}_{i\in\mathbb{N}}$ subordinate to the Whitney decomposition $\{\frac{2}{K}U_i\}_{i\in\mathbb{N}}$ with the following properties:*

(i) $0 \le \omega_i \le 1$, $\omega_i \in C_0^\infty(\frac{2}{K}U_i)$ *for every $i \in \mathbb{N}$ and $\sum_{i\in\mathbb{N}}\omega_i = 1$ on $E(\Lambda)^c$.*
(ii) *There exists a constant $c = c(n,K)$ such that $\|\nabla\omega_j\|_\infty \le cr_i^{-1}$ for every $i \in \mathbb{N}$ and $j \in \mathcal{I}$.*
(iii) *There exists a constant $c = c(n,K)$ such that*

$$\|\partial_t\omega_j\|_\infty \le \begin{cases} cr_i^{-2}\lambda_i^{p-2} & \text{if } U_i = Q_i, \\ cr_i^{-2}g_q(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^q, \\ cr_i^{-2}g_s(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^s, \\ cr_i^{-2}\Lambda\lambda_i^{-2} & \text{if } U_i = Q_i^{q,s} \end{cases}$$

for any $i \in \mathbb{N}$ and $j \in \mathcal{I}$.

Proof. In the previous subsection, we obtain the Whitney decomposition $\{\frac{2}{K}U_i\}_{i\in\mathbb{N}}$. Then, for each $i \in \mathbb{N}$, we choose $\psi_i \in C_0^\infty(\frac{2}{K}U_i)$ satisfying

$$0 \le \psi_i \le 1, \quad \psi_i \equiv 1 \text{ in } U_i, \quad \|\nabla\psi_i\|_\infty \le \frac{2}{K}r_i^{-1}$$

and

$$\|\partial_t\psi_i\|_\infty \le \begin{cases} \frac{2}{K}r_i^{-2}\lambda_i^{p-2} & \text{if } U_i = Q_i, \\ \frac{2}{K}r_i^{-2}g_q(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^q, \\ \frac{2}{K}r_i^{-2}g_s(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^s, \\ \frac{2}{K}r_i^{-2}\Lambda\lambda_i^{-2} & \text{if } U_i = Q_i^{q,s}. \end{cases}$$

Since $E(\Lambda)^c \subset \bigcup_{i\in\mathbb{N}} \frac{2}{K}U_i$ and $|\mathcal{I}|$ is finite for each $i \in \mathbb{N}$, the function

$$\omega_i(z) = \frac{\psi_i(z)}{\sum_{j\in\mathbb{N}}\psi_j(z)} = \frac{\psi_i(z)}{\sum_{j\in\mathcal{I}}\psi_j(z)}$$

is well-defined and satisfies

$$\omega_i \in C_0^\infty\Big(\frac{2}{K}U_i\Big), \quad 0 \le \omega_i \le 1 \text{ in } \frac{2}{K}U_i, \quad \sum_{i\in\mathbb{N}}\omega_i(z) = 1.$$

Thus, the collection $\{\omega_i\}_{i\in\mathbb{N}}$ is a partition of unity subordinate to $\{\frac{2}{K}U_i\}_{i\in\mathbb{N}}$. By Lemmas 2.11, 2.12 and 2.17, we have

$$\|\nabla\omega_j\|_\infty \le cr_i^{-1} \quad \text{and} \quad \|\partial_t\omega_j\|_\infty \le \begin{cases} cr_i^{-2}\lambda_i^{p-2} & \text{if } U_i = Q_i, \\ cr_i^{-2}g_q(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^q, \\ cr_i^{-2}g_s(z_i,\lambda_i)\lambda_i^{-2} & \text{if } U_i = Q_i^s, \\ cr_i^{-2}\Lambda\lambda_i^{-2} & \text{if } U_i = Q_i^{q,s} \end{cases}$$

for any $i \in \mathbb{N}$ and $j \in \mathcal{I}$. □

3 Construction of test function via Lipschitz truncation

The main goal of this section is to construct a Lipschitz function v_h^Λ which can be used as a test function in the proof of energy estimate Theorem 1.2. We begin by defining v_h and v_h^Λ, establishing a crucial Poincaré-type inequality for v_h. By combining this result with the properties of the Whitney decomposition, we conclude the Lipschitz regularity of v_h^Λ.

3.1 Definition of test function

Take $f = \chi_{U_{R_2,S_2}(z_0)}(|\nabla u| + |u - u_0| + |F|) \in L^p(\mathbb{R}^{n+1})$ and $u_0 = (u)_{U_{R_2,S_2}(z_0)}$, where χ is the characteristic function and u, F are extended to zero outside $U_{R_2,S_2}(z_0)$.

Let $0 < h_0 < \frac{S_2-S_1}{4}$ be a sufficiently small number, and let $\eta \in C_0^\infty(B_{R_2}(x_0))$ and $\zeta \in C_0^\infty(\ell_{S_2-h_0}(t_0))$ be standard cutoff functions satisfying $0 \le \eta \le 1$, $0 \le \zeta \le 1$, $\eta \equiv 1$ in $B_{R_1}(x_0)$, $\zeta \equiv 1$ in $\ell_{S_1}(t_0)$ and

$$\|\nabla\eta\|_\infty \le \frac{3}{R_2 - R_1}, \quad \|\partial_t\zeta\|_\infty \le \frac{3}{S_2 - S_1}. \tag{3.1}$$

Now for $0 < h < h_0$, we define the truncated solution

$$v_h(z) = [u(z) - u_0]_h\eta(x)\zeta(t), \quad z = (x,t) \in \mathbb{R}^{n+1}, \tag{3.2}$$

as a suitable candidate for test function to be used to prove energy estimates. For $z \in \mathbb{R}^{n+1}$, we define the Lipschitz truncation of v_h as

$$v_h^\Lambda(z) = v_h(z) - \sum_{i\in\mathbb{N}} (v_h(z) - v_h^i)\omega_i(z), \tag{3.3}$$

where

$$v_h^i = \begin{cases} \displaystyle\fint\!\!\fint_{\frac{2}{K}U_i} v_h(z)\,dz & \text{if } \frac{2}{K}U_i \subset U_{R_2,S_2}(z_0),\\ 0 & \text{elsewhere.}\end{cases}$$

Similarly, we define

$$v(z) = (u(z) - u_0)\eta(x)\zeta(t) \quad \text{and} \quad v^\Lambda(z) = v(z) - \sum_{i\in\mathbb{N}} (v(z) - v^i)\omega_i(z), \tag{3.4}$$

where

$$v^i = \begin{cases} \displaystyle\fint\!\!\fint_{\frac{2}{K}U_i} v(z)\,dz & \text{if } \frac{2}{K}U_i \subset U_{R_2,S_2}(z_0),\\ 0 & \text{elsewhere.}\end{cases}$$

3.2 Preliminary lemmas

In this subsection, we discuss some preparatory machinery to prove Lemma 3.5. We denote a family of parameters as

$$\text{data} \equiv \text{data}(n, N, p, q, s, \alpha, \beta, \nu, L, \|a\|_{L^\infty}, \|b\|_{L^\infty}, [a]_\alpha, [b]_\beta, R_1, R_2, S_1, S_2, K).$$

First, we recall the following lemma from [7, Lemma 8.1].

Lemma 3.1. *Let $f \in L^1(\Omega_T)$ and $h > 0$. Then there exists a constant $c = c(n)$ such that*

$$\fint\!\!\fint_{U_{r_1,r_2}(z_0)} [f]_h\,dz \le c \fint\!\!\fint_{[U_{r_1,r_2}(z_0)]_h} f\,dz,$$

where $[U_{r_1,r_2}(z_0)]_h = U_{r_1,r_2+h}(z_0)$.

Lemma 3.2. *Let $1 \le \gamma \le d$. Then for any cylinder $Q \subset \mathbb{R}^{n+1}$ such that $Q \cap E(\Lambda) \ne \emptyset$, we have*

$$\fint\!\!\fint_{Q} f^\gamma\,dz \le \Lambda^{\frac{\gamma}{p}}. \tag{3.5}$$

Moreover, there exists a constant $c = c(\text{data})$ such that

$$\fint\!\!\fint_{4K_iU_i} f^\gamma\,dz \le c\lambda_i^\gamma, \tag{3.6}$$

where K_iU_i is defined in (2.9)-(2.10).

Proof. Let $w \in Q \cap E(\Lambda)$. Then by (2.1) and (2.2), we have

$$\fint\!\!\fint_{Q} f^\gamma\,dz \le \bigg(\fint\!\!\fint_{Q} f^d\,dz\bigg)^{\frac{\gamma}{d}} \le \Big(M(f^d)(w)\Big)^{\frac{\gamma}{d}} \le \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})(w)\Big)^{\frac{\gamma}{d}} \le \Lambda^{\frac{\gamma}{p}}$$

and that proves (3.5). By Lemma 2.18 (iv), we have $4K_iU_i \cap E(\Lambda) \ne \emptyset$ and it follows that

$$\fint\!\!\fint_{4K_iU_i} f^\gamma\,dz \le \Lambda^{\frac{\gamma}{p}}.$$

Case I: $U_i = Q_i$. In this case, we have $K^2\lambda_i^p \geq a(z_i)\lambda_i^q$ and $K^2\lambda_i^p \geq b(z_i)\lambda_i^s$. Using this, we have

$$\Lambda = \lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s \leq (2K^2+1)\lambda_i^p.$$

Hence, we get

$$\fint\!\!\fint_{4K_iU_i} f^\gamma\,dz \leq \Lambda^{\frac{\gamma}{p}} \leq (2K^2+1)^{\frac{\gamma}{p}}\lambda_i^\gamma = c\lambda_i^\gamma$$

and this proves (3.6) for this case.

Case II: $U_i = Q_i^q$. In this case, we have $K^2\lambda_i^p < a(z_i)\lambda_i^q$ and $K^2\lambda_i^p \geq b(z_i)\lambda_i^s$. By Lemma 2.18 (x), we also have $\frac{a(z_i)}{2} \leq a(z) \leq 2a(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$. Then we have

$$(a(z_i))^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (f^q)^{\frac{d}{p}}\,dz = \fint\!\!\fint_{4K_iU_i} (a(z_i)f^q)^{\frac{d}{p}}\,dz \leq 2^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (a(z)f^q)^{\frac{d}{p}}\,dz. \tag{3.7}$$

By Lemma 2.18 (iv), there exists a $w \in 4K_iU_i \cap E(\Lambda)$ and we obtain

$$\fint\!\!\fint_{4K_iU_i} (a(z)f^q)^{\frac{d}{p}}\,dz \leq M\Big((af^q)^{\frac{d}{p}}\Big)(w) \leq \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})(w)\Big) \leq \Lambda^{\frac{d}{p}}.$$

Hence, from (3.7), we have

$$\begin{aligned} a(z_i)^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (f^q)^{\frac{d}{p}}\,dz &\leq c\Lambda^{\frac{d}{p}} = c\Big(\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s\Big)^{\frac{d}{p}} \\ &\leq c\Big(2a(z_i)\lambda_i^q + K^2\lambda_i^p\Big)^{\frac{d}{p}} \leq (3c)^{\frac{d}{p}}(a(z_i)\lambda_i^q)^{\frac{d}{p}}. \end{aligned}$$

Since $a(z_i) > 0$, we get

$$\fint\!\!\fint_{4K_iU_i} (f^q)^{\frac{d}{p}}\,dz \leq c\lambda_i^{\frac{qd}{p}}$$

and finally, we have

$$\fint\!\!\fint_{4K_iU_i} f^\gamma\,dz \leq \bigg(\fint\!\!\fint_{4K_iU_i} f^{\frac{qd}{p}}\,dz\bigg)^{\frac{\gamma p}{qd}} \leq c\lambda_i^\gamma,$$

which proves (3.6) for this case.

Case III: $U_i = Q_i^s$. In this case, we have $K^2\lambda_i^p \geq a(z_i)\lambda_i^q$ and $K^2\lambda_i^p < b(z_i)\lambda_i^s$. By Lemma 2.18 (x), we also have $\frac{b(z_i)}{2} \leq b(z) \leq 2b(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$. Then we have

$$(b(z_i))^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (f^s)^{\frac{d}{p}}\,dz = \fint\!\!\fint_{4K_iU_i} (b(z_i)f^s)^{\frac{d}{p}}\,dz \leq 2^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (b(z)f^s)^{\frac{d}{p}}\,dz. \tag{3.8}$$

By Lemma 2.18 (iv), there exists a $w \in 4K_iU_i \cap E(\Lambda)$ and we obtain

$$\fint\!\!\fint_{4K_iU_i} (b(z)f^s)^{\frac{d}{p}}\,dz \leq M\Big((bf^s)^{\frac{d}{p}}\Big)(w) \leq \Big(M(f^d + (af^q)^{\frac{d}{p}} + (bf^s)^{\frac{d}{p}})(w)\Big) \leq \Lambda^{\frac{d}{p}}.$$

Hence, from (3.8), we have

$$\begin{aligned} b(z_i)^{\frac{d}{p}} \fint\!\!\fint_{4K_iU_i} (f^q)^{\frac{d}{p}}\,dz &\leq c\Lambda^{\frac{d}{p}} = c\Big(\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s\Big)^{\frac{d}{p}} \\ &\leq c\Big(2b(z_i)\lambda_i^q + K^2\lambda_i^p\Big)^{\frac{d}{p}} \leq (3c)^{\frac{d}{p}}(b(z_i)\lambda_i^q)^{\frac{d}{p}}. \end{aligned}$$

Since $b(z_i) > 0$, we get

$$\fint\!\!\fint_{4K_iU_i} (f^s)^{\frac{d}{p}}\,dz \leq c\lambda_i^{\frac{sd}{p}}$$

and finally, we have

$$\fint\!\!\fint_{4K_iU_i} f^{\gamma}\,dz \le \left(\fint\!\!\fint_{4K_iU_i} f^{\frac{sd}{p}}\,dz\right)^{\frac{\gamma p}{sd}} \le c\lambda_i^{\gamma},$$

which proves (3.6) for this case.

Case IV: $U_i = Q_i^{q,s}$. In this case, we have $K^2\lambda_i^p < a(z_i)\lambda_i^q$ and $K^2\lambda_i^p < b(z_i)\lambda_i^s$. By Lemma 2.18 (x), we also have $\frac{a(z_i)}{2} \le a(z) \le 2a(z_i)$ and $\frac{b(z_i)}{2} \le b(z) \le 2b(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$. Now we consider two cases: either $a(z_i)\lambda_i^q \le b(z_i)\lambda_i^s$ or $a(z_i)\lambda_i^q \ge b(z_i)\lambda_i^s$. The first case corresponds to case III and the second case corresponds to case II and we arrive at the same conclusion. □

Now we prove a parabolic Poincaré-type result.

Lemma 3.3. *Let $U = B_{r_1} \times \ell_{r_2}$ be any cylinder defined in (1.9) and (1.10) satisfying $B_{r_1} \subset B_{R_2}(x_0)$. The the following estimates hold:*

(i) *There exists a constant $c = c(\text{data})$ such that*

$$\fint\!\!\fint_{U} |v_h - (v_h)_U|\,dz \le c\frac{r_2}{r_1}\fint\!\!\fint_{[U]_h} \left(f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1}\right)dz + c(r_1 + r_2)\fint\!\!\fint_{[U]_h} f\,dz. \tag{3.9}$$

(ii) *If in addition $\ell_{r_2} \cap \ell_{S_2}(t_0)^c \neq \emptyset$, then there exists a constant $c = c(\text{data})$ such that*

$$\fint\!\!\fint_{U} |v_h|\,dz \le c\frac{r_2}{r_1}\fint\!\!\fint_{[U]_h} \left(f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1}\right)dz + c(r_1 + r_2)\fint\!\!\fint_{[U]_h} f\,dz. \tag{3.10}$$

In addition, the above estimates hold with v_h and $[U]_h$ replaced by v and U.

Proof. For $t_1, t_2 \in \ell_{r_2}$, $t_1 \le t_2$, let $\zeta_\delta \in W_0^{1,\infty}(\ell_{r_2})$ be a piecewise linear cut-off function defined by

$$\zeta_\delta(t) = \begin{cases} 0, & t \in (-\infty, t_1 - \delta), \\ 1 + \dfrac{t - t_1}{\delta}, & t \in [t_1 - \delta, t_1], \\ 1, & t \in (t_1, t_2), \\ 1 - \dfrac{t - t_2}{\delta}, & t \in [t_2, t_2 + \delta], \\ 0, & t \in (t_2 + \delta, \infty). \end{cases}$$

Furthermore, let $\varphi \in C_0^\infty(B_{r_1})$ be a nonnegative function satisfying

$$\fint_{B_{r_1}} \varphi\,dx = 1, \quad \|\nabla\varphi\|_\infty \le \frac{c(n)}{r_1}, \quad \|\varphi\|_\infty \le c(n).$$

By taking a test function $\psi = \varphi\eta\zeta\zeta_\delta \in W_0^{1,\infty}(U \cap U_{R_2,S_2-h}(z_0))$ in the Steklov averaged weak formulation of (1.1), where standard cutoff functions η and ζ are defined in Section 3.1, we obtain

$$\iint_U -[u - u_0]_h \cdot \varphi\eta\partial_t(\zeta\zeta_\delta)\,dz = \iint_U [-\mathcal{A}(\cdot, \nabla u) + \mathcal{B}(\cdot, F)]_h \cdot \nabla\psi\,dz.$$

By the definition of f, (1.2) and (1.3), we have

$$\begin{aligned} \mathrm{I} &= \left|\iint_U -[u - u_0]_h \cdot \varphi\eta\zeta\partial_t\zeta_\delta\,dz\right| \\ &\le \left|\iint_U [f]_h \cdot \eta\varphi\zeta_\delta\partial_t\zeta\,dz\right| + c(L)\left|\iint_U [f^{p-1} + a(\cdot)f^{q-1} + b(\cdot)f^{s-1}]_h \cdot \nabla\psi\,dz\right| \\ &= \mathrm{II} + \mathrm{III}. \end{aligned}$$

First, we get from Lemma 3.1 that

$$\mathrm{II} \le \iint_U [f]_h|\partial_t\zeta|\|\varphi\|_\infty\,dz \le c(n, S_1, S_2)r_1^n r_2 \fint\!\!\fint_{[U]_h} f\,dz$$

and

$$\begin{aligned} \mathrm{III} &\le c(L) \iint_U [f^{p-1} + a(\cdot)f^{q-1} + b(\cdot)f^{s-1}]_h |\nabla\psi|\, dz \\ &\le c(L) \iint_U [f^{p-1} + a(\cdot)f^{q-1} + b(\cdot)f^{s-1}]_h |\varphi\nabla\eta + \eta\nabla\varphi|\, dz \\ &\le c(L) \iint_U [f^{p-1} + a(\cdot)f^{q-1} + b(\cdot)f^{s-1}]_h (\varphi + \eta)|\nabla\eta + \nabla\varphi|\, dz \\ &\le c(n,L) r_1^n r_2 \Big(\frac{1}{R_2 - R_1} + \frac{1}{r_1}\Big) \fint\!\!\fint_{[U]_h} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\, dz. \end{aligned}$$

Note that $r_1 \le R_2$, and hence

$$\frac{1}{R_2 - R_1} + \frac{1}{r_1} \le 2\max\Big\{\frac{R_2}{R_2 - R_1}, 1\Big\}\frac{1}{r_1}.$$

Therefore, we have

$$\mathrm{III} \le c(n, L, R_1, R_2) r_1^{n-1} r_2 \fint\!\!\fint_{[U]_h} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\, dz.$$

Next, we obtain from the one-dimensional Lebesgue differentiation theorem that

$$\begin{aligned} \lim_{\delta\to 0^+} \mathrm{I} &= \Big| \lim_{\delta\to 0^+} \iint_U [u - u_0]_h \cdot \varphi\eta\zeta\partial_t\zeta_\delta\, dz \Big| \\ &= \Big| \int_{B_{r_1}\times\{t_1\}} v_h\varphi\, dz - \int_{B_{r_1}\times\{t_2\}} v_h\varphi\, dz \Big| \\ &= |B_1| {r_1}^n |(v_h\varphi)_{B_{r_1}}(t_1) - (v_h\varphi)_{B_{r_1}}(t_2)|. \end{aligned}$$

Combining the above inequalities, we conclude that

$$\operatorname*{ess\,sup}_{t_1,t_2\in\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}(t_1) - (v_h\varphi)_{B_{r_1}}(t_2)| \le c r_2 \fint\!\!\fint_{[U]_h} f\, dz + c r_1^{-1} r_2 \fint\!\!\fint_{[U]_h} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\, dz, \tag{3.11}$$

where c depends only on n, L, S_1, S_2, R_1 and R_2.

Now, we estimate the left-hand sides of (3.9) and (3.10). First, we prove (3.9). By the standard Poincaré inequality, we have

$$\begin{aligned} \fint\!\!\fint_U |v_h - (v_h)_U|\, dz &\le \fint\!\!\fint_U |v_h - (v_h)_{B_{r_1}}|\, dz + \fint\!\!\fint_U |(v_h)_{B_{r_1}} - (v_h)_U|\, dz \\ &\le c(n) r_1 \fint\!\!\fint_U |\nabla v_h|\, dz + \fint\!\!\fint_U |(v_h)_{B_{r_1}} - (v_h)_U|\, dz. \end{aligned} \tag{3.12}$$

It follows from the definition of f, Lemma 3.1 and (3.1) that

$$\fint\!\!\fint_U |\nabla v_h|\, dz = \fint\!\!\fint_U |[\nabla u]_h\eta\zeta + [u - u_0]_h\nabla\eta\zeta|\, dz \le c(n, R_1, R_2) \fint\!\!\fint_{[U]_h} f\, dz. \tag{3.13}$$

Moreover, we obtain

$$\begin{aligned} \fint\!\!\fint_U |(v_h)_{B_{r_1}}(\tau) - (v_h)_U|\, dz &= \fint_{\ell_{r_2}} |(v_h)_{B_{r_1}}(\tau) - (v_h)_U|\, d\tau \\ &= \fint_{\ell_{r_2}} \Big| \fint_{\ell_{r_2}} (v_h)_{B_{r_1}}(\tau) - (v_h)_{B_{r_2}}(\sigma)\, d\sigma \Big|\, d\tau \\ &\le 2 \fint_{\ell_{r_2}} |(v_h)_{B_{r_1}}(\tau) - (v_h\varphi)_{B_{r_1}}(\tau)|\, d\tau + \operatorname*{ess\,sup}_{t_1,t_2\in\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}(t_1) - (v_h\varphi)_{B_{r_1}}(t_2)|. \end{aligned} \tag{3.14}$$

To estimate the first term, we use the fact that $(\varphi)_{B_{r_1}} = 1$, the standard Poincaré inequality and (3.13) to obtain

$$\begin{aligned}\fint_{\ell_{r_2}} |(v_h)_{B_{r_1}}(\tau) - (v_h\varphi)_{B_{r_1}}(\tau)|\,d\tau &= \fint_{\ell_{r_2}} \left|(v_h)_{B_{r_1}}(\tau)\fint_{B_{r_1}}\varphi\,dx - \fint_{B_{r_1}\times\{\tau\}} v_h\varphi\,dx\right|d\tau \\ &= \fint_{\ell_{r_2}}\left|\fint_{B_{r_1}\times\{\tau\}}\varphi(v_h-(v_h)_{B_{r_1}})\,dx\right|d\tau \\ &\le c(n)\|\varphi\|_\infty r_1 \iint_U |\nabla v_h|\,dz \\ &\le c(n,R_1,R_2)r_1\iint_{[U]_h} f\,dz.\end{aligned} \tag{3.15}$$

Combining (3.11), (3.12), (3.13), (3.14) and (3.15), we have the conclusion.

Finally, we prove (3.10). As in (3.12), we obtain

$$\iint_U |v_h|\,dz \le \iint_U |v_h - (v_h)_{B_{r_1}}|\,dz + \fint_{\ell_{r_2}} |(v_h)_{B_{r_1}}|\,d\tau \le cr_1\iint_U |\nabla v_h|\,dz + \fint_{\ell_{r_2}}|(v_h)_{B_{r_1}}|\,d\tau. \tag{3.16}$$

Then we have

$$\fint_{\ell_{r_2}} |(v_h)_{B_{r_1}}|\,d\tau \le \fint_{\ell_{r_2}} |(v_h)_{B_{r_1}} - (v_h\varphi)_{B_{r_1}}|\,dt + \fint_{\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}|\,dt. \tag{3.17}$$

Since $\ell_{r_2}\cap\ell_{S_2}(t_0)^c \neq \emptyset$, there exists $t_2\in\ell_{r_2}$ such that $t_2\in\ell_{S_2}(t_0)^c$. Since $\zeta\equiv 0$ in $\ell_{S_2}(t_0)$, we have

$$\fint_{\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}|\,dt \le \operatorname*{ess\,sup}_{t\in\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}(t)| \le \operatorname*{ess\,sup}_{t_1,t_2\in\ell_{r_2}} |(v_h\varphi)_{B_{r_1}}(t_1) - (v_h\varphi)_{B_{r_1}}(t_2)|. \tag{3.18}$$

Thus, combining (3.11), (3.13), (3.15), (3.16), (3.17) and (3.18), we have the conclusion. □

Next, we recall the boundary version of the Poincaré inequality from [29, Theorem 6.22].

Lemma 3.4. *Let $B_\rho(x_0)\subset\mathbb{R}^n$ and $B_r\subset\mathbb{R}^n$ be balls that satisfy $B_r\cap B_\rho(x_0)^c\neq\emptyset$. Assume that $v\in W_0^{1,\eta}(B_\rho(x_0))$ with $1<\eta<\infty$. Moreover, let $1\le\sigma\le\frac{n\eta}{n-\eta}$ for $1<\eta<n$ and $1\le\sigma<\infty$ for $n\le\eta<\infty$. Then there exists a constant $c=c(n,\eta,\sigma)$ such that*

$$\left(\fint_{B_{4r}}|v|^\sigma\,dx\right)^{\frac{1}{\sigma}} \le cr\left(\fint_{B_{4r}}|\nabla v|^\eta\,dx\right)^{\frac{1}{\eta}}.$$

3.3 Poincaré-type inequality for the test function

In this subsection, we prove the Poincaré-type inequality for v_h.

Lemma 3.5. *Let K_iU_i be defined in* (2.10). *Then the following estimates hold:*

(i) *If $K_iU_i\subset U_{R_2,S_2}(z_0)$, then there exists a constant c such that*

$$\iint_{K_iU_i}|v_h-(v_h)_{K_iU_i}|\,dz \le c(\mathrm{data},\Lambda)r_i, \tag{3.19}$$

$$\iint_{K_iU_i}|v-(v)_{K_iU_i}|\,dz \le c(\mathrm{data})\lambda_i r_i. \tag{3.20}$$

(ii) *If $K_iU_i\not\subset U_{R_2,S_2}(z_0)$, then there exists a constant c such that*

$$\iint_{K_iU_i}|v_h|\,dz \le c(\mathrm{data},\Lambda)r_i, \tag{3.21}$$

$$\iint_{K_iU_i}|v|\,dz \le c(\mathrm{data})\lambda_i r_i. \tag{3.22}$$

Proof. We start by proving (3.20). Note that we assume $K_iU_i \subset U_{R_2,S_2}(z_0)$, and hence we can apply Lemma 3.3 with $Q = K_iU_i$. We consider the following cases.

Case I: $U_i = Q_i$. Using Lemma 3.3, we get

$$\iint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \le c\lambda_i^{2-p} r_i \iint_{K_iU_i} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\,dz + c(r_i + \lambda_i^{2-p} r_i^2) \iint_{K_iU_i} f\,dz.$$

Since $r_i \le R_2$ and $p \ge 2$, we have

$$c(r_i + \lambda_i^{2-p} r_i^2) \iint_{K_iU_i} f\,dz \le c(R_2) r_i \iint_{K_iU_i} f\,dz. \tag{3.23}$$

Plugging (3.23) in the above estimate, we get

$$\iint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \le c\lambda_i^{2-p} r_i \iint_{K_iU_i} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\,dz + c r_i \iint_{K_iU_i} f\,dz. \tag{3.24}$$

Since $U_i = Q_i$, using $K^2\lambda_i^p \ge a(z_i)\lambda_i^q$ and $K^2\lambda_i^p \ge b(z_i)\lambda_i^s$, we estimate

$$\begin{aligned}
\iint_{K_iU_i} a(z)f^{q-1} + b(z)f^{s-1}\,dz &\le \iint_{K_iU_i} |a(z) - a(z_i)| f^{q-1}\,dz + a(z_i) \iint_{K_iU_i} f^{q-1}\,dz \\
&\quad + \iint_{K_iU_i} |b(z) - b(z_i)| f^{s-1}\,dz + b(z_i) \iint_{K_iU_i} f^{s-1}\,dz \\
&\le [a]_\alpha (K_i r_i)^\alpha \iint_{K_iU_i} f^{q-1}\,dz + K^2\lambda_i^{p-q} \iint_{K_iU_i} f^{q-1}\,dz \\
&\quad + [b]_\beta (K_i r_i)^\beta \iint_{K_iU_i} f^{s-1}\,dz + K^2\lambda_i^{p-s} \iint_{K_iU_i} f^{s-1}\,dz.
\end{aligned}$$

Using Hölder's inequality and applying (3.6), we further estimate

$$\begin{aligned}
\iint_{K_iU_i} a(z)f^{q-1} + b(z)f^{s-1}\,dz &\le [a]_\alpha (K_i r_i)^\alpha \Big(\iint_{K_iU_i} f^{s-1}\,dz \Big)^{\frac{q-1}{s-1}} + K^2\lambda_i^{p-q} \Big(\iint_{K_iU_i} f^{s-1}\,dz \Big)^{\frac{q-1}{s-1}} \\
&\quad + [b]_\beta (K_i r_i)^\beta \lambda_i^{s-1} + K^2\lambda_i^{p-s}\lambda_i^{s-1} \\
&\le [a]_\alpha (K_i r_i)^\alpha \lambda_i^{q-1} + [b]_\beta (K_i r_i)^\beta \lambda_i^{s-1} + 2K^2\lambda_i^{p-1} \\
&\le c\lambda_i^{p-1},
\end{aligned}$$

where the last inequality follows from Lemma 2.18 (ix). Plugging the above estimate in (3.24), we obtain

$$\iint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \le c(\mathrm{data})\lambda_i r_i.$$

Case II: $U_i = Q_i^q$. Note that, in this case we have $K^2\lambda_i^p < a(z_i)\lambda_i^q$ and $K^2\lambda_i^p \ge b(z_i)\lambda_i^s$. Using Lemma 3.3, we get

$$\iint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \le \underbrace{c\,\frac{\lambda_i^2 r_i}{g_q(z_i,\lambda_i)} \iint_{K_iU_i} (f^{p-1} + a(z)f^{q-1} + b(z)f^{s-1})\,dz}_{J_1} + \underbrace{c\Big(r_i + \frac{\lambda_i^2 r_i^2}{g_q(z_i,\lambda_i)}\Big) \iint_{K_iU_i} f\,dz}_{J_2}.$$

Note that J_2 can be estimated as previous since $g_q(z_i,\lambda_i) > \lambda_i^p$. Now we estimate J_1. First we note that, from Lemma 2.4 we have $\frac{a(z_i)}{2} \le a(z) \le 2a(z_i)$ and $[b]_\beta(50Kr_i)^\beta\lambda_i^s \le (K^2-1)\lambda_i^p$. Hence we have

$$\begin{aligned}
J_1 &\le \frac{\lambda_i^2 r_i}{\lambda_i^p} \iint_{K_iU_i} f^{p-1}\,dz + \frac{\lambda_i^2 r_i}{a(z_i)\lambda_i^q} \iint_{K_iU_i} a(z)f^{q-1}\,dz + \frac{\lambda_i^2 r_i}{\lambda_i^p} \iint_{K_iU_i} b(z)f^{s-1}\,dz \\
&\le c\lambda_i r_i + \frac{2\lambda_i^2 r_i}{\lambda_i^q} \iint_{K_iU_i} f^{q-1}\,dz + \frac{\lambda_i^2 r_i}{\lambda_i^p} \Big([b]_\beta (K_i r_i)^\beta \iint_{K_iU_i} f^{s-1}\,dz + K^2\lambda_i^{p-s} \iint_{K_iU_i} f^{s-1}\,dz \Big) \le c\lambda_i r_i.
\end{aligned}$$

The last part of the above calculation follows from Case I.

Case III: $U_i = Q_i^s$. Note that, in this case we have $K^2\lambda_i^p \ge a(z_i)\lambda_i^q$ and $K^2\lambda_i^p < b(z_i)\lambda_i^s$. Using Lemma 3.3, we get

$$\iint_{K_iU_i} |v-(v)_{K_iU_i}|\,dz \le \underbrace{c\,\frac{\lambda_i^2 r_i}{g_s(z_i,\lambda_i)}\iint_{K_iU_i}(f^{p-1}+a(z)f^{q-1}+b(z)f^{s-1})\,dz}_{J_3} + \underbrace{c\Big(r_i+\frac{\lambda_i^2 r_i^2}{g_s(z_i,\lambda_i)}\Big)\iint_{K_iU_i} f\,dz}_{J_4}.$$

The estimates of J_3 and J_4 are similar to Case II. We use the Hölder regularity of $a(z)$ and bounds on $b(z)$, i.e., $\frac{b(z_i)}{2} \le b(z) \le 2b(z_i)$, and we can arrive at the same conclusion.

Case IV: $U_i = Q_i^{q,s}$. In this case, we have $K^2\lambda_i^p < a(z_i)\lambda_i^q$ and $K^2\lambda_i^p < b(z_i)\lambda_i^s$. and again using Lemma 3.3, we get

$$\iint_{K_iU_i} |v-(v)_{K_iU_i}|\,dz \le \underbrace{c\,\frac{\lambda_i^2 r_i}{g_{q,s}(z_i,\lambda_i)}\iint_{K_iU_i}(f^{p-1}+a(z)f^{q-1}+b(z)f^{s-1})\,dz}_{J_5} + \underbrace{c\Big(r_i+\frac{\lambda_i^2 r_i^2}{g_{q,s}(z_i,\lambda_i)}\Big)\iint_{K_iU_i} f\,dz}_{J_6}.$$

From Lemma 2.18 (x), we have $\frac{a(z_i)}{2} \le a(z) \le 2a(z_i)$ and $\frac{b(z_i)}{2} \le b(z) \le 2b(z_i)$ for every $z \in 200KQ_{r_i}(z_i)$. The estimate of J_6 is same as previous as $\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s > \lambda_i^p$. Next, we estimate J_5. Then we have

$$\begin{aligned} J_5 &\le \frac{\lambda_i^2 r_i}{\lambda_i^p}\iint_{K_iU_i} f^{p-1}\,dz + \frac{\lambda_i^2 r_i}{a(z_i)\lambda_i^q}\iint_{K_iU_i} a(z)f^{q-1}\,dz + \frac{\lambda_i^2 r_i}{b(z_i)\lambda_i^s}\iint_{K_iU_i} b(z)f^{s-1}\,dz \\ &\le \frac{\lambda_i^2 r_i}{\lambda_i^p}\iint_{K_iU_i} f^{p-1}\,dz + \frac{2\lambda_i^2 r_i}{\lambda_i^q}\iint_{K_iU_i} f^{q-1}\,dz + \frac{2\lambda_i^2 r_i}{\lambda_i^s}\iint_{K_iU_i} f^{s-1}\,dz \\ &\le c\lambda_i r_i, \end{aligned}$$

where the last inequality follows from (3.6). This completes the proof of (3.20).

Now we prove (3.22). If $K_iU_i \not\subset U_{R_2,S_2}(z_0)$, then either $K_iB_i \subset B_{R_2}(x_0)$ and $K_iI_i \cap l_{S_2}^c(t_0) \neq \emptyset$, or $K_iB_i \cap B_{R_2}^c \neq \emptyset$. In the first case, we may apply (3.10) of Lemma 3.3. Note that, since the right-hand side of (3.9) and (3.10) are the same, the proof follows from the previous case. In the second case, we apply Lemma 3.4 with $\sigma = 1$ and $\eta = d$ since $v(\cdot,t) \in W_0^{1,d}(B_{R_2}(x_0),\mathbb{R}^N)$. Indeed, we have

$$\iint_{K_iU_i} |v|\,dz \le cr_i\Big(\iint_{4K_iU_i} |\nabla v|^d\,dz\Big)^{\frac{1}{d}}$$

and using $|\nabla v| \le cf$, and (3.6) we obtain

$$\Big(\iint_{4K_iU_i} |\nabla v|^d\,dz\Big)^{\frac{1}{d}} \le c\Big(\iint_{4K_iU_i} f^d\,dz\Big)^{\frac{1}{d}} \le c\lambda_i,$$

which completes the proof of (3.22).

Next, we focus on the estimates involving Steklov averages, i.e., (3.19) and (3.21). If $K_iU_i \subset U_{R_2,S_2}(z_0)$, then from Lemma 3.3, we get

$$\iint_{K_iU_i} |v_h-(v_h)_{K_iU_i}|\,dz \le c\frac{|I_i|}{r_i}\iint_{[K_iU_i]_h}(f^{p-1}+a(z)f^{q-1}+b(z)f^{s-1})\,dz + c(|I_i|+r_i)\iint_{[K_iU_i]_h} f\,dz.$$

We note that $|I_i| \le r_i^2 \le r_iR_2$ and $a(z) \le \|a\|_\infty$, $b(z) \le \|b\|_\infty$. Moreover, we observe that for small $h > 0$, $[K_iU_i]_h$ may intersect $E(\Lambda)$, i.e., $[K_iU_i]_h \cap E(\Lambda) \neq \emptyset$. Using these observations and applying (3.5), we obtain

$$\begin{aligned} \iint_{K_iU_i} |v_h-(v_h)_{K_iU_i}|\,dz &\le cr_i\iint_{[K_iU_i]_h} f^{p-1}\,dz + cr_i\|a\|_\infty\iint_{[K_iU_i]_h} f^{q-1}\,dz + cr_i\|b\|_\infty\iint_{[K_iU_i]_h} f^{s-1}\,dz \\ &\le cr_i\Lambda^{\frac{p-1}{p}} + cr_i\|a\|_\infty\Big(\iint_{[K_iU_i]_h} f^{s-1}\,dz\Big)^{\frac{q-1}{s-1}} + cr_i\|b\|_\infty\Lambda^{\frac{s-1}{p}} \\ &\le cr_i\Big(\Lambda^{\frac{p-1}{p}} + \|a\|_\infty\Lambda^{\frac{q-1}{p}} + \|b\|_\infty\Lambda^{\frac{s-1}{p}}\Big) = c(\mathrm{data},\Lambda)r_i. \end{aligned}$$

This completes the proof of (3.19). To prove (3.21) in the case $K_iB_i \subset B_{R_2}(x_0)$ and $K_iI_i \cap l_{S_2}^c \neq \emptyset$, we need to estimate (3.10) which is the same as estimating (3.9). When $K_iB_i \cap B_{R_2} \neq \emptyset$, we may apply Lemma 3.4 to get

$$\fint\!\!\!\fint_{K_iU_i} |v_h|\,dz \leq cr_i\bigg(\fint\!\!\!\fint_{4K_iU_i} |\nabla v_h|^d\,dz\bigg)^{\frac{1}{d}}.$$

Now using Lemma 3.1 and (3.5), we estimate the right-hand side of the above expression as

$$\bigg(\fint\!\!\!\fint_{4K_iU_i} |\nabla v_h|^d\,dz\bigg)^{\frac{1}{d}} \leq c\bigg(\fint\!\!\!\fint_{[4K_iU_i]_h} |\nabla v|^d dz\bigg)^{\frac{1}{d}} \leq c\bigg(\fint\!\!\!\fint_{[4K_iU_i]_h} f^d\,dz\bigg)^{\frac{1}{d}} \leq c\Lambda^{\frac{1}{p}},$$

which gives the proof of (3.21). □

Corollary 3.6. *We have the following estimates on v_h^i, v_h^j and v^i, v^j for every $i \in \mathbb{N}$ and $j \in \mathcal{I}$:*

(i)
$$\fint\!\!\!\fint_{\frac{2}{K}U_i} |v - v^i|\,dz \leq c(\text{data})\lambda_i r_i, \tag{3.25}$$

(ii)
$$\fint\!\!\!\fint_{\frac{2}{K}U_i} |v_h - v_h^i|\,dz \leq c(\text{data}, \Lambda) r_i, \tag{3.26}$$

(iii)
$$|v_h^i - v_h^j| \leq c(\text{data}, \Lambda) r_i, \tag{3.27}$$

(iv)
$$|v^i - v^j| \leq c(\text{data})\lambda_i r_i. \tag{3.28}$$

Proof. We start by proving (i). Indeed, we obtain

$$\begin{aligned}\fint\!\!\!\fint_{\frac{2}{K}U_i} |v - v^i|\,dz &\leq \fint\!\!\!\fint_{\frac{2}{K}U_i} |v - (v)_{K_iU_i}|\,dz + \fint\!\!\!\fint_{\frac{2}{K}U_i} |(v)_{K_iU_i} - (v)_{\frac{2}{K}U_i}|\,dz \\ &\leq c\fint\!\!\!\fint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz.\end{aligned}$$

If $K_iU_i \subset U_{R_2,S_2}(z_0)$, then from (3.20) we get

$$\fint\!\!\!\fint_{\frac{2}{K}U_i} |v - v^i|\,dz \leq c\fint\!\!\!\fint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \leq c(\text{data})\lambda_i r_i.$$

If $K_iU_i \not\subset U_{R_2,S_2}(z_0)$, from (3.22) we have

$$\fint\!\!\!\fint_{\frac{2}{K}U_i} |v - v^i|\,dz \leq 2\fint\!\!\!\fint_{\frac{2}{K}U_i} |v|\,dz \leq c\fint\!\!\!\fint_{K_iU_i} |v|\,dz \leq c\lambda_i r_i.$$

The proof of (ii) follows from (3.19) and (3.21) as above. Now, let us prove (iv) and the proof of (iii) is similar. First, we assume $K_iU_i, K_jU_j \subset U_{R_2,S_2}(z_0)$. Using Lemma 2.18 (vii) and (viii), we estimate

$$\begin{aligned}|v^i - v^j| &\leq |v^i - (v)_{K_iU_i}| + |v^j - (v)_{K_iU_i}| \\ &= \bigg|\fint\!\!\!\fint_{\frac{2}{K}U_i} v - (v)_{K_iU_i}\,dz\bigg| + \bigg|\fint\!\!\!\fint_{\frac{2}{K}U_j} v - (v)_{K_iU_i}\,dz\bigg| \\ &\leq c\fint\!\!\!\fint_{K_iU_i} |v - (v)_{K_iU_i}|\,dz \leq c\lambda_i r_i,\end{aligned}$$

where the last inequality follows from (3.20). On the other hand, let $K_iU_i \not\subset U_{R_2,S_2}(z_0)$. In this case, again using Lemma 2.18 (vii) and (viii), we obtain

$$|v^i - v^j| \le \iint_{\frac{2}{K}U_i} |v|\,dz + \iint_{\frac{2}{K}U_j} |v|\,dz \le c \iint_{K_iU_i} |v|\,dz \le c\lambda_i r_i.$$

The other case $K_jU_j \not\subset U_{R_2,S_2}(z_0)$ follows from the fact that r_i, r_j and λ_i, λ_j are comparable. □

3.4 Bounds on Lipschitz truncation and its derivatives

In this subsection, we show that v_h^Λ, v^Λ and their gradients are bounded.

Lemma 3.7. *We have* $|v_h^\Lambda(z)| \le c(\text{data}, \Lambda)$ *and* $|v^\Lambda(z)| \le c(\text{data})\lambda_i$ *for every* $z \in U_i$.

Proof. Fix $z \in U_i$. Then, for each $j \in \mathcal{I}$, we obtain from Lemma 2.18 (viii), Lemma 3.2 and the definition of v^j that

$$|v^j| \le \iint_{\frac{2}{K}U_j} |v|\,dz \le c \iint_{K_iU_i} f\,dz \le c\lambda_i.$$

Moreover, we have from Lemma 2.19 that

$$v^\Lambda(z) = \sum_{j\in\mathbb{N}} v^j\omega_j(z) = \sum_{j\in\mathcal{I}} v^j\omega_j(z). \tag{3.29}$$

Therefore, we conclude from Lemma 2.18 (xi) and Lemma 2.19 that

$$|v^\Lambda(z)| \le \sum_{j\in\mathcal{I}} |v^j||\omega_j(z)| \le \sum_{j\in\mathcal{I}} |v^j| \le c\lambda_i.$$

On the other hand, for each $j \in \mathcal{I}$, we get from Lemma 3.1 and Lemma 3.2 that

$$|v_h^i| \le \iint_{\frac{2}{K}U_i} |v_h(z)|\,dz \le c \iint_{\frac{2}{K}[U_i]_h} f\,dz \le c(\text{data}, \Lambda).$$

As in (3.29), we have

$$v_h^\Lambda(z) = \sum_{j\in\mathcal{I}} v_h^j\omega_j(z), \tag{3.30}$$

and hence we have the conclusion. □

Lemma 3.8. *For any* $z \in U_i$, *we have*

$$|\nabla v_h^\Lambda(z)| \le c(\text{data}, \Lambda) \quad \textit{and} \quad |\nabla v^\Lambda(z)| \le c(\text{data})\lambda_i. \tag{3.31}$$

Furthermore, we have

$$|\partial_t v_h^\Lambda(z)| \le c(\text{data}, \Lambda) r_i^{-1} \tag{3.32}$$

and

$$|\partial_t v^\Lambda(z)| \le \begin{cases} c(\text{data}) r_i^{-1}\lambda_i^{p-1} & \textit{if } U_i = Q_i, \\ c(\text{data}) r_i^{-1} g_q(z_i,\lambda_i)\lambda_i^{-1} & \textit{if } U_i = Q_i^q, \\ c(\text{data}) r_i^{-1} g_s(z_i,\lambda_i)\lambda_i^{-1} & \textit{if } U_i = Q_i^s, \\ c(\text{data}) r_i^{-1}\Lambda\lambda_i^{-1} & \textit{if } U_i = Q_i^{q,s}. \end{cases} \tag{3.33}$$

Proof. Fix $z \in U_i$. Then, by (3.29) and (3.30), we get

$$\nabla v_h^\Lambda(z) = \nabla\Big(\sum_{j\in\mathcal{I}} v_h^j\omega_j(z)\Big) = \sum_{j\in\mathcal{I}} v_h^j\nabla\omega_j(z)$$

and

$$\nabla v^\Lambda(z) = \sum_{j\in\mathcal{I}} v^j\nabla\omega_j(z).$$

Note that Lemma 2.19 implies

$$0 = \nabla\Big(\sum_{j\in\mathbb{N}} \omega_j(z)\Big) = \nabla\Big(\sum_{j\in\mathfrak{J}} \omega_j(z)\Big) = \sum_{j\in\mathfrak{J}} \nabla\omega_j(z).$$

Thus, we get from Lemma 2.18 (xi), Lemma 2.19 and Corollary 3.6 that

$$|\nabla v_h^\Lambda(z)| = \Big|\sum_{j\in\mathfrak{J}} (v_h^j - v_h^i)\nabla\omega_j(z)\Big| \le \sum_{j\in\mathfrak{J}} |v_h^j - v_h^i||\nabla\omega_j| \le c(\mathrm{data}, \Lambda)$$

and

$$|\nabla v^\Lambda(z)| \le \sum_{j\in\mathfrak{J}} |v^j - v^i||\nabla\omega_j| \le c(\mathrm{data})\lambda_i.$$

Next, by the above arguments, we get

$$\partial_t v_h^\Lambda(z) = \sum_{j\in\mathfrak{J}} v_h^j \partial_t\omega_j(z) = \sum_{j\in\mathfrak{J}} (v_h^j - v_h^i)\partial_t\omega_j(z)$$

and

$$\partial_t v^\Lambda(z) = \sum_{j\in\mathfrak{J}} v^j \partial_t\omega_j(z) = \sum_{j\in\mathfrak{J}} (v^j - v^i)\partial_t\omega_j(z).$$

Therefore, by Lemma 2.18 (xi), Lemma 2.19 and Corollary 3.6, we have the conclusion. □

Lemma 3.9. *Let $E(\Lambda)$ be defined as in* (2.2)*. Then v_h^Λ, v^Λ, the Lipschitz truncation defined in* (3.3)–(3.4)*, satisfies the following estimates:*

(i)
$$\iint_{E(\Lambda)^c} |v_h - v_h^\Lambda||\partial_t v_h^\Lambda|\,dz \le c(\mathrm{data}, \Lambda)|E(\Lambda)^c|,$$

(ii)
$$\iint_{E(\Lambda)^c} |v - v^\Lambda||\partial_t v^\Lambda|\,dz \le c(\mathrm{data})\Lambda|E(\Lambda)^c|,$$

(iii)
$$H(z, |v^\Lambda|) + H(z, |\nabla v^\Lambda|) \le c(\mathrm{data})\Lambda$$

for almost every $z \in \mathbb{R}^{n+1}$.

Proof. We start with proving (i):

$$\begin{aligned}\iint_{E(\Lambda)^c} |v_h - v_h^\Lambda||\partial_t v_h^\Lambda|\,dz &\le \iint_{E(\Lambda)^c} \sum_{i\in\mathbb{N}} |v_h - v_h^i||\omega_i||\partial_t v_h^\Lambda|\,dz &(3.34)\\ &\le \sum_{i\in\mathbb{N}} \iint_{\frac{2}{K}U_i} |v_h - v_h^i||\omega_i||\partial_t v_h^\Lambda|\,dz\\ &\le \sum_{i\in\mathbb{N}} \|\partial_t v_h^\Lambda\|_{L^\infty(\frac{2}{K}U_i)} \iint_{\frac{2}{K}U_i} |v_h - v_h^i|\,dz. &(3.35)\end{aligned}$$

Now using Corollary 3.6 (ii) and (3.32), we can estimate the last term of the above inequality to obtain

$$\iint_{E(\Lambda)^c} |v_h - v_h^\Lambda||\partial_t v_h^\Lambda|\,dz \le c\sum_{i\in\mathbb{N}} |U_i| = c\sum_{i\in\mathbb{N}} \Big|\frac{1}{6K^6}U_i\Big| = c(\mathrm{data}, \Lambda)|E(\Lambda)^c|.$$

The proof of (ii) can be obtained similarly. To prove (iii), first let $z \in E(\Lambda)$. In this case,

$$\begin{aligned}&|v^\Lambda(z)|^p + a(z)|v^\Lambda(z)|^q + b(z)|v^\Lambda(z)|^s + |\nabla v^\Lambda(z)|^p + a(z)|\nabla v^\Lambda(z)|^q + b(z)|\nabla v^\Lambda(z)|^s\\ &\quad= |v(z)|^p + a(z)|v(z)|^q + b(z)|v(z)|^s + |\nabla v(z)|^p + a(z)|\nabla v(z)|^q + b(z)|\nabla v(z)|^s\\ &\quad\le (f^p + a(z)f^q + b(z)f^s) \le c\Lambda.\end{aligned}$$

Now let us consider $z \in E(\Lambda)^c$. Then $z \in U_i$ for some $i \in \mathbb{N}$. Using Lemma 3.7 and Lemma 3.8, we get

$$\begin{aligned}|v^\Lambda(z)|^p + a(z)|v^\Lambda(z)|^q + b(z)|v^\Lambda(z)|^s + |\nabla v^\Lambda(z)|^p + a(z)|\nabla v^\Lambda(z)|^q + b(z)|\nabla v^\Lambda(z)|^s\\ \le c(\lambda_i^p + a(z)\lambda_i^q + b(z)\lambda_i^s).\end{aligned}$$

Case I: $U_i = Q_i$. Using the Hölder regularity of $a(z)$ and $b(z)$, and Lemma 2.18, (ix) we obtain

$$\begin{aligned}\lambda_i^p + a(z)\lambda_i^q + b(z)\lambda_i^s &\le \lambda_i^p + a(z_i)\lambda_i^q + [a]_\alpha r_i^\alpha \lambda_i^q + b(z_i)\lambda_i^s + [b]_\beta r_i^\beta \lambda_i^s\\ &\le c(\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s) = c\Lambda.\end{aligned}$$

Case II: $U_i = Q_i^q$. In this case, we use Lemma 2.4 to get

$$\begin{aligned}\lambda_i^p + a(z)\lambda_i^q + b(z)\lambda_i^s &\le \lambda_i^p + 2a(z_i)\lambda_i^q + b(z_i)\lambda_i^s + [b]_\beta r_i^\beta \lambda_i^s\\ &\le c(\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s) = c\Lambda.\end{aligned}$$

Case III: $U_i = Q_i^s$. In this case, we use Lemma 2.3 to get

$$\begin{aligned}\lambda_i^p + a(z)\lambda_i^q + b(z)\lambda_i^s &\le \lambda_i^p + a(z_i)\lambda_i^q + 2b(z_i)\lambda_i^s + [a]_\alpha r_i^\alpha \lambda_i^q\\ &\le c(\lambda_i^p + a(z_i)\lambda_i^q + b(z_i)\lambda_i^s) = c\Lambda.\end{aligned}$$

Case IV: $U_i = Q_i^{q,s}$. In this case we use Lemma 2.2 and obtain

$$\lambda_i^p + a(z)\lambda_i^q + b(z)\lambda_i^s \le \lambda_i^p + 2a(z_i)\lambda_i^q + 2b(z_i)\lambda_i^s \le 2\Lambda.$$

This completes the proof. □

3.5 Lipschitz regularity of v_h^Λ

In this subsection, we show that v_h^Λ is Lipschitz continuous with respect to the metric

$$d_{\lambda^p}(z,w) = \max\Big\{|x-y|, \sqrt{\lambda^{p-2}|t-s|}\Big\},$$

where $z, w \in U_{R_2,S_2}(z_0)$ with $z = (x,t)$ and $w = (y,s)$ and λ is chosen such that $\Lambda = \lambda^p + \|a\|_\infty \lambda^q + \|b\|_\infty \lambda^s$. Let us recall the definition of $Q_{l,\lambda}(w)$, that is,

$$Q_{l,\lambda}(w) := B_l(y) \times (s - \lambda^{2-p} l^2, s + \lambda^{2-p} l^2).$$

Lemma 3.10 (Campanato characterization). *Assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Then there exist a constant $c = c(n)$ and a set $E \subset \mathbb{R}^{n+1}$ with $|E| = 0$ such that*

$$|f(z) - f(w)| \le c(n)\, d_{\lambda^p}(z,w) \sup_{l>0} \fint\!\!\!\fint_{Q_{l,\lambda}(w)} \frac{|f(\tilde w) - (f)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde w + c(n)\, d_{\lambda^p}(z,w) \sup_{l>0} \fint\!\!\!\fint_{Q_{l,\lambda}(z)} \frac{|f(\tilde z) - (f)_{Q_{l,\lambda}(z)}|}{l}\, d\tilde z$$

for every $z, w \in \mathbb{R}^{n+1} \setminus E$.

Proof. Since $Q_{d_{\lambda^p}(z,w),\lambda}(z) \subset Q_{2d_{\lambda^p}(z,w),\lambda}(w)$, by replacing $B(x,r)$ with $Q_{l,\lambda}(z)$ and $|x-y|$ with $d_{\lambda^p}(z,w)$, and taking $\beta = 1$ in the proof of [29, Lemma 4.13], we obtain that

$$\begin{aligned}|f(z) - f(w)| &\le c(n)\, d_{\lambda^p}(z,w) \sup_{0<l<4d_{\lambda^p}(z,w)} \fint\!\!\!\fint_{Q_{l,\lambda}(w)} \frac{|f(\tilde w) - (f)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde w\\ &\quad + c(n)\, d_{\lambda^p}(z,w) \sup_{0<l<4d_{\lambda^p}(z,w)} \fint\!\!\!\fint_{Q_{l,\lambda}(z)} \frac{|f(\tilde z) - (f)_{Q_{l,\lambda}(z)}|}{l}\, d\tilde z\\ &\le c(n)\, d_{\lambda^p}(z,w) \sup_{l>0} \fint\!\!\!\fint_{Q_{l,\lambda}(w)} \frac{|f(\tilde w) - (f)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde w\\ &\quad + c(n)\, d_{\lambda^p}(z,w) \sup_{l>0} \fint\!\!\!\fint_{Q_{l,\lambda}(z)} \frac{|f(\tilde z) - (f)_{Q_{l,\lambda}(z)}|}{l}\, d\tilde z.\end{aligned}$$

This completes the proof. □

We remark that the above conclusion holds for every $z, w \in \mathbb{R}^{n+1}$ by [29, Remark 4.14]. Now we are ready to prove the Lipschitz regularity of v_h^Λ.

Lemma 3.11. *There exists a constant $c_\Lambda = c(\text{data}, \Lambda)$ such that*

$$|v_h^\Lambda(z) - v_h^\Lambda(w)| \le c_\Lambda d_{\lambda^p}(z, w)$$

for every $z, w \in \mathbb{R}^{n+1}$.

Proof. We first note that, since $\lambda_i \ge \lambda$ from the definition of d, we have

$$d_{\lambda^p}(z, w) \le d_i(z, w) \quad \text{for all } i \in \mathbb{N}. \tag{3.36}$$

Applying Lemma 3.10, we get

$$\begin{aligned} |v_h^\Lambda(z) - v_h^\Lambda(w)| &\le c(n) d_{\lambda^p}(z, w) \sup_{l>0} \fint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda(\tilde{w}) - (v_h^\Lambda)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \\ &\quad + c(n) d_{\lambda^p}(z, w) \sup_{l>0} \fint_{Q_{l,\lambda}(z)} \frac{|v_h^\Lambda(\tilde{z}) - (v_h^\Lambda)_{Q_{l,\lambda}(z)}|}{l}\, d\tilde{z}. \end{aligned} \tag{3.37}$$

Note that $v_h^\Lambda \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$. Therefore from (3.37), it is enough to show that there exists a constant $c_\Lambda = c_\Lambda(\text{data}, \Lambda)$ such that

$$\fint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda(\tilde{w}) - (v_h^\Lambda)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \le c_\Lambda \quad \text{for all } Q_{l,\lambda}(w) \subset \mathbb{R}^{n+1}. \tag{3.38}$$

We fix the cube $Q_{l,\lambda}(w)$ and prove the above estimate (3.38) when $2Q_{l,\lambda}(w)$ completely lies on the bad set $E(\Lambda)^c$, or $Q_{l,\lambda}(w)$ lies inside the bad set $E(\Lambda)^c$, but $2Q_{l,\lambda}(w)$ meets the good set $E(\Lambda)$ or both $Q_{l,\lambda}(w)$ and $2Q_{l,\lambda}(w)$ meets the good set $E(\Lambda)$.

Case 1: $2Q_{l,\lambda}(\omega) \subset E(\Lambda)^c$. Let $z \in Q_{l,\lambda}(\omega)$. Then, by Lemma 2.18 (i) there exists $i \in \mathbb{N}$ such that $z \in U_i$. Using (3.36), we have

$$l \le d_{\lambda^p}(z, E(\Lambda)) \le d_{\lambda^p}(z, z_i) + d_{\lambda^p}(z_i, E(\Lambda)) \le d_i(z, z_i) + d_i(z_i, E(\Lambda)).$$

Here, z_i is the center of U_i. Since U_i is a ball of radius r_i with respect to the metric $d_i(\cdot,\cdot)$, we get from Lemma 2.18 (iv) that

$$l \le d_i(z, z_i) + d_i(z_i, E(\Lambda)) \le r_i + 5r_i = 6r_i.$$

Lemma 3.8 implies the uniform estimate

$$|\partial_t v_h^\Lambda(z)| \le c(\text{data}, \Lambda) r_i^{-1} \le c(\text{data}, \Lambda) l^{-1} \tag{3.39}$$

for all $z \in Q_{l,\lambda}(w)$.

To prove (3.38), let $z_1, z_2 \in Q_{l,\lambda}(w)$ with $z_1 = (x_1, t_1)$ and $z_2 = (x_2, t_2)$. Since $v_h^\Lambda \in C^\infty(E(\Lambda)^c, \mathbb{R}^N)$ and $Q_{l,\lambda}(w) \subset E(\Lambda)^c$, by the intermediate value theorem, (3.31) and (3.39), we have

$$\begin{aligned} |v_h^\Lambda(z_1) - v_h^\Lambda(z_2)| &\le |v_h^\Lambda(x_1, t_1) - v_h^\Lambda(x_2, t_1)| + |v_h^\Lambda(x_2, t_1) - v_h^\Lambda(x_2, t_2)| \\ &\le cl \sup_{z \in Q_{l,\lambda}(\omega)} |\nabla v_h^\Lambda(z)| + c\lambda^{2-p} l^2 \sup_{z \in Q_{l,\lambda}(\omega)} |\partial_t v_h^\Lambda(z)| \le c(\text{data}, \Lambda) l. \end{aligned}$$

Thus, we conclude

$$\fint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - (v_h^\Lambda)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \le \fint_{Q_{l,\lambda}(w)} \fint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda(z_1) - v_h^\Lambda(z_2)|}{l}\, dz_2\, dz_1 \le c_\Lambda(\text{data}, \Lambda).$$

Case 2: $2Q_{l,\lambda}(w) \cap E(\Lambda) \neq \emptyset$ and $Q_{l,\lambda}(w) \cap E(\Lambda)^c = \emptyset$. In this case $v_h^\Lambda = v_h$ and hence

$$\fint_{Q_{l,\lambda}(\omega)} \frac{|v_h^\Lambda - (v_h^\Lambda)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} = \fint_{Q_{l,\lambda}(\omega)} \frac{|v_h - (v_h)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w}.$$

We denote $w = (y, s)$. If $B_l(y) \subset B_{R_2}(x_0)$, then $l \le R_2$ is satisfied and Lemma 3.2 and Lemma 3.3 imply that

$$\iint_{Q_{l,\lambda}(w)} \frac{|v_h - (v_h)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \le c(\text{data}, \Lambda)(\lambda^{2-p} + \lambda^{2-p} l + 1) \le c(\text{data}, \Lambda).$$

On the other hand, if $B_l(y) \not\subset B_{R_2}(x_0)$, we apply Lemma 3.4 with $\sigma = 1$ and $\eta = d$ to get

$$\iint_{Q_{l,\lambda}(w)} \frac{|v_h - (v_h)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \le 2 \iint_{Q_{l,\lambda}(w)} \frac{|v_h|}{l}\, d\tilde{w} \le c \iint_{4Q_{l,\lambda}(w)} |\nabla v_h|\, d\tilde{w}.$$

Recalling that in this case $2Q_{l,\lambda}(w) \cap E(\Lambda) \neq \emptyset$, we conclude with Lemma 3.2 that

$$\iint_{4Q_{l,\lambda}(w)} |\nabla v_h|\, d\tilde{w} \le c \iint_{[4Q_{l,\lambda}(w)]_h} f\, d\tilde{w} \le c(\text{data}, \Lambda).$$

Case 3: $2Q_{l,\lambda}(w) \cap E(\Lambda) \neq \emptyset$ and $Q_{l,\lambda} \cap E(\Lambda) \neq \emptyset$. We define the index set

$$P = \Big\{ i \in \mathbb{N} : Q_{l,\lambda}(w) \cap \frac{2}{K} U_i \neq \emptyset \Big\}.$$

We want to show that the radii r_i are bounded uniformly by l with respect to $i \in P$. Let $i \in P$, $w_1 \in Q_{l,\lambda}(w) \cap \frac{2}{K} U_i$ and $w_2 \in 2Q_{l,\lambda}(w) \cap E(\Lambda)$ with $w_1 = (y_1, s_1)$ and $w_2 = (y_2, s_2)$. By Lemma 2.18 (iii) and $w_1 \in \frac{2}{K} U_i$, we obtain

$$3r_1 \le d_i(U_i, E(\Lambda)) \le d_i(z_i, w_2) \le d_i(z_i, w_1) + d_i(w_1, w_2) \le 2r_i + d_i(w_1, w_2),$$

and hence $r_i \le d_i(w_1, w_2)$. Moreover, since $\lambda \le \lambda_i$ and $w_1, w_2 \in 2Q_{l,\lambda}(w)$, we get

$$\begin{aligned} d_i(w_1, w_2) &\le \max\Big\{ |y_1 - y_2|, \sqrt{\Lambda \lambda_i^{-2} |s_1 - s_2|} \Big\} \\ &\le \Lambda^{\frac{1}{2}} \max\Big\{ |y_1 - y_2|, \sqrt{\lambda^{p-2} |s_1 - s_2|} \Big\} \\ &= \Lambda^{\frac{1}{2}} d_{\lambda^p}(w_1, w_2) \le 4\Lambda^{\frac{1}{2}} l. \end{aligned}$$

Thus, we have $r_i \le c(\Lambda) l$.

Note that

$$\begin{aligned} \iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - (v_h^\Lambda)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} &\le 2 \iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - (v_h)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w} \\ &\le 2 \iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - v_h|}{l}\, d\tilde{w} + 2 \iint_{Q_{l,\lambda}(w)} \frac{|v_h - (v_h)_{Q_{l,\lambda}(w)}|}{l}\, d\tilde{w}. \end{aligned}$$

As in Case 2, we can estimate the second term on the right-hand side. To estimate the first term, we obtain from the definition of v_h^Λ and the fact that w_i is supported in $\frac{2}{K} U_i$ that

$$\begin{aligned} \iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - v_h|}{l}\, d\tilde{w} &= \iint_{Q_{l,\lambda}(w)} \frac{|\sum_{i\in P} (v_h - v_h^i) w_i|}{l}\, d\tilde{w} \\ &\le \iint_{Q_{l,\lambda}(w)} \sum_{i\in P} \frac{|v_h - v_h^i| w_i}{l}\, d\tilde{w} \\ &= \frac{1}{|Q_{l,\lambda}(w)|} \sum_{i\in P} \iint_{Q_{i,\lambda}(w) \cap \frac{2}{K} U_i} \frac{|v_h - v_h^i| w_i}{l}\, d\tilde{w}. \end{aligned}$$

Since $w_i \le 1$ and $r_i \le c(\Lambda) l$, we have

$$\frac{1}{|Q_{l,\lambda}(w)|} \sum_{i\in P} \iint_{Q_{i,\lambda}(w) \cap \frac{2}{K} U_i} \frac{|v_h - v_h^i| w_i}{l}\, d\tilde{w} \le \frac{c(\Lambda)}{|Q_{l,\lambda}(w)|} \sum_{i\in P} \iint_{\frac{2}{K} U_i} \frac{|v_h - v_h^i|}{r_i}\, d\tilde{w}.$$

Combining the previous inequalities with (3.25), we obtain

$$\iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - v_h|}{l}\, d\tilde{w} \le \frac{c(\text{data}, \Lambda)}{|Q_{l,\lambda}(w)|} \sum_{i\in P} |U_i|.$$

Since $r_i \le c(\Lambda)l$, we get $U_i \subset c(\text{data}, \Lambda)Q_{i,\Lambda}(w)$ for every $i \in P$. By Lemma 2.18 (ii), we have that

$$\iint_{Q_{l,\lambda}(w)} \frac{|v_h^\Lambda - v_h|}{l}\, d\tilde{w} \le \frac{c(\text{data}, \Lambda)}{|Q_{l,\lambda}(w)|} \sum_{i\in P} \left|\frac{1}{6K^6} U_i\right| \le c(\text{data}, \Lambda).$$

Thus, the proof is completed. □

Corollary 3.12. *Let $E(\Lambda)$ be defined in* (2.2). *Then v_h satisfies the estimate*

$$H(z, |v_h^\Lambda(z)|) + H(z, |\nabla v_h^\Lambda(z)|) \le c(\text{data}, \Lambda) \quad \textit{for a.e. } z \in \mathbb{R}^{n+1}.$$

Proof. Since $v_h^\Lambda(z) = 0$ in $U_{R_2,S_2}(z_0)^c$ and v_h^Λ is Lipschitz continuous for almost every $z \in \mathbb{R}^{n+1}$, we get that $|v_h^\Lambda(z)| \le c(\text{data}, \Lambda)$ and $|\nabla v_h^\Lambda(z)| \le c(\text{data}, \Lambda)$ for almost every $z \in \mathbb{R}^{n+1}$. Then we have

$$\begin{aligned} &|v_h^\Lambda(z)|^p + a(z)|v_h^\Lambda(z)|^q + b(z)|v_h^\Lambda(z)|^s + |\nabla v_h^\Lambda(z)|^p + a(z)|\nabla v_h^\Lambda(z)|^q + b(z)|\nabla v_h^\Lambda(z)|^s \\ &\quad \le c(\text{data}, \Lambda)(1 + \|a\|_\infty + \|b\|_\infty). \end{aligned}$$

This completes the proof. □

3.6 Some more properties of Lipschitz truncation

In the following proposition, we collect some more properties of Lipschitz truncation.

Proposition 3.13. *Let $E(\Lambda)$ be as in* (2.2)*, and let η, ζ be the cut-off functions. Then $\{v_h^\Lambda\}_{h>0}$ and a function v^Λ satisfy the following properties:*
(i) $v_h^\Lambda \in W_0^{1,2}(\operatorname{supp}(\zeta); L^2(\operatorname{supp}(\eta), \mathbb{R}^N)) \cap L^\infty(\operatorname{supp}(\zeta); W_0^{1,\infty}(\operatorname{supp}(\eta), \mathbb{R}^N))$.
(ii) $v^\Lambda \in L^\infty(\operatorname{supp}(\zeta) + h_0; W_0^{1,\infty}(\operatorname{supp}(\eta), \mathbb{R}^N))$.
(iii) $v_h^\Lambda = v_h$, $v^\Lambda = v$, $\nabla v_h^\Lambda = \nabla v_h$, $\nabla v^\Lambda = \nabla v$ *a.e. in* $E(\Lambda)$.
(iv) $v_h^\Lambda \to v^\Lambda$ *in* $L^\infty(\Omega, \mathbb{R}^N)$ *as* $h \to 0^+$*, taking a subsequence if necessary.*
(v) $\nabla v_h^\Lambda \to \nabla v^\Lambda$ *and* $\partial_t v_h^\Lambda \to \partial_t v^\Lambda$ *a.e. in* $E(\Lambda)^c$ *as* $h \to 0^+$.

Proof. To show (i), we first note from Definition 1.1 and the definition of Steklov averages that

$$v_h \in W_0^{1,2}(\operatorname{supp}(\zeta); L^2(\operatorname{supp}(\eta), \mathbb{R}^N)).$$

Moreover, the definition of v_h^Λ in (3.3) only matters for finite sum since $|\mathcal{I}| \le c$. Therefore, together with Lemma 3.11, we complete the proof of (i).

The proof of (ii) is obvious from Lemma 3.9 (iii). Indeed, we have $|v^\Lambda(z)| \le c\Lambda^{\frac{1}{p}}$ and $|\nabla v^\Lambda(z)| \le c\Lambda^{\frac{1}{p}}$ for all $z \in \mathbb{R}^{n+1}$.

The proof of (iii) follows from the definitions (3.3) and (3.4). In fact, since $\omega_i(z) = 0$ for $z \in E(\Lambda)$, we have the proof.

Since $v_h^\Lambda(z) = 0$ for $z \in U_{R_2,S_2}^c(z_0)$, using Lemma 3.11 we see that $\{v_h^\Lambda\}_{h>0}$ is equicontinuous and uniformly bounded. By the properties of Steklov averages, we already have $v_h^\Lambda \to v^\Lambda$ as $h \to 0+$. Hence, by Arzela–Ascoli theorem, we get $v_h^\Lambda \to v^\Lambda$ in $L^\infty(\Omega, \mathbb{R}^N)$. This proves (iv).

The proof of (v) follows from $v_h^i \to v^i$ as $h \to 0$ and the expressions computed in Lemma 3.8. □

4 Proof of energy estimate

In this section, we prove the energy estimate in Theorem 1.2 using the Lipschitz truncation v_h^Λ.

4.1 Proof of Theorem 1.2

We closely follow the proof of [28, Subsection 5.1]. For $\tau \in \ell_{S_2-h}(t_0)$ and sufficiently small $\delta > 0$, let

$$\zeta_\delta(t) = \begin{cases} 1, & t \in (-\infty, \tau-\delta), \\ 1 - \dfrac{t-\tau+\delta}{\delta}, & t \in [\tau-\delta, \tau], \\ 0, & t \in (\tau, t_0 + S_2 - h). \end{cases}$$

Note that, by Proposition 3.13 (i), $v_h^\Lambda \eta^s \zeta\zeta_\delta(\,\cdot\,, t) \in W_0^{1,\infty}(B_{R_2}(x_0), \mathbb{R}^N)$ for every $t \in \ell_{S_2-h}(t_0)$. Using this function as the test function of the Steklov averaged weak formulation in (1.1), we obtain

$$\begin{aligned} \mathrm{I} + \mathrm{II} &= \iint_{U_{R_2,S_2}(z_0)} \partial_t [u - u_0]_h \cdot v_h^\Lambda \eta^s \zeta\zeta_\delta \,dz + \iint_{U_{R_2,S_2}(z_0)} [\mathcal{A}(z, \nabla u)]_h \cdot \nabla(v_h^\Lambda \eta^s \zeta\zeta_\delta)\,dz \\ &= \iint_{U_{R_2,S_2}(z_0)} [\mathcal{B}(z, F)]_h \cdot \nabla(v_h^\Lambda \eta^s \zeta\zeta_\delta)\,dz = \mathrm{III}. \end{aligned}$$

Now we estimate the each term above by dividing the integral domain into $E(\Lambda)$ and $E(\Lambda)^c$.

Estimate of I. By integration by parts and the product rule, we obtain

$$\mathrm{I} = \iint_{U_{R_2,S_2}(z_0)} (-v_h \cdot v_h^\Lambda \eta^{s-1} \partial_t \zeta_\delta - v_h \cdot \partial_t v_h^\Lambda \eta^{s-1} \zeta_\delta)\,dz + \iint_{U_{R_2,S_2}(z_0)} -[u - u_0]_h \cdot v_h^\Lambda \eta^s \zeta_\delta \partial_t \zeta \,dz = \mathrm{I}_1 + \mathrm{I}_2.$$

First, we consider I_1. Then we have

$$\begin{aligned} \mathrm{I}_1 = &\iint_{U_{R_2,S_2}(z_0)} -|v_h|^2 \eta^{s-1} \partial_t \zeta_\delta \,dz + \iint_{U_{R_2,S_2}(z_0)} v_h \cdot (v_h - v_h^\Lambda) \eta^{s-1} \partial_t \zeta_\delta \,dz \\ &- \iint_{U_{R_2,S_2}(z_0)} (v_h - v_h^\Lambda) \cdot \partial_t v_h^\Lambda \eta^{s-1} \zeta_\delta \,dz - \iint_{U_{R_2,S_2}(z_0)} v_h^\Lambda \cdot \partial_t v_h^\Lambda \eta^{s-1} \zeta_\delta \,dz. \end{aligned}$$

Note that integration by parts implies

$$\iint_{U_{R_2,S_2}(z_0)} v_h^\Lambda \cdot \partial_t v_h^\Lambda \eta^{s-1} \zeta_\delta \,dz = -\frac{1}{2} \iint_{U_{R_2,S_2}(z_0)} |v_h^\Lambda|^2 \eta^{s-1} \partial_t \zeta_\delta \,dz.$$

Since $v_h^\Lambda = v_h$ a.e. in $E(\Lambda)$ and $v_h = v_h^\Lambda = 0$ in $U_{R_2,S_2}(z_0)^c$, we obtain

$$\mathrm{I}_1 = \iint_{U_{R_2,S_2}(z_0)} -(|v_h|^2 - \tfrac{1}{2}|v_h^\Lambda|^2) \eta^{s-1} \partial_t \zeta_\delta \,dz + \iint_{E(\Lambda)^c} v_h \cdot (v_h - v_h^\Lambda) \eta^{s-1} \partial_t \zeta_\delta \,dz - \iint_{E(\Lambda)^c} (v_h - v_h^\Lambda) \cdot \partial_t v_h^\Lambda \eta^{s-1} \zeta_\delta \,dz.$$

Letting $h \to 0^+$, we obtain from the properties of Steklov averages, Proposition 3.13 (iv) and Lemma 3.9 (i) that

$$\begin{aligned} \lim_{h \to 0^+} \mathrm{I}_1 &= \iint_{U_{R_2,S_2}(z_0)} -(|v|^2 - \tfrac{1}{2}|v^\Lambda|^2) \eta^{s-1} \partial_t \zeta_\delta \,dz + \iint_{E(\Lambda)^c} v \cdot (v - v^\Lambda) \eta^{s-1} \partial_t \zeta_\delta \,dz - \iint_{E(\Lambda)^c} (v - v^\Lambda) \cdot \partial_t v^\Lambda \eta^{s-1} \zeta_\delta \,dz \\ &= \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{13}. \end{aligned}$$

Since the Lipschitz truncation is done only in the bad set $E(\Lambda)^c$, we get

$$\mathrm{I}_{11} = \iint_{E(\Lambda)} -\frac{1}{2}|v|^2 \eta^{s-1} \partial_t \zeta_\delta \,dz - \iint_{E(\Lambda)^c} \Big(|v|^2 - \frac{1}{2}|v^\Lambda|^c\Big) \eta^{s-1} \partial_t \zeta_\delta \,dz$$

Since $v \in L^2(U_{R_2,S_2}(z_0), \mathbb{R}^N)$, we obtain from the absolute continuity, Lemma 3.9 (iii) and (2.3) that

$$\lim_{\Lambda \to \infty} \mathrm{I}_{11} = \iint_{U_{R_2,S_2}(z_0)} -\frac{1}{2}|v|^2 \eta^{s-1} \partial_t \zeta_\delta \,dz.$$

For the same reason, we also have $\lim_{\Lambda\to\infty} \mathrm{I}_{12} = 0$. Finally, Lemma 3.9 (ii) and (2.3) imply that $\lim_{\Lambda\to\infty} \mathrm{I}_{13} = 0$. Thus, we have

$$\lim_{\Lambda\to\infty}\lim_{h\to0^+} \mathrm{I}_1 = \iint\limits_{U_{R_2,S_2}(z_0)} -\frac{1}{2}|v|^2\eta^{s-1}\partial_t\zeta_\delta\,dz.$$

Now, we estimate I_2. By the properties of Steklov averages and Proposition 3.13 (iv) and by dividing into the good and bad sets, we get

$$\begin{aligned}\lim_{h\to0} \mathrm{I}_2 &= -\iint\limits_{U_{R_2,S_2}(z_0)} (u-u_0)\cdot v^\Lambda\eta^s\zeta_\delta\partial_t\zeta\,dz\\ &\geq -\iint\limits_{U_{R_2,S_2}(z_0)\cap E(\Lambda)} |u-u_0|^2|\partial_t\zeta|\,dz - \iint\limits_{E(\Lambda)^c} |u-u_0||v^\Lambda||\partial_t\zeta|\,dz.\end{aligned}$$

Then Hölder's inequality implies

$$\iint\limits_{E(\Lambda)^c} |u-u_0||v^\Lambda||\partial_t\zeta|\,dz \leq \Bigg(\iint\limits_{U_{R_2,S_2}(z_0)\cap E(\Lambda)^c} |u-u_0|^2|\partial_t\zeta|^2\,dz\Bigg)^{\frac12}\Bigg(\iint\limits_{E(\Lambda)^c} |v^\Lambda|^2\,dz\Bigg)^{\frac12}.$$

Since $|u| \in L^2(\Omega_T)$, the first integral vanishes as $\Lambda\to\infty$. By Lemma 3.9 (iii) and (2.3), we get

$$\lim_{\Lambda\to\infty}\iint\limits_{E(\Lambda)^c} |v^\Lambda|^2\,dz \leq \lim_{\Lambda\to\infty} c\Lambda^{\frac{2}{p}}|E(\Lambda)^c| \leq \lim_{\Lambda\to\infty} c\Lambda|E(\Lambda)^c| = 0.$$

Thus, we get

$$\lim_{\Lambda\to\infty}\lim_{h\to0^+} \mathrm{I}_2 \geq -\iint\limits_{U_{R_2,S_2}(z_0)} |u-u_0|^2|\partial_t\zeta|\,dz,$$

and hence we conclude

$$\lim_{\Lambda\to\infty}\lim_{h\to0^+} \mathrm{I} \geq \iint\limits_{U_{R_2,S_2}(z_0)} -\frac{1}{2}|v|^2\eta^{s-1}\partial_t\zeta_\delta\,dz - \iint\limits_{U_{R_2,S_2}(z_0)} |u-u_0|^2|\partial_t\zeta|\,dz.$$

Estimate of II. As in [28], we obtain

$$\lim_{h\to0^+} \mathrm{II} = \iint\limits_{U_{R_2,S_2}(z_0)\cap E(\Lambda)} \mathcal{A}(z,\nabla u)\cdot\nabla((u-u_0)\eta^{s+1}\zeta^2\zeta_\delta)\,dz + \iint\limits_{U_{R_2,S_2}(z_0)\cap E(\Lambda)^c} \mathcal{A}(z,\nabla u)\cdot\nabla(v^\Lambda\eta^s\zeta\zeta_\delta)\,dz = \mathrm{II}_1+\mathrm{II}_2.$$

Since the integral within the good set does not contain Λ, letting $\Lambda\to\infty$, we see from (1.2) and (3.1) that

$$\begin{aligned}\lim_{\Lambda\to\infty}\lim_{h\to0^+} \mathrm{II}_1 &= \iint\limits_{U_{R_2,S_2}(z_0)} (\mathcal{A}(z,\nabla u)\cdot\nabla u)\eta^{s+1}\zeta^2\zeta_\delta\,dz + \iint\limits_{U_{R_2,S_2}(z_0)} \mathcal{A}(z,\nabla u)\cdot(u-u_0)\nabla(\eta^{s+1})\zeta^2\zeta_\delta\,dz\\ &\geq c\iint\limits_{U_{R_2,S_2}(z_0)} (|\nabla u|^p + a(z)|\nabla u|^q + b(z)|\nabla u|^s)\eta^{s+1}\zeta^2\zeta_\delta\,dz\\ &\quad - c\iint\limits_{U_{R_2,S_2}(z_0)} (|\nabla u|^{p-1} + a(z)|\nabla u|^{q-1} + b(z)|\nabla u|^{s-1})\frac{|u-u_0|}{R_2-R_1}\eta^s\zeta^2\zeta_\delta\,dz,\end{aligned}$$

where $c = c(s,\nu,L)$. Then, Young's inequality with conjugate $(\frac{p}{p-1},p)$, $(\frac{q}{q-1},q)$ and $(\frac{s}{s-1},s)$, respectively, gives

$$\begin{aligned}&c\iint\limits_{U_{R_2,S_2}(z_0)} (|\nabla u|^{p-1} + a(z)|\nabla u|^{q-1} + b(z)|\nabla u|^{s-1})\frac{|u-u_0|}{R_2-R_1}\eta^s\zeta^2\zeta_\delta\,dz\\ &\quad\leq \frac{\nu}{2}\iint\limits_{U_{R_2,S_2}(z_0)} (|\nabla u|^p + a(z)|\nabla u|^q + b(z)|\nabla u|^s)\eta^{s+1}\zeta^2\zeta_\delta\,dz\\ &\qquad + c\iint\limits_{U_{R_2,S_2}(z_0)} \bigg(\frac{|u-u_0|^p}{(R_2-R_1)^p} + a(z)\frac{|u-u_0|^q}{(R_2-R_1)^q} + b(z)\frac{|u-u_0|^s}{(R_2-R_1)^s}\bigg)\,dz,\end{aligned}$$

where c depends only on p, q, s, ν and L. Hence we have

$$\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{II}_1 \geq \frac{\nu}{2} \iint_{U_{R_2,S_2}(z_0)} (|\nabla u|^p + a(z)|\nabla u|^q + b(z)|\nabla u|^s)\eta^{s+1}\zeta^2\zeta_\delta \, dz - c \iint_{U_{R_2,S_2}(z_0)} \Big(\frac{|u-u_0|^p}{(R_2-R_1)^p} + a(z)\frac{|u-u_0|^q}{(R_2-R_1)^q} + b(z)\frac{|u-u_0|^s}{(R_2-R_1)^s} \Big) \, dz.$$

Moreover, it is easy to show $\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{II}_2 = 0$ by using (1.2), Young's inequality, Lemma 3.9 (iii) and (2.3). For detailed calculations, refer to [28]. Thus, we conclude

$$\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{II} \geq \frac{\nu}{2} \iint_{U_{R_2,S_2}(z_0)} H(z,|\nabla u|)\eta^{s+1}\zeta^2\zeta_\delta \, dz - c \iint_{U_{R_2,S_2}(z_0)} H\Big(\frac{|u-u_0|}{R_2-R_1}\Big) \, dz.$$

Estimate of III. The estimate for III follows a process similar to that for II. Again, we divide into the good and bad parts to obtain

$$\lim_{h\to 0^+} \mathrm{III} = \iint_{U_{R_2,S_2}(z_0)\cap E(\Lambda)} \mathcal{B}(z,F)\cdot\nabla((u-u_0)\eta^{s+1}\zeta^2\zeta_\delta)\, dz + \iint_{U_{R_2,S_2}(z_0)\cap E(\Lambda)^c} \mathcal{B}(z,F)\cdot\nabla(v_h^\Lambda\eta^s\zeta\zeta_\delta)\, dz = \mathrm{III}_1 + \mathrm{III}_2.$$

Applying (1.3) and Young's inequality, we get

$$\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{III}_1 \leq c \iint_{U_{R_2,S_2}(z_0)} \Big(H\Big(z, \frac{|u-u_0|}{R_2-R_1}\Big) + H(z,|F|)\Big) \, dz + \frac{\nu}{4} \iint_{U_{R_2,S_2}(z_0)} H(z,|\nabla u|)\eta^{s+1}\zeta^2\zeta_\delta) \, dz$$

for some $c = c(p,q,s,\nu,L)$. In the same reason as II_2, we get $\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{III}_2 = 0$. Thus, we conclude that

$$\lim_{\Lambda\to\infty}\lim_{h\to 0^+} \mathrm{III} \leq c \iint_{U_{R_2,S_2}(z_0)} \Big(H\Big(z, \frac{|u-u_0|}{R_2-R_1}\Big) + H(z,|F|)\Big) \, dz + \frac{\nu}{4} \iint_{U_{R_2,S_2}(z_0)} H(z,|\nabla u|)\eta^{s+1}\zeta^2\zeta_\delta) \, dz.$$

Combining all the estimates for I, II and III, we obtain

$$\begin{aligned} &\iint_{U_{R_2,S_2}(z_0)} -\frac{1}{2}|v|^2\eta^{s-1}\partial_t\zeta_\delta \, dz + \frac{\nu}{4}\iint_{U_{R_2,S_2}(z_0)} H(z,|\nabla u|)\eta^{s+1}\zeta^2\zeta_\delta) \, dz \\ &\quad \leq c \iint_{U_{R_2,S_2}(z_0)} \Big(H\Big(z, \frac{|u-u_0|}{R_2-R_1}\Big) + |u-u_0|^2|\partial_t\zeta| + H(z,|F|)\Big) \, dz. \end{aligned}$$

Finally, we complete the proof by letting $\delta \to 0$, recalling that $\tau \in \ell_{S_1}(t_0)$ is arbitrary, and replacing u_0 with $(u)_{U_{R_1,S_1}(z_0)}$. □

Acknowledgment: The authors would like to express their sincere gratitude to the anonymous referee who provided valuable comments and suggestions on the earlier version, which have greatly improved the quality and clarity of the manuscript.

Funding: Bogi Kim is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (Grant No. RS-2023-00217116). Jehan Oh is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (Grant No. RS-2025-00555316). Abhrojyoti Sen is supported by research grants from the Alexander von Humboldt Foundation for postdocs.

References

[1] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Ration. Mech. Anal.* **86** (1984), no. 2, 125–145.

[2] E. Acerbi and N. Fusco, An approximation lemma for $W^{1,p}$ functions, in: *Material Instabilities in Continuum Mechanics* (Edinburgh 1985–1986), Oxford Sci. Publ., Oxford University, New York (1988), 1–5.

[3] S. Baasandorj, S.-S. Byun and J. Oh, Calderón–Zygmund estimates for generalized double phase problems, *J. Funct. Anal.* **279** (2020), no. 7, Article ID 108670.

[4] S. Baasandorj, S.-S. Byun and J. Oh, Gradient estimates for multi-phase problems, *Calc. Var. Partial Differential Equations* **60** (2021), no. 3, Paper No. 104.

[5] P. Baroni, M. Colombo and G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal.* **121** (2015), 206–222.

[6] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57** (2018), no. 2, Paper No. 62.

[7] V. Bögelein, Very weak solutions of higher-order degenerate parabolic systems, *Adv. Differential Equations* **14** (2009), no. 1–2, 121–200.

[8] S.-S. Byun and H.-S. Lee, Calderón–Zygmund estimates for elliptic double phase problems with variable exponents, *J. Math. Anal. Appl.* **501** (2021), no. 1, Article ID 124015.

[9] S.-S. Byun and H.-S. Lee, Gradient estimates of ω-minimizers to double phase variational problems with variable exponents, *Q. J. Math.* **72** (2021), no. 4, 1191–1221.

[10] S.-S. Byun and J. Oh, Global gradient estimates for non-uniformly elliptic equations, *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Paper No. 46.

[11] S.-S. Byun and J. Oh, Regularity results for generalized double phase functionals, *Anal. PDE* **13** (2020), no. 5, 1269–1300.

[12] I. Chlebicka, P. Gwiazda and A. Zatorska-Goldstein, Parabolic equation in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentiev's phenomenon, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **36** (2019), no. 5, 1431–1465.

[13] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), no. 1, 219–273.

[14] M. Colombo and G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.* **215** (2015), no. 2, 443–496.

[15] M. Colombo and G. Mingione, Calderón–Zygmund estimates and non-uniformly elliptic operators, *J. Funct. Anal.* **270** (2016), no. 4, 1416–1478.

[16] C. De Filippis, Optimal gradient estimates for multi-phase integrals, *Math. Eng.* **4** (2022), no. 5, Paper No. 043.

[17] C. De Filippis and G. Mingione, A borderline case of Calderón–Zygmund estimates for nonuni formly elliptic problems, *St. Petersburg Math. J.* **31** (2019), no. 3, 82–115.

[18] C. De Filippis and J. Oh, Regularity for multi-phase variational problems, *J. Differential Equations* **267** (2019), no. 3, 1631–1670.

[19] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with (p, q) growth, *J. Differential Equations* **204** (2004), no. 1, 5–55.

[20] Y. Fang, V. D. Rădulescu, C. Zhang and X. Zhang, Gradient estimates for multi-phase problems in Campanato spaces, *Indiana Univ. Math. J.* **71** (2022), no. 3, 1079–1099.

[21] I. Fonseca, J. Malý and G. Mingione, Scalar minimizers with fractal singular sets, *Arch. Ration. Mech. Anal.* **172** (2004), no. 2, 295–307.

[22] P. Hästö and J. Ok, Maximal regularity for local minimizers of non-autonomous functionals, *J. Eur. Math. Soc. (JEMS)* **24** (2022), no. 4, 1285–1334.

[23] P. Hästö and J. Ok, Regularity theory for non-autonomous partial differential equations without Uhlenbeck structure, *Arch. Ration. Mech. Anal.* **245** (2022), no. 3, 1401–1436.

[24] B. Kim and J. Oh, Higher integrability for weak solutions to parabolic multi-phase equations, *J. Differential Equations* **409** (2024), 223–298.

[25] B. Kim and J. Oh, Regularity for double phase functionals with two modulating coefficients, *J. Geom. Anal.* **34** (2024), no. 5, Paper No. 134.

[26] W. Kim, Calderón–Zygmund type estimate for the parabolic double-phase system, preprint (2023), https://arxiv.org/abs/2304.10615; *Ann. Sc. Norm. Super. Pisa.* (2024), DOI 10.2422/2036-2145.202307_017.

[27] W. Kim, J. Kinnunen and K. Moring, Gradient higher integrability for degenerate parabolic double-phase systems, *Arch. Ration. Mech. Anal.* **247** (2023), no. 5, Paper No. 79.

[28] W. Kim, J. Kinnunen and L. Särkiö, Lipschitz truncation method for parabolic double-phase systems and applications, *J. Funct. Anal.* **288** (2025), no. 3, Article ID 110738.

[29] J. Kinnunen, J. Lehrbäck and A. Vähäkangas, *Maximal Function Methods for Sobolev Spaces*, Math. Surveys Monogr. 257, American Mathematical Society, Providence, 2021.

[30] J. Kinnunen and J. L. Lewis, Very weak solutions of parabolic systems of p-Laplacian type, *Ark. Mat.* **40** (2002), no. 1, 105–132.

[31] J. Ok, Regularity of ω-minimizers for a class of functionals with non-standard growth, *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Paper No. 48.

[32] J. Ok, Regularity for double phase problems under additional integrability assumptions, *Nonlinear Anal.* **194** (2020), Article ID 111408.

[33] A. Sen, Gradient higher integrability for degenerate/ singular parabolic multi-phase problems, preprint (2024), https://arxiv.org/abs/2406.00763v2; to appear in *J. Geom. Anal.*

[34] T. Singer, Existence of weak solutions of parabolic systems with p, q-growth, *Manuscripta Math.* **151** (2016), no. 1–2, 87–112.

[35] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 4, 675–710, 877.

[36] V. V. Zhikov, Lavrentiev phenomenon and homogenization for some variational problems, *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), no. 5, 435–439.

[37] V. V. Zhikov, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **3** (1995), no. 2, 249–269.

[38] V. V. Zhikov, On some variational problems, *Russian J. Math. Phys.* **5** (1997), no. 1, 105–116.