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Research Article

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Stepanov differentiability theorem for intrinsic graphs in Heisenberg groups

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Abstract: We prove a Stepanov differentiability type theorem for intrinsic graphs in sub-Riemannian Heisenberg groups.

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1 Introduction

The classical Rademacher theorem states that if $\Omega \subset \mathbb{R}^n$ is an open set and $f\colon \Omega \to \mathbb{R}^m$ is Lipschitz continuous, then f is differentiable almost everywhere in Ω. One important generalization of the Rademacher theorem is the following result due to Stepanov (see e.g. [14, Section 3.1.9] and [30]).

Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $f\colon \Omega \to \mathbb{R}^m$. Consider the set*

$$S_f := \left\{ a \in \Omega\colon \limsup_{x \to a} \frac{|f(x) - f(a)|}{|x - a|} < +\infty \right\}. \tag{1.1}$$

Then f is differentiable almost everywhere on S_f.

The classical proof of the Stepanov differentiability theorem can be found in many textbooks (see e.g. [14]) and it is essentially based on density theorems and the application of Rademacher's theorem to a Lipschitz extension of $f|_{G_n}$, where G_n are suitable measurable sets on which f is Lipschitz. In 1999, J. Malý [25] proposed, for *real-valued* functions defined on separable Banach spaces, an alternative and elegant proof without using any density theorem. In [4, 13], using differentiability points of the distance functions instead of density theorems, the authors were able to prove a Stepanov type theorem (in the Gâteux sense) for functions between infinite-dimensional Banach spaces. Later, in [26], Malý and Zajíček, presented a new approach which shows how a Stepanov type theorem (in the Frechet sense) can be inferred from the corresponding theorem of Rademacher type.

In recent years, there has been significant and ongoing research aimed at extending classical analysis results from Euclidean spaces to more general metric-measure spaces (see e.g. [20, 22] and references therein). A major result is due to Cheeger [6], who found a deep generalization of Rademacher's theorem in the context of doubling metric measure spaces that satisfy a Poincaré inequality. More recently, the approach in [25] has been used in [3] to prove a Stepanov type theorem for real-valued maps defined on metric spaces endowed with a doubling Borel measure, and later, in [35], the result has been further generalized to maps between metric measure

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spaces and Banach spaces. Building on the foundation established in [6], a substantial body of literature has emerged; see e.g. [21, 22] for a complete overview. Let us also mention the recent [10].

The notion of Lipschitz submanifolds in sub-Riemannian geometry was introduced, at least in the setting of Carnot groups, by B. Franchi, R. Serapioni and F. Serra Cassano in [16, 17, 19] through the theory of intrinsic Lipschitz graphs. Roughly speaking, a subset $S \subset \mathbb{G}$ of a Carnot group $\mathbb{G}$ is intrinsic Lipschitz if, at each point $P \in S$, there is an intrinsic cone with vertex P and fixed opening intersecting S only at P. Remarkably, this notion turned out to be the right one in the setting of the intrinsic rectifiability in the simplest Carnot group, namely the Heisenberg group $\mathbb{H}^n$. Indeed, it was proved in [16, 32] that the notion of rectifiable set in terms of an intrinsic regular hypersurfaces is equivalent to the one in terms of intrinsic Lipschitz graphs. Recently, the theory of intrinsic Lipschitz functions has played a crucial role in the study of quantitative rectifiability [7, 8] and has even been applied to problems in information theory [27, 28]. See also [9, 29] for further applications.

The main open question in this area of research is whether a Rademacher type theorem holds for intrinsically Lipschitz functions between homogeneous subgroups of a Carnot group. Specifically, consider a splitting $\mathbb{G} = \mathbb{W}\mathbb{V}$ of a Carnot group $\mathbb{G}$ and let $\phi\colon \mathbb{W} \to \mathbb{V}$ be an intrinsically Lipschitz function. The question is whether such a function is intrinsically differentiable almost everywhere (see Definition 2.23 below). This question has been affirmatively answered when $\mathbb{V} \equiv \mathbb{R}$ and $\mathbb{G}$ is either the Heisenberg group [18] (see also [5]) or a step 2 Carnot group [15] or a Carnot group of type $\star$ (see [19]) or a Carnot group of type $\diamond$ (see [24]). More recently, the third named author [32] has proved that the answer is also affirmative in the case of the Heisenberg group, even without assuming any prior splitting condition. When $\mathbb{W}$ is a Carnot subgroup and $\mathbb{V}$ is a normal subgroup, the Rademacher theorem is proved in [1]. Remarkably, in [23], the authors constructed intrinsic Lipschitz graphs of codimension 2 in Carnot groups which are not intrinsically differentiable almost everywhere, thus discovering a deep connection between the notion of intrinsic differentiability and the geometry of the underlying Carnot group.

In the present paper, we prove an analogue of the Stepanov differentiability theorem for intrinsic graphs in Heisenberg groups. More precisely, we prove (see Theorems 3.1 and 3.3 below) that, given complementary subgroups $\mathbb{W}, \mathbb{V}$ of $\mathbb{H}^n$, then every function $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable almost everywhere on the set of points where its pointwise intrinsic Lipschitz constant is finite. The proof of both results, although inspired by that of [14], differs in some fundamental points. First of all, the notion of intrinsic Lipschitz continuity differs from the classical metric one (see Definition 2.12). This means that an intrinsic Lipschitz function $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$ need not to be metric Lipschitz (with respect to the distances induced on $\mathbb{W}$ and $\mathbb{V}$); see [18]. Therefore, our result does not fit into the classical framework of Lipschitz maps between metric measure spaces. Secondly, the notions of differentiability and Lipschitz continuity are indeed intrinsic geometric properties of ϕ, which take into account not only the structure of $\mathbb{W}, \mathbb{V}$ but also how they interact inside $\mathbb{H}^n$. To overcome these problems, we revisited the proof provided in [14] from a new geometric perspective; specifically, instead of working with the function ϕ, we considered its intrinsic graph. Finally, in the last part of the paper, we provide an alternative proof of the theorem in the case of codimension one, using the approach developed in [25]. We point out that the proof of our main results, i.e., Theorems 3.1 and 3.3, are not dependent on the particular structure of $\mathbb{H}^n$. They can be extended to intrinsic graphs in general Carnot groups $\mathbb{G}$ endowed with a splitting $\mathbb{W}\mathbb{V}$ for which a Rademacher theorem holds.

2 Notation and preliminary results

Definition 2.1. For $n \geq 1$, we denote by $\mathbb{H}^n$ the n-th *Heisenberg group*, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. We denote a point $p \in \mathbb{H}^n$ as $p = (x, y, t)$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If

$$p = (x, y, t),\ q = (x', y', t') \in \mathbb{H}^n,$$

the group operation is defined as

$$p \cdot q := \Big(x + x', y + y', t + t' + \frac{1}{2}\langle x, y'\rangle_{\mathbb{R}^n} - \frac{1}{2}\langle x', y\rangle_{\mathbb{R}^n}\Big).$$

If $p = (x, y, t) \in \mathbb{H}^n$, its inverse is $p^{-1} = (-x, -y, -t)$ and $0 = (0, 0, 0) \in \mathbb{H}^n$ is the identity of the group.

For $\lambda > 0$, we denote by $\delta_\lambda \colon \mathbb{H}^n \to \mathbb{H}^n$ the *intrinsic dilations* of the Heisenberg group defined by

$$\delta_\lambda(x, y, t) := (\lambda x, \lambda y, \lambda^2 t) \quad \text{for } p = (x, y, t) \in \mathbb{H}^n.$$

Observe that dilations form a one-parameter family of group isomorphisms. We say that a subgroup of $\mathbb{H}^n$ is *homogeneous* if it is closed under intrinsic dilations.

The Heisenberg group $\mathbb{H}^n$ admits the structure of a Lie group of *topological dimension* $2n + 1$. We denote by $Q := 2n + 2$ the *homogeneous dimension* of $\mathbb{H}^n$. The Lebesgue measure $\mathcal{L}^{2n+1}$ is the Haar measure on $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and it is Q-homogeneous with respect to dilations.

Definition 2.2. We denote by $\mathfrak{h}^n$ (or by $\mathfrak{h}$ when the dimension n is clear) the $(2n + 1)$-dimensional Lie algebra of left invariant vector fields in $\mathbb{H}^n$. The algebra $\mathfrak{h}$ is generated by the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$, where (for $1 \le j \le n$)

$$X_j := \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j := \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad T := \partial_t.$$

We denote by $\mathfrak{h}_1$ the horizontal subspace of $\mathfrak{h}$, i.e.,

$$\mathfrak{h}_1 := \operatorname{span}(X_1, \dots, X_n, Y_1, \dots, Y_n),$$

and by $\mathfrak{h}_2$ the linear span of T. Since $[X_j, Y_j] = T$, the Lie algebra $\mathfrak{h}$ admits the 2-step stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Note also that, since $\mathbb{H}^n$ is simply connected and nilpotent, the exponential map $\exp \colon \mathfrak{h} \to \mathbb{H}^n$ is a global diffeomorphism.

Definition 2.3. We say that a distance function $d \colon \mathbb{H}^n \times \mathbb{H}^n \to [0, +\infty)$ is a *left invariant and homogeneous distance* if
(i) $d(p, q) = d(r \cdot p, r \cdot q)$ for all $p, q, r \in \mathbb{H}^n$,
(ii) $d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p, q)$ for all $p, q \in \mathbb{H}^n$ and $\lambda > 0$.
We define the norm $\|\cdot\|$ associated to d as

$$\|p\| := d(0, p) \quad \text{for every } p \in \mathbb{H}^n.$$

Moreover, if, for every $(x, y, t), (x', y', t) \in \mathbb{H}^n$ such that $|(x, y)|_{\mathbb{R}^{2n}} = |(x', y')|_{\mathbb{R}^{2n}}$, we have $\|(x, y, t)\| = \|(x', y', t)\|$, we say that d is a *rotationally invariant* distance.

Example 2.4. There are numerous examples of left invariant, homogeneous and rotationally invariant distances on $\mathbb{H}^n$, the most noteworthy being the following.
(i) The *Carnot–Carathéodory* distance d_{cc} defined for $p \in \mathbb{H}^n$ as

$$d_{cc}(0, p) := \inf\Big\{\|h\|_{L^1([0,1],\mathbb{R}^{2n})} : \text{the curve } \gamma_h \colon [0, 1] \to \mathbb{H}^n \text{ defined by } \gamma_h(0) = 0,\ \dot\gamma_h = \sum_{j=1}^n (h_j X_j + h_{j+n} Y_j)(\gamma_h) \text{ has final point } \gamma_h(1) = p\Big\}.$$

(ii) The *infinity* distance d_∞ defined for $(x, y, t) \in \mathbb{H}^n$ as

$$d_\infty(0, (x, y, t)) := \max\{|(x, y)|_{\mathbb{R}^{2n}}, 2|t|_{\mathbb{R}}^{\frac{1}{2}}\}.$$

(iii) The *Korányi (or Cygan–Korányi)* distance d_K defined for $(x, y, t) \in \mathbb{H}^n$ as

$$d_K(0, (x, y, t)) := \big((|x|^2_{\mathbb{R}^n} + |y|^2_{\mathbb{R}^n})^2 + 16t^2\big)^{\frac{1}{4}}.$$

In view of the following proposition, we will denote by d a fixed left invariant, homogeneous and rotationally invariant distance on $\mathbb{H}^n$, by $\|\cdot\|$ its associated norm and, for any $p \in \mathbb{H}^n$, $r > 0$, we will denote by $B(p, r)$ the corresponding open balls.

Proposition 2.5 ([11, Proposition 1.3.15]). *Let d_1 and d_2 be left invariant and homogeneous distances on $\mathbb{H}^n$. Then they are bi-Lipschitz equivalent, i.e., there exists $C > 0$ such that, for all $p, q \in \mathbb{H}^n$,*

$$\frac{1}{C} d_2(p, q) \le d_1(p, q) \le C d_2(p, q).$$

In particular, every left invariant and homogeneous distance induces the Euclidean topology on $\mathbb{H}^n$.

We say that a homogeneous subgroup $\mathbb{V} \subset \mathbb{H}^n$ is *horizontal* if it is contained in the *horizontal fiber*, i.e., $\mathbb{V} \subseteq \exp(\mathfrak{h}_1)$. Horizontal subgroups can be identified with $(\mathbb{R}^k, |\cdot|)$: more precisely, if $V_1, \dots, V_k \in \mathfrak{h}_1$ are such that $\mathbb{V} = \exp(\operatorname{span}(V_1, \dots, V_k))$, then the map $\mathbb{R}^k \ni x \mapsto \exp(x_1 V_1 + \dots + x_k V_k)$ is a bi-Lipschitz diffeomorphism between $(\mathbb{R}^k, |\cdot|)$ and $(\mathbb{V}, d)$. In particular, the Hausdorff dimension of $\mathbb{V}$ equals the topological dimension k.

On the other hand, we will say that a subgroup $\mathbb{W}$ is *vertical* if it contains the *center* of the group, i.e., $\exp(\mathfrak{h}_2) \subseteq \mathbb{W}$. In this case, the Hausdorff dimension of $\mathbb{W}$ is greater than the topological one: for instance, the metric dimension of $\mathbb{H}^n$ coincides with the homogeneous dimension $Q = 2n + 2$.

All homogeneous subgroups of the Heisenberg group are either horizontal (and in this case, they are *abelian*) or vertical (and in this case, they are *normal*).

Definition 2.6. Let $\mathbb{W}, \mathbb{V}$ be homogeneous subgroups of $\mathbb{H}^n$. We say that $\mathbb{W}, \mathbb{V}$ are *complementary subgroups* in $\mathbb{H}^n$ if $\mathbb{W} \cap \mathbb{V} = \{0\}$ and $\mathbb{W} \cdot \mathbb{V} = \mathbb{H}^n$.

All possible couples of complementary subgroups in $\mathbb{H}^n$ are formed by an (abelian) horizontal subgroup $\mathbb{V}$ of dimension k, for $1 \le k \le n$, and by a (normal) vertical subgroup $\mathbb{W}$ of topological dimension $2n + 1 - k$.

Remark 2.7. If $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups in $\mathbb{H}^n$, then each element $p \in \mathbb{H}^n$ can be written in a unique way as $p = w \cdot v$, for $w \in \mathbb{W}$, $v \in \mathbb{V}$. The elements w, v are called the *components* (or the *projections*) of p with respect to the decomposition $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ and we will use the notation $w = p_{\mathbb{W}}$, $v = p_{\mathbb{V}}$. Let us stress that the components of a point $p \in \mathbb{H}^n$ depend on both the complementary subgroups and also on the order in which they are taken.

Some properties of the projections are described in [19, Section 2.2]. For convenience, we collect below the ones that we will use in the paper.

Proposition 2.8. *Let $\mathbb{H}^n = \mathbb{W}\mathbb{V}$, where $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups. Let us denote by $P_{\mathbb{W}}$ and $P_{\mathbb{V}}$ the* projection maps *onto $\mathbb{W}$ and $\mathbb{V}$ respectively, namely*

$$P_{\mathbb{W}}\colon \mathbb{H}^n \to \mathbb{W}, \quad P_{\mathbb{V}}\colon \mathbb{H}^n \to \mathbb{V},$$
$$P_{\mathbb{W}}(p) := p_{\mathbb{W}}, \qquad P_{\mathbb{V}}(p) := p_{\mathbb{V}}.$$

(1) *If $\mathbb{W}$ is a normal subgroup, then $P_{\mathbb{V}}$ is a Lipschitz homomorphism of groups. Similarly, if $\mathbb{V}$ is normal, then $P_{\mathbb{W}}$ is a Lipschitz homomorphism of groups.*
(2) *If $\mathbb{W}$ is a normal subgroup, then the following identities hold:*

$$P_{\mathbb{W}}(p \cdot q) = P_{\mathbb{W}}(p) \cdot P_{\mathbb{V}}(p) \cdot P_{\mathbb{W}}(q) \cdot P_{\mathbb{V}}(p)^{-1}, \quad P_{\mathbb{V}}(p \cdot q) = P_{\mathbb{V}}(p) \cdot P_{\mathbb{V}}(q),$$
$$P_{\mathbb{W}}(p^{-1}) = P_{\mathbb{V}}(p)^{-1} \cdot P_{\mathbb{W}}(p)^{-1} \cdot P_{\mathbb{V}}(p), \qquad P_{\mathbb{V}}(p^{-1}) = P_{\mathbb{V}}(p)^{-1}.$$

(3) *There exists a constant $\tilde{C} = \tilde{C}(\mathbb{W}, \mathbb{V}) > 0$ such that*

$$\tilde{C}(\|P_{\mathbb{W}}(p)\| + \|P_{\mathbb{V}}(p)\|) \le \|p\| \le \|P_{\mathbb{W}}(p)\| + \|P_{\mathbb{V}}(p)\| \quad \textit{for all } p \in \mathbb{H}^n.$$

Definition 2.9. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$. Given a function $\phi\colon A \subset \mathbb{W} \to \mathbb{V}$, we define its *intrinsic graph* as the set

$$\operatorname{gr}_\phi := \{w \cdot \phi(w) : w \in A\}.$$

Definition 2.10. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$. If $\beta \ge 0$, we define the *(intrinsic) cone* $C_{\mathbb{W},\mathbb{V}}(0, \beta)$ of vertex 0, base $\mathbb{W}$, axis $\mathbb{V}$ and opening β as

$$C_{\mathbb{W},\mathbb{V}}(0, \beta) := \{p \in \mathbb{H}^n : \|p_{\mathbb{W}}\| \le \beta \|p_{\mathbb{V}}\|\}.$$

Moreover, for every $p \in \mathbb{H}^n \setminus \{0\}$, we define the *(intrinsic) cone* of vertex p, base $\mathbb{W}$, axis $\mathbb{V}$ and opening β as

$$C_{\mathbb{W},\mathbb{V}}(p, \beta) := p \cdot C_{\mathbb{W},\mathbb{V}}(0, \beta). \tag{2.1}$$

For the sake of brevity, when there is no risk of confusion, in the following, we will denote such cone as $C_\beta(p)$.

Remark 2.11. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$. For every $p \in \mathbb{H}^n$, $0 < \alpha < \beta$, we have $C_\alpha(p) \subset C_\beta(p)$. Moreover, $\delta_\lambda(C_\beta(0)) = C_\beta(0)$ for every $\lambda > 0$. Finally, $C_0(0) = \mathbb{V}$, while $\overline{\bigcup_{\beta>0} C_\beta(0)} = \mathbb{H}^n$.

Definition 2.12. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and let $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$. We say that ϕ is an *intrinsic Lipschitz map* if there exists $M > 0$ such that, for all $p \in \mathrm{gr}_\phi$,

$$C_{\frac{1}{M}}(p) \cap \mathrm{gr}_\phi = \{p\}.$$

If this is the case, we will say that ϕ is intrinsic M-Lipschitz. Moreover, for $E \subseteq A$, we define the *Lipschitz constant* of ϕ on E as

$$\mathrm{Lip}(\phi, E) := \inf\{M > 0 : C_{\frac{1}{M}}(p) \cap \mathrm{gr}_{\phi|_E} = \{p\} \text{ for all } p \in \mathrm{gr}_{\phi|_E}\}.$$

Definition 2.13. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and let $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$. We define the set

$$S_\phi := \{w \in A : \text{there exist } \beta > 0 \text{ and } U \text{ open in } A \text{ with } w \in U,\ C_\beta(w \cdot \phi(w)) \cap \mathrm{gr}_{\phi|_U} = \{w \cdot \phi(w)\}\}.$$

If $w \in S_\phi$, we say that ϕ is *locally intrinsic Lipschitz* at w and we define the *pointwise intrinsic Lipschitz constant* of ϕ at w as

$$\mathrm{lip}(\phi, w) := \inf\{M > 0 : \text{there exists } r > 0 \text{ such that } C_{\frac{1}{M}}(w \cdot \phi(w)) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(w,r)}} = \{w \cdot \phi(w)\}\},$$

where, with $B_{\mathbb{W}}(w, r)$, we intend, here and in the following, the ball depending on the distance induced on $\mathbb{W}$ by d.

Remark 2.14. By definition, intrinsic cones depend on the chosen distance d. However, the intrinsic Lipschitz continuity property of a given map ϕ does not depend on the choice of d; also, the set S_ϕ does not depend on d.

Definition 2.15. Let $m \geq 0$. Given any left invariant, homogeneous and rotationally invariant distance on $\mathbb{H}^n$, we denote by $\mathcal{S}^m$ the *spherical Hausdorff measure* on $\mathbb{H}^n$ defined for $E \subset \mathbb{H}^n$ by

$$\mathcal{S}^m(E) := \lim_{r\to 0^+} \inf\Big\{\sum_{i\in\mathbb{N}} (2r_i)^m : \text{there exists } (p_i)_i \subset \mathbb{H}^n, \text{there exists } (r_i)_i \text{ with } 0 < r_i < r \text{ and } E \subset \bigcup_{i\in\mathbb{N}} B(p_i, r_i)\Big\}.$$

Definition 2.16. Let μ be a measure on $\mathbb{H}^n$, $E \subseteq \mathbb{H}^n$ a measurable set for μ and $p \in E$. We say that p is a *point of density* of E with respect to μ if

$$\lim_{r\to 0^+} \frac{\mu(E \cap B(p, r))}{\mu(B(p, r))} = 1.$$

Theorem 2.17 ([19, Theorem 3.9]). *Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and let $\phi\colon \mathbb{W} \to \mathbb{V}$ be an intrinsic Lipschitz map. Denote by k the (metric) dimension of $\mathbb{V}$. Then $\mathcal{S}^{Q-k} \llcorner \mathrm{gr}_\phi$ is $(Q-k)$-Ahlfors regular on gr_ϕ.*

Definition 2.18. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$. We define the *graph distance* $\rho_\phi\colon A \times A \to [0, +\infty)$ as

$$\rho_\phi(w_1, w_2) := \frac{1}{2}(\|(p_1^{-1} \cdot p_2)_{\mathbb{W}}\| + \|(p_2^{-1} \cdot p_1)_{\mathbb{W}}\|) \quad \text{for all } w_1, w_2 \in A,$$

where $p_i = w_i \cdot \phi(w_i)$ for $i = 1, 2$.

Proposition 2.19 ([29, Proposition 4.59] and [19, Remark 3.6]). *Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and let $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$ be intrinsic M-Lipschitz. Then the graph distance ρ_ϕ is a quasi-distance, i.e.,*
(i) $\rho_\phi(w_1, w_2) = \rho_\phi(w_2, w_1)$ *for every* $w_1, w_2 \in A$,
(ii) $\rho_\phi(w_1, w_2) = 0$ *if and only if* $w_1 = w_2$,
(iii) *there exists a constant* $C > 1$ *such that*

$$\rho_\phi(w_1, w_2) \leq C(\rho_\phi(w_1, w_3) + \rho_\phi(w_3, w_2)) \quad \text{for every } w_1, w_2, w_3 \in A.$$

Moreover, there exists a positive constant $C = C(M) > 1$ *such that*

$$\frac{1}{C}\rho_\phi(w_1, w_2) \leq d(w_1 \cdot \phi(w_1), w_2 \cdot \phi(w_2)) \leq C\rho_\phi(w_1, w_2)$$

for every $w_1, w_2 \in A$. *This means that* ρ_ϕ *is equivalent to the distance on the graph of* ϕ.

Remark 2.20. If $\mathbb{W}$ is a normal subgroup of $\mathbb{H}^n$, then Definition 2.13 is equivalent to the following one, which generalizes (1.1):

$$S_\phi = \left\{ w \in A \colon \limsup_{\mathbb{W} \ni y \to w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} < +\infty \right\}.$$

In fact, on one side, assume that there exist $\beta > 0$ and U open in A with $w \in U$ and

$$C_\beta(w \cdot \phi(w)) \cap \mathrm{gr}_{\phi|U} = \{w \cdot \phi(w)\}.$$

Then, for every $y \in U$ with $y \neq w$, it holds $y \cdot \phi(y) \notin C_\beta(w \cdot \phi(w))$. Denoting $p := w \cdot \phi(w)$ and $q := y \cdot \phi(y)$, by definition of cone, we get that $p^{-1} \cdot q \notin C_\beta(0)$ and so

$$\|(p^{-1} \cdot q)_{\mathbb{W}}\| > \beta \|(p^{-1} \cdot q)_{\mathbb{V}}\| = \beta \|\phi(w)^{-1} \cdot \phi(y)\|,$$

where the last identity follows by the fact that $P_{\mathbb{V}}$ is a group homomorphism (see Proposition 2.8). Hence

$$d(\phi(y), \phi(w)) = \|\phi(w)^{-1} \cdot \phi(y)\| < \frac{1}{\beta} \|(p^{-1} \cdot q)_{\mathbb{W}}\| \leq \frac{1}{\beta} [\|(p^{-1} \cdot q)_{\mathbb{W}}\| + \|(q^{-1} \cdot p)_{\mathbb{W}}\|] = \frac{2}{\beta} \rho_\phi(y, w)$$

for every $y \in U$ with $y \neq w$. It follows that

$$\limsup_{\mathbb{W} \ni y \to w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} \leq \frac{2}{\beta} < +\infty.$$

On the other hand, assume that

$$\limsup_{\mathbb{W} \ni y \to w} \frac{d(\phi(y), \phi(w))}{\rho_\phi(y, w)} < +\infty,$$

which implies that there exist $L > 0$ and $U \ni w$ open in A such that, for every $y \in U$,

$$\|\phi^{-1}(w) \cdot \phi(y)\| = d(\phi(y), \phi(w)) \leq L \rho_\phi(y, w). \tag{2.2}$$

For every $y \in U$, let us denote as before $q := y \cdot \phi(y)$ and $p := w \cdot \phi(w)$.

Claim. *There exists $C > 0$ (not depending on y) such that $\rho_\phi(w, y) \leq C \|(p^{-1} \cdot q)_{\mathbb{W}}\|$ for every $y \in U$.*

Assuming the claim, thanks to (2.2), we get that, for a suitable constant $\tilde{L}$ and for every $y \in U$,

$$\|(p^{-1} \cdot q)_{\mathbb{V}}\| = \|\phi(w)^{-1} \cdot \phi(y)\| \leq \tilde{L} \|(p^{-1} \cdot q)_{\mathbb{W}}\|,$$

which means that the point $p^{-1} q$ does not belong to the cone $C_{\frac{1}{\tilde{L}}}(0)$. Therefore, $q \notin C_{\frac{1}{\tilde{L}}}(p)$ for every $y \in U$ and

$$C_{\frac{1}{\tilde{L}}}(p) \cap \mathrm{gr}_{\phi|U} = \{p\},$$

concluding the proof of the equivalence.

It remains to show the validity of the claim: such a conclusion can be obtained proceeding as in the second part of the proof of [29, Theorem 4.60]. In particular, by [29, formula (100)], one gets that, for every $\varepsilon > 0$, there exists a constant $\overline{C} = \overline{C}(\varepsilon)$ such that, for every $y \in A$,

$$\rho_\phi(w, y) \leq \overline{C}(\varepsilon) \|(p^{-1} \cdot q)_{\mathbb{W}}\| + \varepsilon \|(p^{-1} \cdot q)_{\mathbb{V}}\|.$$

From (2.2), if $y \in U$, we get

$$\rho_\phi(w, y) \leq \overline{C}(\varepsilon) \|(p^{-1} \cdot q)_{\mathbb{W}}\| + \varepsilon L \rho_\phi(w, y).$$

Fixing $\varepsilon < 1/L$, we finally get

$$\rho_\phi(w, y) \leq \frac{\overline{C}}{1 - \varepsilon L} \|(p^{-1} \cdot q)_{\mathbb{W}}\|$$

for every $y \in U$, proving the claim.

Definition 2.21. Let $\mathbb{W}$, $\mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and $\phi \colon A \subseteq \mathbb{W} \to \mathbb{V}$. We say that ϕ is an *intrinsic linear* map if its graph gr_ϕ is a homogeneous subgroup of $\mathbb{H}^n$.

Remark 2.22. Another characterization of the notion of intrinsic linear map can be given as follows; see [32]. Let $k = \dim(\mathbb{V})$. For every $w \in \mathbb{W}$, define $w_H \in \mathbb{R}^{2n+1-k}$ as

$$w_H := (x_{k+1}, \dots, y_n) \quad \text{if } k < n \text{ and } w = (x_{k+1}, \dots, y_n, t),$$
$$w_H := (y_1, \dots, y_n) \quad \text{if } k = n \text{ and } w = (y_1, \dots, y_n, t).$$

Then ϕ is intrinsic linear if and only if there exists a $k \times (2n-k)$ matrix M such that, for every $w \in \mathbb{W}$, $\phi(w) = Mw_H$ (identifying M with a linear map $M\colon \mathbb{R}^{2n-k} \to \mathbb{R}^k \equiv \mathbb{V}$).

Definition 2.23. Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$ and $\phi\colon A \subseteq \mathbb{W} \to \mathbb{V}$, where A is a relatively open set. We say that ϕ is *intrinsically differentiable* at $\bar{w} \in A$ if there exists an intrinsic linear map $d\phi_{\bar{w}}\colon \mathbb{W} \to \mathbb{V}$ such that

$$\lim_{\mathbb{W} \ni w \to 0} \frac{d(\phi_{\bar{w}}(w), d\phi_{\bar{w}}(w))}{\|w\|} = 0,$$

where $\phi_{\bar{w}}$ is the map whose intrinsic graph is given by $(\bar{w} \cdot \phi(\bar{w}))^{-1} \cdot \mathrm{gr}_\phi$. The map $d\phi_{\bar{w}}$ is called the *intrinsic differential* of ϕ at $\bar{w}$.

Remark 2.24. If $\mathbb{W}$ is a normal subgroup, then the explicit expression of $\phi_{\bar{w}}$ is given by

$$\phi_{\bar{w}}(w) := \phi(\bar{w})^{-1} \cdot \phi(\bar{w} \cdot \phi(\bar{w}) \cdot w \cdot \phi(\bar{w})^{-1}). \tag{2.3}$$

If instead $\mathbb{V}$ is normal, then $\phi_{\bar{w}}$ can be written as follows:

$$\phi_{\bar{w}}(w) := w^{-1} \cdot \phi(\bar{w})^{-1} \cdot w \cdot \phi(\bar{w} \cdot w). \tag{2.4}$$

A general expression for the map $\phi_{\bar{w}}$, which has as special cases formulas (2.3) and (2.4), can be found for instance in [19, Proposition 2.21].

In the proof of the Stepanov differentiability theorem, we will need two main results: an equivalent characterization of intrinsic differentiability and the Rademacher theorem for intrinsic graphs. We now state both.

Theorem 2.25 ([18, Theorem 4.15]; see also [15, Theorem 3.2.8]). *Fix $1 \le k \le n$ and let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension k. Let $A \subseteq \mathbb{W}$ be an open set and $\phi\colon A \to \mathbb{V}$. Fix $\bar{w} \in A$ and define $\bar{p} := \bar{w} \cdot \phi(\bar{w})$. Then the following statements are equivalent:*
(i) *ϕ is intrinsically differentiable at $\bar{w}$;*
(ii) *there exists a vertical subgroup $\mathbb{T}_{\phi,\bar{p}}$, complementary to $\mathbb{V}$, such that, for every $\alpha > 0$, there exists*

$$\bar{r} = \bar{r}(\phi, \bar{w}, \alpha) > 0$$

such that

$$C_{\mathbb{T}_{\phi,\bar{p}},\mathbb{V}}(\bar{p}, \alpha) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(\bar{w},\bar{r})}} = \{\bar{p}\}.$$

Theorem 2.26 ([32, Theorem 1.1 and Theorem 1.5]). *Fix $1 \le k \le n$ and let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension k. Let $A \subseteq \mathbb{W}$ be a set and let $\phi\colon A \to \mathbb{V}$ be intrinsic Lipschitz. Then there exists an intrinsic Lipschitz map $\tilde{\phi}\colon \mathbb{W} \to \mathbb{V}$ such that $\tilde{\phi}|_A \equiv \phi$ and $\tilde{\phi}$ is intrinsically differentiable almost everywhere on $\mathbb{W}$.*

3 Proof of Stepanov theorem

3.1 Proof of Stepanov theorem for intrinsic graphs of arbitrary codimension

In this subsection, we provide the proof of the Stepanov differentiability theorem starting with the case of maps from horizontal subgroups to normal subgroups and then our main result which involves maps going from normal subgroups to horizontal subgroups.

The first case is a fairly easy consequence of the "classical" Stepanov theorem for maps between Carnot groups [33]. Indeed, by applying [2, Proposition 3.25], we can reduce the intrinsic differentiability of a map to the Pansu differentiability of its graph map.

Theorem 3.1. *Fix $1 \leq k \leq n$ and let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension k. Let $A \subseteq \mathbb{V}$ be an open set and $\phi\colon A \to \mathbb{W}$. Then ϕ is intrinsically differentiable almost everywhere on S_ϕ.*

Proof. Let $\phi\colon A \subset \mathbb{V} \to \mathbb{W}$ be as in the assumptions. Define $\Phi_\phi\colon A \to \mathbb{H}^n$ to be the graph map

$$\Phi_\phi(v) := v \cdot \phi(v).$$

We start by proving the following elementary fact: $\bar{v} \in S_\phi$ if and only if Φ_ϕ is pointwise (metric) Lipschitz continuous at $\bar{v}$, that is

$$\limsup_{A \ni v \to \bar{v}} \frac{\|(\Phi_\phi(\bar{v}))^{-1} \cdot \Phi_\phi(v)\|}{\|\bar{v}^{-1} \cdot v\|} < +\infty. \tag{3.1}$$

Suppose first $\bar{v} \in S_\phi$. This means that there exist $\beta > 0$ and $U \subset A$ open containing $\bar{v}$ such that

$$C_{\mathbb{V},\mathbb{W}}(\Phi_\phi(\bar{v}), \beta) \cap \mathrm{gr}_{\phi|_U} = \{\Phi_\phi(\bar{v})\}.$$

Equivalently, after a left translation and (2.1),

$$C_{\mathbb{V},\mathbb{W}}(0, \beta) \cap (\Phi_\phi(\bar{v}))^{-1} \cdot \mathrm{gr}_{\phi|_U} = \{0\}.$$

For notational simplicity, let us write $\bar{p} := \Phi_\phi(\bar{v})$. By Definition 2.10, we get, for every point $p = v \cdot \phi(v) \in \mathrm{gr}_{\phi|_U}$,

$$\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| \leq \frac{1}{\beta}\|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|.$$

Since $\mathbb{W}$ is normal, the projection on $\mathbb{V}$ is a group homomorphism (Proposition 2.8): hence $(\bar{p}^{-1} \cdot p)_{\mathbb{V}} = \bar{v}^{-1} \cdot v$. Therefore, we get

$$\|\bar{p}^{-1} \cdot p\| \leq \|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| + \|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\| \leq \left(1 + \frac{1}{\beta}\right)\|\bar{v}^{-1} \cdot v\|$$

for every $v \in U$, which implies (3.1).

On the other hand, assuming (3.1) and keeping the same notation as before, there exists $U \subset A$ open with $\bar{v} \in U$ and $L > 0$ such that

$$\|\bar{p}^{-1} \cdot p\| \leq L\|\bar{v}^{-1} \cdot v\| = L\|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|$$

for all $v \in U$. By Proposition 2.8, there exists a constant $\tilde{C} > 0$ such that $\|\bar{p}^{-1} \cdot p\| \geq \tilde{C}\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\|$. We deduce that

$$\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\| \leq \frac{L}{\tilde{C}}\|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|,$$

which implies, arguing as in the first part of the proof, that $\bar{v} \in S_\phi$.

We move now to the proof of the theorem: since $\mathbb{V}$ is horizontal, we can identify $\mathbb{V} \equiv \mathbb{R}^k$ for some $1 \leq k \leq n$. Hence, by applying the "classical" Stepanov differentiability theorem for maps between Carnot groups (see [33, Theorem 3.1] and [34, Theorem 1]), we deduce that the graph map $\Phi_\phi\colon A \subset \mathbb{V} \equiv \mathbb{R}^k \to \mathbb{H}^n$ is Pansu-differentiable almost everywhere in the set of points where Φ_ϕ is pointwise Lipschitz continuous, which coincides by the previous argument with S_ϕ. Then we conclude by applying [2, Proposition 3.25]: a map $\phi\colon A \subset \mathbb{V} \to \mathbb{W}$ is intrinsically differentiable at $\bar{v}$ if and only if the graph map $\Phi_\phi\colon A \subset \mathbb{V} \to \mathbb{H}^n$ is Pansu-differentiable at $\bar{v}$. □

Remark 3.2. We point out that all the results used in the proof of Theorem 3.1 hold in general Carnot groups $\mathbb{G}$, which can be written as $\mathbb{G} = \mathbb{V}\mathbb{W}$, where $\mathbb{V}$ is a horizontal subgroup and $\mathbb{W}$ is normal; see, in particular, [1]. Therefore, Theorem 3.1 holds even in the generality described above for maps $\phi\colon A \subseteq \mathbb{V} \to \mathbb{W}$.

Let us now move to the main result of this paper, concerning the proof of the Stepanov differentiability theorem for maps from a normal subgroup to an abelian one. Theorem 3.3 below combined with Theorem 3.1 completes the proof of the Stepanov differentiability theorem.

Theorem 3.3. *Fix $1 \leq k \leq n$ and let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension k. Let $A \subseteq \mathbb{W}$ be an open set and $\phi\colon A \to \mathbb{V}$. Then ϕ is intrinsically differentiable almost everywhere on S_ϕ.*

Proof. For convenience, we split the proof into several steps.

Step 1: Split S_ϕ into countably many sets, where ϕ is intrinsic Lipschitz. For $j \in \mathbb{N}$, we define

$$E_j := \{w \in A : C_{1/j}(w \cdot \phi(w)) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(w,1/j)}} = \{w \cdot \phi(w)\}\}. \tag{3.2}$$

Then each E_j is measurable and it is clear that $S_\phi = \bigcup_{j\in\mathbb{N}} E_j$. Then we express each E_j as the union of measurable sets $E_{j,1}, E_{j,2}, \ldots$ such that $\mathrm{diam}(E_{j,i}) < \frac{1}{j}$ for every $i, j \in \mathbb{N}$. We can do that for example intersecting E_j with countably many balls of diameter smaller than $\frac{1}{j}$. Then we have that $S_\phi = \bigcup_{j,i\in\mathbb{N}} E_{j,i}$ and $\phi|_{E_{j,i}}$ is intrinsic Lipschitz.

Step 2: Use Theorem 2.26 to extend each $\phi|_{E_{j,i}}$. By Theorem 2.26, for every $j, i \in \mathbb{N}$, there exists an intrinsic Lipschitz map $\tilde\phi\colon \mathbb{W} \to \mathbb{V}$ such that $\tilde\phi|_{E_{j,i}} \equiv \phi|_{E_{j,i}}$ and $\tilde\phi$ is intrinsically differentiable almost everywhere on $\mathbb{W}$. Fix $\bar w \in E_{j,i}$ such that $\bar w$ is a point of intrinsic differentiability for $\tilde\phi$ and $\bar w \cdot \phi(\bar w)$ is a density point of $\mathrm{gr}_{\phi|E_{j,i}}$ with respect to $\mathcal{S}^{Q-k} \llcorner \mathrm{gr}_{\tilde\phi}$ (recall that, from Theorem 2.17, $\mathcal{S}^{Q-k} \llcorner \mathrm{gr}_{\tilde\phi}$ is a $(Q-k)$-Ahlfors regular measure on $\mathrm{gr}_{\tilde\phi}$, so that the Lebesgue density theorem holds). By [32, Remark 4.6], there exists a constant $\overline{C} > 0$ such that, denoting by Φ the graph map $\Phi(w) := w \cdot \tilde\phi(w)$,

$$\overline{C}^{-1}\mathcal{S}^{Q-k} \llcorner \mathrm{gr}_{\tilde\phi} \le \Phi_\#(\mathcal{L}^{2n+1-k} \llcorner \mathbb{W}) \le \overline{C}\mathcal{S}^{Q-k} \llcorner \mathrm{gr}_{\tilde\phi}\,.$$

This implies that the set of points $\bar w$ with the previous properties is a full-measure set in $E_{j,i}$.

If we prove that $\bar w$ is also a point of intrinsic differentiability for ϕ, then we are done.

Step 3: Without loss of generality, one can assume $\bar w = 0$ and $\phi(\bar w) = 0$. Assuming that $\bar w = 0$ and $\phi(\bar w) = 0$ is equivalent to replacing the function ϕ with the translated function $\phi_{\bar w}$ (see (2.3)) since $\phi_{\bar w}(0) = 0$. Notice that, by definition, ϕ is intrinsically differentiable at $\bar w$ if and only if $\phi_{\bar w}$ is intrinsically differentiable at 0. Hence it suffices to show that all the properties we will use of the map ϕ are true also for $\phi_{\bar w}$. Again, the differentiability of $\tilde\phi$ is preserved by translation of the graph. The same holds for the intrinsic Lipschitz property of $\tilde\phi$. Moreover, $\bar w \cdot \phi(\bar w)$ is a density point of $\mathrm{gr}_{\phi|E_{j,i}}$ if and only if 0 is a density point of $(\bar w \cdot \phi(\bar w))^{-1} \cdot \mathrm{gr}_{\phi|E_{j,i}}$ (by invariance of the distance and the measure). The last condition to be verified is to show that there exists $\delta > 0$ such that, for all $w \in E_{j,i}$, one has

$$B_{\mathbb{W}}(\phi(\bar w)^{-1} \cdot \bar w^{-1} \cdot w \cdot \phi(\bar w), \delta) \subseteq \phi(\bar w)^{-1} \cdot \bar w^{-1} \cdot B_{\mathbb{W}}\Big(w, \frac{1}{j}\Big) \cdot \phi(\bar w). \tag{3.3}$$

In fact, if (3.3) holds, then we obtain Step 3 upon noticing that the set corresponding to $B_{\mathbb{W}}(w, \frac{1}{j})$ after the translation of the graph of ϕ is exactly $\phi(\bar w)^{-1} \cdot \bar w^{-1} \cdot B_{\mathbb{W}}(w, \frac{1}{j}) \cdot \phi(\bar w)$. Hence the property defining E_j in (3.2) remains true for the function $\phi_{\bar w}$ replacing $\frac{1}{j}$ with δ.

We are left to prove (3.3). Since

$$\lim_{\mathbb{W}\ni a\to 0} \|\phi(\bar w) \cdot a \cdot \phi(\bar w)^{-1}\| = 0,$$

there exists $\delta > 0$ such that if $\|a\| < \delta$, then

$$\|\phi(\bar w) \cdot a \cdot \phi(\bar w)^{-1}\| < \frac{1}{j}. \tag{3.4}$$

We define $\tilde w := \phi(\bar w)^{-1} \cdot \bar w^{-1} \cdot w \cdot \phi(\bar w)$ and we claim that

$$\phi(\bar w) \cdot B_{\mathbb{W}}(\tilde w, \delta) \cdot \phi(\bar w)^{-1} \subseteq B_{\mathbb{W}}\Big(\phi(\bar w) \cdot \tilde w \cdot \phi(\bar w)^{-1}, \frac{1}{j}\Big). \tag{3.5}$$

If (3.5) holds, then

$$\begin{aligned} B_W(\tilde w, \delta) &\subseteq \phi(\bar w)^{-1} \cdot B_{\mathbb{W}}\Big(\phi(\bar w) \cdot \tilde w \cdot \phi(\bar w)^{-1}, \frac{1}{j}\Big) \cdot \phi(\bar w) \\ &= \phi(\bar w)^{-1} B_{\mathbb{W}}\Big(\bar w^{-1} \cdot w, \frac{1}{j}\Big) \cdot \phi(\bar w) \\ &= \phi(\bar w)^{-1} \cdot \bar w^{-1} \cdot B_{\mathbb{W}}\Big(w, \frac{1}{j}\Big) \cdot \phi(\bar w), \end{aligned}$$

proving (3.3). We are left to prove (3.5). Let $y \in \phi(\bar{w}) \cdot B_{\mathbb{W}}(\tilde{w}, \delta) \cdot \phi(\bar{w})^{-1}$. Then $y = \phi(\bar{w}) \cdot x \cdot \phi(\bar{w})^{-1}$ for some $x \in B_{\mathbb{W}}(\tilde{w}, \delta)$. Since $d(x, \tilde{w}) < \delta$, by (3.4), we have $\|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot x \cdot \phi(\bar{w})^{-1}\| < \frac{1}{j}$. The latter implies that

$$d(y, \phi(\bar{w}) \cdot \tilde{w} \cdot \phi(\bar{w})^{-1}) = \|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot \phi(\bar{w})^{-1} \cdot y\| = \|\phi(\bar{w}) \cdot \tilde{w}^{-1} \cdot x \cdot \phi(\bar{w})^{-1}\| < \frac{1}{j},$$

finally proving (3.5).

Step 4: Use the equivalent characterization from Theorem 2.25. Since $\tilde{\phi}$ is intrinsically differentiable at 0, by Theorem 2.25, there exists a vertical subgroup $\mathbb{T}_{\tilde{\phi},0}$ such that, for every $\alpha > 0$, there exists $\tilde{r} = \tilde{r}(\tilde{\phi}, 0, \alpha) > 0$ such that

$$C_{\mathbb{T}_{\tilde{\phi},0},\mathbb{V}}(0, \alpha) \cap \mathrm{gr}_{\tilde{\phi}|_{B_{\mathbb{W}}(0,\tilde{r})}} = \{0\}. \tag{3.6}$$

Using again Theorem 2.25, if we show that, for every $\alpha > 0$, there exists $\bar{r} = \bar{r}(\phi, 0, \alpha) > 0$ such that

$$C_{\mathbb{T}_{\tilde{\phi},0},\mathbb{V}}(0, \alpha) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(0,\bar{r})}} = \{0\},$$

then we get that ϕ is intrinsically differentiable at 0. Assume not: then there exists a certain $\alpha > 0$ such that, for every $\bar{r} > 0$, one has

$$C_{\mathbb{T},\mathbb{V}}(0, \alpha) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(0,\bar{r})}} \neq \{0\}, \tag{3.7}$$

where, for the sake of brevity, we write $\mathbb{T} := \mathbb{T}_{\tilde{\phi},0}$. From (3.7), we obtain that there is a sequence of points $(x_h)_{h\in\mathbb{N}} \subseteq \mathbb{W}$ such that $x_h \xrightarrow{h\to\infty} 0$ and

$$p_h := x_h \cdot \phi(x_h) \in C_{\mathbb{T},\mathbb{V}}(0, \alpha). \tag{3.8}$$

On the other hand, $\tilde{\phi}$ is intrinsically differentiable at 0, so, by (3.6), for h sufficiently large,

$$\widetilde{p_h} := x_h \cdot \tilde{\phi}(x_h) \notin C_{\mathbb{T},\mathbb{V}}(0, 2\alpha). \tag{3.9}$$

Step 5: Prove that $(p_h)_{\mathbb{T}} = (\widetilde{p_h})_{\mathbb{T}}$. Here and in the following of the proof, we will use the following notation: in order to indicate the components of a point $q \in \mathbb{H}^n$ with respect to the splitting $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, we will use $q = q_{\mathbb{W}} \cdot q_{\mathbb{V}}$; in order to indicate the components of a point $q \in \mathbb{H}^n$ with respect to the splitting $\mathbb{H}^n = \mathbb{T} \cdot \mathbb{V}$, we will use $q = q_{\mathbb{T}} \cdot q_{\mathbb{V}}^{\mathbb{T}}$. Notice that, in general, $q_{\mathbb{V}} \neq q_{\mathbb{V}}^{\mathbb{T}}$. We observe that

$$(p_h)_{\mathbb{W}} \cdot (p_h)_{\mathbb{V}} = p_h = (p_h)_{\mathbb{T}} \cdot (p_h)_{\mathbb{V}}^{\mathbb{T}} = ((p_h)_{\mathbb{T}})_{\mathbb{W}} \cdot \underbrace{((p_h)_{\mathbb{T}})_{\mathbb{V}} \cdot (p_h)_{\mathbb{V}}^{\mathbb{T}}}_{\in\mathbb{V}}.$$

By the uniqueness of the components (see Remark 2.7), we conclude that $(p_h)_{\mathbb{W}} = ((p_h)_{\mathbb{T}})_{\mathbb{W}}$, so $x_h = ((p_h)_{\mathbb{T}})_{\mathbb{W}}$. In the same fashion, we obtain that $x_h = ((\widetilde{p_h})_{\mathbb{T}})_{\mathbb{W}}$ and so $((p_h)_{\mathbb{T}})_{\mathbb{W}} = ((\widetilde{p_h})_{\mathbb{T}})_{\mathbb{W}}$. Now we observe that

$$\mathbb{T} \ni ((p_h)_{\mathbb{T}})^{-1} \cdot (\widetilde{p_h})_{\mathbb{T}} = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \underbrace{(((p_h)_{\mathbb{T}})_{\mathbb{W}})^{-1} \cdot ((\widetilde{p_h})_{\mathbb{T}})_{\mathbb{W}}}_{=0} \cdot ((\widetilde{p_h})_{\mathbb{T}})_{\mathbb{V}} \in \mathbb{V},$$

but $\mathbb{T}$ and $\mathbb{V}$ are complementary, so $\mathbb{T} \cap \mathbb{V} = \{0\}$, implying that $(p_h)_{\mathbb{T}} = (\widetilde{p_h})_{\mathbb{T}}$.

Step 6: Prove that $d(\phi(x_h), \tilde{\phi}(x_h)) \geq K\|x_h\|$ for some $K > 0$ and for h sufficiently large. We observe preliminarily that, by the fact that $((p_h)_{\mathbb{T}})_{\mathbb{W}} = x_h$,

$$(p_h)_{\mathbb{V}}^{\mathbb{T}} = ((p_h)_{\mathbb{T}})^{-1} \cdot p_h = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot x_h^{-1} \cdot x_h \cdot \phi(x_h) = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \phi(x_h) \tag{3.10}$$

and, in the same fashion, since $(p_h)_{\mathbb{T}} = (\tilde{p}_h)_{\mathbb{T}}$,

$$(\widetilde{p_h})_{\mathbb{V}}^{\mathbb{T}} = (((\tilde{p}_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \tilde{\phi}(x_h) = (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \tilde{\phi}(x_h). \tag{3.11}$$

From (3.8) and (3.9), we obtain, for h sufficiently large,

$$\begin{cases} \|(p_h)_{\mathbb{T}}\| \leq \alpha\|(p_h)_{\mathbb{V}}^{\mathbb{T}}\|, \\ \|(\widetilde{p_h})_{\mathbb{T}}\| > 2\alpha\|(\widetilde{p_h})_{\mathbb{V}}^{\mathbb{T}}\|. \end{cases} \tag{3.12}$$

By the left-invariance of the distance, we have

$$d(\phi(x_h), \tilde{\phi}(x_h)) = d((((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \phi(x_h), (((p_h)_{\mathbb{T}})_{\mathbb{V}})^{-1} \cdot \tilde{\phi}(x_h)) = d((p_h)_{\mathbb{V}}^{\mathbb{T}}, (\widetilde{p_h})_{\mathbb{V}}^{\mathbb{T}})$$
$$= \|((\widetilde{p_h})_{\mathbb{V}}^{\mathbb{T}})^{-1} \cdot (p_h)_{\mathbb{V}}^{\mathbb{T}}\| \geq \|(p_h)_{\mathbb{V}}^{\mathbb{T}}\| - \|(\widetilde{p_h})_{\mathbb{V}}^{\mathbb{T}}\| \geq \frac{1}{2\alpha}\|(p_h)_{\mathbb{T}}\|.$$

where we used the left-invariance of the distance, (3.10) and (3.11), the definition of norm associated to the distance, the reverse triangular inequality and, finally, (3.12) combined with Step 5, respectively. From Proposition 2.8, we obtain that

$$\|(p_h)_{\mathbb{T}}\| = \|x_h \cdot ((p_h)_{\mathbb{T}})_{\mathbb{V}}\| \geq \tilde{C}\|x_h\|, \tag{3.13}$$

where $\tilde{C}$ is a constant only depending on $\mathbb{W}$ and $\mathbb{V}$. The latter proves that $d(\phi(x_h), \tilde{\phi}(x_h)) \geq K\|x_h\|$ for some $K > 0$.

Step 7: Construction of an auxiliary sequence $(q_h)_{h\in\mathbb{N}}$ with certain properties. Since we are assuming that 0 is a density point of $\mathrm{gr}_{\phi|_{E_{j,i}}}$ in $\mathrm{gr}_{\tilde{\phi}}$, by [3, p. 409], there exists a sequence $(q_h)_{h\in\mathbb{N}} \subseteq \mathrm{gr}_{\phi|_{E_{j,i}}}$ such that

$$d(q_h, \widetilde{p_h}) = o(d(0, \widetilde{p_h})) = o(\|\widetilde{p_h}\|).$$

In other words, for every $\varepsilon > 0$, there exists $\bar{h} \in \mathbb{N}$ such that, for every $h > \bar{h}$,

$$d(q_h, \widetilde{p_h}) \leq \varepsilon\|\widetilde{p_h}\|. \tag{3.14}$$

We observe that

$$\|\widetilde{p_h}\| = \|x_h \cdot \tilde{\phi}(x_h)\| \leq \|x_h\| + \|\tilde{\phi}(x_h)\|, \tag{3.15}$$

and since $\tilde{\phi}$ is intrinsically Lipschitz and $\tilde{\phi}(0) = \phi(0) = 0$, we obtain $\|\tilde{\phi}(x_h)\| \leq L\|x_h\|$ for some constant $L > 0$. The latter together with (3.14) and (3.15) implies that, for every $\varepsilon > 0$, there exists $\bar{h}_1 \in \mathbb{N}$ such that, for every $h > \bar{h}_1$,

$$d(q_h, \widetilde{p_h}) \leq \varepsilon\|x_h\|. \tag{3.16}$$

Moreover, for every $h \in \mathbb{N}$, there exists $y_h \in E_{j,i}$ such that

$$q_h = y_h \cdot \phi(y_h) = y_h \cdot \tilde{\phi}(y_h),$$

so that we can rewrite (3.16) as

$$d(y_h \cdot \tilde{\phi}(y_h), x_h \cdot \tilde{\phi}(x_h)) \leq \varepsilon\|x_h\|. \tag{3.17}$$

Since $\mathbb{W}$ is normal, the projection on $\mathbb{V}$ is Lipschitz continuous (again Proposition 2.8) and there exists a constant $D > 0$ such that

$$d(\tilde{\phi}(y_h), \tilde{\phi}(x_h)) \leq Dd(y_h \cdot \tilde{\phi}(y_h), x_h \cdot \tilde{\phi}(x_h)).$$

The latter together with (3.17) implies that, for every $\varepsilon > 0$, there exists $\bar{h}_2 \in \mathbb{N}$ such that, for every $h > \bar{h}_2$,

$$d(\tilde{\phi}(y_h), \tilde{\phi}(x_h)) \leq \varepsilon\|x_h\|. \tag{3.18}$$

Moreover, since $y_h \in E_{j,i} \subseteq E_j$, it follows from (3.2) and (3.3) that

$$C_{1/j}(q_h) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(y_h,\delta)}} = \{q_h\}, \tag{3.19}$$

where $\delta > 0$ is found as in (3.3).

Step 8: Prove that $d(x_h, y_h) \xrightarrow{h\to+\infty} 0$. By (3.16) and the fact that $x_h \to 0$, we know that $d(q_h, \tilde{p}_h) \xrightarrow{h\to+\infty} 0$. If $\tilde{C}$ is as in (3.13), by the explicit expression of the components given in Proposition 2.8 (2), we have

$$\tilde{C}\|\tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h)\| = \tilde{C}\|(\widetilde{p_h}^{-1} \cdot q_h)_{\mathbb{W}}\| \leq \|\widetilde{p_h}^{-1} \cdot q_h\| = d(q_h, \widetilde{p_h}),$$

which implies

$$\|\tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h)\| \xrightarrow{h\to+\infty} 0. \tag{3.20}$$

Recall that $\tilde{\phi}$ is intrinsic Lipschitz, therefore continuous, so $x_h \xrightarrow{h\to+\infty} 0$ implies $\tilde{\phi}(x_h) \to \tilde{\phi}(0) = \phi(0) = 0$. We conclude that $d(x_h, y_h) \xrightarrow{h\to+\infty} 0$ upon observing that, from (3.20), we get

$$d(x_h, y_h) = \|x_h^{-1} \cdot y_h\| = \|\tilde{\phi}(x_h) \cdot \tilde{\phi}(x_h)^{-1} \cdot x_h^{-1} \cdot y_h \cdot \tilde{\phi}(x_h) \cdot \tilde{\phi}(x_h)^{-1}\| \xrightarrow{h\to+\infty} 0.$$

The latter implies that, for h sufficiently large, $x_h \in B_{\mathbb{W}}(y_h, \delta)$.

Step 9: Conclude the proof obtaining a contradiction. Because of Step 8 and (3.19), we have, for h large enough,

$$p_h = x_h \cdot \phi(x_h) \notin C_{1/j}(q_h) = q_h \cdot C_{1/j}(0). \tag{3.21}$$

The latter is true unless $p_h = q_h$, but in that case, we would get $x_h = y_h$ by uniqueness of the components. Therefore, $x_h \in E_{j,i}$ and hence $\phi(x_h) = \tilde{\phi}(x_h)$: so we would obtain a contradiction with Step 6. From (3.21), we obtain

$$q_h^{-1} \cdot p_h \notin C_{1/j}(0) \implies \|(q_h^{-1} \cdot p_h)_{\mathbb{W}}\| > \frac{1}{j}\|(q_h^{-1} \cdot p_h)_{\mathbb{V}}\|. \tag{3.22}$$

From an explicit computation of the projections (see Proposition 2.8), we get

$$\begin{cases} (q_h^{-1} \cdot p_h)_{\mathbb{V}} = \tilde{\phi}(y_h)^{-1} \cdot \phi(x_h), \\ (q_h^{-1} \cdot p_h)_{\mathbb{W}} = \tilde{\phi}(y_h)^{-1} \cdot y_h^{-1} \cdot x_h \cdot \tilde{\phi}(y_h). \end{cases} \tag{3.23}$$

In particular, $(q_h^{-1} \cdot p_h)_{\mathbb{W}} = (q_h^{-1} \cdot \tilde{p}_h)_{\mathbb{W}}$. From (3.23), (3.22) and Proposition 2.8 (3), we then obtain

$$\|\tilde{\phi}(y_h)^{-1} \cdot \phi(x_h)\| = \|(q_h^{-1} \cdot p_h)_{\mathbb{V}}\| < j\|(q_h^{-1} \cdot p_h)_{\mathbb{W}}\| = j\|(q_h^{-1} \cdot \tilde{p}_h)_{\mathbb{W}}\| \le Cj\|q_h^{-1} \cdot \tilde{p}_h\|. \tag{3.24}$$

Finally,

$$\begin{aligned} \|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\| &\le \|\tilde{\phi}(x_h)^{-1} \cdot \tilde{\phi}(y_h)\| + \|\tilde{\phi}(y_h)^{-1} \cdot \phi(x_h)\| \\ &\le \varepsilon\|x_h\| + jCd(\tilde{p}_h, q_h) \le \varepsilon\|x_h\| + jC\varepsilon\|x_h\| = \underbrace{(1 + jC)}_{M}\varepsilon\|x_h\|, \end{aligned}$$

where we used the triangle inequality, (3.18) and (3.24) and, finally, (3.16), respectively. Combining the latter with Step 6 (that is, $d(\phi(x_h), \tilde{\phi}(x_h)) \ge K\|x_h\|$), we get

$$\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\| \le M\varepsilon\frac{d(\phi(x_h), \tilde{\phi}(x_h))}{K} = M\varepsilon\frac{\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\|}{K}, \tag{3.25}$$

where K is the same constant coming from Step 6. Simplifying $\|\tilde{\phi}(x_h)^{-1} \cdot \phi(x_h)\|$ from both sides of (3.25), we obtain

$$1 \le \frac{M\varepsilon}{K},$$

but then we get a contradiction from the arbitrariness of ε, concluding the proof. □

Remark 3.4. We emphasize that the proof of Theorem 3.3 is not dependent on the particular structure of $\mathbb{H}^n$. It can be extended to general Carnot groups $\mathbb{G} = \mathbb{W}\mathbb{V}$, where $\mathbb{W}$ is a normal subgroup and $\mathbb{V}$ is horizontal, provided that a Rademacher type theorem holds. This is particularly relevant for graphs of codimension 1 either in step 2 Carnot groups or, more generally, in Carnot groups of type $\star$. However, it is important to note that the validity of a Rademacher type theorem for intrinsic graphs of arbitrary codimension has only been established in the Heisenberg case. For this reason, we have chosen to set this paper within the context of $\mathbb{H}^n$.

4 Alternative proof of Stepanov theorem for graphs of codimension 1

In this section, we study the case of *1-codimensional* graphs in $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, which means that we restrict to the case $\dim(\mathbb{V}) = 1$. Under this assumption, $\mathbb{V} = \{\exp(tV) : t \in \mathbb{R}\}$ for a fixed $V \in \mathfrak{h}_1$ (the first layer of the stratification of the Lie algebra of $\mathbb{H}^n$) and we can naturally identify $\mathbb{V} \equiv \mathbb{R}$ with its usual order relation. In particular, we can define *infimum* and *supremum* of functions: if $\phi_\alpha : \mathbb{W} \to \mathbb{V}$ are such that $\phi_\alpha(w) = \exp(g_\alpha(w)V)$, for some $g_\alpha : \mathbb{W} \to \mathbb{R}$, we define

$$\inf_{\alpha \in I} \phi_\alpha(w) := \exp(\inf_{\alpha \in I} g_\alpha(w)V).$$

In the same way, we can define the supremum of a family of maps from $\mathbb{W}$ to $\mathbb{V}$.

Infimum and supremum of intrinsic Lipschitz maps are themselves intrinsic Lipschitz maps.

Lemma 4.1 ([18, Proposition 4.24]). *Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension* 1. *Then, for all $L > 0$, there exists $\tilde{L} \geq L$ with the following property: if $\{\phi_\alpha : \mathbb{W} \to \mathbb{V}\}$ is a family of intrinsic L-Lipschitz maps, then the function $\phi := \inf_\alpha \phi_\alpha$ is either well defined and intrinsic $\tilde{L}$-Lipschitz, or $\phi \equiv -\infty$. The same property holds for the supremum.*

Before presenting the alternative proof, inspired by the proof of J. Malý [25], we need an auxiliary lemma.

Lemma 4.2. *Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension* 1. *Let $A \subset \mathbb{W}$ be open and $\bar{w} \in A$. Suppose $\psi, \phi, \eta : A \to \mathbb{V}$ are such that $\psi \leq \phi \leq \eta$ on A (where we identify $\mathbb{V} \equiv \mathbb{R}$), $\psi(\bar{w}) = \phi(\bar{w}) = \eta(\bar{w})$ and assume also that the functions ψ and η are intrinsically differentiable at $\bar{w}$. Then ϕ is intrinsically differentiable at $\bar{w}$ and $d\psi_{\bar{w}} \equiv d\phi_{\bar{w}} \equiv d\eta_{\bar{w}}$.*

Proof. For convenience, we split the proof into four steps.

Step 1. We start by proving that if a map $\varphi : A \subset \mathbb{W} \to \mathbb{V}$ is intrinsically differentiable at $\bar{w} = 0$, $\varphi(0) = 0$ and $\varphi(w) \geq 0$ for every $w \in A$, then $d\varphi_0 \equiv 0$. Notice that, in this case, $\varphi_0(w) = \varphi(w)$. By differentiability, we know that

$$\frac{\|d\varphi_0(w)^{-1} \cdot \varphi(w)\|}{\|w\|} \to 0 \quad \text{if } w \to 0. \tag{4.1}$$

Since $\mathbb{V} \equiv \mathbb{R}$ is horizontal, we can rewrite (4.1) as

$$\frac{\varphi(w) - d\varphi_0(w)}{\|w\|} \to 0 \quad \text{if } w \to 0.$$

By definition of limit, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|w\| < \delta$, then

$$\frac{\varphi(w)}{\|w\|} < \frac{d\varphi_0(w)}{\|w\|} + \varepsilon.$$

Since $\varphi(w) \geq 0$ and by homogeneity of $d\varphi_0$, we get

$$d\varphi_0(\delta_{\frac{1}{\|w\|}} w) > -\varepsilon.$$

Notice that $\delta_{\frac{1}{\|w\|}} w \in \partial B(0,1) \cap \mathbb{W}$, independently on δ. By arbitrariness of ε, we infer that $d\varphi_0(u) \geq 0$ for every $u \in \partial B(0,1) \cap \mathbb{W}$, hence also for every $u \in \mathbb{W}$. This is possible only if $d\varphi_0 \equiv 0$, because an intrinsic linear map from $\mathbb{W}$ to $\mathbb{V}$ is actually H-linear and, if $d\varphi_0(u) > 0$ for some u, then $d\varphi_0(u^{-1}) = -d\varphi_0(u) < 0$.

Step 2. We show now that if $\rho, \sigma : A \subset \mathbb{W} \to \mathbb{V}$ are intrinsically differentiable at 0 with $\rho(0) = \sigma(0) = 0$, then $\rho^{-1}\sigma$ is also intrinsically differentiable at 0 and moreover $d(\rho^{-1}\sigma)_0 = d\rho_0^{-1} \cdot d\sigma_0$.

It is easy to check that $d\rho_0^{-1} \cdot d\sigma_0$ is an intrinsic linear map, since in our setting intrinsic linear maps are exactly H-linear maps (see [2, Proposition 3.23]). So we are left to prove that

$$\frac{\|(d\rho_0^{-1}(w) \cdot d\sigma_0(w))^{-1} \cdot (\rho^{-1}\sigma)_0(w)\|}{\|w\|} \to 0 \quad \text{as } w \to 0.$$

Since $\rho(0) = \sigma(0) = 0$, then $(\rho^{-1}\sigma)_0 = \rho^{-1}\sigma$. Hence, by commutativity of $\mathbb{V}$ and the triangular inequality,

$$\frac{\|(d\rho_0^{-1}(w) \cdot d\sigma_0(w))^{-1} \cdot (\rho^{-1}\sigma)_0(w)\|}{\|w\|} \leq \frac{\|d\rho_0(w)^{-1} \cdot \rho(w)\|}{\|w\|} + \frac{\|d\sigma_0(w)^{-1} \cdot \sigma(w)\|}{\|w\|}.$$

The last two quantities tend to 0 as $w \to 0$ since ρ, σ are intrinsically differentiable at 0 and we conclude.

Step 3. Let us now prove that if ψ and η are as in the statement of the lemma, then $d\psi_{\bar{w}} \equiv d\eta_{\bar{w}}$. Consider the translated functions

$$\psi_{\bar{w}}(w) = \psi(\bar{w})^{-1}\psi(\bar{w}\psi(\bar{w})w\psi(\bar{w})^{-1}),$$
$$\eta_{\bar{w}}(w) = \eta(\bar{w})^{-1}\eta(\bar{w}\eta(\bar{w})w\eta(\bar{w})^{-1}).$$

Since $\psi \leq \eta$ and $\psi(\bar{w}) = \eta(\bar{w})$, we get $\psi_{\bar{w}} \leq \eta_{\bar{w}}$ and clearly $\psi_{\bar{w}}(0) = \eta_{\bar{w}}(0) = 0$. Notice also that, since ψ and η are intrinsically differentiable at $\bar{w}$, so are $\psi_{\bar{w}}$ and $\eta_{\bar{w}}$ at 0. Hence, by Step 2, the map $\psi_{\bar{w}}^{-1}\eta_{\bar{w}}$ is intrinsically differentiable at 0 and $d(\psi_{\bar{w}}^{-1}\eta_{\bar{w}})_0 = d(\psi_{\bar{w}})_0^{-1} \cdot d(\eta_{\bar{w}})_0$. Moreover, $\psi_{\bar{w}}^{-1}\eta_{\bar{w}} \geq 0$. Thus, by Step 1, $d(\psi_{\bar{w}}^{-1}\eta_{\bar{w}})_0 \equiv 0$, which implies $d(\psi_{\bar{w}})_0 \equiv d(\eta_{\bar{w}})_0$. Hence, by definition, $d\psi_{\bar{w}} = d\eta_{\bar{w}}$.

Step 4. Let us finally prove the main conclusion. Let ψ, ϕ, η be as in the statement. As before, we observe that $\psi_{\bar{w}} \leq \phi_{\bar{w}} \leq \eta_{\bar{w}}$. Now let $\theta := d\psi_{\bar{w}} \equiv d\eta_{\bar{w}}$. Then, for every $w \in \mathbb{W}$ near 0, we have

$$\frac{\psi_{\bar{w}}(w) - d\psi_{\bar{w}}(w)}{d(0,w)} \leq \frac{\phi_{\bar{w}}(w) - \theta(w)}{d(0,w)} \leq \frac{\eta_{\bar{w}}(w) - d\eta_{\bar{w}}(w)}{d(0,w)}.$$

Then the left-hand side and the right-hand side go to 0 when $w \in B(0,s) \cap \mathbb{W}$ for $s \to 0$. This concludes the proof. □

Now we can present the alternative proof for the Stepanov theorem for 1-codimensional intrinsic graphs, that we restate.

Theorem 4.3. *Let $\mathbb{W}, \mathbb{V}$ be complementary subgroups of $\mathbb{H}^n$, where $\mathbb{V}$ is a horizontal subgroup of dimension 1. Let $A \subseteq \mathbb{W}$ be an open set and $\phi\colon A \to \mathbb{V}$. Then ϕ is intrinsically differentiable almost everywhere on S_ϕ.*

Proof. Let $\{U_j\}_{j\in\mathbb{N}}$ be an enumeration of all rational balls (i.e., Euclidean balls with rational center and rational radius) contained in A such that ϕ is bounded on U_j (here we are identifying $\mathbb{W} \equiv \mathbb{R}^{2n}$). Is it clear that $S_\phi \subseteq \bigcup_{i\in\mathbb{N}} U_i$. For each $j \in \mathbb{N}$, we define two intrinsic Lipschitz functions η_j and ψ_j on U_j by setting

$$\begin{aligned} \eta_j(w) &:= \inf\{\eta(w) : \eta \geq \phi \text{ on } B_j,\ \mathrm{Lip}(\eta, U_j) \leq j\}, \\ \psi_j(w) &:= \sup\{\psi(w) : \psi \leq \phi \text{ on } B_j,\ \mathrm{Lip}(\psi, U_j) \leq j\}. \end{aligned} \tag{4.2}$$

By Lemma 4.1 (combined with the extension theorem in [31, Proposition 3.4] or [19, Theorem 4.1]), for every $j \in \mathbb{N}$, there exists $\tilde{j} \geq j$ such that η_j and ψ_j are intrinsic $\tilde{j}$-Lipschitz on U_j. Define now

$$N = \bigcup_{j\in\mathbb{N}} \{w \in U_j : \eta_j \text{ or } \psi_j \text{ is not intrinsically differentiable at } w\}.$$

By Theorem 2.26, we have that $\mathcal{L}^{2n}(N) = 0$. Let $\bar{w} \in S_\phi \setminus N$: we will prove that ϕ is intrinsically differentiable at $\bar{w}$, concluding the proof. By definition of S_ϕ, there exist $\beta > 0$ and $r > 0$ such that

$$C_\beta(\bar{w} \cdot \phi(\bar{w})) \cap \mathrm{gr}_{\phi|_{B_{\mathbb{W}}(\bar{w},r)}} = \{\bar{w} \cdot \phi(\bar{w})\}.$$

Since $\mathbb{V}$ has dimension 1, we can write $\mathbb{V} = \{\exp(tV) \mid t \in \mathbb{R}\}$. Moreover, the "positive part" of the cone

$$C^+_\beta(\bar{w} \cdot \phi(\bar{w})) := C_\beta(\bar{w} \cdot \phi(\bar{w})) \cap \exp(\{Z \mid \langle Z, V\rangle \geq 0\})$$

is the graph of an intrinsic M-Lipschitz function $\gamma\colon \mathbb{W} \to \mathbb{V}$ for some $M > 0$ (see [18, Lemma 4.20] and also [19]). Consider now $i \geq M$ such that

$$B(\bar{w}, r/2) \subseteq U_i \subseteq B(\bar{w}, r).$$

Clearly, $\phi(\bar{w}) \leq \eta_i(\bar{w})$. On the other hand, γ is a suitable competitor in the family defined in (4.2): hence $\eta_i(\bar{w}) \leq \gamma(\bar{w}) = \phi(\bar{w})$. The same argument works for ψ_i and we deduce $\psi_i(\bar{w}) = \phi(\bar{w}) = \eta_i(\bar{w})$. Hence we conclude using Lemma 4.2. □

Remark 4.4. Similarly to Theorem 3.1 and Theorem 3.3, the proof of Theorem 4.3 can be extended to a more general context. Specifically, the same approach can be applied to one-codimensional graphs within general Carnot groups that satisfy a Rademacher type theorem. The equivalence between intrinsic linear and H-linear maps (from normal to abelian subgroups), utilized in Lemma 4.2, can be derived as shown in [2], using, for example, [12, Proposition 3.4].

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