

Research Article

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On prescribing the number of singular points in a Cosserat-elastic solid

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Abstract: In a geometrically non-linear Cosserat model for micro-polar elastic solids, we prove that critical points of the Cosserat energy functional with an arbitrary large (finite) number of singularities do exist, whereas Cosserat energy minimizers are known to be locally Hölder continuous. To reach that goal, we first develop a technique to insert dipole pairs of singularities into smooth maps while controlling the amount of Cosserat energy needed to do so. We then use this method to force an arbitrary number of singular points into (weak) Cosserat-elastic solids by prescribing smooth boundary data. The boundary data themselves are given in such a way, that they contain no topological obstruction to regularity. Throughout this paper, we often exploit connections between harmonic maps and Cosserat-elastic solids, so that we are able to adapt and incorporate ideas of R. Hardt and F.-H. Lin for harmonic maps with singularities, as well as of F. Béthuel for dipole pairs of singularities.

Keywords: Non-linear Cosserat elasticity, micro-polar elasticity, regularity, harmonic maps

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1 Introduction and statement of results

Cosserat elasticity is a well-known class of models in elasticity theory, whose foundations were laid at the beginning of the 20th century by the Cosserat brothers. The geometrically non-linear model for micro-polar elastic solids being discussed in this paper is a type of Cosserat elasticity that has first been studied in the context of calculus of variations by P. Neff, for example in [13]. Its basic concept is the following.

An elastic body in its original state is described as a subset $\Omega \subset \mathbb{R}^3$. It can be deformed by shifting each point $x \in \Omega$ to its new location $\varphi(x) \in \mathbb{R}^3$. Moreover, the micro-polar structure of the body allows each point to undergo some micro-rotation (without deforming the body any further), meaning that to each point x , there is attached an orthonormal frame, which is free to rotate by an orthogonal matrix $R(x) \in \text{SO}(3)$. The micro-rotation being in $\text{SO}(3)$, rather than using infinitesimal rotations in the corresponding Lie-algebra of skew-symmetric matrices, ultimately leads to the geometric non-linearities in the Euler–Lagrange equations of the model. Both deformation and micro-rotation cause material stresses, measured in terms of $R^T \cdot D\varphi - I_3$ and $R^T \cdot DR$, respectively. Leaving additional external forces and moments aside (as it was discussed in [6]), summing up the energy stored in the body, the *Cosserat energy functional* is given by

$$\mathcal{J}_\Omega(\varphi, R) = \|P(R^T \cdot D\varphi - I_3)\|_{L^2(\Omega)}^2 + \lambda \|R^T \cdot DR\|_{L^p(\Omega)}^p,$$

with constant $\lambda > 0$, parameter $p \geq 2$ and linear operator $P: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$, describing a weighted sum of the deviatoric symmetric part and the skew-symmetric part of a matrix as well as a diagonal matrix of its trace.

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With material constants $\mu_1, \mu_c, \mu_2 > 0$,

$$P(A) = \sqrt{\mu_1} \operatorname{devsym}(A) + \sqrt{\mu_c} \operatorname{skew}(A) + \frac{\sqrt{\mu_2}}{3} \operatorname{tr}(A) \cdot I_3.$$

The existence of minimizers of this Cosserat energy on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ was proven in [14]. Further aspects of the model and the existence of Cosserat energy minimizers are discussed in [15].

When studying regularity of minimizers, Gastel recently observed a connection between the Cosserat problem and p -harmonic maps, which is a well-studied area in Geometric Analysis. In the case $p = 2$ ($\lambda = 1$ without loss of generality), when all constants are assumed equal ($\mu_1 = \mu_c = \mu_2$), he found the following (cf. [6]): On one hand, he showed Hölder-continuity for all minimizers on the whole domain Ω . On the other hand, he gave an example of a critical point (meaning a weak solution of the Euler–Lagrange equations) of the Cosserat energy for $\Omega = B^3$, $p = 2$ and $\mu_1 = \mu_c = \mu_2 = 1$, whose micro-rotational part exhibits a point singularity at the origin. So in contrast to minimizers, regularity of critical points should be an issue.

Note that with this particular choice of constants $P(\cdot)$ becomes the identity and

$$\mathcal{J}_\Omega(\varphi, R) = \int_\Omega |R^T \cdot D\varphi - I_3|^2 + |DR|^2 dx.$$

In Geometric Analysis, many results are known about (non-)regularity of harmonic mappings (i.e. weak solutions for the Euler–Lagrange equations of the Dirichlet integral). Having in mind several of them, concerning harmonic mappings into the standard sphere S^2 , the starting point of our research is the question: How “big” can the singular set $\operatorname{Sing}(f)$ of a critical point $f = (\varphi, R)$ of the Cosserat energy get? (In this situation, $\operatorname{Sing}(f)$ denotes the set of points, where f fails to be locally in $C^{1,\mu} \times C^{0,\mu}$ for any $\mu \in (0, 1)$, its elements are called singularities. Similarly, $\operatorname{Sing}(R)$ denotes the set, where R fails to be locally in $C^{0,\mu}$ for any $\mu \in (0, 1)$.)

An idea for being able to use the vast machinery of results about the regularity of harmonic mappings into S^2 is to observe a connection between S^2 and the set

$$\mathcal{S} := \{A \in \operatorname{SO}(3) : A \text{ describes a } 180^\circ\text{-rotation around some axis in } \mathbb{R}^3\}.$$

By identifying each rotation in $\mathcal{S} \subset \operatorname{SO}(3)$ with its axis of rotation, we obtain a two-fold covering of the manifold $\mathcal{S}$, given by $F: S^2 \rightarrow \mathcal{S}$, $q \mapsto 2q \otimes q - I_3$. A quick calculation in local coordinates shows that F is locally isometric up to the factor $\sqrt{8}$. Moreover, a well-known fact from Algebraic Topology implies that, if the domain Ω is simply connected and locally path-connected, any continuous mapping can be lifted [8, Theorem 6.1 and Corollary 6.4, p. 26f]. To be precise, for the covering F and any continuous mapping $R: \Omega \rightarrow \mathcal{S}$, there exist two continuous mappings $\eta_{1,2}: \Omega \rightarrow S^2$, $\eta_1 = -\eta_2$ such that $R = F \circ \eta_{1,2}$, as long as Ω is simply connected and locally path-connected.

So instead of looking at the full variational Cosserat problem

$$\mathcal{J}_\Omega(\varphi, R) = \int_\Omega |R^T \cdot D\varphi - I_3|^2 + |DR|^2 dx \rightarrow \min \quad \text{in } H^1(\Omega, \mathbb{R}^3 \times \operatorname{SO}(3)), \quad (\mathcal{P})$$

we mostly work with the *restricted Cosserat problem*

$$\mathcal{J}_\Omega(\varphi, R) = \int_\Omega |R^T \cdot D\varphi - I_3|^2 + |DR|^2 dx \rightarrow \min \quad \text{in } H^1(\Omega, \mathbb{R}^3 \times \mathcal{S}). \quad (\mathcal{P}^*)$$

Often, restricting a variational problem to a submanifold changes the Euler–Lagrange equations and thus is not a suitable method for finding results for the general problem. But here $\mathcal{S}$ is a totally geodesic submanifold of $\operatorname{SO}(3)$. This fact implies (just like it is proven for harmonic mappings), that *restricted minimizers* (i.e. minimizers of the restricted Cosserat problem $(\mathcal{P}^*)$) are at least still critical points of the full Cosserat problem $(\mathcal{P})$. In general, they are not minimizers of $(\mathcal{P})$.

In [6], Gastel showed that the (interior) singular set of a Cosserat energy minimizer of the full problem $(\mathcal{P})$ is a discrete set and in fact empty. But the line of reasoning made there to show discreteness holds true for restricted minimizers, cf. [11, Section 4.1]. With similar arguments, following the suggestions from [18], based on [17] in the context of harmonic maps, one can even show full boundary regularity given C^1 -Dirichlet bound-

ary data, see also [11, Section 4.2]. So at most, we expect only isolated point singularities in the interior for restricted minimizers. In contrast to the local Hölder continuity for Cosserat energy minimizers in [6], as well as in contrast to smoothness for energy minimizers of a (two-dimensional) flat Cosserat micropolar membrane shell model developed via dimensional decent (recently shown by A. Gastel and P. Neff in [7]), yet at the same time in analogy to a result by R. Hardt and F.-H. Lin [9] for harmonic maps $u: B^3 \rightarrow S^2$, we derive the following statement. It shows that critical points of the Cosserat energy can be forced to have an arbitrary large number of singularities, by prescribing suitable smooth boundary data.

Theorem 1. *For every $N \in \mathbb{N}$ there exist smooth boundary data $g_0 = (\varphi_0, R_0) \in C^\infty(\partial B^3, \mathbb{R}^3 \times \mathbb{S})$ with $\deg(R_0) = 0$ such that each (restricted) minimizer $f = (\varphi, R)$ of the Cosserat energy $\mathcal{J}_{B^3}(\cdot)$ in the class $H_{g_0}^1(B^3, \mathbb{R}^3 \times \mathbb{S}) := \{g \in H^1(B^3, \mathbb{R}^3 \times \mathbb{S}) : g|_{\partial B^3} = g_0\}$ must have at least N singularities in its micro-rotational part R .*

Remark 1. The property $\deg(R_0) = 0$ emphasizes that the singularities, which we are about to enforce, do not appear simply due to elementary topological reasons, see the discussion in [3, p. 15] for example, in regard to harmonic mappings $u: \Omega \rightarrow S^2$. But, as $\mathbb{S} \simeq \mathbb{RP}^2$ is a non-orientable manifold, the concept of the classical Brouwer-degree $\deg(\psi)$ of a mapping ψ between orientable manifolds, which is used for the deformation component φ , needs to be modified for the micro-rotational component R . Inspired by observations in [16] and [12, Section 4], we define the (mod 2)-degree as follows.

Definition 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected and locally path-connected set and let f be a map with components $f = (\varphi, R): \Omega \rightarrow \mathbb{R}^3 \times \mathbb{S}$.

(i) For $R \in C^0(\Omega, \mathbb{S})$, there exists a lift $n: \Omega \rightarrow S^2$, which means $F \circ n = R$. Then the (mod 2)-degree of R is given by

$$\deg(R) := \deg(n) \bmod 2.$$

(ii) For an isolated singularity $a \in \text{Sing}(R)$, we define

$$\deg_a(R) := \deg(R|_{S_r^2(a)}) = \deg_a(n_{(a)}) \bmod 2,$$

where $S_r^2(a) \subset \Omega$ is an arbitrary sphere of radius $r > 0$ around a such that the corresponding ball $\overline{B_r^3}(a)$ does not contain any other singularities of f , and $n_{(a)}$ denotes the lift of R existing on $\overline{B_r^3}(a) \setminus \{a\}$.

In both cases, the (mod 2)-degree lies in $\mathbb{Z}/2\mathbb{Z}$. This definition has the advantage, that nice properties of the classical Brouwer-degree, like additivity and homotopy invariance, continue to hold.

Because we are going to use the concept of dipoles a lot throughout this paper, we also have to modify the original definition of a dipole as introduced in [5] to fit into the situation of (restricted) Cosserat solids.

Definition 2. Let $\Omega \subset \mathbb{R}^3$ and $f = (\varphi, R): \Omega \rightarrow \mathbb{R}^3 \times \mathbb{S}$ be as in Definition 1. A pair of singularities (P, N) of R is called a *dipole* for R if there is an open bounded cylinder $Z_r^3(q) \subset \Omega$, rotationally symmetric (of radius $r > 0$) around the line segment $[P, N]$ such that

- (i) $[P, N] \subset Z_r^3(q)$ and $Z_r^3(q)$ is centered at the center q of $[P, N]$,
- (ii) $Z_r^3(q)$ does not contain any further singularities of f ,
- (iii) $\deg_P(R) = 1 = \deg_N(R)$ and the lift $n_{(q)}$ of R (existing on $\overline{Z_r^3(q)} \setminus \{P, N\}$) has a classical dipole (P, N) , i.e. $\deg_P(n_{(q)}) = d = -\deg_N(n_{(q)})$ for a $d \in \mathbb{Z} \setminus \{0\}$.

A central method to prove Theorem 1 in Section 3 is inserting dipoles into a given smooth mapping while controlling the energy needed to do so. The details are stated in the following theorem, which will be proven in Section 2.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected and locally path-connected set. Let P, N be two distinct points in Ω such that the line segment $[P, N]$ lies fully in Ω . For any smooth mapping $f = (\varphi, R) \in C^\infty(\Omega, \mathbb{R}^3 \times \mathbb{S})$, there exists a sequence of mappings*

$$f_m = (\varphi_m, R_m) \in H^1(\Omega, \mathbb{R}^3 \times \mathbb{S}) \cap C^\infty(\Omega \setminus \{P, N\}, \mathbb{R}^3 \times \mathbb{S})$$

with the following three properties. First, the pair (P, N) is a dipole for each R_m , i.e. in particular it holds

$$\deg_P(R_m) = 1 = \deg_N(R_m).$$

Second, all mappings f_m agree with f outside of a small neighborhood K_m of $[P, N]$ which itself fulfils $K_m \rightarrow [P, N]$, $m \rightarrow \infty$, in Hausdorff-distance. Third,

$$\lim_{m \rightarrow \infty} \mathcal{J}_\Omega(f_m) \leq \mathcal{J}_\Omega(f) + 64\pi|P - N|.$$

2 Construction of dipoles

As mentioned above, a key ingredient in the construction of suitable boundary data for the proof of Theorem 1 is the insertion of dipole pairs of singularities, each with $(\text{mod } 2)$ -degree 1, into smooth maps. Theorem 2 gives us a tool for doing so while using a controlled amount of Cosserat energy, depending only on the dipole's length. The main part of this paper will consist of its proof, as it contains some technical intricacies.

Proof of Theorem 2. This proof is divided into three steps: First, we present a construction, that was used by F. Béthuel in [1] to remove a dipole from a given map with a controlled amount of (Dirichlet) energy. Working in the other direction, it gives rise to a sequence of Lipschitz mappings with the desired singularities of $(\text{mod } 2)$ -degree 1 inserted. Additionally, the mappings of the sequence exhibit further singularities of degree 0. Second, we calculate the estimates for the Cosserat energy needed. During the last step, we use some approximation results from [2] to replace each Lipschitz mapping of the sequence with an approximation in order to get rid of the additional singularities of degree 0 and to gain the desired smoothness (except in P, N) without affecting the Cosserat energy.

Step 1 (Construction). In [1, Lemma 2], F. Béthuel uses a cuboid construction together with a cube lemma [1, Lemma 3] plus some calculations from the two-dimensional case in [4]. We can use exactly the same cuboid construction, together with the following modified cube lemma for the Cosserat energy which itself will be proved after having completed the proof of Theorem 2.

Lemma 1 (Cube Lemma). *For $\nu > 0$, let $C_\nu = [-\nu, \nu]^2 \times [-2\nu, 0]$ be a cube. On the cubes' boundary, consider a Lipschitz mapping $f = (\varphi, R): \partial C_\nu \rightarrow \mathbb{R}^3 \times \mathcal{S}$ with $\deg(R) = d_0$, $d_0 \in \mathbb{Z}/2\mathbb{Z}$. Then, for each $\varepsilon > 0$, there exists a constant $a_0 \in (0, \nu)$ such that for any $0 < a < a_0$, there exists a Lipschitz mapping $f_a = (\varphi, R_a): \partial C_\nu \rightarrow \mathbb{R}^3 \times \mathcal{S}$ with*

$$\deg(R_a) = d_0 + 1 \pmod{2},$$

$$f_a = (\varphi, R_a) = (\varphi, R) = f \text{ in } \partial C_\nu \setminus (B_a^2 \times \{0\})$$

and

$$\int_{B_a^2 \times \{0\}} 2 \cdot |R_a^T \cdot D\varphi - I_3|^2 + |DR_a|^2 d\mathcal{H}^2 < 64\pi + \varepsilon. \quad (2.1)$$

Moreover, on $(B_a^2 \setminus B_{\frac{a}{2}}^2) \times \{0\}$ we have

$$|DR_a| \leq \text{const}, \quad (2.2)$$

and on $B_{\frac{a}{2}}^2 \times \{0\}$

$$|DR_a(x, y, 0)|^2 = \frac{64a^4}{(a^4 + x^2 + y^2)^2}. \quad (2.3)$$

Following the notation from [1] for the cuboid construction, with

- $d := |P - N|$,
- $a_m := \frac{d}{2(m-1)}$, $m \in \mathbb{N}_{>1}$,
- K_m the cuboid around $[P, N]$: $K_m := [-a_m, a_m]^2 \times [-a_m, d + a_m]$,
- K_m divided into m cubes: $C_m^j := [-a_m, a_m]^2 \times [(-1 + 2j)a_m, (1 + 2j)a_m]$, $j = 0, \dots, m-1$,
- c_j the barycenter of C_m^j and
- $\pi_m^j: C_m^j \rightarrow \partial C_m^j$ the radial retraction with center c_j , given by

$$\pi_m^j(x) = \frac{x - c_j}{|x - c_j|_\infty} \cdot a_m + c_j,$$

where $|x - c_j|_\infty = \max_{i=1,2,3} (|x_i - (c_j)_i|)$,

we iteratively use Lemma 1 (for m fixed) on the boundaries of each of the $(m-1)$ single lower cubes C_m^j , $j = 0, \dots, m-2$. For the boundary of the uppermost cube C_m^{m-1} , we set $f_{m,a}$ to be equal to f on all faces except for the bottom one. As this bottom face of the last cube C_m^{m-1} simultaneously is the upper face of the cube C_m^{m-2} , we set $f_{m,a}$ to have the same values there as constructed by using Lemma 1 on C_m^{m-2} , in order to obtain matching boundary values. Extending everything to the cubes' interiors by means of radial retraction π_m^j , the whole process implies that for each $m \geq m_0 \gg 1$, there exists a sequence of Lipschitz mappings $(\tilde{f}_{m,a})_a \in H^1(\Omega, \mathbb{R}^3 \times \mathbb{S})$ (with $a \searrow 0$) given by

$$\tilde{f}_{m,a} = (\tilde{\varphi}_{m,a}, \tilde{R}_{m,a}) := \begin{cases} (\varphi, R) = f & \text{in } \Omega \setminus K_m, \\ f_{m,a} \circ \pi_m^j & \text{in } K_m, \end{cases}$$

where

$$f_{m,a}: \bigcup_{j=0}^{m-1} \partial C_m^j \rightarrow \mathbb{R}^3 \times \mathbb{S}, \quad f_{m,a} = (\varphi, R_{m,a}),$$

with $\text{Sing}(\tilde{f}_{m,a}) = \{P = c_0, c_1, \dots, c_{m-2}, c_{m-1} = N\}$. For this sequence, we have

$$\begin{aligned} \deg_P(\tilde{R}_{m,a}) &= 1 = \deg_N(\tilde{R}_{m,a}), \\ \deg_{C_j}(\tilde{R}_{m,a}) &= 0 & \text{for } j = 1, \dots, m-2, \\ \deg_{C_j}(\tilde{\varphi}_{m,a}) &= 0 & \text{for } j = 0, \dots, m-1, \end{aligned}$$

as well as

$$\tilde{f}_{m,a} = f \quad \text{on } \partial K_m,$$

due to the fact that during the construction, changes of the original mapping only happen on little discs (of radius $\alpha < a_m$) on the upper faces of the lower $m-1$ cubes $C_m^0, \dots, C_m^{m-2}$, so that the values on ∂K_m remain unaffected. The degree of $\tilde{R}_{m,a}$ in the inner singularities $c_1, \dots, c_{m-2}$ vanishes, because for the corresponding cubes' boundaries, the original map was changed both on the upper and the lower face. Also, by definition and known properties of the classical topological degree for the deformation component, we have $\deg_{C_j}(\tilde{\varphi}_{m,a}) = \deg(\varphi) \cdot \deg(\pi_m^j) = 0 \cdot 1 = 0$.

Step 2 (Calculation of Cosserat energy cost). Many of the calculations in this step follow ideas and estimates carried out in [19], where in the context of removing dipoles from given maps $u: \Omega \rightarrow S^2$, [1, Lemma 2] was generalized to the case that S^2 is equipped with an arbitrary Riemannian metric.

So similar to [19], in order to calculate the Cosserat energy of $\tilde{f}_{m,a}$ on K_m , we start by dividing each cube C_m^j into disjoint sets

$$\begin{aligned} B_{a_m}^3(c_j), \quad A_m^j &= (C_m^j \setminus B_{a_m}^3(c_j)) \cap (\pi_m^j)^{-1}(B_{\frac{a}{2}}^2 \times \{(-1+2j)a_m\}), \\ D_m^j &= (C_m^j \setminus B_{a_m}^3(c_j)) \cap (\pi_m^j)^{-1}(B_{\frac{a}{2}}^2 \times \{(1+2j)a_m\}), \\ E_m^j &= (C_m^j \setminus B_{a_m}^3(c_j)) \cap (\pi_m^j)^{-1}((B_{\frac{a}{2}}^2 \setminus B_{\frac{a}{2}}^2) \times \{(-1+2j)a_m\}), \\ F_m^j &= (C_m^j \setminus B_{a_m}^3(c_j)) \cap (\pi_m^j)^{-1}((B_{\frac{a}{2}}^2 \setminus B_{\frac{a}{2}}^2) \times \{(1+2j)a_m\}) \end{aligned}$$

and the rest

$$G_m^j = C_m^j \setminus (B_{a_m}^3(c_j) \cup A_m^j \cup D_m^j \cup E_m^j \cup F_m^j).$$

Note that different constants appearing in the following estimates are always denoted by the same $\gamma \in \mathbb{R}_{\geq 0}$. They only depend on a_m , Lipschitz constants of $\pi_m^j|_{S_{\rho}^2(c_j)}$ (see below) and the suprema of $|\mathcal{D}\varphi|^2$ and $|\mathcal{D}R|^2$ for the original smooth (φ, R) in the compact set K_m .

Starting with $B_{a_m}^3(c_j)$, we can estimate the Cosserat energy in terms of

$$\begin{aligned} \mathcal{J}_{B_{a_m}^3(c_j)}(\tilde{f}_{m,a}) &= \int_{B_{a_m}^3(c_j)} |(R_{m,a} \circ \pi_m^j)^T \cdot \mathcal{D}(\varphi \circ \pi_m^j) - I_3|^2 + |\mathcal{D}(R_{m,a} \circ \pi_m^j)|^2 dx \\ &= \int_0^{a_m} \int_{S_{\rho}^2(c_j)} |(R_{m,a} \circ \pi_m^j)^T \cdot \mathcal{D}(\varphi \circ \pi_m^j) - I_3|^2 + |\mathcal{D}(R_{m,a} \circ \pi_m^j)|^2 d\mathcal{H}^2 d\rho \end{aligned}$$

$$\begin{aligned}
&\leq \gamma a_m^3 + \text{Lip}^2(\pi_m^j|_{S_\rho^2(c_j)}) \int_0^{a_m} \int_{\partial C_m^j} \{2|R_{m,a}^T \cdot D\varphi - I_3|^2 + |DR_{m,a}|^2\} \text{Jac}^{-1}(\pi_m^j|_{S_\rho^2(c_j)}) d\mathcal{H}^2 d\rho \\
&\leq \gamma a_m^3 + 9a_m \cdot 2 \cdot \int_{\partial C_m^j \setminus (B_a^2 \times \{(-1+2j)a_m, (1+2j)a_m\})} |R^T \cdot D\varphi - I_3|^2 + |DR|^2 d\mathcal{H}^2 \\
&\quad + a_m \int_{B_a^2 \times \{(-1+2j)a_m, (1+2j)a_m\}} \left(1 + \frac{|y|^2}{a_m^2}\right)^2 \{2|R_{m,a}^T \cdot D\varphi - I_3|^2 + |DR_{m,a}|^2\} d\mathcal{H}^2(y) \\
&\leq \gamma a_m^3 + 18a_m \int_{\partial C_m^j} |R^T \cdot D\varphi - I_3|^2 + |DR|^2 d\mathcal{H}^2 + 2a_m \left(1 + \frac{a^2}{a_m^2}\right)^2 (64\pi + \varepsilon) \\
&\leq \gamma a_m^3 + 2a_m \left(1 + \frac{a^2}{a_m^2}\right)^2 (64\pi + \varepsilon), \tag{2.4}
\end{aligned}$$

because of (2.1) and because the original $f = (\varphi, R)$ is smooth in all of Ω . Therefore, $|R^T \cdot D\varphi - I|^2$ and $|DR|^2$ are bounded on K_m , thus for each $j = 0, \dots, m-1$,

$$\int_{\partial C_m^j} |R^T \cdot D\varphi - I_3|^2 + |DR|^2 d\mathcal{H}^2 \leq \gamma \cdot a_m^2.$$

In the set G_m^j , because each $x \in G_m^j$ gets projected by π_m^j onto the boundary of C_m^j outside of the small discs $B_a^2 \times \{(-1+2j)a_m; (1+2j)a_m\}$, we have $\tilde{f}_{m,a}(x) = (f_{m,a} \circ \pi_m^j)(x) = (f \circ \pi_m^j)(x)$. Hence,

$$\mathcal{J}_{G_m^j}(\tilde{f}_{m,a}) = \int_{G_m^j} |(R \circ \pi_m^j)^T \cdot D(\varphi \circ \pi_m^j) - I_3|^2 + |D(R \circ \pi_m^j)|^2 dx \leq \gamma \cdot a_m^3, \tag{2.5}$$

as $(R \circ \pi_m^j) \in \mathcal{S} \subset \text{SO}(3)$, $D\varphi$ and I_3 , as well as DR are bounded and π_m^j is Lipschitz in $C_m^j \setminus B_{a_m}^3$. Similarly, we have

$$\begin{aligned}
\tilde{f}_{m,a} &= f \circ \pi_m^0 && \text{in } A_m^0 \cup E_m^0, \\
\tilde{f}_{m,a} &= f \circ \pi_m^{m-1} && \text{in } D_m^{m-1} \cup F_m^{m-1}
\end{aligned}$$

by construction, and therefore

$$\mathcal{J}_{A_m^0 \cup E_m^0 \cup D_m^{m-1} \cup F_m^{m-1}}(\tilde{f}_{m,a}) \leq \gamma \cdot a_m^3. \tag{2.6}$$

Along the same line of reasoning, which is possible because of (2.2), we find

$$\mathcal{J}_{E_m^j}(\tilde{f}_{m,a}) \leq \gamma \cdot a_m^3, \quad j = 1, \dots, m-1, \tag{2.7}$$

and

$$\mathcal{J}_{F_m^j}(\tilde{f}_{m,a}) \leq \gamma \cdot a_m^3, \quad j = 0, \dots, m-2. \tag{2.8}$$

We now proceed with estimates on A_m^j ($j \neq 0$), and note that D_m^j ($j \neq m-1$) can be treated analogously by symmetry. While using the Cube Lemma (Lemma 1) in the construction's background, we changed the original lift $n: \Omega \rightarrow S^2$ of R into a Lipschitz mapping $n_{m,a}: \Omega \rightarrow S^2$ in order to get the new $R_{m,a} = F \circ n_{m,a}$ having m singularities. Here again, F denotes the two-fold covering map between S^2 and $\mathcal{S}$ as introduced in the explanation just above (P). The deformation part of the Cosserat energy on A_m^j ($j \neq 0$) thus is bounded once again by $\gamma \cdot a_m^3$ with the same argument as for estimate (2.5). Additionally, for the micro-rotational part we have the following, notating $\tilde{R}_{m,a} = R_{m,a} \circ \pi_m^j = F \circ n_{m,a} \circ \pi_m^j = F \circ \tilde{n}_{m,a}$ and with the fact that $\tilde{n}_{m,a}$ is constant in $(x_1, x_2, 2ja_m - x_3)$ -direction. It holds

$$\frac{\partial \tilde{n}_{m,a}}{\partial x_3} = \frac{x_1}{x_3 - 2ja_m} \frac{\partial \tilde{n}_{m,a}}{\partial x_1} + \frac{x_2}{x_3 - 2ja_m} \frac{\partial \tilde{n}_{m,a}}{\partial x_2}$$

and thus

$$|D\tilde{n}_{m,a}(x)|^2 = \left(1 + \left(\frac{x_1}{x_3 - 2ja_m}\right)^2\right) \left|\frac{\partial \tilde{n}_{m,a}}{\partial x_1}\right|^2 + \left(1 + \left(\frac{x_2}{x_3 - 2ja_m}\right)^2\right) \left|\frac{\partial \tilde{n}_{m,a}}{\partial x_2}\right|^2 + 2 \frac{x_1 x_2}{(x_3 - 2ja_m)^2} \frac{\partial \tilde{n}_{m,a}}{\partial x_1} \frac{\partial \tilde{n}_{m,a}}{\partial x_2}.$$

As $x_1^2 + x_2^2 \leq (x_3 - 2ja_m)^2$ in this regime, it directly follows that

$$\begin{aligned} |D(n_{m,a} \circ \pi_m^j)(x)|^2 &\leq 3 \cdot \left(\left| \frac{\partial \tilde{n}_{m,a}}{\partial x_1} \right|^2 + \left| \frac{\partial \tilde{n}_{m,a}}{\partial x_2} \right|^2 \right) \\ &= 3 \cdot \left(\frac{a_m}{x_3 - 2ja_m} \right)^2 \cdot |Dn_{m,a}|_{B_{\frac{a_m}{2}}^2 \times \{(-1+2j)a_m\}}(\pi_m^j(x))|^2 \\ &\leq 6 \cdot \frac{8a^4}{(\alpha^4 + (\pi_m^j(x))_1^2 + (\pi_m^j(x))_2^2)^2}, \end{aligned}$$

using (2.3) and $R_{m,a} = F \circ n_{m,a}$ in combination with the fact, that the covering map F is homothetic.

With the transformation to polar coordinates (η, ϑ, ξ) for the cone $(\pi_m^j)^{-1}(B_{\frac{a_m}{2}}^2 \times \{(-1+2j)a_m\})$ translated in x_3 -direction, with η denoting the radius, $\vartheta \in [0, 2\pi)$ denoting the angle and ξ denoting the translated height-level of the disc of points (η, ϑ) , i.e.

$$\begin{aligned} \xi &= \sqrt{x_1^2 + x_2^2 + (x_3 - 2ja_m)^2}, \\ y_1 &= (\pi_m^j(x))_1 = \frac{a_m}{|x_3 - 2ja_m|} \cdot x_1 = \frac{a_m}{\sqrt{\xi^2 - x_1^2 - x_2^2}} \cdot x_1, \\ y_2 &= (\pi_m^j(x))_2 = \frac{a_m}{|x_3 - 2ja_m|} \cdot x_2 = \frac{a_m}{\sqrt{\xi^2 - x_1^2 - x_2^2}} \cdot x_2, \\ \eta^2 &= y_1^2 + y_2^2 \\ dx_1 dx_2 dx_3 &= \frac{\xi^2 a_m \eta}{(a_m^2 + \eta^2)^{\frac{3}{2}}} d\eta d\vartheta d\xi, \end{aligned}$$

we finally get the estimate

$$\begin{aligned} \mathcal{J}_{A_m^j}(\tilde{f}_{m,a}) &= \int_{A_m^j} |(R_{m,a} \circ \pi_m^j)^T \cdot D(\varphi \circ \pi_m^j) - I_3|^2 + |D(R_{m,a} \circ \pi_m^j)|^2 dx \\ &\leq \gamma \cdot a_m^3 + \int_{A_m^j} |D(F \circ (n_{m,a} \circ \pi_m^j))|^2 dx = \gamma \cdot a_m^3 + 8 \int_{A_m^j} |D(n_{m,a} \circ \pi_m^j)|^2 dx \\ &\leq \gamma \cdot a_m^3 + 8 \cdot 48a^4 \cdot \int_0^{\frac{a}{2}} \int_0^{\frac{a}{2}} \int_{a_m}^{\sqrt{\eta^2 + a_m^2}} \frac{1}{(\alpha^4 + \eta^2)^2} \cdot \frac{\xi^2 a_m \eta}{(a_m^2 + \eta^2)^{\frac{3}{2}}} d\xi d\eta d\vartheta \\ &= \gamma \cdot a_m^3 + 8 \cdot 32\pi a^4 \cdot \int_0^{\frac{a}{2}} \frac{a_m \eta}{(\alpha^4 + \eta^2)^2} \cdot \frac{(a_m^2 + \eta^2)^{\frac{3}{2}} - a_m^3}{(a_m^2 + \eta^2)^{\frac{3}{2}}} d\eta \\ &= \gamma \cdot a_m^3 + 8 \cdot 32\pi a^4 \cdot \int_0^{\frac{a}{2}} \frac{a_m \eta}{(\alpha^4 + \eta^2)^2} \cdot \left(1 - \frac{1}{(1 + (\frac{\eta}{a_m})^2)^{\frac{3}{2}}} \right) d\eta \\ &\leq \gamma \cdot a_m^3 + \gamma \cdot \alpha^4 \cdot \int_0^{\frac{a}{2}} \frac{a_m \eta}{(\alpha^4 + \eta^2)^2} \cdot \frac{\eta^2}{a_m^2} d\eta, \end{aligned} \tag{2.9}$$

since $\eta \leq \frac{a}{2} < a_m$. Hence for $j = 1, \dots, m-1$, (2.9) becomes

$$\begin{aligned} \mathcal{J}_{A_m^j}(\tilde{f}_{m,a}) &\leq \gamma \cdot a_m^3 + \gamma \alpha^4 \cdot \frac{1}{a_m} \int_0^{\frac{a}{2}} \frac{\eta^3}{(\alpha^4 + \eta^2)^2} d\eta \\ &\leq \gamma \cdot a_m^3 + \gamma \cdot \frac{\alpha^4}{a_m} \cdot \ln \left(1 + \frac{1}{4\alpha^2} \right) \\ &\leq \gamma \cdot (a_m^3 + a_m^2). \end{aligned} \tag{2.10}$$

The last inequality holds because of $\alpha < a_m$ and the estimate

$$\ln(1+x) = 2 \ln(\sqrt{1+x}) \leq 2 \ln(1+\sqrt{x}) \leq 2\sqrt{x} \quad \text{for } x > 0,$$

as $1+x \leq 1+2\sqrt{x}+x \leq (1+\sqrt{x})^2$ for non-negative x .

As mentioned above, we also find

$$\mathcal{J}_{D_m^j}(\tilde{f}_{m,a}) \leq \gamma \cdot (a_m^3 + a_m^2) \quad (2.11)$$

for $j = 0, \dots, m-2$ by symmetry.

Combining (2.4)–(2.8), (2.10) and (2.11), we have

$$\begin{aligned} \mathcal{J}_{K_m}(\tilde{f}_{m,a}) &= \sum_{j=0}^{m-1} \mathcal{J}_{C_m^j}(\tilde{f}_{m,a}) \\ &= \sum_{j=0}^{m-1} (\mathcal{J}_{B_{a_m}^3(c_j)}(\tilde{f}_{m,a}) + \mathcal{J}_{G_m^j}(\tilde{f}_{m,a})) + \sum_{j=1}^{m-1} (\mathcal{J}_{A_m^j}(\tilde{f}_{m,a}) + \mathcal{J}_{E_m^j}(\tilde{f}_{m,a})) \\ &\quad + \sum_{j=0}^{m-2} (\mathcal{J}_{D_m^j}(\tilde{f}_{m,a}) + \mathcal{J}_{F_m^j}(\tilde{f}_{m,a})) + \mathcal{J}_{A_m^0 \cup E_m^0 \cup D_m^{m-1} \cup F_m^{m-1}}(\tilde{f}_{m,a}) \\ &\leq 2m \cdot a_m \cdot \left(1 + \frac{\alpha^2}{a_m^2}\right)^2 (64\pi + \varepsilon) + \gamma \cdot (a_m^3 + a_m) \xrightarrow{\alpha \searrow 0} 2m \cdot a_m (64\pi + \varepsilon) + \gamma \cdot (a_m^3 + a_m). \end{aligned}$$

In other words, for any $\varepsilon > 0$ and any $m \geq m_0$, there is a number $\tilde{\alpha}$ and a corresponding mapping $\tilde{f}_{m,\tilde{\alpha}}$, which we simply call $\tilde{f}_m$, with

$$\mathcal{J}_{K_m}(\tilde{f}_m) \leq 2m \cdot a_m (64\pi + \varepsilon) + \gamma \cdot (a_m^3 + a_m) + \varepsilon \xrightarrow{m \rightarrow \infty} d \cdot 64\pi + (d+1)\varepsilon.$$

Hence, for each $\varepsilon > 0$ we have a number $m (= m(\varepsilon)) \geq m_0$ and a mapping $\tilde{f}_m$ that fulfils

$$\mathcal{J}_{K_m}(\tilde{f}_m) \leq 64\pi \cdot d + (d+2)\varepsilon.$$

As a consequence, for any sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ with $\varepsilon_m \searrow 0$, we have constructed a sequence of Lipschitz mappings $\tilde{f}_m \in H^1(\Omega, \mathbb{R}^3 \times S)$ such that

(i)

$$\tilde{f}_m = (\tilde{\varphi}_m, \tilde{R}_m) = \begin{cases} (\varphi, R) = f & \text{in } \Omega \setminus K_m, \\ (\varphi \circ \pi_m^j, R_m \circ \pi_m^j) & \text{in } K_m, \end{cases}$$

where $K_m \rightarrow [P, N]$, $m \rightarrow \infty$, in Hausdorff-distance,

(ii) $\text{Sing}(\tilde{f}_m) = \{P = c_0, c_1, \dots, c_{m-2}, c_{m-1} = N\}$ with

$$\begin{cases} \deg_{C_j}(\tilde{\varphi}_m) = 0, & j = 0, \dots, m-1, \\ \deg_{C_j}(\tilde{R}_m) = 0, & j = 1, \dots, m-2, \\ \deg_P(\tilde{R}_m) = 1 = \deg_N(\tilde{R}_m), \end{cases}$$

(iii)

$$\limsup_{m \rightarrow \infty} \mathcal{J}_{\Omega}(\tilde{f}_m) \leq \mathcal{J}_{\Omega}(f) + 64\pi \cdot |P - N|.$$

Step 3 (Approximation). Finally, we need suitable approximation arguments to achieve smoothness except in P, N for each $\tilde{f}_m$ without affecting the Cosserat energy estimates. Also, (P, N) is not yet a dipole for $\tilde{R}_m$ according to Definition 2. Luckily, we are able to use several methods developed in [2]. During the construction in Step 1, we changed the original smooth mapping f in the cuboid K_m only. Since $\mathbb{R}^3$ is contractible, we can use [2, Theorem 1 bis]. Therefore it is possible to approximate the changed deformation component $\tilde{\varphi}_{m|K_m}$ in H^1 -topology with mappings

$$\varphi_{m,s} \in C^\infty(K_m, \mathbb{R}^3), \quad s \in \mathbb{N},$$

while the boundary values remain $\varphi_{m,s} = \tilde{\varphi}_m = \varphi$ in ∂K_m . Replacing $\varphi_{m,s}$ by a subsequence, we may assume $\varphi_{m,s} \rightarrow \tilde{\varphi}_m$, $s \rightarrow \infty$, pointwise almost everywhere.

For the micro-rotational component $\tilde{R}_{m|K_m} = (R_m \circ \pi_m^j) \in H^1(K_m, \mathbb{S})$ we first note again that

$$\tilde{R}_{m|K_m} = F \circ n_m \circ \pi_m^j = F \circ \tilde{n}_m,$$

with $\tilde{n}_m \in H^1(K_m, S^2)$,

$$\deg_p(\tilde{n}_m) = 2k + 1 = -\deg_N(\tilde{n}_m) \quad \text{for some } k \in \mathbb{Z}, \quad \text{and} \quad \deg_{c_j}(\tilde{n}_m) = 0$$

for each $j = 1, \dots, m-2$, having smooth boundary values $\tilde{n}_{m|\partial K_m} = n_{|\partial K_m} \in C^\infty(\partial K_m, S^2)$ due to the underlying construction making use of the smooth lift n of the original smooth R . Applying [2, Theorem 2 bis], since S^2 is a compact manifold without boundary and K_m is dividable into cubes (cf. “cubeulation” in [2]), there exists a sequence

$$n_{m,t} \in H^1(K_m, S^2) \cap C^\infty(K_m \setminus \{P, N, c_1, \dots, c_{m-2}\}, S^2)$$

with

- (i) $n_{m,t} \rightarrow \tilde{n}_m, t \rightarrow \infty$, in $H^1(K_m, S^2)$,
- (ii) $\deg_p(n_{m,t}) = 2k + 1 = -\deg_N(n_{m,t})$ and $\deg_{c_j}(n_{m,t}) = 0, j = 1, \dots, m-2$,
- (iii) $n_{m,t|\partial K_m} = \tilde{n}_{m|\partial K_m} = n_{|\partial K_m}$.

Now we can use the technique from the proof of [2, Lemma 1 bis] to get rid of those singularities $c_1, \dots, c_{m-2}$, in which the homotopy class of $n_{m,t}$ is trivial. For $n_{m,t}$ in $Q_m := \bigcup_{j=1}^{m-2} C_m^j$, each $n_{m,t|Q_m}$ (subject to their own boundary values $g := n_{m,t|Q_m}$) can be approximated in H^1 -topology by mappings

$$n_{m,t,s} \in H_g^1(Q_m, S^2) \cap C^\infty(Q_m, S^2),$$

that agree with $n_{m,t}$ outside of $\bigcup_{j=1}^{m-2} B_{1/s}^3(c_j)$.

That is why for each $\tilde{n}_m: K_m \rightarrow S^2$, there exists a sequence of mappings

$$n_{m,t,s} \in H^1(K_m, S^2) \cap C^\infty(K_m \setminus \{P, N\}, S^2), \quad s \in \mathbb{N},$$

by defining

$$n_{m,t,s} = \begin{cases} n_{m,t,s} & \text{in } Q_m, \\ n_{m,t} & \text{in } C_m^0 \cup C_m^{m-1}, \end{cases}$$

with $n_{m,t,s} \rightarrow \tilde{n}_m$ in $H^1(K_m, S^2)$, $\deg_p(n_{m,t,s}) = 2k + 1 = -\deg_N(n_{m,t,s})$ and smooth values on the boundary of K_m which are given by $n_{m,t,s|\partial K_m} = \tilde{n}_{m|\partial K_m} = n_{|\partial K_m}$.

Finally, we project everything back from S^2 to $\mathbb{S}$. For each of the mappings $\tilde{R}_{m|K_m} = F \circ \tilde{n}_m: K_m \rightarrow \mathbb{S}$, we thus have a sequence of mappings

$$R_{m,t,s} := F \circ n_{m,t,s} \in H^1(K_m, \mathbb{S}) \cap C^\infty(K_m \setminus \{P, N\}, \mathbb{S}),$$

which approximate $\tilde{R}_{m|K_m}$ in H^1 -topology, because (after passing to an a.e.-pointwise convergent subsequence) it holds that

$$\begin{aligned} \int_{K_m} |R_{m,t,s} - \tilde{R}_m|^2 dx &= \int_{K_m} |F \circ n_{m,t,s} - F \circ \tilde{n}_m|^2 dx \\ &= \int_{K_m} |2[n_{m,t,s} \otimes n_{m,t,s} - \tilde{n}_m \otimes \tilde{n}_m]|^2 dx \\ &\leq \sum_{i,j=1}^3 \int_{K_m} \left[\underbrace{|n_{m,t,s}^i - \tilde{n}_m^i|}_{\rightarrow 0} \cdot \underbrace{|n_{m,t,s}^j + \tilde{n}_m^j|}_{\leq 2} + \underbrace{|n_{m,t,s}^i + \tilde{n}_m^i|}_{\leq 2} \cdot \underbrace{|n_{m,t,s}^j - \tilde{n}_m^j|}_{\rightarrow 0} \right]^2 dx \xrightarrow{s \rightarrow \infty} 0, \end{aligned}$$

and

$$\begin{aligned} \int_{K_m} |DR_{m,t,s} - D\tilde{R}_m|^2 dx &= \int_{K_m} |DF \circ n_{m,t,s} \cdot Dn_{m,t,s} - DF \circ \tilde{n}_m \cdot D\tilde{n}_m|^2 dx \\ &\leq 2 \int_{K_m} |DF \circ n_{m,t,s} \cdot Dn_{m,t,s} - DF \circ \tilde{n}_m \cdot Dn_{m,t,s}|^2 + |DF \circ \tilde{n}_m \cdot Dn_{m,t,s} - DF \circ \tilde{n}_m \cdot D\tilde{n}_m|^2 dx \\ &\leq 2 \int_{K_m} |DF \circ n_{m,t,s} - DF \circ \tilde{n}_m|^2 \cdot |Dn_{m,t,s}|_\infty^2 + |DF \circ \tilde{n}_m|_\infty^2 \cdot |Dn_{m,t,s} - D\tilde{n}_m|^2 dx \xrightarrow{s \rightarrow \infty} 0. \end{aligned}$$

Summarizing, for any sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ with $\varepsilon_m \searrow 0$, there exists a sequence of Sobolev mappings $(\tilde{f}_m)_m$ with

$$\mathcal{J}_\Omega(\tilde{f}_m) \leq \mathcal{J}_\Omega(f) + 64\pi \cdot d + (d+2)\varepsilon_m,$$

as well as another sequence of mappings $(f_{m,t,s})_s$,

$$f_{m,t,s} \in H^1(\Omega, \mathbb{R}^3 \times \mathcal{S}) \cap C^\infty(\Omega \setminus \{P, N\}, \mathbb{R}^3 \times \mathcal{S}), \quad f_{m,t,s} := \begin{cases} (\varphi, R) = f & \text{in } \Omega \setminus K_m, \\ (\varphi_{m,t,s}, R_{m,t,s}) & \text{in } K_m, \end{cases}$$

with $f_{m,t,s} \rightarrow \tilde{f}_m$, $s \rightarrow \infty$, in $H^1(\Omega, \mathbb{R}^3 \times \mathcal{S})$. Moreover, $R_{m,t,s}$ has a dipole (P, N) . Hence for each ε_m , by dominated convergence, we get the existence of a mapping f_m with the desired properties and

$$\mathcal{J}_{K_m}(f_m) \leq \mathcal{J}_{K_m}(\tilde{f}_m) + \varepsilon_m \leq 64\pi \cdot d + (d+3)\varepsilon_m,$$

meaning we have found the sequence $(f_m)_{m \in \mathbb{N}}$ of mappings, which are smooth except for a dipole in the micro-rotation and whose Cosserat energy fulfils

$$\lim_{m \rightarrow \infty} \mathcal{J}_\Omega(f_m) \leq \mathcal{J}_\Omega(f) + 64\pi \cdot |P - N|. \quad \square$$

Having finished the proof of Theorem 2, it remains to prove the Cube Lemma for the Cosserat energy.

Proof of Lemma 1. Since the set ∂C_ν is bounded, simply connected and locally path-connected, the Lipschitz mapping $R: \partial C_\nu \rightarrow \mathcal{S}$ can be lifted, which means for the covering map F of $\mathcal{S}$ ($F: S^2 \rightarrow \mathcal{S}$, $q \mapsto 2q \otimes q - I_3$), there exist exactly two Lipschitz mappings

$$n_i: \partial C_\nu \rightarrow S^2 \quad \text{with } R = F \circ n_i,$$

$i = 1, 2$ and $n_1 = -n_2$ (cf. Section 1). We choose one of these mappings and keep it fixed ($n := n_1$). Additionally, there is a number $\deg(n) = c_0 \in \mathbb{Z}$ such that $d_0 = \deg(R) = c_0 \bmod 2$. Moreover, F is homothetic, i.e. for any tangent vector $V \in T_p(\partial C_\nu)$ we have

$$|DR_p(V)|^2 = |DF_{n(p)}(Dn_p(V))|^2 = 8 \cdot |Dn_p(V)|^2. \quad (2.12)$$

We perform the modifications used in [4] on our n , to the following effect. The construction from [4], which will shortly be described below, implies that for each $\varepsilon > 0$ there is a constant $0 < \alpha_0 < \nu$ such that for each $0 < \alpha < \alpha_0$, there exists a Lipschitz mapping $\tilde{n}_\alpha: \partial C_\nu \rightarrow S^2$, which fulfils

$$\begin{aligned} \deg(\tilde{n}_\alpha) &= c_0 + 1, \\ \tilde{n}_\alpha &= n \quad \text{on } \partial C_\nu \setminus (B_\alpha^2 \times \{0\}) \end{aligned}$$

and

$$\int_{B_\alpha^2 \times \{0\}} |D\tilde{n}_\alpha|^2 d\mathcal{H}^2 = 8\pi + O(\alpha^2) \quad \text{with } \alpha \searrow 0, \quad (2.13)$$

as well as

$$|D\tilde{n}_\alpha| \leq \text{const} \quad \text{in } (B_\alpha^2 \setminus B_{\frac{\alpha}{2}}^2) \times \{0\} \quad (2.14)$$

and

$$|D\tilde{n}_\alpha(x, y, 0)|^2 = \frac{8\alpha^4}{(\alpha^4 + x^2 + y^2)^2} \quad \text{in } B_{\frac{\alpha}{2}}^2 \times \{0\}. \quad (2.15)$$

That is to say, we change the degree of the original n by $+1$, without changing the map n outside of the disc $B_\alpha^2 \times \{0\}$. For doing so, we only need a controlled amount of (Dirichlet) energy. All this is achieved by the construction mentioned above taken from [4]: For polar coordinates (r, ϕ) in (x_1, x_2) -plane, we set

$$\tilde{n}_\alpha(x_1, x_2, x_3) = \begin{cases} \frac{2\alpha^2}{\alpha^4 + r^2} \begin{pmatrix} x_1 \\ x_2 \\ -\alpha^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & r < \frac{\alpha}{2}, x_3 = 0, \\ \begin{pmatrix} A_1 r + B_1 \\ A_2 r + B_2 \\ \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2} \end{pmatrix}, & \frac{\alpha}{2} \leq r < \alpha, x_3 = 0, \\ n(x_1, x_2, x_3), & \text{else.} \end{cases}$$

Basically, in the small disc $B_{\frac{\alpha}{2}}^2 \times \{0\}$, we stretch by a factor of $\frac{1}{\alpha^2}$ and then use stereographic projection from the North-pole to S^2 . In the annulus $(B_{\frac{\alpha}{2}}^2 \setminus B_{\frac{\alpha}{4}}^2) \times \{0\}$, we interpolate (linearly in r) between the values of the original n on $\partial B_{\frac{\alpha}{4}}^2 \times \{0\}$ and the values of the stereographic projection after stretching on $\partial B_{\frac{\alpha}{2}}^2 \times \{0\}$, so that the new map $\tilde{n}_\alpha$ is continuous. This means that the interpolating functions A_i, B_i ($i = 1, 2$, depending on ϕ) are chosen such that

$$\begin{aligned}\frac{\alpha}{2}A_1 + B_1 &= \frac{4\alpha}{4\alpha^2 + 1} \cos(\phi), \\ \frac{\alpha}{2}A_2 + B_2 &= \frac{4\alpha}{4\alpha^2 + 1} \sin(\phi), \\ \alpha A_i + B_i &= n_i(\alpha \cos(\phi), \alpha \sin(\phi), 0) \quad \text{for } i = 1, 2.\end{aligned}$$

With further calculations to be found in [4], we then get the statement made above, including (2.13)–(2.15).

Coming back to our micro-rotation R , we make use of the covering map F again. Now we define $\tilde{R}_\alpha := F \circ \tilde{n}_\alpha$ such that

$$\tilde{R}_\alpha = F \circ \tilde{n}_\alpha = F \circ n = R \quad \text{on } \partial C_V \setminus (B_{\frac{\alpha}{2}}^2 \times \{0\})$$

and

$$\deg(\tilde{R}_\alpha) = \deg(\tilde{n}_\alpha) \bmod 2 = c_0 + 1 \bmod 2 = d_0 + 1 \bmod 2.$$

The properties (2.2) and (2.3) follow directly from (2.14) and (2.15) respectively, because of (2.12). For the other part of the Cosserat energy of $\tilde{f}_\alpha$, coming from the deformation, we easily see $|\tilde{R}_\alpha^T \cdot D\varphi - I_3|^2 = O(1)$, $\alpha \searrow 0$, as $\tilde{R}_\alpha \in S \subset SO(3)$ and φ is Lipschitz by assumption.

Using (2.12) and (2.13), we therefore find

$$\begin{aligned}\int_{B_{\frac{\alpha}{2}}^2 \times \{0\}} 2|\tilde{R}_\alpha^T \cdot D\varphi - I_3|^2 + |D\tilde{R}_\alpha|^2 d\mathcal{H}^2 &= \int_{B_{\frac{\alpha}{2}}^2 \times \{0\}} 2|\tilde{R}_\alpha^T \cdot D\varphi - I_3|^2 + 8 \cdot |D\tilde{n}_\alpha|^2 d\mathcal{H}^2 \\ &= 64\pi + O(\alpha^2) \quad \text{with } \alpha \searrow 0.\end{aligned}$$

Thus we can choose $\alpha_0 \ll 1$ sufficiently small such that

$$\int_{B_{\frac{\alpha}{2}}^2 \times \{0\}} 2|\tilde{R}_\alpha^T \cdot D\varphi - I_3|^2 + |D\tilde{R}_\alpha|^2 d\mathcal{H}^2 < 64\pi + \varepsilon$$

holds for every $\alpha < \alpha_0$. □

3 Prescribing the number of singular points in a Cosserat-elastic solid (proof)

Now that we have provided a tool for constructing dipoles with a controlled amount of Cosserat energy, we can use it to prove Theorem 1.

Proof of Theorem 1. We use a combination of ideas from [3] and [9], adapted to the restricted Cosserat problem ($\mathcal{P}^*$). First, we define the desired smooth boundary conditions following an idea from [3, Section II.4]. Then we show that each restricted minimizer of the Cosserat energy in the corresponding class must at least have a given number of point singularities, just as it was done in [9] for harmonic mappings.

For $N \in \mathbb{N}$, we define numbers $\lambda_i := \frac{i}{2N}$, $i = 1, \dots, N$, points

$$\xi_i = (0, \sqrt{1 - \lambda_i^2}, \lambda_i) \quad \text{and} \quad \eta_i = (0, -\sqrt{1 - \lambda_i^2}, -\lambda_i),$$

as well as pairs of points

$$\begin{cases} P_{i,\varepsilon}^+ = (1 - \varepsilon)\xi_i, \\ N_{i,\varepsilon}^+ = (1 + \varepsilon)\xi_i, \end{cases} \quad \text{and} \quad \begin{cases} P_{i,\varepsilon}^- = (1 + \varepsilon)\eta_i, \\ N_{i,\varepsilon}^- = (1 - \varepsilon)\eta_i, \end{cases}$$

for each $\varepsilon > 0$. By $Z_{i,\varepsilon}^+$ and $Z_{i,\varepsilon}^-$, we denote the ε -tubular neighborhood of the line segment $[P_{i,\varepsilon}^+, N_{i,\varepsilon}^+]$ and $[P_{i,\varepsilon}^-, N_{i,\varepsilon}^-]$ respectively.

Without loss of generality, let $\varepsilon \ll 1$ such that

$$|z - \hat{z}| \geq \frac{1}{4N} \quad \text{for all } (x, y, z) \in Z_{i,\varepsilon}^+, (\hat{x}, \hat{y}, \hat{z}) \in Z_{j,\varepsilon}^+ \text{ with } i \neq j, \quad (3.1)$$

meaning the neighborhoods $Z_{i,\varepsilon}^+$ are separated by circular slices with a height of at least $\frac{1}{4N}$. Moreover, let $g = (\vartheta, S) \in C^\infty(B_2^3, \mathbb{R}^3 \times \mathbb{S})$ be given as

$$\vartheta(x, y, z) \equiv \begin{pmatrix} -x \\ -y \\ z \end{pmatrix} \quad \text{and} \quad S(x, y, z) \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

At first, Theorem 2 gives us a mapping $\tilde{g} = (\tilde{\vartheta}, \tilde{S})$ in

$$H^1(B_2^3, \mathbb{R}^3 \times \mathbb{S}) \cap C^\infty\left(B_2^3 \setminus \left(\bigcup_{i=1}^N \{P_{i,\varepsilon}^+, N_{i,\varepsilon}^+, P_{i,\varepsilon}^-, N_{i,\varepsilon}^-\}\right), \mathbb{R}^3 \times \mathbb{S}\right),$$

which agrees with g outside of $Z := \bigcup_{i=1}^N (Z_{i,\varepsilon}^+ \cup Z_{i,\varepsilon}^-)$, with $\tilde{S}$ having only the dipoles $(P_{i,\varepsilon}^\pm, N_{i,\varepsilon}^\pm)$ as singularities,

$$\deg_{P_{i,\varepsilon}^+}(\tilde{S}) = 1 = \deg_{N_{i,\varepsilon}^+}(\tilde{S}),$$

$$\deg_{P_{i,\varepsilon}^-}(\tilde{S}) = 1 = \deg_{N_{i,\varepsilon}^-}(\tilde{S}),$$

and

$$\mathcal{J}_{B^3}(\tilde{g}) < 64\pi \cdot 2\varepsilon \cdot 2N + \varepsilon = (64\pi \cdot 4N + 1) \cdot \varepsilon. \quad (3.2)$$

Now, the boundary values we prescribe are

$$g_0 = (\varphi_0, R_0) := \tilde{g}|_{\partial B^3} \in C^\infty(\partial B^3, \mathbb{R}^3 \times \mathbb{S}),$$

with $\deg(R_0) = 0$ and

$$\tilde{g}|_{\overline{B^3} \setminus Z} \equiv (\vartheta, S).$$

Due to (3.2), combined with (3.1), it is possible to choose $\varepsilon \ll 1$ such that

$$\mathcal{J}_{B^3}(\tilde{g}) < \frac{\pi}{N}. \quad (3.3)$$

For the rest of this proof $\tilde{g}$ always means $\tilde{g}|_{\overline{B^3}}$ and therefore, $\tilde{g} \in H_{g_0}^1(B^3, \mathbb{R}^3 \times \mathbb{S})$. Thus, let $f = (\varphi, R)$ be a restricted minimizer, i.e. a minimizer of the problem $(\mathcal{P}^*)$ in the (non-empty) class $H_{g_0}^1(B^3, \mathbb{R}^3 \times \mathbb{S})$. Because of (3.3), we also have

$$\mathcal{J}_{B^3}(f) < \frac{\pi}{N}. \quad (3.4)$$

From here, following [9], with (3.1) and (3.4), we find $N + 1$ numbers $\mu_0, \dots, \mu_N \in \mathbb{R}$,

$$0 < \mu_0 < \lambda_1 < \mu_1 < \lambda_2 < \dots < \lambda_N < \mu_N < 1,$$

such that for each $i = 0, \dots, N$ the disc $D_i := \{(x, y, z) \in B^3 : z = \mu_i\}$ satisfies $\overline{D_i} \cap Z = \emptyset$ and

$$\int_{\overline{D_i}} |DR|^2 d\mathcal{H}^2 < 4\pi. \quad (3.5)$$

Suppose (3.5) was not possible. Then we would get

$$\mathcal{J}_{B^3}(f) \geq \frac{1}{4N} \cdot 4\pi = \frac{\pi}{N},$$

because of (3.1), contradicting (3.4).

As we mentioned in the introduction, following the line of reasoning from [6] and carrying out similar arguments at the boundary, we get discreteness of the singular set for restricted Cosserat energy minimizers in the interior and full boundary regularity, cf. [11, Section 4]. Moreover, this is in accordance to full boundary

regularity for (not restricted) Cosserat energy minimizers in a more general setting, as recently shown also with similar arguments, but technically more involved [10]. In particular, singularities cannot lie at the boundary. This implies that also $\text{Sing}(R) \subseteq \text{Sing}(f) \subset B^3$ is a discrete set in (the interior of) B^3 . Hence we can assume without loss of generality, that all isolated singularities of R are in $B^3 \setminus (\bigcup_{i=1}^N D_i)$. Then for each compact subset C of $B^3 \setminus \text{Sing}(R)$ and each $i = 0, \dots, N$, we have (with the area-formula, because R is Lipschitz continuous in $D_i \cap C$)

$$\int_{\mathcal{S}} \mathcal{H}^0(D_i \cap C \cap R^{-1}(p)) \, d\mu(p) = \int_{D_i \cap C} \text{Jac}(R|_{D_i \cap C}) \, d\mathcal{H}^2,$$

where μ denotes the natural Riemannian measure on $\mathcal{S}$. Approximating the open set B^3 from within by a monotone sequence of compact sets $C_k \subset B^3$, meaning $C_k \subset C_{k+1}$ for all $k \in \mathbb{N}_0$, we know that the sequences $\mathcal{H}^0(D_i \cap C_k \cap R^{-1}(p))$ and $\text{Jac}(R|_{D_i \cap C_k})$ are hence also monotone in k . Thus from monotone convergence (for $C_k \nearrow B^3$), together with (3.5), we infer for each $i = 0, \dots, N$,

$$\begin{aligned} \mu(R(D_i)) &\leq \int_{\mathcal{S}} \mathcal{H}^0(D_i \cap R^{-1}(p)) \, d\mu(p) = \lim_{k \rightarrow \infty} \int_{\mathcal{S}} \mathcal{H}^0(D_i \cap C_k \cap R^{-1}(p)) \, d\mu(p) \\ &= \lim_{k \rightarrow \infty} \int_{D_i \cap C_k} \text{Jac}(R|_{D_i \cap C_k}) \, d\mathcal{H}^2 \leq \lim_{k \rightarrow \infty} \int_{D_i} \text{Jac}(R|_{D_i \cap C_k}) \, d\mathcal{H}^2 \\ &= \int_{D_i} \text{Jac}(R|_{D_i}) \, d\mathcal{H}^2 \leq \frac{1}{2} \int_{D_i} |DR|^2 \, d\mathcal{H}^2 < 2\pi = \mu(\mathcal{S}). \end{aligned}$$

So each image $R(D_i)$ is a proper subset of $\mathcal{S}$. On one hand, as R is (Hölder-) continuous on D_i and $D_i \cap Z = \emptyset$, combined with $R|_{\partial B^3} = R_0 = \tilde{S}|_{\partial B^3}$ and

$$\tilde{S}|_{B^3 \setminus Z} = S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it holds that $R|_{D_i}$ is homotopic to S relative to ∂D_i , i.e.

$$R|_{D_i} \simeq \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ rel. } \partial D_i. \quad (3.6)$$

And on the other hand, we infer

$$\tilde{S}|_{D_i} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.7)$$

Finally, for each $i = 1, \dots, N$, we consider the slice $\Omega_i := \{(x, y, z) \in B^3 : \mu_{i-1} < z < \mu_i\}$. Then $\partial\Omega_i = D_{i-1} \cup D_i \cup (\partial B^3 \cap \overline{\Omega_i})$ is homeomorphic to S^2 . By construction, the only singularity of $\tilde{S}$ in Ω_i is the point $P_{i,\varepsilon}^+$ with

$$\deg_{P_{i,\varepsilon}^+}(\tilde{S}) = 1.$$

We also know that no singularities lie in $\partial\Omega_i$ and that the interior of Ω_i contains only isolated singularities (namely only $P_{i,\varepsilon}^+$). Thus the degree of $\tilde{S}|_{\partial\Omega_i}$ is obtained by adding up the degrees of $\tilde{S}$ at all isolated interior singularities (mod 2) in Ω_i and we get

$$\deg(\tilde{S}|_{\partial\Omega_i}) = \left(\sum_{x \in \text{Sing}(\tilde{S} \cap \Omega_i)} \deg_x(\tilde{S}) \right) \bmod 2 = \deg_{P_{i,\varepsilon}^+}(\tilde{S}) = 1.$$

Moreover, (3.6), (3.7) and $R|_{\partial B^3} = \tilde{S}|_{\partial B^3}$ imply $R|_{\partial\Omega_i} \simeq \tilde{S}|_{\partial\Omega_i}$. Hence, due to homotopy invariance of $\deg(\cdot)$, we also find

$$\deg(R|_{\partial\Omega_i}) = \deg(\tilde{S}|_{\partial\Omega_i}) = 1.$$

This proves that R must have at least one singularity in Ω_i for each $i = 1, \dots, N$. □

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References

- [1] F. Bethuel, A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **7** (1990), no. 4, 269–286.
- [2] F. Bethuel, The approximation problem for Sobolev maps between two manifolds, *Acta Math.* **167** (1991), no. 3–4, 153–206.
- [3] H. Brezis, S^k -valued maps with singularities, in: *Topics in Calculus of Variations* (Montecatini Terme 1987), Lecture Notes in Math. 1365, Springer, Berlin (1989), 1–30.
- [4] H. Brezis and J.-M. Coron, Large solutions for harmonic maps in two dimensions, *Comm. Math. Phys.* **92** (1983), no. 2, 203–215.
- [5] H. Brezis, J.-M. Coron and E. H. Lieb, Harmonic maps with defects, *Comm. Math. Phys.* **107** (1986), no. 4, 649–705.
- [6] A. Gastel, Regularity issues for Cosserat continua and p -harmonic maps, *SIAM J. Math. Anal.* **51** (2019), no. 6, 4287–4310.
- [7] A. Gastel and P. Neff, Regularity for a geometrically nonlinear flat Cosserat micropolar membrane shell with curvature, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* (2024), DOI 10.4171/AIHPC/108.
- [8] M. J. Greenberg and J. R. Harper, *Algebraic Topology*, Math. Lecture Note Ser. 58, Benjamin/Cummings, Boca Raton, 1981.
- [9] R. Hardt and F.-H. Lin, A remark on H^1 mappings, *Manuscripta Math.* **56** (1986), no. 1, 1–10.
- [10] V. Hüsken, Full boundary regularity for minimizers of a Cosserat energy functional, preprint (2023).
- [11] V. Hüsken, *Sehr singuläre Lösungen eines geometrisch nichtlinearen Cosserat-Modells für mikropolare Festkörper*, PhD Thesis, Universität Duisburg-Essen, 2023.
- [12] J. Milnor, *Topology from the Differentiable Viewpoint. Based on notes by David W. Weaver*, revised 2nd ed., Princeton Landmarks in Math. Phys., Princeton University press, Princeton, 1997.
- [13] P. Neff, *Geometrically exact Cosserat theory for bulk behaviour and thin structures: Modelling and mathematical analysis*, Habilitation Thesis, TU Darmstadt, 2004.
- [14] P. Neff, Existence of minimizers for a finite-strain micromorphic elastic solid, *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), no. 5, 997–1012.
- [15] P. Neff, M. Bîrsan and F. Osterbrink, Existence theorem for geometrically nonlinear Cosserat micropolar model under uniform convexity requirements, *J. Elasticity* **121** (2015), no. 1, 119–141.
- [16] P. Olum, Mappings of manifolds and the notion of degree, *Ann. of Math.* **58** (1953), no. 3, 458–480.
- [17] R. Schoen and K. Uhlenbeck, Boundary regularity and the Dirichlet problem for harmonic maps, *J. Differential Geom.* **18** (1983), no. 2, 253–268.
- [18] K. Steffen, An introduction to harmonic mappings, Report, Universität Bonn. SFB 256. Nichtlineare Partielle Differentialgleichungen, Bonn, 1991.
- [19] U. Tarp, *Singuläre harmonische Abbildungen*, Diploma Thesis, Mathematisches Institut der Heinrich-Heine-Universität Düsseldorf, 2000.