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Research Article

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A weakly coupled system of p-Laplace type in a heat conduction problem

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Abstract: Given is a bounded domain $\Omega \subset \mathbb{R}^n$, and a vector-valued function defined on $\partial\Omega$ (depicting temperature distributions from different sources), our objective is to study the mathematical model of a physical problem of enclosing $\partial\Omega$ with a specific volume of insulating material to reduce heat loss in a stationary scenario. Mathematically, this task involves identifying a vector-valued function $\mathbf{u} = (u^1, \dots, u^m)$ $(m \geq 1)$ that represents the temperature within Ω and gives rise to a free boundary, somehow reminiscent of, but not equivalent to, the Bernoulli free boundary problem.

Keywords: Free boundary, heat conduction, p-Laplacian

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1 Introduction

1.1 Background

In this paper we consider an extension of a classical optimization problem in heat conduction, described as follows: given a surface $\partial\Omega$ (boundary of a domain $\Omega \subset \mathbb{R}^n$) and positive functions defined on it (each representing temperature distribution), the aim is to enclose $\partial\Omega$ with a prescribed volume of insulating material to minimize heat loss in a stationary scenario. Mathematically, the objective is to discover a vector-valued function $\mathbf{u} = (u^1, \dots, u^m)$ $(m \geq 1)$ that corresponds to the temperature within Ω. Whenever the components of $\mathbf{u}$ are nonnegative and the volume of its support is equal to 1, they become p-harmonic. The target is to minimize the heat flow, which can be regarded as a continuous family of convex functions dependent on $\nabla\mathbf{u}$ along $\partial\Omega$.

Our research was inspired by a series of papers [2–4] and their generalization presented in [16]. The initial two articles focused on studying constant temperature distributions, specifically in the linear case where $\Gamma(x, t) = t$. This linear setting enabled [2, 4] to reduce the quantity to be minimized to the Dirichlet integral. However, even within the linear case, the problem of nonconstant temperature distribution, examined in [3], introduced various new challenges.

The main objective of our article is to explore the system version of the nonlinear case with a nonconstant temperature distribution, wherein the equation is governed by the p-Laplacian. The nonlinearity addressed in this paper holds significant physical importance, as problems involving monotone operators, akin to those studied in [16], arise in the optimization of domains for electrostatic configurations.

The nonlinearity associated with $\nabla\mathbf{u}$ introduces various new challenges. For instance, computing normal derivatives of $W^{1,p}$-functions becomes problematic, leading to difficulties in providing a reasonable mathe-

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matical model. In [3], this challenge was overcome by minimizing the total mass of Δu, which can be treated as a nonnegative measure when u is subharmonic. However, in the present case, there is no representation available for

$$\int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}(x))\, d\sigma$$

as an integral over Ω. To address this issue, similarly to [16], we solve appropriate auxiliary variational problems and compare them with the minimizer.

Now let us introduce the problem in mathematical framework. Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded open set with smooth boundary whose volume $|\Omega| > 1$. Consider the p-Laplace differential operator ($1 < p < \infty$)

$$\Delta_p u^i = \operatorname{div}(|\nabla u^i|^{p-2}\, \nabla u^i) = \operatorname{div}(A[u^i]),$$

where we set $A[u^i] = A(\nabla u^i) := |\nabla u^i|^{p-2}\, \nabla u^i$ to simplify the notation.

Let $\varphi : \partial\Omega \to \mathbb{R}^m$ be a C^1 function with positive components $\varphi^i > 0$. For $\mathbf{u} : \Omega \to \mathbb{R}^m$ ($m \geq 1$) satisfying

$$\begin{cases} \Delta_p u^i = 0 & \text{in } \{|\mathbf{u}| > 0\}, \\ u^i = \varphi^i & \text{on } \partial\Omega, \\ \operatorname{vol}(\operatorname{spt} |\mathbf{u}|) = 1, \end{cases} \tag{1.1}$$

we want to minimize the functional

$$J(\mathbf{u}) := \int_{\partial\Omega} \Gamma(x, A_\nu u^1(x), \ldots, A_\nu u^m(x))\, d\sigma(x),$$

where ν is the outward normal vector on $\partial\Omega$,

$$A_\nu u^i := |\nabla u^i|^{p-2}\, \partial_\nu u^i,$$

and $\Gamma(x, \xi) : \partial\Omega \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function that satisfies:
(1) For each fixed x, $\Gamma(x, \cdot)$ is a convex function.
(2) For every i, $\partial_{\xi_i}\Gamma(\,\cdot\,, \cdot\,)$ is positive and has a positive lower bound on any set of the form $\{(x, \xi) : \xi_i \geq a\}$. In addition, $\partial_{\xi_i}\Gamma(\,\cdot\,, \cdot\,)$ is bounded above on any set of the form $\{(x, \xi) : \xi_i \leq b\}$. (The bounds can depend on a, b.)
(3) For each fixed ξ, $\partial_{\xi_i}\Gamma(\cdot, \xi)$ is a C^1 function.

Note that, as a result, for every ξ we have

$$\Gamma(x, \xi_1, \ldots, \xi_m) \geq \sum_{i \leq m} \partial_{\xi_i}\Gamma(x, 0)\xi_i + \Gamma(x, 0) \geq \sum_{i \leq m} \psi_i(x)\xi_i - C,$$

where $\psi_i(x) := \partial_{\xi_i}\Gamma(x, 0) > 0$ are positive C^1 functions and C is a constant. In particular, we have

$$\Gamma(x, A_\nu u_1, \ldots, A_\nu u_m) \geq \sum_{i=1}^{m} \psi_i(x) A_\nu u^i - C. \tag{1.2}$$

A typical example of Γ is

$$\Gamma(x, \xi) = \psi_1(x)\gamma_1(\xi_1) + \cdots + \psi_m(x)\gamma_m(\xi_m),$$

where the ψ_i are C^1 and positive, and the γ_i are C^1 increasing convex functions with positive derivative.

Remark. It might be worth remarking that this problem has some fundamental differences with the well-known Bernoulli problem [4], singular perturbation [8], or volume constraint problems [1], where in all these problems the Dirichlet integral is part of the cost functional to be minimized under constraints. There is a vast literature around these problems, and we refrain ourselves to get into. To see connection between our problem at hand and the aforementioned ones, we consider the energy above in a simple scalar case such as $\Gamma(x, A_\nu \mathbf{u}(x)) = \partial_\nu u(x)$, with u being constant, say $u = 1$ on $\partial\Omega$. Alternatively we may consider a u-dependent function $u\partial_\nu u(x)$. The drill is now simple:

$$\int_{\partial\Omega} u\partial_\nu u = \frac{1}{2}\int_{\partial\Omega} \partial_\nu u^2 = \int_{\Omega} u\Delta u + |\nabla u|^2 = \int_{\Omega} |\nabla u|^2,$$

upon assuming u would be harmonic in its support.

1.2 Structure of the paper

The structure of our paper is as follows: In Section 2, we introduce the physical problem under consideration. We then formulate a penalized version of the variational problem for the temperature $\mathbf{u}$ and define suitable constraint sets as part of our strategy to overcome the challenges arising from the nonlinearity. We solve the optimization problem over weakly closed subsets of $W^{1,p}$ (the sets V_δ), establishing the optimal regularity properties of the minimizers, including Lipschitz regularity. These results are crucial for proving the existence of an optimal configuration for the original penalized problem, as discussed in Section 3. Here we also present fundamental geometric-measure properties of the optimal configuration, such as linear growth away from the free boundary and uniformly positive density. These properties allow us to establish a representation theorem following the framework of [4].

In Section 4, we recover the original physical problem from the penalized problem by showing that for sufficiently small ε, the volume of $\{|\mathbf{u}_\varepsilon| > 0\}$ automatically adjusts to be equal to 1.

Section 5 is dedicated to the optimal regularity of the free boundary, for the case $p = 2$. We demonstrate that the normal derivative of the minimizer along the free boundary is a Hölder continuous function, leading to the conclusion that the free boundary is a $C^{1,\alpha}$ surface. Furthermore, using the free boundary condition obtained during the proof of Hölder continuity, we establish that the free boundary is an analytic surface, except for a small singular set.

2 The penalized problem

Let $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ and

$$V_\delta := \{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^m) : u^i \geq 0,\ \Delta_p u^i \geq 0,\ \Delta_p u^i = 0 \text{ in } \Omega_\delta,\ u^i = \varphi^i \text{ on } \partial\Omega\}.$$

Furthermore, we set

$$V := \bigcup_{\delta>0} V_\delta.$$

Observe that the above definition is consistent due to the assumption $\varphi^i > 0$ on $\partial\Omega$. Also, by $\Delta_p u^i \geq 0$ we mean that for any test function $\zeta \in C_c^\infty(\Omega)$ with $\zeta \geq 0$ we have

$$-\int_\Omega \nabla\zeta \cdot |\nabla u^i|^{p-2}\nabla u^i \, dx \geq 0.$$

This implies that there is a Radon measure μ^i such that for any test function $\zeta \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \zeta \, d\mu^i = -\int_\Omega \nabla\zeta \cdot |\nabla u^i|^{p-2}\nabla u^i \, dx.$$

To simplify the notation, we denote μ^i by $\Delta_p u^i$, and $d\mu^i$ by $\Delta_p u^i \, dx$. (It should be noted that this notation is not meant to imply μ^i is absolutely continuous with respect to the Lebesgue measure. In fact, for the minimizer, the two measures are mutually singular as we will see in Theorem 3.5.) It is also worth noting that

$$u^i > 0 \quad \text{in } \Omega_\delta \tag{2.1}$$

by the strong maximum principle, since u^i is p-harmonic in Ω_δ, and while it is positive on $\partial\Omega$, it is nonnegative everywhere.

Let $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ be

$$f_\varepsilon(t) := \begin{cases} 1 + \frac{1}{\varepsilon}(t-1), & t \geq 1, \\ 1 + \varepsilon(t-1), & t < 1. \end{cases}$$

We are interested in minimizing the penalized functional

$$J_\varepsilon(\mathbf{u}) := \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}(x)) \, d\sigma + f_\varepsilon(|\{|\mathbf{u}| > 0\}|)$$

over V. The significance of the above penalization is that it forces the volume $|\{|\mathbf{u}| > 0\}|$ to be 1 for small enough ε; see Theorem 4.3. (Notice that the components of $\mathbf{u} \in V$ are p-harmonic near $\partial\Omega$; therefore they are smooth enough near the boundary, and it makes sense to compute their derivatives along $\partial\Omega$.) We first consider the minimizer of J_ε over V_δ.

Lemma 2.1. *Let $\mathbf{u} \in V$. Then we have*

$$\int_\Omega \psi_i \Delta_p u^i \, dx + \int_\Omega \nabla\psi_i \cdot A[u^i] \, dx = \int_{\partial\Omega} \psi_i A_\nu u^i \, d\sigma, \tag{2.2}$$

where the ψ_i are C^1 functions.

Proof. Let $\phi_k \in C^\infty(\Omega)$ be such that $\phi_k \equiv 1$ on $\tilde{\Omega}_k := \Omega - \Omega_{1/k}$ and $\phi_k \equiv 0$ on $\partial\Omega$. We know that $\mathbf{u} \in V_\delta$ for some δ. Suppose k is large enough so that $\frac{1}{k} < \delta$, and thus $\Omega_{1/k} \subset \Omega_\delta$. Then we have

$$\begin{aligned}\int_\Omega \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx &= \int_{\Omega_{1/k}} \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx + \int_{\tilde{\Omega}_k} \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx \\ &= \int_{\Omega_{1/k}} \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx + \int_{\tilde{\Omega}_k} \nabla\psi_i \cdot A[u^i] \, dx.\end{aligned}$$

Now, noting that $\partial\Omega_{1/k} = \partial\tilde{\Omega}_k \cup \partial\Omega$, and by using the integration by parts formula proved in [6], we get

$$\begin{aligned}\int_{\Omega_{1/k}} \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx &= \int_{\Omega_{1/k}} \nabla(\phi_k\psi_i) \cdot A[u^i] + \phi_k\psi_i\Delta_p u^i \, dx \\ &= -\int_{\partial\tilde{\Omega}_k} \phi_k\psi_i A_\nu u^i \, d\sigma + \int_{\partial\Omega} \phi_k\psi_i A_\nu u^i \, d\sigma \\ &= -\int_{\partial\tilde{\Omega}_k} \psi_i A_\nu u^i \, d\sigma \xrightarrow[k\to\infty]{} -\int_{\partial\Omega} \psi_i A_\nu u^i \, d\sigma.\end{aligned}$$

In addition, we have

$$\int_{\tilde{\Omega}_k} \nabla\psi_i \cdot A[u^i] \, dx \xrightarrow[k\to\infty]{} \int_\Omega \nabla\psi_i \cdot A[u^i] \, dx,$$

and

$$\int_\Omega \nabla(\phi_k\psi_i) \cdot A[u^i] \, dx = -\int_\Omega \phi_k\psi_i \, \Delta_p u^i \, dx \xrightarrow[k\to\infty]{} -\int_\Omega \psi_i\Delta_p u^i \, dx,$$

which together give the desired result. □

We can similarly show that

$$\int_\Omega \Delta_p u^i \, dx = \int_{\partial\Omega} A_\nu u^i \, d\sigma.$$

In addition, note that $\int_\Omega u^i\Delta_p u^i \, dx$ is meaningful (since $u^i - \varphi^i \in W_0^{1,p}$ while $\Delta_p u^i \in W^{-1,p/(p-1)}$ and φ^i is continuous) and we can similarly show that

$$\int_\Omega u^i\Delta_p u^i + |\nabla u^i|^p \, dx = \int_{\partial\Omega} \varphi^i A_\nu u^i \, d\sigma. \tag{2.3}$$

Note that $u^i = \varphi^i$ on $\partial\Omega$.

Lemma 2.2. *For $\mathbf{u} \in V$ we have*

$$\sum_{i=1}^m \int_\Omega |\nabla u^i|^p \, dx \le C + C\int_{\partial\Omega} \sum_{i\le m} \psi_i(x) A_\nu u^i \, d\sigma.$$

Remark. As we will see, the above inequality actually holds for each summand. Furthermore, with a slight modification of the last part of the proof we obtain that

$$\int_\Omega \Delta_p u^i \, dx \le C + C\int_{\partial\Omega} \psi_i(x) A_\nu u^i \, d\sigma.$$

Proof. Let $\mathbf{h}_0$ be the vector-valued function in Ω satisfying $\Delta_p h_0^i = 0$, and taking the boundary values φ on $\partial\Omega$. Note that h_0^i is C^1 and we can plug it in (2.2). By subtracting the resulting relation from (2.3) we get

$$\int_\Omega (u^i - h_0^i)\Delta_p u^i \, dx + \int_\Omega \nabla(u^i - h_0^i)\cdot A[u^i]\, dx = 0.$$

Hence

$$\begin{aligned}\int_\Omega |\nabla u^i|^p \, dx = \int_\Omega \nabla u^i \cdot A[u^i]\, dx &= \int_\Omega (h_0^i - u^i)\Delta_p u^i \, dx + \int_\Omega \nabla h_0^i \cdot A[u^i]\, dx \\ &\le \int_\Omega h_0^i \Delta_p u^i \, dx + C\int_\Omega |\nabla h_0^i|^p \, dx + \frac{1}{2}\int_\Omega |A[u^i]|^{\frac{p}{p-1}}\, dx \\ &\le C\int_\Omega \Delta_p u^i \, dx + C + \frac{1}{2}\int_\Omega |\nabla u^i|^p \, dx,\end{aligned}$$

where we have used the facts that $u^i, \Delta_p u^i \ge 0$ and $|A[u^i]|^{\frac{p}{p-1}} = |\nabla u^i|^p$. Thus we have

$$\int_\Omega |\nabla u^i|^p \, dx \le C\int_\Omega \Delta_p u^i \, dx.$$

But since $\psi_i, \Delta_p u^i \ge 0$ we get

$$\int_\Omega |\nabla u^i|^p \, dx \le C\int_\Omega \Delta_p u^i \, dx \le CC_i \int_\Omega \psi_i \, \Delta_p u^i \, dx,$$

where $C_i = \max_{\overline{\Omega}} \frac{1}{\psi_i} > 0$. Hence by (2.2) we get

$$\begin{aligned}\int_\Omega |\nabla u^i|^p \, dx &\le C\int_\Omega \psi_i \, \Delta_p u^i \, dx \\ &= -C\int_\Omega \nabla\psi_i \cdot A[u^i]\, dx + C\int_{\partial\Omega} \psi_i A_\nu u^i \, d\sigma \\ &\le \tilde{C}\int_\Omega |\nabla\psi_i|^p \, dx + \frac{1}{2}\int_\Omega |A[u^i]|^{\frac{p}{p-1}}\, dx + C\int_{\partial\Omega} \psi_i A_\nu u^i \, d\sigma \\ &\le C + \frac{1}{2}\int_\Omega |\nabla u^i|^p \, dx + C\int_{\partial\Omega} \psi_i A_\nu u^i \, d\sigma,\end{aligned}$$

which gives the desired. □

Theorem 2.3. *There exists a minimizer $\mathbf{u}_\varepsilon^\delta \in V_\delta$ for J_ε.*

Proof. Let $\{\mathbf{u}_k\} \subset V_\delta$ be a minimizing sequence. Then by the above lemma and (1.2) we have

$$\begin{aligned}\sum_{i=1}^m \int_\Omega |\nabla u_k^i|^p \, dx &\le C + C\int_{\partial\Omega} \sum_{i\le m} \psi_i A_\nu u_k^i \, d\sigma \\ &\le C + C\int_{\partial\Omega} \Gamma(x, A_\nu u_k^1, \dots, A_\nu u_k^m) + C\, d\sigma \\ &\le C + CJ_\varepsilon(\mathbf{u}_k).\end{aligned}$$

Hence $\|\nabla \mathbf{u}_k\|_{L^p}$ is bounded. In addition, for the dual exponent $q = \frac{p}{p-1}$ we can see that $\|A[\mathbf{u}_k]\|_{L^q} = \|\nabla \mathbf{u}_k\|_{L^p}^{p-1}$ is also bounded. Hence, up to a subsequence, we can assume that $\nabla u_k^i \rightharpoonup \nabla u^i$ in L^p, $A[u_k^i] \rightharpoonup A[u^i]$ in L^q, and $\mathbf{u}_k \to \mathbf{u}$ a.e. in Ω. Thus we have $u^i \ge 0$. Also, $u^i = \varphi^i$ on $\partial\Omega$, since $u_k^i - \varphi^i \in W_0^{1,p}(\Omega)$, which is a closed and convex set, hence weakly closed. Finally, to see that $\Delta_p u^i$ has the desired properties, notice that for an appropriate test function ϕ we have

$$\int_\Omega \nabla\phi \cdot A[u^i]\, dx = \lim_{k\to\infty} \int_\Omega \nabla\phi \cdot A[u_k^i]\, dx$$

due to the weak convergence of $A[\mathbf{u}_k]$. Therefore $\mathbf{u} \in V_\delta$. Now we can repeat the proof of [16, Lemma 3.3] to deduce the weak lower semicontinuity of J_ε with respect to this sequence, and conclude the proof (the convexity of Γ is needed here). □

Although Hopf's lemmas for p-harmonic functions are well known (see for example [15]), we include the proof of the following version as we need a specific form for the constant.

Lemma 2.4 (Hopf's lemma for p-harmonic functions). *Suppose h is a p-harmonic function on $B_1(0)$ with nonnegative boundary values on ∂B_1. Then we have*

$$h(x) \geq c(n,p)\operatorname{dist}(x,\partial B_1)\sup_{B_{1/2}} h.$$

Proof. Consider the function $g(x) = e^{-\lambda|x|^2} - e^{-\lambda}$ for some $\lambda > 0$. Note that $g = 0$ on ∂B_1, and $0 < g < 1$ on B_1. We also have

$$\partial_i g = -2\lambda x_i e^{-\lambda|x|^2}, \quad \partial_{ij} g = (4\lambda^2 x_i x_j - 2\lambda\delta_{ij})e^{-\lambda|x|^2}.$$

Now we have $\Delta g = (4\lambda^2|x|^2 - 2n\lambda)e^{-\lambda|x|^2}$, and

$$\Delta_\infty g := \sum_{i,j} \partial_i g \partial_j g \partial_{ij} g = \sum_{i,j} 4\lambda^2(4\lambda^2 x_i^2 x_j^2 - 2\lambda\delta_{ij}x_i x_j)e^{-3\lambda|x|^2} = 4\lambda^2(4\lambda^2|x|^4 - 2\lambda|x|^2)e^{-3\lambda|x|^2}.$$

Therefore

$$\begin{aligned}
\Delta_p g &= \operatorname{div}(|\nabla g|^{p-2}\nabla g) = |\nabla g|^{p-4}(|\nabla g|^2\Delta g + (p-2)\Delta_\infty g)\\
&= (2\lambda|x|)^{p-4}(4\lambda^2|x|^2(4\lambda^2|x|^2 - 2n\lambda) + (p-2)4\lambda^2(4\lambda^2|x|^4 - 2\lambda|x|^2))e^{-(p-1)\lambda|x|^2}\\
&= (2\lambda)^{p-1}|x|^{p-2}(2\lambda|x|^2 - n + (p-2)(2\lambda|x|^2 - 1))e^{-(p-1)\lambda|x|^2}\\
&= (2\lambda)^{p-1}|x|^{p-2}(2(p-1)\lambda|x|^2 - n - p + 2)e^{-(p-1)\lambda|x|^2}.
\end{aligned}$$

Thus for $\frac{1}{2} \leq |x| \leq 1$ and large enough λ we have

$$\Delta_p g \geq 2\lambda^{p-1}\Big((p-1)\frac{\lambda}{2} - n - p + 2\Big)e^{-(p-1)\lambda} > 0.$$

Now we have $h \geq \inf_{\overline{B}_{1/2}} h > (\inf_{\overline{B}_{1/2}} h)g$ on $\overline{B}_{1/2}$ (note that h is positive on B_1 by maximum principle), and on $B_1 - \overline{B}_{1/2}$ we have $\Delta_p h = 0 < \Delta_p g$. Also on ∂B_1 we have $h \geq 0 = g$. Hence by the maximum principle we have $h(x) \geq g(x)(\inf_{\overline{B}_{1/2}} h)$ for $x \in B_1$. But by the Harnack's inequality we have

$$\inf_{\overline{B}_{1/2}} h \geq C \sup_{\overline{B}_{1/2}} h$$

for some constant C which does not depend on h. Hence we obtain

$$h(x) \geq Cg(x)\sup_{\overline{B}_{1/2}} h.$$

On the other hand note that

$$\begin{aligned}
g(x) = g(x) - g(x/|x|) &= \int_{\frac{1}{|x|}}^{1} \frac{d}{dt} g(tx)\,dt = \int_{\frac{1}{|x|}}^{1} x\cdot\nabla g(tx)\,dt\\
&= \int_{\frac{1}{|x|}}^{1} -2\lambda t|x|^2 e^{-\lambda t^2|x|^2}\,dt = 2\lambda|x|^2\int_{1}^{\frac{1}{|x|}} te^{-\lambda t^2|x|^2}\,dt\\
&\geq 2\lambda|x|^2\int_{1}^{\frac{1}{|x|}} te^{-\lambda}\,dt = \lambda e^{-\lambda}|x|^2\Big(\frac{1}{|x|^2} - 1\Big) = \lambda e^{-\lambda}(1 - |x|^2)\\
&\geq \lambda e^{-\lambda}(1 - |x|) = \lambda e^{-\lambda}\operatorname{dist}(x,\partial B_1),
\end{aligned}$$

which gives the desired. □

If h is a p-harmonic function on $B_r(x_0)$, then $\tilde{h}(x) := h(x_0 + rx)$ is a p-harmonic function on $B_1(0)$. Hence we have

$$\begin{aligned} h(x_0 + rx) = \tilde{h}(x) &\geq c(n,p)\,\mathrm{dist}(x, \partial B_1) \sup_{B_{1/2}} \tilde{h} \\ &= c(n,p)\,(1 - |x|) \sup_{B_{1/2}} \tilde{h} = c(n,p)\,\frac{r - r|x|}{r} \sup_{B_{r/2}(x_0)} h \\ &= c(n,p)\,\mathrm{dist}(x_0 + rx, \partial B_r(x_0)) \frac{1}{r} \sup_{B_{r/2}(x_0)} h. \end{aligned}$$

Lemma 2.5. *Let $w \in W^{1,p}(\Omega)$ be a nonnegative function. Then there exists $c > 0$, depending only on p and the dimension, such that for any ball $\overline{B}_r(x_0) \subset \Omega$ we have*

$$\Big(\frac{1}{r} \sup_{B_{r/2}(x_0)} h\Big)^p \cdot |B_r(x_0) \cap \{w = 0\}| \leq c \int_{B_r(x_0)} |\nabla(w - h)|^p \, dy,$$

where h satisfies $\Delta_p h = 0$ in $B_r(x_0)$ taking boundary values equal to w on $\partial B_r(x_0)$.

Proof. Let $\tau \in (0,1)$ be fixed. For ξ with $|\xi| = 1$ we set

$$t_\xi := \inf\{t \in [\tau r, r] : w(x_0 + t\xi) = 0\}$$

provided that this set is nonempty. Otherwise we set $t_\xi := r$. Now note that $w - h$ and w are absolutely continuous in almost every direction ξ; in particular we have $w(x_0 + t_\xi \xi) = 0$ (note that this will not be necessarily true if we allow τ to be zero). Also $w - h$ is $\mathcal{H}^{n-1}$-a.e. zero on $\partial B_r(x_0)$ as its trace is zero there, so $(w - h)(x_0 + r\xi) = 0$. Thus for almost every ξ for which $t_\xi < r$ we have

$$\begin{aligned} h(x_0 + t_\xi \xi) &= (w - h)(x_0 + r\xi) - (w - h)(x_0 + t_\xi \xi) \\ &= \int_{t_\xi}^{r} \frac{d}{dt}((w - h)(x_0 + t\xi)) \, dt = \int_{t_\xi}^{r} \nabla_\xi (w - h)(x_0 + t\xi) \, dt \\ &\leq (r - t_\xi)^{\frac{p-1}{p}} \Big(\int_{t_\xi}^{r} |\nabla(w - h)(x_0 + t\xi)|^p \, dt \Big)^{\frac{1}{p}}. \end{aligned}$$

On the other hand, using Hopf's lemma we get

$$h(x_0 + t_\xi \xi) \geq c(n,p)\,\mathrm{dist}(x_0 + t_\xi \xi, \partial B_r(x_0)) \frac{1}{r} \sup_{B_{r/2}(x_0)} h = c(n,p)(r - t_\xi) \frac{1}{r} \sup_{B_{r/2}(x_0)} h.$$

Hence we obtain

$$(r - t_\xi) \Big(\frac{1}{r} \sup_{B_{r/2}(x_0)} h \Big)^p \leq C(n,p) \int_{t_\xi}^{r} |\nabla(w - h)(x_0 + t\xi)|^p \, dt.$$

Note that this inequality is trivially satisfied if $t_\xi = r$.

Now by integrating with respect to $d\xi$ we get

$$\begin{aligned} C(n,p) \int_{B_r(x_0)} |\nabla(w - h)(x)|^p \, dx &\geq C(n,p) \int_{\partial B_1(0)} \int_{t_\xi}^{r} |\nabla(w - h)(x_0 + t\xi)|^p \, dt \, d\xi \\ &\geq \Big(\frac{1}{r} \sup_{B_{r/2}(x_0)} h \Big)^p \int_{\partial B_1(0)} (r - t_\xi) \, d\xi \\ &= \Big(\frac{1}{r} \sup_{B_{r/2}(x_0)} h \Big)^p \int_{\partial B_1(0)} \int_{t_\xi}^{r} 1 \, dt \, d\xi \\ &\geq \Big(\frac{1}{r} \sup_{B_{r/2}(x_0)} h \Big)^p \int_{B_r(x_0) - B_{\tau r}(x_0)} \chi_{\{w=0\}} \, dx, \end{aligned}$$

where the last inequality follows from the definition of t_ξ. Finally, we get the desired by letting $\tau \to 0$. □

Lemma 2.6. *Let* $\mathbf{u} = \mathbf{u}_\varepsilon^\delta$ *be a minimizer of* J_ε *over* V_δ*, and let* $B \subset \Omega$ *be a ball. Then there exists a unique* $v^i \in W^{1,p}(\Omega)$ *that minimizes the functional*

$$\int_\Omega |\nabla v^i|^p \, dx$$

among all functions with $v^i = \varphi^i$ *on* $\partial\Omega$ *and* $v^i \le 0$ *on* $\{u^i = 0\} - B$*. The functions* v^i *also satisfy:*

(1) $v^i = 0$ *on* $\{u^i = 0\} - B$,
(2) $\mathbf{v} = (v^1, \dots, v^m) \in V_\delta$,
(3) $0 \le u^i \le v^i \le C_0 = \max_{\partial\Omega} |\varphi|$,
(4) $\int_\Omega v^i \Delta_p v^i \, dx = 0$.

Remark. Instead of a ball B, we can also use other open subsets of Ω in the above lemma. Essentially, all we need is that the p-energy functional has a minimum over the corresponding set K in the following proof; so no regularity assumption is actually needed regarding such open sets.

Proof. It is easy to see that

$$K := \{v \in W^{1,p}(\Omega) : v = \varphi^i \text{ on } \partial\Omega \text{ and } v \le 0 \text{ on } \{u^i = 0\} - B\}$$

is a closed convex subset of $W^{1,p}(\Omega)$. It is nonempty too as $u^i \in K$. So there exists a unique $v^i \in K$ minimizing the strictly convex and coercive functional $\int_\Omega |\nabla v|^p \, dx$. Then for every $v \in K$ we have

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega |\nabla(v^i + t(v - v^i))|^p \, dx \ge 0,$$

and hence v^i satisfies the variational inequality

$$\int_\Omega |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla(v - v^i) \, dx \ge 0. \tag{2.4}$$

Now note that $v = v^i - \zeta \in K$ for any test function $\zeta \in C_c^\infty(\Omega)$ with $\zeta \ge 0$. Therefore

$$-\int_\Omega |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla\zeta \, dx \ge 0,$$

which means $\Delta_p v^i \ge 0$. As a result, $v^i \le \max_{\partial\Omega} \varphi^i \le C_0$ by the maximum principle.

Next note that if spt ζ does not intersect $\{u^i = 0\} - B$, then we also have $v^i + \zeta \in K$. Thus we also get

$$\int_\Omega |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla\zeta \, dx \ge 0,$$

which together with the previous inequality implies

$$-\int_\Omega |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla\zeta \, dx = 0.$$

Therefore $\Delta_p v^i = 0$ in the interior of

$$\Omega - (\{u^i = 0\} - B) = (\Omega - \{u^i = 0\}) \cup B.$$

In particular, $\Delta_p v^i = 0$ in Ω_δ since $u^i > 0$ in Ω_δ by (2.1).

In addition, for $\epsilon > 0$ we have $v = \max(v^i, -\epsilon) \in K$. By plugging this test function in (2.4) we get

$$0 \le \int_\Omega |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla(v - v^i) \, dx = \int_{\{v^i < -\epsilon\}} |\nabla v^i|^{p-2} \nabla v^i \cdot \nabla(-\epsilon - v^i) \, dx = -\int_{\{v^i < -\epsilon\}} |\nabla v^i|^p \, dx.$$

By letting $\epsilon \to 0$ we obtain $\int_{\{v^i < 0\}} |\nabla v^i|^p \, dx = 0$, and hence $v^i \ge 0$. In particular, we must have $v^i = 0$ on $\{u^i = 0\} - B$ as v^i is assumed to be nonpositive there. Furthermore, note that we have so far shown $\mathbf{v} = (v^1, \dots, v^m) \in V_\delta$.

Next, since $\Delta_p v^i = 0$ in the exterior of $\{u^i = 0\} - B$, and $\Delta_p u^i \geq 0$, the maximum principle implies that $u^i \leq v^i$ (note that u^i, v^i have the same boundary values in the exterior of $\{u^i = 0\} - B$).

Finally, for $\zeta \in C_c^\infty(\Omega)$ with $\zeta \geq 0$ and small enough ϵ we have

$$v^i \pm \epsilon\zeta v^i = (1 \pm \epsilon\zeta)v^i \in K.$$

By plugging this test function in (2.4) we get

$$0 \leq \pm\epsilon \int_\Omega |\nabla v^i|^{p-2}\nabla v^i \cdot \nabla(\zeta v^i)\, dx \implies \int_\Omega |\nabla v^i|^{p-2}\nabla v^i \cdot \nabla(\zeta v^i)\, dx = 0.$$

In other words

$$\int_\Omega \zeta v^i \Delta_p v^i\, dx = 0.$$

By letting $\zeta \to 1$ we obtain $\int_\Omega v^i \Delta_p v^i\, dx = 0$, as desired. Alternatively, we can take ζ to be 1 over a neighborhood of $\{u^i = 0\} - B$. From this and that $\Delta_p v^i = 0$ in the exterior of $\{u^i = 0\} - B$, we obtain $\int_\Omega v^i \Delta_p v^i\, dx = 0$. □

Theorem 2.7. *Let $\mathbf{u} = \mathbf{u}_\varepsilon^\delta$ be a minimizer of J_ε over V_δ. There exists a constant $M = M_\varepsilon$, independent of δ, such that if for some j we have*

$$\frac{1}{r} \sup_{B_{r/2}(x)} u^j \geq M,$$

then $B_r(x) \subset \{|\mathbf{u}| > 0\}$, and $\Delta_p u^i = 0$ in $B_r(x)$ for every i.

Proof. Let $\mathbf{v} \in V_\delta$ be the function given by Lemma 2.6 for $B_r(x)$. Then we have

$$J_\varepsilon(\mathbf{u}) \leq J_\varepsilon(\mathbf{v}).$$

Let $\mathbf{h}_0$ be the vector-valued function in Ω satisfying $\Delta_p h_0^i = 0$, and taking the boundary values $\boldsymbol{\varphi}$ on $\partial\Omega$. Since $0 \leq u^i \leq v^i \leq h_0^i$, for each $z \in \partial\Omega$ we have

$$\partial_\nu h_0^i(z) \leq \partial_\nu v^i(z) \leq \partial_\nu u^i(z) \leq 0.$$

Then by using the fact that $\mathbf{u}, \mathbf{v}, \mathbf{h}_0$ take the same boundary values and therefore have equal tangential derivatives on $\partial\Omega$, we deduce that

$$a \leq A_\nu h_0^i(z) \leq A_\nu v^i(z) \leq A_\nu u^i(z),$$

where a is a lower bound for $A_\nu h_0^i$ (note that a does not depend on δ).

Hence by property (2) of Γ we have

$$\begin{aligned}\int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}(x)) - \Gamma(x, A_\nu \mathbf{v}(x))\, d\sigma &= \sum_{i=1}^m \int_{\partial\Omega} \Gamma(x, A_\nu u^1, \dots, A_\nu u^{i-1}, A_\nu u^i, A_\nu v^{i+1}, \dots, A_\nu v^m) \\ &\qquad - \Gamma(x, A_\nu u^1, \dots, A_\nu u^{i-1}, A_\nu v^i, A_\nu v^{i+1}, \dots, A_\nu v^m)\, d\sigma \\ &\geq C_a \sum_{i=1}^m \int_{\partial\Omega} A_\nu u^i - A_\nu v^i\, d\sigma,\end{aligned} \tag{2.5}$$

where $C_a > 0$ is the lower bound of the $\partial_{\xi_i}\Gamma$ on the set $\{(x, \xi) : \xi_i \geq a\}$. On the other hand, using the identity (2.3) we get

$$\begin{aligned}C_0 \int_{\partial\Omega} A_\nu u^i - A_\nu v^i\, d\sigma &\geq \int_{\partial\Omega} \varphi^i (A_\nu u^i - A_\nu v^i)\, d\sigma \\ &= \int_\Omega u^i\, \Delta_p u^i + |\nabla u^i|^p\, dy - \int_\Omega v^i\, \Delta_p v^i + |\nabla v^i|^p\, dy \\ &\geq \int_\Omega |\nabla u^i|^p\, dy - \int_\Omega |\nabla v^i|^p\, dy,\end{aligned} \tag{2.6}$$

where $C_0 = \max_{\partial\Omega} |\boldsymbol{\varphi}|$, and in the last line we used the facts that $\int_\Omega v^i\, \Delta_p v^i\, dy = 0$ and $u^i, \Delta_p u^i \geq 0$. Now consider the function h^i in $B_r(x)$ satisfying $\Delta_p h^i = 0$, and taking boundary values equal to u^i. We extend h^i to be equal to u^i outside of $B_r(x)$. Then we have $\mathbf{h} = (h_1, \dots, h_m) \in V_\delta$. In addition, $h^i = u^i = \varphi^i$ on $\partial\Omega$ and $h^i = u^i = 0$ on $\{u^i = 0\} - B_r(x)$. Hence due to the minimality property of v^i given by Lemma 2.6 we have

$$\int_\Omega |\nabla v^i|^p \, dy \le \int_\Omega |\nabla h^i|^p \, dy.$$

Combining this with the above inequality we get

$$C_0 \int_{\partial\Omega} A_\nu u^i - A_\nu v^i \, d\sigma \ge \int_\Omega |\nabla u^i|^p - |\nabla v^i|^p \, dy \ge \int_\Omega |\nabla u^i|^p - |\nabla h^i|^p \, dy \ge C \int_{B_r(x)} |\nabla(u^i - h^i)|^p \, dy,$$

where the last inequality can be proved similarly to the proof of [9, Lemma 3.1]. (Note that in the last line we have also used the fact that $u^i = h^i$ outside $B_r(x)$.)

Summing the above inequality for each i, and using the facts that $J_\varepsilon(\mathbf{u}) \le J_\varepsilon(\mathbf{v})$, and f_ε has Lipschitz constant equal to $\frac{1}{\varepsilon}$, we get

$$\begin{aligned}
\frac{C_a}{C_0} \sum_{i \le m} \int_{B_r(x)} |\nabla(u^i - h^i)|^p \, dy &\le C_a \int_{\partial\Omega} \sum_{i \le m} (A_\nu u^i - A_\nu v^i) \, d\sigma \\
&\le \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}(x)) - \Gamma(x, A_\nu \mathbf{v}(x)) \, d\sigma \\
&\le f_\varepsilon(|\{|\mathbf{v}| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|) \\
&\le \frac{1}{\varepsilon} |B_r(x) \cap \{|\mathbf{u}| = 0\}|,
\end{aligned}$$

since $0 \le u^i \le v^i$, and outside of $B_r(x)$, $|\mathbf{u}| = 0$ implies $|\mathbf{v}| = 0$. Therefore by Lemma 2.5 applied to u^j we obtain

$$\begin{aligned}
|B_r(x) \cap \{|\mathbf{u}| = 0\}| &\ge \frac{\varepsilon C_a}{C_0} \sum_{i \le m} \int_{B_r(x)} |\nabla(u^i - h^i)|^p \, dy \\
&\ge \frac{\varepsilon C_a}{C_0} \int_{B_r(x)} |\nabla(u^j - h^j)|^p \, dy \\
&\ge \frac{\varepsilon C_a}{c C_0} \Big(\frac{1}{r} \sup_{B_{r/2}(x)} h^j \Big)^p \cdot |B_r(x) \cap \{u^j = 0\}| \\
&\ge \frac{\varepsilon C_a}{c C_0} \Big(\frac{1}{r} \sup_{B_{r/2}(x)} u^j \Big)^p \cdot |B_r(x) \cap \{u^j = 0\}| \\
&\ge \frac{\varepsilon C_a M^p}{c C_0} |B_r(x) \cap \{|\mathbf{u}| = 0\}|,
\end{aligned}$$

since $|\mathbf{u}| = 0$ implies $u^j = 0$, and $h^j \ge u^j$ as u^j is p-subharmonic. Hence if $M > (\frac{cC_0}{\varepsilon C_a})^{\frac{1}{p}}$, then $|B_r(x) \cap \{|\mathbf{u}| = 0\}|$ must be zero, as desired. Note that in this case the above inequality also implies that $u^i = h^i$ in $B_r(x)$ for each i; so u^i satisfies the equation in $B_r(x)$. □

Corollary 2.8. *All minimizers $\mathbf{u}_\varepsilon^\delta$ are Lipschitz, and for every $\Omega' \subset\subset \Omega$ there exists a constant $K_\varepsilon = K_\varepsilon(\Omega')$, independent of δ, such that*

$$\|\mathbf{u}_\varepsilon^\delta|_{\Omega'}\|_{\mathrm{Lip}} \le K_\varepsilon.$$

In addition, $\Delta_p (u_\varepsilon^\delta)^i = 0$ in the open set $\{|\mathbf{u}_\varepsilon^\delta| > 0\}$.

Proof. For simplicity we set $\mathbf{u} = \mathbf{u}_\varepsilon^\delta$. First let us show that $\{|\mathbf{u}| > 0\}$ is an open set. Suppose $x \in \{|\mathbf{u}| > 0\}$. Then $u^j(x) > 0$ for some j. Then for small enough r we must have

$$\frac{1}{r} \sup_{B_{r/2}(x)} u^j \ge \frac{1}{r} u^j(x) \ge M.$$

Hence the previous theorem implies that $B_r(x) \subset \{|\mathbf{u}| > 0\}$ and we have $\Delta_p u^i = 0$ in $B_r(x)$.

Next note that $\nabla \mathbf{u} = 0$ a.e. in $\{|\mathbf{u}| = 0\}$. So suppose $x \in \{|\mathbf{u}| > 0\} \cap \Omega'$. Let $\Omega' \subset\subset \tilde{\Omega} \subset\subset \Omega$, and $B = B_d(x)$, where $d = \operatorname{dist}(x, \partial(\{|\mathbf{u}| > 0\} \cap \tilde{\Omega}))$. If ∂B touches $\partial\{|\mathbf{u}| = 0\}$ then $B_{d+d'}(x)$ intersects $\{|\mathbf{u}| = 0\}$, and by previous theorem we have

$$\frac{1}{d+d'} \sup_{B_{(d+d')/2}(x)} u^i \le M$$

for every i. Hence in the limit $d' \to 0$ we get

$$\frac{1}{d} \sup_{B_{d/2}(x)} u^i \le M.$$

Now since the u^i are p-harmonic in B, as shown in the proof of [7, Lemma 3.1], we have

$$|\nabla u^i(x)| \le C\frac{1}{d} \sup_{B_{d/2}(x)} u^i \le CM,$$

where the constant C depends only on p and the dimension n. On the other hand, if ∂B touches $\partial\tilde{\Omega}$ then, by the interior derivative estimate of [11], we obtain (the dependence on d follows from the proof of this estimate; see [11, equation (3.4)])

$$|\nabla u^i(x)| \le \frac{C(n,p)}{d^n}\|\mathbf{u}\|_{W^{1,p}} \le C,$$

since $d \ge \operatorname{dist}(\Omega', \partial\tilde{\Omega})$, and $\|\mathbf{u}\|_{W^{1,p}}$ is bounded independently of δ as will be shown now. Let $\Omega' \subset\subset \Omega$ be a smooth open set with $|\Omega - \Omega'| = 1$. Let $\mathbf{u}_0$ be a vector-valued function on $\Omega - \Omega'$ that satisfies the equation $\Delta_p u_0^i = 0$, and takes the boundary values $\boldsymbol{\varphi}$ on $\partial\Omega$ and 0 on $\partial\Omega'$. Then for every small enough δ we have $\mathbf{u}_0 \in V_\delta$. Hence (remember that $\mathbf{u} = \mathbf{u}_\varepsilon^\delta$)

$$\begin{aligned} C = J_\varepsilon(\mathbf{u}_0) \ge J_\varepsilon(\mathbf{u}_\varepsilon^\delta) &\ge \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}_\varepsilon^\delta(x))\, d\sigma \\ &\ge \int_{\partial\Omega} \sum_{i=1}^m \psi_i(x) A_\nu (u_\varepsilon^\delta)^i - C\, d\sigma, \end{aligned}$$

where we used (1.2) in the last line. Thus by Lemma 2.2 the $\|\nabla \mathbf{u}_\varepsilon^\delta\|_{L^p(\Omega;\mathbb{R}^m)}$ is bounded as $\delta \to 0$, and the boundedness of $\|\mathbf{u}_\varepsilon^\delta\|_{W^{1,p}}$ follows from Poincaré inequality and the fact that all of $\mathbf{u}_\varepsilon^\delta$'s have the same boundary values.

Finally, to see that $\mathbf{u}$ is Lipschitz continuous on all of Ω, note that $\mathbf{u}$ has p-harmonic components near the smooth boundary $\partial\Omega$, attaining smooth boundary conditions $\boldsymbol{\varphi}$; hence the gradient of $\mathbf{u}$ is bounded near the boundary too. □

Lemma 2.9. *There exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for every δ we have $|\mathbf{u}_\varepsilon^\delta| > 0$ in Ω_{δ_0}.*

Remark. Note that as a consequence, $\Delta_p (u_\varepsilon^\delta)^i = 0$ on Ω_{δ_0} for every δ (by Theorem 2.7). In other words, $\mathbf{u}_\varepsilon^\delta \in V_{\delta_0}$ for every δ.

Proof. Suppose to the contrary that there is a sequence $\mathbf{u}_k = \mathbf{u}_\varepsilon^{\delta_k}$ for which we have

$$2d_k := \operatorname{dist}(\{|\mathbf{u}_k| = 0\}, \partial\Omega) \to 0.$$

Then the midpoint of the closest points on $\{|\mathbf{u}_k| = 0\}$ and $\partial\Omega$, which we call x_k, has distance d_k from both of these sets. So the boundary of the ball $B_{d_k}(x_k)$ touches both of these sets. In addition, by Theorem 2.7, for every $t > 0$ we must have

$$\frac{1}{d_k} \sup_{B_{d_k/2}(x_k + t\nu_k)} u_k^i \le M_\varepsilon$$

for every i (here ν_k is the direction of the line segment from x_k to its closest point on $\{|\mathbf{u}_k| = 0\}$). So in the limit $t \to 0$ we get

$$\sup_{B_{d_k/2}(x_k)} u_k^i \le M_\varepsilon d_k. \tag{2.7}$$

We also have

$$\sup_{B_{d_k}(x_k)} |\mathbf{u}_k| \ge c_0,$$

where $c_0 = \min_i \min_{\partial\Omega} \varphi^i > 0$. Because at the point $y_k \in \partial B_{d_k}(x_k) \cap \partial\Omega$ we have $u_k^i(y_k) = \varphi^i(y_k) \ge c_0$ (note that u_k^i is continuous up to the boundary).

Next consider the functions

$$\hat{\mathbf{u}}_k(x) := \frac{\mathbf{u}(x_k + d_k x)}{\sup_{B_{d_k}(x_k)} |\mathbf{u}_k|}$$

on B_1. Then $\hat{u}_k^i$ is positive and p-harmonic on B_1, and we have $\sup_{B_1} |\hat{\mathbf{u}}_k| = 1$. In addition, by (2.7) we have

$$\sup_{B_{1/2}} \hat{u}_k^i = \frac{\sup_{B_{d_k/2}(x_k)} u_k^i}{\sup_{B_{d_k}(x_k)} |\mathbf{u}_k|} \leq \frac{M_\varepsilon d_k}{c_0} \xrightarrow[k\to\infty]{} 0.$$

Furthermore, note that $\hat{u}_k^i$ is a uniformly bounded sequence of p-harmonic functions on B_1, so there is $\alpha > 0$ such that for all $r < 1$ the Hölder norms $\|\hat{u}_k^i\|_{C^{0,\alpha}(\overline{B}_r)}$ are uniformly bounded (see [10, p. 251]). Hence, by a diagonal argument, we can construct a subsequence of $\hat{u}_k^i$, which we still denote by $\hat{u}_k^i$, that locally uniformly converges to a nonnegative p-harmonic function $\hat{u}_\infty^i$ on B_1. In addition, $\hat{u}_\infty^i$ must vanish on $B_{1/2}$ by the above estimate. Thus by the strong maximum principle we must have $\hat{\mathbf{u}}_\infty \equiv 0$ on B_1.

Now for $y_k \in \partial B_{d_k}(x_k) \cap \partial\Omega$ and $r < d_k$ we have

$$\operatorname*{osc}_{B_r(y_k)\cap\Omega} u_k^i \leq C(n,p)\Big(r^\alpha + \operatorname*{osc}_{B_r(y_k)\cap\partial\Omega} \varphi^i\Big) \leq Cr^\alpha$$

for some $\alpha \in (0,1)$. This estimate holds by [14, Theorem 4.19] when $1 < p \leq n$. And when $p > n$ this estimate holds due to the uniform Hölder continuity of u_k^i on $\overline{\Omega}$, since $\|\mathbf{u}_k\|_{W^{1,p}(\Omega)}$ is uniformly bounded as we have seen in the proof of Corollary 2.8. Hence for $r = d_k/2$ we have

$$\min_{B_{d_k/2}(y_k)\cap B_{d_k}(x_k)} u_k^i \geq \min_{B_{d_k/2}(y_k)\cap\Omega} u_k^i \geq \frac{1}{2}c_0,$$

where $c_0 = \min_i \min_{\partial\Omega} \varphi^i$. Therefore for $\hat{y}_k = \frac{1}{d_k}(y_k - x_k) \in \partial B_1$ we have

$$\min_{B_{1/2}(\hat{y}_k)\cap B_1} \hat{u}_k^i = \frac{1}{\sup_{B_{d_k}(x_k)} |\mathbf{u}_k|} \min_{B_{d_k/2}(y_k)\cap B_{d_k}(x_k)} u_k^i \geq c > 0,$$

since $\sup_{B_{d_k}(x_k)} |\mathbf{u}_k| \leq mC_0$ where $C_0 = \max_{\partial\Omega} |\varphi|$. Thus $\hat{u}_k^i$ has a uniform positive lower bound on a subset of B_1 with positive volume (where the volume is independent of k). So no subsequence of $\hat{\mathbf{u}}_k$ can converge locally uniformly to $\hat{\mathbf{u}}_\infty \equiv 0$, because otherwise they will uniformly converge to 0 outside a set of small volume, contradicting the uniform boundedness from below. ☐

Now we can find a minimizer for J_ε over V.

Theorem 2.10. *There exists a minimizer $\mathbf{u}_\varepsilon \in V$ for J_ε. Moreover, $\mathbf{u}_\varepsilon$ is a Lipschitz function, and $\Delta_p u_\varepsilon^i = 0$ in the open set $\{|\mathbf{u}_\varepsilon| > 0\}$.*

Remark. As we will see in the following proof, $\mathbf{u}_\varepsilon^\delta \in V_{\delta_0}$ for $\delta_0 = \delta_0(\varepsilon)$ given by the above lemma. So in fact $\mathbf{u}_\varepsilon$ is a minimizer of J_ε over some V_δ, and therefore it has all the properties of $\mathbf{u}_\varepsilon^\delta$'s that we have proved so far. In particular, we have $|\mathbf{u}_\varepsilon| > 0$ on Ω_{δ_0}.

Proof. As we have shown in the proof of Corollary 2.8, $\|\nabla \mathbf{u}_\varepsilon^\delta\|_{L^p(\Omega;\mathbb{R}^m)}$ is bounded as $\delta \to 0$. Hence there is a subsequence such that $\mathbf{u}_\varepsilon^\delta \rightharpoonup \mathbf{u}_\varepsilon$ weakly in $W^{1,p}$ (and also a.e.) with $A(\nabla(u_\varepsilon^\delta)^i) \rightharpoonup A(\nabla u_\varepsilon^i)$ in L^q as $\delta \to 0$. So, in particular, $u_\varepsilon^i \geq 0$, u_ε^i is p-subharmonic, and attains the boundary condition φ^i. Furthermore, by Corollary 2.8, $\mathbf{u}_\varepsilon^\delta \to \mathbf{u}_\varepsilon$ uniformly on compact subsets of Ω. Hence for each ball $\overline{B} \subset \{|\mathbf{u}_\varepsilon| > 0\}$ and all small enough δ we have $\overline{B} \subset \{|\mathbf{u}_\varepsilon^\delta| > 0\}$. Therefore by using test functions with support in B together with $A(\nabla(u_\varepsilon^\delta)^i) \rightharpoonup A(\nabla u_\varepsilon^i)$ we can conclude that u_ε^i is p-harmonic in B.

The same reasoning applied to test functions with support in Ω_{δ_0}, for δ_0 given by the previous lemma, implies that u_ε^i is p-harmonic in Ω_{δ_0}, and thus $\mathbf{u}_\varepsilon \in V_{\delta_0} \subset V$. In particular, u_ε^i is p-harmonic near the smooth boundary $\partial\Omega$, attaining smooth boundary conditions φ^i, so it is Lipschitz near $\partial\Omega$. Moreover, $\mathbf{u}_\varepsilon$ is Lipschitz inside Ω away from its boundary, because it is the uniform limit of a sequence of Lipschitz functions with uniformly bounded Lipschitz constants. Hence $\mathbf{u}_\varepsilon$ is Lipschitz on all of Ω.

Finally, note that $\mathbf{u}_\varepsilon$ minimizes J_ε over V, since for every $\mathbf{w} \in V$ we have $\mathbf{w} \in V_\delta$ for some δ. Thus we obtain $J_\varepsilon(\mathbf{u}_\varepsilon^\delta) \leq J_\varepsilon(\mathbf{w})$. However, $\mathbf{u}_\varepsilon^\delta \to \mathbf{u}_\varepsilon$, so we get $J_\varepsilon(\mathbf{u}_\varepsilon) \leq J_\varepsilon(\mathbf{w})$ due to the semicontinuity of J_ε. ☐

3 Regularity of solutions to the penalized problem

To simplify the notation, throughout this section we will suppress the index ε in $\mathbf{u}_\varepsilon$.

Theorem 3.1. *For $\tau \in (0, \frac{1}{4})$ there exists $m_\varepsilon(\tau)$ such that if for each i we have*

$$\frac{1}{r}\sup_{B_{r/2}(x)} u^i \le m_\varepsilon(\tau),$$

then $B_{\tau r}(x) \subset \{|\mathbf{u}| = 0\}$.

Proof. Similarly to Lemma 2.6, we can show that there is $v^i \in W^{1,p}(\Omega)$ that minimizes the functional $\int_\Omega |\nabla v^i|^p\,dx$ among all functions with $v^i = \varphi^i$ on $\partial\Omega$ and $v^i \le 0$ on $\{u^i = 0\} \cup \overline{B}_{\tau r}(x)$. The function v^i also satisfies

$$\Delta_p v^i \ge 0, \qquad \int_\Omega v^i\,\Delta_p v^i\,dx = 0, \qquad u^i \ge v^i \ge 0$$

(to see this, note that $\Delta_p v^i \ge \Delta_p u^i$ on $\Omega - (\{u^i = 0\} \cup \overline{B}_{\tau r}(x)) \subset \{|\mathbf{u}| > 0\}$, and $v^i - u^i \le 0$ on $\{u^i = 0\} \cup \overline{B}_{\tau r}(x)$ or $\partial\Omega$). In addition, we have $\mathbf{v} = (v_1, \dots, v_m) \in V_{\delta_1} \subset V$ (where δ_1 is small enough so that $\overline{B}_{\tau r}(x) \subset \Omega - \Omega_{\delta_1}$). Thus $J_\varepsilon(\mathbf{u}) \le J_\varepsilon(\mathbf{v})$. Let us assume that δ_1 is small enough so that $\overline{B}_r(x) \subset \Omega - \Omega_{\delta_1}$ and $\mathbf{u} \in V_{\delta_1}$. Let $\mathbf{w}$ be a vector-valued p-harmonic function in Ω_{δ_1} with boundary values equal to $\boldsymbol{\varphi}$ on $\partial\Omega$ and equal to 0 on $\partial\Omega_{\delta_1} - \partial\Omega$. Then we have $u^i \ge v^i \ge w^i \ge 0$ (since $\mathbf{u}, \mathbf{v}$ are also p-harmonic on Ω_{δ_1}, and nonnegative everywhere). Thus for each $z \in \partial\Omega$ we have

$$0 \ge \partial_\nu w^i(z) \ge \partial_\nu v^i(z) \ge \partial_\nu u^i(z).$$

Next using the fact that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ take the same boundary values on $\partial\Omega$, and therefore have equal tangential derivatives on $\partial\Omega$, we deduce that

$$0 \ge A_\nu w^i(z) \ge A_\nu v^i(z) \ge A_\nu u^i(z).$$

Now similar to (2.5) we can show that

$$\int_{\partial\Omega} \Gamma(x, A_\nu\mathbf{v}(x)) - \Gamma(x, A_\nu\mathbf{u}(x))\,d\sigma \le C_1 \sum_{i=1}^m \int_{\partial\Omega} A_\nu v^i - A_\nu u^i\,d\sigma,$$

where $C_1 > 0$ is the upper bound of $\partial_{\xi_i}\Gamma$'s on the set $\{(x,\xi) : \xi_i \le 0\}$. On the other hand, using the identity (2.3) we obtain (using the notation $c_0 = \min_i \min_{\partial\Omega} \varphi^i$)

$$\begin{aligned}
c_0 \int_{\partial\Omega} A_\nu v^i - A_\nu u^i\,d\sigma &\le \int_{\partial\Omega} \varphi^i (A_\nu v^i - A_\nu u^i)\,d\sigma \\
&= \int_\Omega v^i\,\Delta_p v^i + |\nabla v^i|^p\,dy - \int_\Omega u^i\,\Delta_p u^i + |\nabla u^i|^p\,dy \\
&= \int_\Omega |\nabla v^i|^p\,dy - \int_\Omega |\nabla u^i|^p\,dy,
\end{aligned} \tag{3.1}$$

where in the last line we used the facts that $\int_\Omega v^i\,\Delta_p v^i\,dy = 0$, and $\Delta_p u^i = 0$ on $\{u^i \ne 0\} \subset \{|\mathbf{u}| > 0\}$.

Summing the above inequality for each i, and using the facts that $J_\varepsilon(\mathbf{u}) \le J_\varepsilon(\mathbf{v})$, and the derivative of f_ε is bounded below by ε, we get

$$\begin{aligned}
\frac{C_1}{c_0} \sum_{i \le m} \int_\Omega |\nabla v^i|^p - |\nabla u^i|^p\,dy &\ge C_1 \int_{\partial\Omega} \sum_{i \le m} (A_\nu v^i - A_\nu u^i)\,d\sigma \\
&\ge \int_{\partial\Omega} \Gamma(x, A_\nu\mathbf{v}(x)) - \Gamma(x, A_\nu\mathbf{u}(x))\,d\sigma \\
&\ge f_\varepsilon(|\{|\mathbf{u}| > 0\}|) - f_\varepsilon(|\{|\mathbf{v}| > 0\}|) \\
&\ge \varepsilon|\{|\mathbf{u}| > 0\} \cap \{|\mathbf{v}| = 0\}| \\
&\ge \varepsilon|\{|\mathbf{u}| > 0\} \cap B_{\tau r}(x)|,
\end{aligned} \tag{3.2}$$

since $u^i \ge v^i \ge 0$ and $v^i = 0$ in $B_{\tau r}(x)$.

Next we define $g:(0,\infty)\to\mathbb{R}$ by

$$g(t):=\begin{cases} t^{\frac{p-n}{p-1}}-(\tau r)^{\frac{p-n}{p-1}}, & p>n,\\ \log t-\log(\tau r), & p=n,\\ (\tau r)^{\frac{p-n}{p-1}}-t^{\frac{p-n}{p-1}}, & p<n.\end{cases}$$

Note that g is an increasing function that vanishes at $t=\tau r$, and is negative for $t<\tau r$. In addition, $g(|x|)$ is a p-harmonic function in $\mathbb{R}^n-\{0\}$, which is negative on $B_{\tau r}(x)$ and vanishes on $\partial B_{\tau r}(x)$. Now let us define $h^i: B_{\sqrt{\tau}r}(x)\to\mathbb{R}$ by

$$h^i(y):=\min\Big\{u^i(y),\frac{s_i}{g(\sqrt{\tau}r)}(g(|y-x|))^+\Big\},$$

where $s_i:=\max_{\overline{B}_{\sqrt{\tau}r}(x)}u^i$. We extend h^i by u^i outside of $B_{\sqrt{\tau}r}(x)$. Note that we have $h^i=0$ on $\{u^i=0\}\cap\overline{B}_{\tau r}(x)$ and $h^i=u^i=\varphi^i$ on $\partial\Omega$. Hence h^i competes with v^i, and we have $\int_\Omega|\nabla v^i|^p\,dx\le\int_\Omega|\nabla h^i|^p\,dx$. Therefore we can exchange v^i by h^i in inequality (3.2) to get

$$\frac{\varepsilon c_0}{C_1}|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)|\le\sum_{i\le m}\int\limits_{B_{\sqrt{\tau}r}(x)}|\nabla h^i|^p-|\nabla u^i|^p\,dy.$$

Now since $h^i=0$ on $B_{\tau r}(x)$, we can rewrite the above inequality as

$$\frac{\varepsilon c_0}{C_1}|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)|+\sum_{i\le m}\int\limits_{B_{\tau r}(x)}|\nabla u^i|^p\,dy\le\sum_{i\le m}\int\limits_{B_{\sqrt{\tau}r}(x)-B_{\tau r}(x)}|\nabla h^i|^p-|\nabla u^i|^p\,dy. \tag{3.3}$$

But

$$|\nabla h^i|^p-|\nabla u^i|^p\le -p|\nabla h^i|^{p-2}\nabla h^i\cdot\nabla(u^i-h^i),$$

since for two vectors a,b we have $|a|^p-|b|^p\le -p|a|^{p-2}a\cdot(b-a)$ due to the convexity of the function $\cdot\mapsto|\cdot|^p$ (see for example [12]). So we can estimate the right-hand side of (3.3) as follows (using integration by parts, and the facts that $\Delta_p h^i=0$ on $\{u^i>h^i\}$, $h^i=0$ on $\partial B_{\tau r}(x)$, and $h^i=u^i$ on $\partial B_{\sqrt{\tau}r}(x)$):

$$\begin{aligned}\int\limits_{B_{\sqrt{\tau}r}(x)-B_{\tau r}(x)}|\nabla h^i|^p-|\nabla u^i|^p\,dy&\le -p\int\limits_{B_{\sqrt{\tau}r}(x)-B_{\tau r}(x)}|\nabla h^i|^{p-2}\nabla(u^i-h^i)\cdot\nabla h^i\,dy\\ &=p\int\limits_{\partial B_{\tau r}(x)}(u^i-h^i)|\nabla h^i|^{p-2}\nabla h^i\cdot\nu\,d\sigma-p\int\limits_{\partial B_{\sqrt{\tau}r}(x)}(u^i-h^i)|\nabla h^i|^{p-2}\nabla h^i\cdot\nu\,d\sigma\\ &=p\int\limits_{\partial B_{\tau r}(x)}u^i|\nabla h^i|^{p-2}\nabla h^i\cdot\nu\,d\sigma\\ &=C(n,p,\tau)\frac{s_i^{p-1}}{r^{p-1}}\int\limits_{\partial B_{\tau r}(x)}u^i\,d\sigma,\end{aligned}$$

where the last equality is calculated using the fact $h^i(y)=\frac{s_i}{g(\sqrt{\tau}r)}(g(|y-x|))^+=0$ on $\overline{B}_{\tau r}(x)$; hence on $\partial B_{\tau r}(x)$ we have

$$\nabla h^i=C(\tau)s_i r^{\frac{n-p}{p-1}}\begin{cases}\dfrac{2|p-n|}{p-1}|y-x|^{\frac{2-n-p}{p-1}}(y-x), & p\ne n,\\ |y-x|^{-2}(y-x), & p=n,\end{cases}$$

and thus

$$\begin{aligned}|\nabla h^i|^{p-2}\nabla h^i\cdot\nu&=C(n,p,\tau)s_i^{p-1}r^{n-p}|y-x|^{2-n-p}|y-x|^{p-2}(y-x)\cdot\frac{(y-x)}{\tau r}\\ &=C(n,p,\tau)s_i^{p-1}r^{n-p}|y-x|^{-n+2}\frac{1}{\tau r}=C(n,p,\tau)\frac{s_i^{p-1}}{r^{p-1}}.\end{aligned}$$

Hence (3.3) becomes

$$\frac{\varepsilon c_0}{C_1}|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)|+\sum_{i\le m}\int\limits_{B_{\tau r}(x)}|\nabla u^i|^p\,dy\le C(n,p,\tau)\sum_{i\le m}\frac{s_i^{p-1}}{r^{p-1}}\int\limits_{\partial B_{\tau r}(x)}u^i\,d\sigma. \tag{3.4}$$

On the other hand we have

$$\int_{\partial B_{\tau r}(x)} u^i \, d\sigma \le c(n,\tau)\Big(\int_{B_{\tau r}(x)} u^i \, dy + \int_{B_{\tau r}(x)} |\nabla u^i| \, dy\Big)$$
$$\le c(n,\tau)\Big((s_i+1)\cdot|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)| + \int_{B_{\tau r}(x)} |\nabla u^i|^p \, dy\Big), \tag{3.5}$$

where in the last line we estimated u^i, $|\nabla u^i|$ from above by s_i, $1+|\nabla u^i|^p$ on the set $\{u^i>0\}\subset\{|\mathbf{u}|>0\}$. Next note that

$$s_i = \max_{\overline{B}_{\sqrt{\tau}r}(x)} u^i \le \sup_{B_{r/2}(x)} u^i \le r m_\varepsilon(\tau), \tag{3.6}$$

since $\sqrt{\tau} < \frac{1}{2}$. Combining inequalities (3.4), (3.5), and (3.6), we get

$$\frac{\varepsilon c_0}{C_1}|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)| + \sum_{i\le m}\int_{B_{\tau r}(x)} |\nabla u^i|^p \, dy \le cC\sum_{i\le m}\frac{s_i^{p-1}}{r^{p-1}}\Big((s_i+1)\cdot|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)| + \int_{B_{\tau r}(x)} |\nabla u^i|^p \, dy\Big)$$
$$\le cC\, m_\varepsilon^{p-1}(\tau)\Big(|\{|\mathbf{u}|>0\}\cap B_{\tau r}(x)|\sum_{i\le m}(s_i+1) + \sum_{i\le m}\int_{B_{\tau r}(x)} |\nabla u^i|^p \, dy\Big).$$

Now if $m_\varepsilon(\tau)$ is small enough, we must necessarily have $|\mathbf{u}|=0$ on $B_{\tau r}(x)$, as desired. □

Now let us set

$$U := \{x\in\Omega : |\mathbf{u}(x)|>0\},$$
$$E := \{x\in\Omega : |\mathbf{u}(x)|=0\}.$$

Lemma 3.2. *For every i we have*

$$U = \{x\in\Omega : u^i(x)>0\}, \quad E = \{x\in\Omega : u^i(x)=0\}.$$

Proof. By Theorem 2.10, each u^i is p-harmonic in the open set U. So in each component of U either $u^i>0$ or $u^i\equiv 0$ (by the strong maximum principle). Now consider a component of U, say U_1. If ∂U_1 does not intersect $\partial\Omega$, then it must be a subset of E. Therefore every u^i vanishes on ∂U_1, and hence every u^i vanishes on U_1 by the maximum principle. So we would have $U_1\subset E$, which is a contradiction. Thus ∂U_1 must intersect $\partial\Omega$. Hence each $u^i>0$ on U_1, since they are positive on $\partial\Omega$. Therefore each u^i is positive on every component of U, as desired. □

Corollary 3.3. *There are $c, C>0$ such that for $x\in U$ near ∂E we have*

$$c\cdot \operatorname{dist}(x,\partial E) \le |\mathbf{u}(x)| \le C\cdot\operatorname{dist}(x,\partial E).$$

Proof. The right-hand side inequality holds according to the Lipschitz regularity of the solutions, Theorem 2.10. To see the left-hand side inequality, we argue indirectly. Assume to the contrary that there exists a sequence $x_k\in U$ such that

$$|\mathbf{u}(x_k)| \le \frac{1}{k}\operatorname{dist}(x_k,\partial E). \tag{3.7}$$

Let $r_k=\operatorname{dist}(x_k,\partial E)$ and define

$$\mathbf{u}_k(x) = \frac{\mathbf{u}(x_k+r_k x)}{r_k}.$$

The sequence $\mathbf{u}_k$ is uniformly bounded and uniformly Lipschitz in B_1 due to Lipschitz regularity of $\mathbf{u}$ and assumption (3.7).

Recall that $\Delta_p u_k^i = 0$ in U, then we may choose a converging subsequence $\mathbf{u}_k\to\mathbf{u}_0$ such that u_0^i is also p-harmonic. Furthermore, by Theorem 3.1 we get that

$$\sup_{B_{1/2}(0)} |\mathbf{u}_0| = \lim_{k\to\infty}\sup_{B_{1/2}(0)} |\mathbf{u}_k| \ge m_\varepsilon > 0,$$

since $|\mathbf{u}_k(0)|>0$. Also, (3.7) yields that $\mathbf{u}_0(0)=0$, which contradicts the maximum (minimum) principle; remember that each component of $\mathbf{u}_0$ is nonnegative. □

Corollary 3.4. *There exists $c = c_\varepsilon \in (0, 1)$ such that for any $x \in \partial U$ and small enough r we have*

$$c \le \frac{|E \cap B_r(x)|}{|B_r(x)|} \le 1 - c. \tag{3.8}$$

Proof. The proof is similar to the proof of [9, Theorem 4.2]. By Theorem 3.1, there exists $z \in B_{r/2}(x)$ such that $|\mathbf{u}(z)| \ge m_\varepsilon r > 0$. Now for any $y \in B_{\tau r}(z)$ we have

$$|\mathbf{u}(y) - \mathbf{u}(z)| \le \operatorname{Lip}(\mathbf{u})|y - z| < \operatorname{Lip}(\mathbf{u})\tau r < \frac{m_\varepsilon r}{2},$$

provided that τ is small enough. Hence we must have $|\mathbf{u}(y)| > \frac{m_\varepsilon r}{2} > 0$. This gives the upper estimate in (3.8).

To prove the estimate from below, suppose to the contrary that there exists a sequence of points $x_k \in \partial U$ and radii $r_k \to 0$ such that

$$|\{|\mathbf{u}| = 0\} \cap B_{r_k}(x_k)| < \frac{1}{k}|B_{r_k}(x_k)| = \frac{1}{k} r_k^n |B_1|.$$

Now let us define

$$\mathbf{u}_k(x) = \frac{\mathbf{u}(x_k + r_k x)}{r_k}.$$

Note that $\mathbf{u}_k(0) = \mathbf{u}(x_k) = 0$, and thus $\mathbf{u}_k$ is uniformly bounded and uniformly Lipschitz in $B_1 = B_1(0)$ due to Lipschitz regularity of $\mathbf{u}$. Also

$$|\{|\mathbf{u}_k| = 0\} \cap B_1| = \frac{1}{r_k^n}|\{|\mathbf{u}| = 0\} \cap B_{r_k}(x_k)| \xrightarrow[k\to\infty]{} 0.$$

Let v_k^i be a p-harmonic function in $B_{1/2}$ with boundary data $v_k^i = u_k^i$ on $\partial B_{1/2}$. Then $h_k^i(x) = r_k v_k^i(\frac{x - x_k}{r_k})$ is a p-harmonic function in $B_{r_k/2}(x_k)$ with boundary data $h_k^i = u^i$ on $\partial B_{r_k/2}(x_k)$. Now, similarly to the proof of Theorem 2.7, we can show that

$$\begin{aligned}\int_{B_{1/2}} |\nabla(u_k^i - v_k^i)|^p \, dx &= \frac{1}{r_k^n} \int_{B_{r_k/2}(x_k)} |\nabla(u^i - h_k^i)|^p \, dx \\ &\le \frac{C}{\varepsilon}\frac{1}{r_k^n}|\{|\mathbf{u}| = 0\} \cap B_{r_k}(x_k)| \xrightarrow[k\to\infty]{} 0.\end{aligned} \tag{3.9}$$

(Note that the constant C does not depend on the radius r_k or the point x_k.)

Since u_k^i and therefore v_k^i are uniformly Lipschitz in $B_{1/4}$, we may assume that $u_k^i \to u_0^i$ and $v_k^i \to v_0^i$ uniformly in $B_{1/4}$. Observe that $\Delta_p v_0^i = 0$, and (3.9) implies that $u_0^i = v_0^i + C$ for some constant C. Thus $\Delta_p u_0^i = 0$ in $B_{1/4}$ and from the strong minimum principle it follows $u_0^i \equiv 0$ in $B_{1/4}$, since $u_0^i \ge 0$ and $u_0^i(0) = \lim u_k^i(0) = 0$. On the other hand the nondegeneracy property, Theorem 3.1, implies that (since x_k is not in the interior of $\{|\mathbf{u}| = 0\}$)

$$\|\mathbf{u}_k\|_{L^\infty(B_{1/4})} = \frac{1}{r_k}\|\mathbf{u}\|_{L^\infty(B_{r_k/4}(x_k))} \ge \frac{m_\varepsilon}{2} > 0.$$

Therefore we get $\|\mathbf{u}_0\|_{L^\infty(B_{1/4})} \ge m_\varepsilon/2$, which is a contradiction. □

Hence we can apply the results in [4, Section 4] and in [5, Section 3] to conclude (see also [9, Sections 5 and 6])

Theorem 3.5. *Let $\mathbf{u} = \mathbf{u}_\varepsilon$ be a minimizer of J_ε over V. Then we have:*

(1) *The $(n-1)$-dimensional Hausdorff measure of ∂E is locally finite, i.e. $\mathcal{H}^{n-1}(\Omega' \cap \partial E) < \infty$ for every $\Omega' \subset\subset \Omega$. Moreover, there exist positive constants c_ε, C_ε, depending on $n, p, \Omega, \Omega', \varepsilon$, such that for each ball $B_r(x) \subset \Omega'$ with $x \in \partial E$ we have*

$$c_\varepsilon r^{n-1} \le \mathcal{H}^{n-1}(B_r(x) \cap \partial E) \le C_\varepsilon r^{n-1}.$$

(2) *There exist Borel functions $q^i = q_\varepsilon^i$ such that*

$$\Delta_p u^i = q^i \mathcal{H}^{n-1} \llcorner \partial E,$$

that is, for any $\zeta \in C_0^\infty(\Omega)$ we have

$$-\int_\Omega A[u^i] \cdot \nabla\zeta \, dy = \int_{\partial E} \zeta q^i \, d\mathcal{H}^{n-1}.$$

(3) *For $\mathcal{H}^{n-1}$-a.e. points $x \in \partial E$ we have*

$$c_\varepsilon \le \sum_{i=1}^{m} q^i(x) \le C_\varepsilon.$$

(4) *For $\mathcal{H}^{n-1}$-a.e. points $x \in \partial E$ an outward unit normal $\nu = \nu_E(x)$ is defined, and*

$$u^i(x+y) = (q^i(x))^{\frac{1}{p-1}}(y \cdot \nu)^+ + o(|y|),$$

which allows us to define $A_\nu u^i(x) = q^i(x)$ at those points.

(5) *The reduced boundary $\partial_{\mathrm{red}} E$ satisfies $\mathcal{H}^{n-1}(\partial E - \partial_{\mathrm{red}} E) = 0$.*

4 The original problem

In this section we will show that for $\varepsilon > 0$ small enough, a minimizer of J_ε over V satisfies $|\{|\mathbf{u}_\varepsilon| > 0\}| = 1$, and hence it can be regarded as a solution to our original problem (1.1). Remember that

$$U = U_\varepsilon = \{|\mathbf{u}_\varepsilon| > 0\}, \quad E = E_\varepsilon = \{|\mathbf{u}_\varepsilon| = 0\}.$$

Note that by Lemma 2.9, the free boundary ∂E has a positive distance from the fixed boundary $\partial\Omega$. We say $x \in \partial E$ is a *regular point* of the free boundary if it satisfies (3) and (4) in Theorem 3.5. The set of such regular points of the free boundary will be denoted by $\mathcal{R} = \mathcal{R}_\varepsilon$; Theorem 3.5 shows that $\mathcal{H}^{n-1}(\partial E - \mathcal{R}) = 0$.

Lemma 4.1. *There is a constant $C > 0$, independent of ε, such that*

$$\inf_{\mathcal{R}_\varepsilon}\Big(\sum_{i\le m} q^i_\varepsilon\Big) \le C.$$

Remark. Note that $\sum_{i\le m} q^i_\varepsilon \ge c_\varepsilon > 0$ by Theorem 3.5.

Proof. Let $\Omega' \subset\subset \Omega$ be a smooth open set with $|\Omega - \Omega'| = 1$. Let $\mathbf{u}_0$ be a vector-valued function on $\Omega - \Omega'$ that satisfies the equation $\Delta_p u^i_0 = 0$, and takes the boundary values $\boldsymbol{\varphi}$ on $\partial\Omega$ and 0 on $\partial\Omega'$. Then for some small enough δ_0 we have $\mathbf{u}_0 \in V_{\delta_0} \subset V$; hence

$$\begin{aligned}
C = \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}_0)\, d\sigma + 1 = J_\varepsilon(\mathbf{u}_0) \ge J_\varepsilon(\mathbf{u}_\varepsilon)
&= \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}_\varepsilon)\, d\sigma + f_\varepsilon(|\{|\mathbf{u}_\varepsilon| > 0\}|) \\
&\ge \int_{\partial\Omega} \sum_{i=1}^{m} \psi_i A_\nu u^i_\varepsilon - C\, d\sigma + f_\varepsilon(|\{|\mathbf{u}_\varepsilon| > 0\}|) \\
&\ge C \sum_{i=1}^{m} \int_{\Omega} |\nabla u^i_\varepsilon|^p\, dx - C + f_\varepsilon(|\{|\mathbf{u}_\varepsilon| > 0\}|) \\
&\ge -C + \frac{1}{\varepsilon}(|\{|\mathbf{u}_\varepsilon| > 0\}| - 1),
\end{aligned}$$

where we have used (1.2) and Lemma 2.2. Thus we get the bound

$$|U| = |\{|\mathbf{u}_\varepsilon| > 0\}| \le 1 + C\varepsilon.$$

Note that $J_\varepsilon(\mathbf{u}_0)$, and thus C, does not depend on ε due to the definition of f_ε. As a result, we have a lower bound for the volume of E. Hence, by the isoperimetric inequality, we have a lower bound for $\mathcal{H}^{n-1}(\partial E)$, independent of ε. Now note that (keep in mind that ν_E points to the interior of U)

$$\int_{\partial\Omega} A_\nu u^i_\varepsilon\, d\sigma - \int_{\partial E} A_\nu u^i_\varepsilon\, d\mathcal{H}^{n-1} = \int_U \Delta_p u^i_\varepsilon\, dx = 0.$$

Therefore we get

$$\int_{\partial E} A_\nu u_\varepsilon^i \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} A_\nu u_\varepsilon^i \, d\sigma = \int_{\Omega} \Delta_p u_\varepsilon^i \, dx \le C + C \int_{\partial\Omega} \psi_i A_\nu u_\varepsilon^i \, d\sigma,$$

where the last inequality follows from the remark below Lemma 2.2. Thus we have

$$\begin{aligned}\inf_{\mathcal{R}_\varepsilon}\Big(\sum_{i\le m} A_\nu u_\varepsilon^i\Big)\mathcal{H}^{n-1}(\partial E) &\le \int_{\partial E} \sum_{i\le m} A_\nu u_\varepsilon^i \, d\mathcal{H}^{n-1} \\ &\le C + C\int_{\partial\Omega} \sum_{i\le m} \psi_i A_\nu u_\varepsilon^i \, d\sigma \\ &\le C + C\int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}_\varepsilon)\, d\sigma \quad \text{(by (1.2))} \\ &\le C + CJ_\varepsilon(\mathbf{u}_\varepsilon) \le C + CJ_\varepsilon(\mathbf{u}_0) \le C,\end{aligned}$$

which gives the desired (noting that $q^i = A_\nu u_\varepsilon^i$ by Theorem 3.5). □

Lemma 4.2. *For small enough ε we have*

$$|\{|\mathbf{u}_\varepsilon| > 0\}| \ge 1.$$

Proof. Consider a point $z_0 \in \Omega^c$ which has distance δ_0 from $\partial\Omega$. Then the ball $B_{\delta_0}(z_0)$ is an exterior tangent ball to $\partial\Omega$. Let $t = t(\varepsilon)$ be the first time at which $\partial B_{\delta_0+t}(z_0)$ intersects $\partial\{|\mathbf{u}_\varepsilon| = 0\}$, at a point $x_0 = x_0(\varepsilon)$. Now let v be a p-harmonic function in $B_{\delta_0+t}(z_0) - \overline{B}_{\delta_0}(z_0)$ with boundary values 0 on $\partial B_{\delta_0+t}(z_0)$ and c_0 on $\partial B_{\delta_0}(z_0)$, where $c_0 = \min_i \min_{\partial\Omega} \varphi^i > 0$. Then on $\partial(\Omega \cap B_{\delta_0+t}(z_0))$ we have $v \le u^i$; so by the maximum principle we have $v \le u^i$ in $\Omega \cap B_{\delta_0+t}(z_0)$. However, by an easy modification of the proof of Hopf's lemma (Lemma 2.4), we can see that

$$v(x) \ge cc_0 \operatorname{dist}(x, \partial B_{\delta_0+t}(z_0)),$$

where the constant c only depends on n, p, δ_0. Therefore, for points x in the line segment between x_0, z_0 we have

$$u^i(x) \ge v(x) \ge cc_0 \operatorname{dist}(x, \partial B_{\delta_0+t}(z_0)) = cc_0|x - x_0|.$$

Now consider the ball $B_r(x_0)$ for small enough r. Then we have

$$\frac{1}{r}\sup_{B_{r/2}(x_0)} u^i \ge \frac{1}{r} cc_0 \frac{r}{2} = \frac{cc_0}{2},$$

independently of ε.

Let $\mathbf{h}$ be the vector-valued function which satisfies $\Delta_p h^i = 0$ in $B_r(x_0)$, and is equal to $\mathbf{u}$ in $\Omega - B_r(x_0)$. By Lemma 2.5 and the fact that $h^i \ge u^i$ we have

$$\begin{aligned}\int_{B_r(x_0)} |\nabla(u^i - h^i)|^p \, dy &\ge C\Big(\frac{1}{r}\sup_{B_{r/2}(x_0)} h^i\Big)^p \cdot |B_r(x_0) \cap \{u^i = 0\}| \\ &\ge C\Big(\frac{1}{r}\sup_{B_{r/2}(x_0)} u^i\Big)^p \cdot |B_r(x_0) \cap \{u^i = 0\}| \\ &\ge C|B_r(x_0) \cap \{u^i = 0\}| \ge C|B_r(x_0) \cap \{|\mathbf{u}| = 0\}|.\end{aligned}$$

Next let $\mathbf{v}$ be the function given by Lemma 2.6 for $B_r(x_0)$. We know that $J_\varepsilon(\mathbf{u}) \le J_\varepsilon(\mathbf{v})$. Then similarly to the proof of Theorem 2.7 we can see that

$$\begin{aligned}C\sum_{i\le m}\int_{B_r(x_0)} |\nabla(u^i - h^i)|^p \, dy &\le \int_{\partial\Omega} \Gamma(x, A_\nu \mathbf{u}) - \Gamma(x, A_\nu \mathbf{v})\, d\sigma \\ &\le f_\varepsilon(|\{|\mathbf{v}| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|).\end{aligned}$$

A closer inspection of the proof of Theorem 2.7 reveals that the constant C in the above estimate only depends on $n, p, \Omega, \boldsymbol{\varphi}, \Gamma$.

Now suppose to the contrary that $|\{|\mathbf{u}| > 0\}| < 1$. Then, since $0 \le u^i \le v^i$, and outside of $B_r(x_0)$, $|\mathbf{u}| = 0$ implies $|\mathbf{v}| = 0$, we have

$$|\{|\mathbf{v}| > 0\}| \le |\{|\mathbf{u}| > 0\}| + |B_r(x_0) \cap \{|\mathbf{u}| = 0\}| < 1$$

for small enough r. Hence, using the monotonicity of f_ε, we have

$$\begin{aligned} f_\varepsilon(|\{|\mathbf{v}| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|) &\le f_\varepsilon\Big(|\{|\mathbf{u}| > 0\}| + |B_r(x_0) \cap \{|\mathbf{u}| = 0\}|\Big) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|) \\ &= \varepsilon|B_r(x_0) \cap \{|\mathbf{u}| = 0\}|. \end{aligned}$$

Combining this estimate with the estimates of the above paragraph, and using (3.8), we obtain

$$0 < C|B_r(x_0) \cap \{|\mathbf{u}| = 0\}| \le \varepsilon|B_r(x_0) \cap \{|\mathbf{u}| = 0\}|,$$

which gives a positive lower bound for ε, and results in a contradiction. □

Theorem 4.3. *When ε is small enough, we have*

$$|\{|\mathbf{u}_\varepsilon| > 0\}| = 1.$$

Proof. By the above lemma we only need to show that $|\{|\mathbf{u}_\varepsilon| > 0\}| \le 1$. To this end, we will compare $\mathbf{u}_\varepsilon$ with a suitable perturbation of itself. Let $x_0 \in \mathcal{R}$, and let $\rho : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function supported in $(0, 1)$. For small enough $r, \lambda > 0$ we consider the vector field

$$T_r(x) := \begin{cases} x + r\lambda\rho\Big(\dfrac{|x - x_0|}{r}\Big)\nu(x_0) & \text{if } x \in B_r(x_0), \\ x & \text{elsewhere.} \end{cases}$$

Here, $\nu(x_0)$ is the outward normal vector provided in (4) of Theorem 3.5. We can easily see that for x in $B_r(x_0)$ we have

$$DT_r(x) \cdot = I \cdot + \lambda\rho'\Big(\frac{|x - x_0|}{r}\Big)\frac{\langle x - x_0, \cdot\rangle}{|x - x_0|}\nu(x_0), \tag{4.1}$$

where I is the identity matrix. Hence, if λ is small enough, T_r is a diffeomorphism that maps $B_r(x_0)$ onto itself.

Now consider

$$\mathbf{v}_r(x) := \mathbf{u}(T_r^{-1}(x))$$

for $r > 0$ small enough. Similarly to the proof of Theorem 3.1, we consider the vector-valued function $\mathbf{w}$ whose components minimize the Dirichlet p-energy subject to the condition

$$w^i \le 0 \quad \text{on } \{\mathbf{u} = 0\} \cup (\overline{B}_r(x_0) \cap \{\mathbf{v}_r = 0\}).$$

With a calculation similar to (3.1) and (2.5) we get

$$\begin{aligned} 0 \le J_\varepsilon(\mathbf{w}) - J_\varepsilon(\mathbf{u}) &\le C\sum_{i=1}^m \int_\Omega |\nabla w^i|^p - |\nabla u^i|^p \, dx + f_\varepsilon(|\{|\mathbf{w}| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|) \\ &\le C\sum_{i=1}^m \int_{B_r(x_0)} |\nabla v_r^i|^p - |\nabla u^i|^p \, dx + f_\varepsilon(|\{|\mathbf{w}| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|), \end{aligned} \tag{4.2}$$

where in the last inequality we have compared the Dirichlet p-energy of $\mathbf{w}$ with that of $\mathbf{v}_r\chi_{B_r(x_0)} + \mathbf{u}\chi_{\Omega - B_r(x_0)}$.

Now notice that

$$\begin{aligned} \int_{B_r(x_0)} |\nabla v_r^i|^p \, dx &= \int_{B_r(x_0)} |DT_r(T_r^{-1}(x))^{-1}\nabla u^i(T_r^{-1}(x))|^p \, dx \\ &= \int_{B_r(x_0)} |DT_r(y)^{-1}\nabla u^i(y)|^p |\det DT_r(y)| \, dy \\ &= r^n \int_{B_1} |DT_r(y)^{-1}\nabla u^i(y)|^p |\det DT_r(y)| \, dz, \quad z = \frac{y - x_0}{r}. \end{aligned}$$

From (4.1), for small enough λ we can write

$$
\begin{aligned}
DT_r(y)^{-1} &= I + \bigg(\sum_{k=1}^{\infty} (-1)^k \lambda^k \rho'(|z|)^k \frac{\langle z, \nu\rangle^{k-1}}{|z|^{k-1}} \bigg) \frac{\langle z, \cdot\rangle}{|z|} \nu(x_0) \\
&= I - \lambda\rho'(|z|) \frac{\langle z, \cdot\rangle}{|z|} \nu(x_0) + \lambda^2 g(\lambda, z) \frac{\langle z, \cdot\rangle}{|z|} \nu(x_0)
\end{aligned}
\tag{4.3}
$$

for some g. Hence we have

$$DT_r(y)^{-1} \nabla u^i(y) = \nabla u^i(y) - \lambda\rho'(|z|) \frac{\langle z, \nabla u^i(y)\rangle}{|z|} \nu(x_0) + O(\lambda^2).$$

Thus

$$|DT_r(y)^{-1} \nabla u^i(y)|^2 = |\nabla u^i(y)|^2 - 2\lambda\rho'(|z|) \frac{\langle z, \nabla u^i(y)\rangle}{|z|} \langle \nu(x_0), \nabla u^i(y)\rangle + O(\lambda^2),$$

and therefore

$$|DT_r(y)^{-1} \nabla u^i(y)|^p = |\nabla u^i(y)|^p \bigg(1 - p\lambda\rho'(|z|) \frac{\langle z, \nabla u^i(y)\rangle}{|z||\nabla u^i(y)|^2} \langle \nu(x_0), \nabla u^i(y)\rangle \bigg) + O(\lambda^2).$$

Also, we have (noting that DT_r is the identity matrix plus a rank 1 matrix)

$$|\det DT_r(y)| = 1 + \lambda\rho'(|z|) \frac{\langle z, \nu(x_0)\rangle}{|z|}.$$

All these together, we obtain (remember that $y = x_0 + rz$)

$$r^{-n} \int_{B_r(x_0)} |\nabla v_r^i|^p - |\nabla u^i|^p \, dx = \lambda \int_{B_1} |\nabla u^i(y)|^p \rho'(|z|) \bigg(\frac{\langle z, \nu(x_0)\rangle}{|z|} - p \frac{\langle z, \nabla u^i(y)\rangle \langle \nabla u^i(y), \nu(x_0)\rangle}{|z||\nabla u^i(y)|^2} \bigg) \, dz + O(\lambda^2).$$

Now consider the blowup sequence $\mathbf{u}_r(z) := \frac{1}{r}\mathbf{u}(x_0 + rz)$. We know that as $r \to 0$ (see [5])

$$
\begin{aligned}
\{u_r^i > 0\} \cap B_1 &\to \{z : z \cdot \nu(x_0) > 0\} \cap B_1, \\
\nabla u^i(y) = \nabla u_r^i(z) &\to (q^i(x_0))^{\frac{1}{p-1}} \nu(x_0) \chi_{\{z \cdot \nu(x_0) > 0\}} \quad \text{a.e. in } B_1.
\end{aligned}
$$

Therefore we get

$$r^{-n} \int_{B_r(x_0)} |\nabla v_r^i|^p - |\nabla u^i|^p \, dx \xrightarrow[r \to 0]{} -(p-1)\lambda |q^i(x_0)|^{\frac{p}{p-1}} \int_{B_1 \cap \{z \cdot \nu(x_0) > 0\}} \rho'(|z|) \frac{\langle z, \nu(x_0)\rangle}{|z|} \, dz + O(\lambda^2).$$

Note that formula (4.3) for $(DT_r)^{-1}$ does not depend on r, and the function $\cdot \mapsto |\cdot|^p$ is continuous; so the $O(\lambda^2)$ term converges to an $O(\lambda^2)$ term as $r \to 0$. Next note that

$$\operatorname{div}(\rho(|z|)\nu) = \frac{\rho'(|z|)}{|z|} \langle z, \nu\rangle.$$

Thus (noting that $\rho(|z|)$ is zero near ∂B_1)

$$
\begin{aligned}
\int_{B_1 \cap \{z \cdot \nu(x_0) > 0\}} \rho'(|z|) \frac{\langle z, \nu(x_0)\rangle}{|z|} \, dz &= - \int_{B_1 \cap \{z \cdot \nu(x_0) = 0\}} \rho(|z|) \, dz \\
&= -\omega_{n-1} \int_0^1 \rho(t) t^{n-1} \, dt = -C_\rho \omega_{n-1},
\end{aligned}
$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional ball of radius 1, and C_ρ depends only on ρ. Hence we can write

$$\int_{B_r(x_0)} |\nabla v_r^i|^p - |\nabla u^i|^p \, dx = [(p-1)\lambda C_\rho \omega_{n-1} |q^i(x_0)|^{\frac{p}{p-1}} + O(\lambda^2)] r^n + o(r^n).$$

On the other hand,

$$\begin{aligned}\lim_{r\to 0} r^{-n}|B_r(x_0)\cap\{|\mathbf{v}_r|>0\}| &= \lim_{r\to 0} r^{-n}\int\limits_{\{|\mathbf{v}_r|>0\}\cap B_r(x_0)} dx\\ &= \lim_{r\to 0} r^{-n}\int\limits_{\{|\mathbf{u}|>0\}\cap B_r(x_0)} |\det DT_r(y)|\,dy\\ &= \int\limits_{B_1\cap\{z\cdot\nu(x_0)>0\}} 1+\lambda\rho'(|z|)\frac{\langle z,\nu(x_0)\rangle}{|z|}\,dz\\ &= \frac{1}{2}\omega_n - \lambda\omega_{n-1}\int\limits_0^1 \rho(t)t^{n-1}\,dt = \frac{1}{2}\omega_n - \lambda C_\rho\omega_{n-1}.\end{aligned}$$

Thus for $A_0 := (\{|\mathbf{u}|>0\} - B_r(x_0)) \cup (\{|\mathbf{v}_r|>0\}\cap B_r(x_0))$ we have

$$|A_0| - |\{|\mathbf{u}|>0\}| = |B_r(x_0)\cap\{|\mathbf{v}_r|>0\}| - |B_r(x_0)\cap\{|\mathbf{u}|>0\}| = -\lambda C_\rho\omega_{n-1}r^n + o(r^n).$$

In addition, it is easy to see that $\{|\mathbf{w}|>0\}\subset A_0$.

Now suppose to the contrary that $|\{|\mathbf{u}|>0\}|>1$. Then we can choose r small enough so that

$$|A_0| = |\{|\mathbf{u}|>0\}| - \lambda C_\rho\omega_{n-1}r^n + o(r^n) > 1.$$

Therefore, using the monotonicity of f_ε we get

$$\begin{aligned}f_\varepsilon(|\{|\mathbf{w}|>0\}|) - f_\varepsilon(|\{|\mathbf{u}|>0\}|) &\le f_\varepsilon(|A_0|) - f_\varepsilon(|\{|\mathbf{u}|>0\}|)\\ &= \frac{1}{\varepsilon}(|A_0| - |\{|\mathbf{u}|>0\}|) = -\frac{1}{\varepsilon}\lambda C_\rho\omega_{n-1}r^n + o(r^n).\end{aligned}$$

Finally, by putting all these estimates in (4.2), we obtain

$$\begin{aligned}0 &\le C\sum_{i=1}^m \int\limits_{B_r(x_0)} |\nabla v_r^i|^p - |\nabla u^i|^p\,dx + f_\varepsilon(|A_0|) - f_\varepsilon(|\{|\mathbf{u}|>0\}|)\\ &= \Big[(p-1)\lambda C_\rho\omega_{n-1}\sum_{i=1}^m |q^i(x_0)|^{\frac{p}{p-1}} + O(\lambda^2)\Big]r^n - \frac{1}{\varepsilon}\lambda C_\rho\omega_{n-1}r^n + o(r^n).\end{aligned}$$

Dividing by r^n and letting $r\to 0$, and then dividing by λ and letting $\lambda\to 0$, we get

$$\frac{1}{\varepsilon} \le (p-1)\sum_{i=1}^m |q^i(x_0)|^{\frac{p}{p-1}}.$$

Now if we choose x_0 such that

$$\sum_{i\le m} q^i(x_0) \le \inf_{\mathcal{R}_\varepsilon}\Big(\sum_{i\le m} q^i\Big) + 1,$$

then by Lemma 4.1 (and the equivalence of all norms on the finite-dimensional space $\mathbb{R}^m$) we have

$$\sum_{i\le m} |q^i(x_0)|^{\frac{p}{p-1}} \le C,$$

independently of ε. However, this implies that ε has a positive lower bound, which is a contradiction. □

5 Regularity of the free boundary (case $p=2$)

We are going to show that $\mathcal{R}$ is an analytic hypersurface when $p=2$. To see this, we first derive the free boundary condition, also known as the optimality condition, in the following lemma. We perturb the optimal set Ω and compute the first variation of the energy functional J_ε. To perform this computation, it is crucial to ensure that the p-harmonic solution within the perturbed domain is differentiable with respect to the perturbation parameter. When $p = 2$, this can be established through the implicit function theorem . However, it is noteworthy that for $p \neq 2$ the following proof breaks down, primarily due to the ill-posedness of the derivative of the map $u \mapsto \Delta_p u$. Nevertheless, we believe a different approach may give a direct proof of the smoothness of the free boundary. This is left to future investigations.

Lemma 5.1. *Let* $\mathbf{u}$ *be a solution of the minimization problem* (1.1) *for* $p = 2$. *Let* h^i *be the solution of*

$$\begin{cases} \Delta h^i = 0 & \text{in } \Omega - E, \\ h^i = 0 & \text{on } E, \\ h^i = \partial_{\xi_i}\Gamma(x, \partial_\nu \mathbf{u}) & \text{on } \partial\Omega. \end{cases}$$

Then, on the regular part of the free boundary, we have

$$\sum_{i=1}^{m} \partial_\nu h^i \partial_\nu u^i = C \tag{5.1}$$

for some positive constant C.

Proof. Let x_1 and x_2 be two regular points in $\mathcal{R}$ with corresponding unit normal vectors $\nu(x_1)$ and $\nu(x_2)$. Also, let $\rho : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function supported in $(0, 1)$. Similarly to the proof of Theorem 4.3 we define the vector field

$$T_{r,\lambda}(x) := \begin{cases} x - r\lambda\rho\Big(\frac{|x - x_1|}{r}\Big)\nu(x_1) & \text{if } x \in B_r(x_1), \\ x + r\lambda\rho\Big(\frac{|x - x_2|}{r}\Big)\nu(x_2) & \text{if } x \in B_r(x_2), \\ x & \text{elsewhere}, \end{cases}$$

for small enough $r, \lambda > 0$ (which makes $T_{r,\lambda}$ a diffeomorphism from $B_r(x_a)$ onto itself for $a = 1, 2$).

Now for some fixed $r > 0$ let $E_\lambda = T^{-1}_{r,\lambda}(E)$, and assume that $\mathbf{w}_\lambda$ solves

$$\begin{cases} \Delta w^i_\lambda = 0 & \text{in } \Omega - E_\lambda, \\ w^i_\lambda = \varphi^i & \text{on } \partial\Omega, \\ w^i_\lambda = 0 & \text{on } \partial E_\lambda. \end{cases}$$

Define $\mathbf{v}_\lambda(y) := \mathbf{w}_\lambda(T^{-1}_{r,\lambda}(y))$. We are going to show that $\lambda \mapsto \mathbf{v}_\lambda$ is a C^1 map from a neighborhood of $\lambda = 0$ into $W^{1,2}(\Omega - E)$. We know that each v^i_λ satisfies an elliptic PDE of the form

$$F[v, \lambda] = F(D^2_y v, \nabla_y v, y, \lambda) = 0 \quad \text{in } U = \Omega - E.$$

We also know that $F = \Delta$ when $y \notin B_r(x_1) \cup B_r(x_2)$ or when $\lambda = 0$. In addition, we can consider F as a C^1 map

$$\begin{aligned} F : W^{1,2}(U) \times \mathbb{R} &\to W^{-1,2}(U), \\ (v, \lambda) &\mapsto F[v, \lambda], \end{aligned}$$

where $U = \Omega - E$.

Now we employ the implicit function theorem to show that $\lambda \mapsto \mathbf{v}_\lambda$ is C^1. This can be readily deduced from the fact that

$$\partial_v F|_{\lambda=0} : W^{1,2}_0(U) \to W^{-1,2}(U)$$

is invertible, since we have

$$\partial_v F|_{\lambda=0} \cdot = \frac{d}{ds}\Big|_{s=0} F[v + s\,\cdot\,, 0] = \frac{d}{ds}\Big|_{s=0} \Delta(v + s\,\cdot) = \Delta\,\cdot\,.$$

Therefore, $\mathbf{v}_\lambda = \mathbf{u} + \lambda\mathbf{u}_0 + o(\lambda)$ in $W^{1,2}(U)$, where $\mathbf{u}_0 \in W^{1,2}_0(U)$ solves

$$0 = \frac{d}{d\lambda}\Big|_{\lambda=0} F[v^i_\lambda, \lambda] = \partial_v F u^i_0 + \partial_\lambda F.$$

In other words

$$\Delta u_0^i = -\partial_\lambda F|_{v=u^i,\,\lambda=0}.$$

Note that we also have $\nabla \mathbf{v}_\lambda = \nabla \mathbf{u} + \lambda \nabla \mathbf{u}_0 + o(\lambda)$, since $\lambda \mapsto \mathbf{v}_\lambda$ is a C^1 map into $W^{1,2}(U)$; so $\lambda \mapsto \nabla \mathbf{v}_\lambda$ is a C^1 map into $L^2(U)$.

Now let h^i be the solution of $\Delta h^i = 0$ in $U = \Omega - E$ with boundary data $h^i = \partial_{\xi_i}\Gamma(x, \partial_\nu \mathbf{u})$ on $\partial\Omega$ and $h^i = 0$ on ∂E. Then for small $\lambda > 0$ we have (note that for $p = 2$ we have $A_\nu = \partial_\nu$)

$$\begin{aligned}
\int_{\partial\Omega} \Gamma(x, \partial_\nu \mathbf{v}_\lambda) - \Gamma(x, \partial_\nu \mathbf{u})\, d\sigma &= \int_{\partial\Omega} \sum_i \partial_i \Gamma(x, \partial_\nu \mathbf{u})(\partial_\nu v_\lambda^i - \partial_\nu u^i)\, d\sigma + o(\lambda)\\
&= \lambda \int_{\partial\Omega} \sum_i \partial_i \Gamma(x, \partial_\nu \mathbf{u}) \partial_\nu u_0^i\, d\sigma + o(\lambda)\\
&= \lambda \int_{\partial\Omega} \sum_i h^i \partial_\nu u_0^i\, d\sigma + o(\lambda)\\
&= \lambda \sum_i \int_U \nabla h^i \cdot \nabla u_0^i + h^i \Delta u_0^i\, dx + o(\lambda)\\
&= -\lambda \sum_i \int_{(B_r(x_1)\cup B_r(x_2))-E} h^i \partial_\lambda F|_{v=u^i,\,\lambda=0}\, dx + o(\lambda).
\end{aligned}$$

Note that in the last line we have used the facts that $\Delta h^i = 0$ in U and $u_0^i = 0$ on $\partial U = \partial\Omega \cup \partial E$. Also, we have $\partial_\lambda F|_{v=u^i,\,\lambda=0} = 0$ outside $B_r(x_1) \cup B_r(x_2)$, because in that region $F = \Delta$ for all λ.

Now let us extend $\mathbf{w}_\lambda$ to all of Ω by setting it equal to 0 on E_λ. Note that w_λ^i is positive on $\Omega - E_\lambda$ by the maximum principle. Hence

$$\{|\mathbf{w}_\lambda| > 0\} = \Omega - E_\lambda.$$

Furthermore, similarly to the proof of Theorem 4.3, we obtain

$$\begin{aligned}
f_\varepsilon(|\{|\mathbf{w}_\lambda| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|) &\le \frac{1}{\varepsilon}(|E| - |E_\lambda|)\\
&= \frac{\lambda}{\varepsilon}\Bigg(\int_{B_r(x_2)\cap\{|\mathbf{u}|>0\}} \rho'(|x - x_2|) \frac{\langle x - x_2, \nu(x_2)\rangle}{|x - x_2|}\, dx\\
&\qquad - \int_{B_r(x_1)\cap\{|\mathbf{u}|>0\}} \rho'(|x - x_1|) \frac{\langle x - x_1, \nu(x_1)\rangle}{|x - x_1|}\, dx \Bigg)\\
&= \frac{\lambda}{\varepsilon} o(r^n).
\end{aligned}$$

Therefore if we compare the energy of $\mathbf{u}$ with $\mathbf{w}_\lambda$ (it is easy to see that $\mathbf{w}_\lambda \in V$) we get (in the second equality below we use the fact that $\mathbf{v}_\lambda = \mathbf{w}_\lambda$ near $\partial\Omega$)

$$\begin{aligned}
0 \le J_\varepsilon(\mathbf{w}_\lambda) - J_\varepsilon(\mathbf{u}) &= \int_{\partial\Omega} \Gamma(x, \partial_\nu \mathbf{w}_\lambda) - \Gamma(x, \partial_\nu \mathbf{u})\, d\sigma + f_\varepsilon(|\{|\mathbf{w}_\lambda| > 0\}|) - f_\varepsilon(|\{|\mathbf{u}| > 0\}|)\\
&= \int_{\partial\Omega} \Gamma(x, \partial_\nu \mathbf{v}_\lambda) - \Gamma(x, \partial_\nu \mathbf{u})\, d\sigma + \frac{\lambda}{\varepsilon} o(r^n)\\
&= -\lambda \sum_i \int_{(B_r(x_1)\cup B_r(x_2))-E} h^i \partial_\lambda F|_{v=u^i,\,\lambda=0}\, dx + o(\lambda) + \frac{\lambda}{\varepsilon} o(r^n).
\end{aligned}$$

Hence if we divide by λ and let $\lambda \to 0$ we obtain

$$0 \le -\sum_i \int_{(B_r(x_1)\cup B_r(x_2))-E} h^i \partial_\lambda F|_{v=u^i,\,\lambda=0}\, dx + o(r^n). \tag{5.2}$$

So we need to compute $\partial_\lambda F|_{v=u^i,\,\lambda=0}$.

Next let us compute F explicitly. Set $x = T_{r,\lambda}^{-1}(y)$ so that $y = T_{r,\lambda}(x)$. To simplify the notation we suppress the λ or r in the indexes. We have $v^i(T(x)) = v^i(y) = w^i(x)$.

Hence

$$\begin{aligned}\partial_{x_k} w^i &= \sum_j \partial_{y_j} v^i \partial_{x_k} T^j,\\ \partial^2_{x_k x_k} w^i &= \sum_j \partial_{x_k}(\partial_{y_j} v^i \partial_{x_k} T^j)\\ &= \sum_{j,\ell} \partial^2_{y_j y_\ell} v^i \partial_{x_k} T^j \partial_{x_k} T^\ell + \sum_j \partial_{y_j} v^i \partial^2_{x_k x_k} T^j.\end{aligned}$$

Therefore

$$0 = \Delta w^i = \sum_{j,\ell,k} \partial^2_{y_j y_\ell} v^i \partial_{x_k} T^j \partial_{x_k} T^\ell + \sum_{j,k} \partial_{y_j} v^i \partial^2_{x_k x_k} T^j.$$

It is easy to see that inside $B_r(x_a)$ ($a = 1, 2$) we have

$$\begin{aligned}\partial_{x_k} T^j &= \delta_{jk} + (-1)^a \lambda \rho'(|z|)\frac{z_k}{|z|} v^j(x_a), \qquad z = \frac{x - x_a}{r},\\ \partial^2_{x_k x_k} T^j &= (-1)^a \lambda \partial_{x_k}\Big(\rho'(|z|)\frac{z_k}{|z|}\Big) v^j(x_a).\end{aligned}$$

Thus

$$\begin{aligned}F[v,\lambda] &= \sum_{j,\ell,k} \partial^2_{y_j y_\ell} v \partial_{x_k} T^j \partial_{x_k} T^\ell + \sum_{j,k} \partial_{y_j} v \partial^2_{x_k x_k} T^j\\ &= \sum_{j,\ell,k}\Big[\delta_{jk} + (-1)^a\lambda\rho'(|z|)\frac{z_k}{|z|}v^j(x_a)\Big]\Big[\delta_{\ell k} + (-1)^a\lambda\rho'(|z|)\frac{z_k}{|z|}v^\ell(x_a)\Big]\partial^2_{y_j y_\ell} v\\ &\quad + \sum_{j,k}[(-1)^a\lambda\partial_{x_k}\Big(\rho'(|z|)\frac{z_k}{|z|}\Big)v^j(x_a)]\partial_{y_j} v\end{aligned}$$

in $B_r(x_a)$ for $a = 1, 2$, and $F[v, \lambda] = \Delta v$ elsewhere. Now note that

$$\sum_k \partial_{x_k}\Big(\rho'(|z|)\frac{z_k}{|z|}\Big) = \sum_k\Big(\rho''(|z|)\frac{z_k^2}{r|z|^2} + \rho'(|z|)\frac{1}{r|z|} - \rho'(|z|)\frac{z_k^2}{r|z|^3}\Big) = \frac{1}{r}\rho''(|z|).$$

Hence we get

$$\partial_\lambda F|_{v=u^i,\,\lambda=0} = (-1)^a\Big(2\rho'(|z|)\sum_{j,k}\frac{z_k}{|z|}v^j(x_a)\partial^2_{jk}u^i + \frac{1}{r}\rho''(|z|)\sum_j v^j(x_a)\partial_j u^i\Big)$$

in $B_r(x_a)$ for $a = 1, 2$. Note that although a priori z, u^i in the above equation are functions of y, at $\lambda = 0$ we have $y = x$, and thus we can regard them as functions of x too.

Let $\mathbf{u}_r(z) = \frac{1}{r}\mathbf{u}(x_a + rz) = \frac{1}{r}\mathbf{u}(x)$ and $h_r^i(z) = \frac{1}{r}h^i(x_a + rz) = \frac{1}{r}h^i(x)$. Putting all these in (5.2), we get (note that in the following integration by parts the boundary term is zero, since ρ is 0 for z near ∂B_1 and h^i is 0 on ∂E)

$$\begin{aligned}0 &\le -\sum_i \int\limits_{(B_r(x_1)\cup B_r(x_2))-E} h^i \partial_\lambda F|_{v=u^i,\,\lambda=0}\,dx + o(r^n)\\ &= \sum_{a,i}(-1)^{a+1}\int\limits_{B_r(x_a)-E} h^i\Big(2\rho'(|z|)\sum_{j,k}\frac{z_k}{|z|}v^j(x_a)\partial^2_{jk}u^i + \frac{1}{r}\rho''(|z|)\sum_j v^j(x_a)\partial_j u^i\Big)dx + o(r^n)\\ &= \sum_{a,i}(-1)^{a+1}\int\limits_{B_r(x_a)-E}\Big(-2\sum_k \partial_k\Big[h^i\rho'(|z|)\frac{z_k}{|z|}\Big]\sum_j v^j(x_a)\partial_j u^i + \frac{1}{r}h^i\rho''(|z|)\sum_j v^j(x_a)\partial_j u^i\Big)dx + o(r^n)\\ &= \sum_{a,i}(-1)^{a+1}\int\limits_{B_r(x_a)-E}\Big(-2\sum_k\Big[\partial_k h^i\rho'(|z|)\frac{z_k}{|z|} + h^i\partial_k(\rho'(|z|)\frac{z_k}{|z|})\Big] + \frac{1}{r}h^i\rho''(|z|)\Big)\sum_j v^j(x_a)\partial_j u^i\,dx + o(r^n)\\ &= \sum_{a,i}(-1)^{a+1}\int\limits_{B_r(x_a)-E}\Big(-2\sum_k\Big[\partial_k h^i\rho'(|z|)\frac{z_k}{|z|}\Big] - \frac{1}{r}h^i\rho''(|z|)\Big)\sum_j v^j(x_a)\partial_j u^i\,dx + o(r^n)\\ &= \sum_{a,i}(-1)^{a+1}r^n\int\limits_{B_1\cap\{|\mathbf{u}_r|>0\}}\Big(-2\sum_k\Big[\partial_k h_r^i\rho'(|z|)\frac{z_k}{|z|}\Big] - \frac{1}{r}rh_r^i\rho''(|z|)\Big)\sum_j v^j(x_a)\partial_j u_r^i\,dz + o(r^n).\end{aligned}$$

Now note that $\partial_j u_r^i(z) \to q^i(x_a)\nu^j(x_a) = \partial_j u^i(x_a)$ when $z \cdot \nu(x_a) > 0$ by the results of [5]. Next note that h^i is Lipschitz continuous, since u^i is Lipschitz and we have $0 \le h^i \le cu^i$ for some constant c. To see this note that the function $\partial_{\xi_i}\Gamma(x, \partial_\nu \mathbf{u})$ is positive and continuous on the compact set $\partial\Omega$, so it is bounded there, and thus for some $c > 0$ we have $h^i = \partial_{\xi_i}\Gamma(x, \partial_\nu \mathbf{u}) \le c\varphi^i = cu^i$ on $\partial\Omega$. Hence the claim follows by the maximum principle. Therefore, by Lemma B.1 in [9], we also have $\partial_k h_r^i(z) \to p^i(x_a)\nu^k(x_a) = \partial_k h^i(x_a)$ for some function p^i, and $h_r^i(z) \to \nabla h^i(x_a) \cdot z$ as $h^i(x_a) = 0$. Thus if we divide the above expression by r^n and let $r \to 0$ we obtain

$$
\begin{aligned}
0 &\le \sum_{a,i} (-1)^{a+1} \int_{B_1 \cap \{z \cdot \nu(x_a) > 0\}} \Big(-2 \sum_k \Big[\partial_k h^i(x_a) \rho'(|z|) \frac{z_k}{|z|} \Big] z) \rho''(|z|) \Big) \sum_j \nu^j(x_a) \partial_j u^i(x_a)\, dz \\
&= \sum_{a,i} (-1)^{a+1} \int_{B_1 \cap \{z \cdot \nu(x_a) > 0\}} \Big(-2 \sum_k \Big[p^i(x_a)\nu^k(x_a)\rho'(|z|)\frac{z_k}{|z|} \Big] - p^i(x_a)(\nu(x_a)\cdot z)\rho''(|z|) \Big) \partial_\nu u^i(x_a)\, dz \\
&= \sum_{a,i} (-1)^{a} \int_{B_1 \cap \{z \cdot \nu(x_a) > 0\}} \Big(\frac{2}{|z|}\rho'(|z|) + \rho''(|z|) \Big)(\nu(x_a)\cdot z) p^i(x_a) \partial_\nu u^i(x_a)\, dz \\
&= \sum_{a,i} (-1)^{a} \partial_\nu h^i(x_a) \partial_\nu u^i(x_a) \int_{B_1 \cap \{z \cdot \nu(x_a) > 0\}} \Big(\frac{2}{|z|}\rho'(|z|) + \rho''(|z|) \Big)(\nu(x_a)\cdot z)\, dz \\
&= C_\rho \Big(\sum_i \partial_\nu h^i(x_2)\partial_\nu u^i(x_2) - \sum_i \partial_\nu h^i(x_1)\partial_\nu u^i(x_1) \Big),
\end{aligned}
$$

where

$$
C_\rho = \int_{B_1 \cap \{z \cdot \nu(x_a) > 0\}} \Big(\frac{2}{|z|}\rho'(|z|) + \rho''(|z|) \Big)(\nu(x_a)\cdot z)\, dz
$$

does not depend on x_a; we have also used the fact that $p^i(x_a) = \partial_\nu h^i(x_a)$. By switching the role of x_1, x_2 we conclude that

$$
\sum_i \partial_\nu h^i(x_2)\partial_\nu u^i(x_2) - \sum_i \partial_\nu h^i(x_1)\partial_\nu u^i(x_1)
$$

must be zero, as desired. □

The main idea to show the regularity of the free boundary lies in utilizing the boundary Harnack principle, which allows us to reduce the system into a scalar problem. The key tool in employing this approach is non-tangential accessibility of the domain; for the definition of non-tangentially accessible (NTA) domains we refer to [3].

Lemma 5.2. *Let $\mathbf{u}$ be a solution of the minimization problem* (1.1) *for* $p = 2$. *Then $U = \{x : |\mathbf{u}(x)| > 0\}$ is a non-tangentially accessible domain.*

Proof. This result follows from the same analysis as of [3, Theorem 4.8] for the function $\mathfrak{u} = u^1 + \cdots + u^m$. Note that $\mathfrak{u}$ is harmonic in $\{|\mathbf{u}| > 0\} = \{\mathfrak{u} > 0\}$ (these two sets are equal due to Lemma 3.2), and the function $\mathfrak{u}$ is also Lipschitz continuous and satisfies the nondegeneracy property by Corollary 3.3. □

Theorem 5.3. *Let $x_0 \in \mathcal{R}$ be a regular point of the free boundary. Then there is $r > 0$ such that $B_r(x_0) \cap \partial\{|\mathbf{u}| > 0\}$ is a $C^{1,\alpha}$ hypersurface for some $\alpha > 0$.*

Proof. We may assume that $u^1 > 0$ in $B_{r_0}(x_0) \cap \{|\mathbf{u}| > 0\}$ for some $r_0 > 0$. First we show that for some $0 < r \le r_0$ there is a Hölder continuous function g defined on $B_r(x_0) \cap \partial\{|\mathbf{u}| > 0\}$, such that in the viscosity sense we have

$$
\partial_\nu h^1 \partial_\nu u^1 = g \quad \text{on } \partial E,
$$

where h^1 is defined in Lemma 5.1. Since $B_{r_0}(x_0) \cap \{|\mathbf{u}| > 0\}$ is an NTA domain, the boundary Harnack inequality implies that $G^i := \frac{u^i}{u^1}$ and $H^i = \frac{h^i}{h^1}$ are Hölder continuous functions in $B_r(x_0) \cap \overline{\{|\mathbf{u}| > 0\}}$ for some $0 < r \le r_0$. Now if we consider a one-sided tangent ball at some point $y \in B_r(x_0) \cap \partial\{|\mathbf{u}| > 0\}$, we have asymptotic developments (see [9, Lemma B.1], noting that h^i is Lipschitz as we have shown in the proof of Lemma 5.1)

$$
\begin{aligned}
u^i(y + x) &= q^i(y)(x \cdot \nu(y))^+ + o(|x|), \\
h^i(y + x) &= p^i(y)(x \cdot \nu(y))^+ + o(|x|).
\end{aligned}
$$

Therefore $G^i(y) = \frac{q^i(y)}{q^1(y)}$ and $H^i(y) = \frac{p^i(y)}{p^1(y)}$. Thus from (5.1) we can infer that

$$p^1(y)q^1(y)\Big(1 + \sum_{i>1} G^i(y)H^i(y)\Big) = \sum_i p^i(y)q^i(y) = \sum_i \partial_\nu h^i \partial_\nu u^i$$

is constant for every $y \in B_r(x_0) \cap \partial\{|\mathbf{u}| > 0\}$. Note that $G^i, H^i > 0$ at y as $p^i, q^i > 0$. Hence by applying [13, Theorem 3.1] we get the desired result. □

Corollary 5.4. *Let* $\mathbf{u}$ *be a solution of the minimization problem* (1.1) *for* $p = 2$. *Then the regular part of the free boundary,* $\mathcal{R}$, *is analytic.*

Proof. Suppose $0 \in \mathcal{R}$ and $u^1 > 0$ in $B_r \cap \{|\mathbf{u}| > 0\}$. Then we apply the hodograph-Legendre transformation $x \mapsto y = (x_1, \dots, x_{n-1}, u^1)$. Next we define the partial Legendre functions

$$\begin{aligned} v^1(y) := x_n, \quad v^i(y) &:= u^i(x) \quad \text{for } i = 2, \dots, m, \\ w^i(y) &:= h^i(x) \quad \text{for } i = 1, \dots, m. \end{aligned}$$

As $\mathcal{R}$ is $C^{1,\alpha}$, it follows that u^i and h^i are in $C^{1,\alpha}(\overline{B_r \cap \{|\mathbf{u}| > 0\}})$. So, v^i and w^i are $C^{1,\alpha}$ in a neighborhood of the origin in $\{y_n \geq 0\}$. Now we have verified all the hypothesis of Theorem 7.1 in [3], and through a similar argument we can obtain the analyticity of $\mathcal{R}$. □

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References

[1] A. Acker, A free boundary optimization problem, *SIAM J. Math. Anal.* **9** (1978), no. 6, 1179–1191.
[2] N. Aguilera, H. W. Alt and L. A. Caffarelli, An optimization problem with volume constraint, *SIAM J. Control Optim.* **24** (1986), no. 2, 191–198.
[3] N. E. Aguilera, L. A. Caffarelli and J. Spruck, An optimization problem in heat conduction, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **14** (1987), no. 3, 355–387.
[4] H. W. Alt and L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, *J. Reine Angew. Math.* **325** (1981), 105–144.
[5] H. W. Alt, L. A. Caffarelli and A. Friedman, A free boundary problem for quasilinear elliptic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* **11** (1984), no. 1, 1–44.
[6] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, *Ann. Mat. Pura Appl. (4)* **135** (1983), 293–318.
[7] D. Danielli and A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, *Calc. Var. Partial Differential Equations* **23** (2005), no. 1, 97–124.
[8] D. Danielli, A. Petrosyan and H. Shahgholian, A singular perturbation problem for the p-Laplace operator, *Indiana Univ. Math. J.* **52** (2003), no. 2, 457–476.
[9] M. Fotouhi and H. Shahgholian, A minimization problem with free boundary for p-Laplacian weakly coupled system, *Adv. Nonlinear Anal.* **13** (2024), no. 1, Article ID 20230138.
[10] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
[11] J. L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, *Indiana Univ. Math. J.* **32** (1983), no. 6, 849–858.
[12] P. Lindqvist, Notes on the p-Laplace equation, Report No. 161, University of Jyväskylä, Jyväskylä, 2017.
[13] F. P. Maiale, G. Tortone and B. Velichkov, Epsilon-regularity for the solutions of a free boundary system, *Rev. Mat. Iberoam.* **39** (2023), no. 5, 1947–1972.
[14] J. Malý and W. P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys Monogr. 51, American Mathematical Society, Providence, 1997.
[15] H. Mikayelyan and H. Shahgholian, Hopf's lemma for a class of singular/degenerate PDE-s, *Ann. Acad. Sci. Fenn. Math.* **40** (2015), no. 1, 475–484.
[16] E. V. Teixeira, The nonlinear optimization problem in heat conduction, *Calc. Var. Partial Differential Equations* **24** (2005), no. 1, 21–46.