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Research Article

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Boundary behavior of solutions to fractional p-Laplacian equation

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Abstract: In this work, a generalized Hopf's lemma and a global boundary Harnack inequality are proved for solutions to fractional p-Laplacian equations. Then the isolation of the first (s,p)-eigenvalue is shown in bounded open sets satisfying the Wiener criterion.

Keywords: (s,p)-eigenvalue problem, Hopf's Lemma, fractional p-Laplacian

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1 Introduction

We maintain the previous work in [1] and prove the boundary properties of solutions to fractional p-Laplacian equations.

The first result is a generalized Hopf's Lemma. To bring the result, we need the notion of Wiener regular boundaries, δ-neighborhoods, and the torsion function. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p > 1$, and $0 < s < 1$. Denote $(-\Delta_p)^s$ as the s-fractional p-Laplacian, which satisfies

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,\mathrm{d}y,$$

pointwise for $x \in \mathbb{R}^n$. We say that Ω has regular boundary for the s-fractional p-Laplacian if for every $f \in \mathrm{L}^\infty(\Omega)$, $g \in C(\mathbb{R}^n)$, and every weak solution u of $(-\Delta_p)^s u = f$ in Ω with $u = g$ in $\mathbb{R}^n \setminus \Omega$, we have $u \in C(\mathbb{R}^n)$, see Section 3 for more details. Now, assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set, which has Wiener regular boundary for the s-fractional p-Laplacian. For $\delta > 0$, the δ-neighborhood of Ω, denoted by Ω_δ, is defined by $\{x \in \mathbb{R}^n : \operatorname{dist}(x,\overline{\Omega}) < \delta\}$. The torsion function $u_{\mathrm{tor}} \in \mathrm{L}^{p-1}_{ps}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ satisfies

$$\begin{aligned} u_{\mathrm{tor}} &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,\\ (-\Delta_p)^s u_{\mathrm{tor}} &= 1 \quad \text{in } \Omega, \end{aligned}$$

in the viscosity sense, see Proposition 2.8 and Proposition 2.7 for the existence of u_{tor}. We say that $K \Subset \Omega$ if $\overline{K} \subset \Omega$.

Lemma 1.1. *Let $u \in \mathrm{L}^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ be a non-negative function for a $\delta > 0$ and $K \Subset \Omega$. Assume that $(-\Delta_p)^s u \geq f$ in Ω in the viscosity sense, where $f \in C(\Omega)$ satisfies*

$$f(x_0) > -2 \int\limits_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x_0-y|^{n+ps}}\,\mathrm{d}y \quad \text{if } x_0 \in \Omega,\ u(x_0) = 0,$$

$$\limsup_{\Omega \ni x \to x_0} f(x) \geq -2 \int\limits_{K} \frac{u^{p-1}(y)}{|x_0-y|^{n+ps}}\,\mathrm{d}y \quad \text{if } x_0 \in \partial\Omega,\ u(x_0) = 0.$$

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Then $u > 0$ in Ω and

$$u \geq C u_{\mathrm{tor}} \quad \text{in } \Omega$$

for a constant $C > 0$.

Setting $f = g(u)$, we arrive at the following result:

Corollary 1.2. *Let $u \in \mathrm{L}^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ be a non-negative viscosity supersolution of $(-\Delta_p)^s u = g(u)$ in Ω, where $\delta > 0$, $\nu g \in C([0,\infty))$, $\nu g(0) = 0$. Then either $u = 0$ a.e. in $\mathbb{R}^n$, or $u > 0$ in Ω and $u \geq C u_{\mathrm{tor}}$ for a constant $C > 0$.*

If Ω has a $C^{1,1}$ boundary, then, by [16, Lemma 2.3], $u_{\mathrm{tor}}(x) \geq C \operatorname{dist}(x, \partial\Omega)^s$ for $x \in \Omega$, where $C > 0$ is a constant. Hence, the above result generalizes the previous versions of Hopf's lemma for the s-fractional p-Laplacian in [8, 9, 13, 14, 20, 25].

We remark that we could not verify the argument in [9, Lemma 4.1], which considers the case that $g(u) = c|u|^{p-2}u$ in a ball $B \subset \mathbb{R}^n$ of radius R, where $c \in C(\overline{\Omega})$ is negative. To elaborate on the issue, the authors take the set $B_\rho := \{x \in B : \operatorname{dist}(x, \partial B) < \rho\}$ and a compact subset $K \subset B \setminus B_\rho$, where ρ is taken small enough such that $(-\Delta_p)^s \operatorname{dist}(x, \partial\Omega)^s \in \mathrm{L}^\infty(B_\rho)$, see [15, Theorem 2.3]. They choose α large enough such that $(-\Delta_p)^s(d^s + \alpha 1_K)$ becomes very small in compare to $c|u|^{p-1}u$ in B_ρ, where d is the function $\operatorname{dist}(x, \partial B)$. Then they consider $0 < \epsilon < 1$ small enough such that $\epsilon(R^s + \alpha) \leq \inf_{B \setminus B_\rho} u$ and define $v := \epsilon(d^s + \alpha 1_K)$. Finally, they claim that, since $v \leq u$ in $\mathbb{R}^n \setminus B_\rho$ and $(-\Delta_p)^s v \leq (-\Delta_p)^s u$ in B_ρ, one can apply comparison principle to obtain $v \leq u$ in B_ρ. However, $(-\Delta_p)^s v = \epsilon^{p-1}(-\Delta_p)^s(d^s + \alpha 1_K)$ and the decrease of $\epsilon > 0$ increases the value of $(-\Delta_p)^s v$, since $d^s + \alpha 1_K \leq c|u|^{p-1}u \leq 0$ by the maximum principle, see [15, Theorem 1.2]. Hence, it is not clear that $(-\Delta_p)^s v$ remains below $(-\Delta_p)^s u$ in B_ρ.

Note that as it is mentioned in [1] and [14, Remark 2.8], Corollary 1.2 does not hold for the local p-Laplacian, see [24, 28] for the necessary assumptions on g to have strong maximum property. Hence, the nonlocal property of fractional p-Laplacian plays a key role in the proof.

We observe that unlike [1, Lemma 1.2], to prove Lemma 1.1, we need a stronger continuity of u around a neighborhood of $\overline{\Omega}$.

The second result is a global boundary Harnack theorem. We briefly mention that $V^{s,p}_g(\Omega|\mathbb{R}^n)$ is the fractional Sobolev space on $\mathbb{R}^n$ with the boundary value g in the trace space $V^{s,p}(\Omega|\mathbb{R}^n)$, see Section 2.1 for more details.

Theorem 1.3. *Let $\delta > 0$, $u \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_u}(\Omega|\mathbb{R}^n)$, $v \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_v}(\Omega|\mathbb{R}^n)$ satisfy*

$$\begin{aligned} &u > 0, \quad v > 0 && \text{in } \Omega, \\ &0 \leq \frac{1}{B} g_v \leq g_u \leq B g_v \leq M && \text{in } \mathbb{R}^n \setminus \Omega, \end{aligned}$$

for $B > 0$, $\nu M \geq 0$, and

$$\begin{aligned} -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K u^{p-1}(y)\,\mathrm{d}y \leq (-\Delta_p)^s u \leq 1 \quad \text{in } \Omega, \\ -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K v^{p-1}(y)\,\mathrm{d}y \leq (-\Delta_p)^s v \leq 1 \quad \text{in } \Omega, \end{aligned}$$

in the locally weak sense, where $K \Subset \Omega$. If either $u(x_0) \geq D$, $v(x_0) \geq D$ or $\|u\|_{\mathrm{L}^q(\Omega \setminus K)} \geq D$, $\|v\|_{\mathrm{L}^q(\Omega \setminus K)} \geq D$ for a fixed point $x_0 \in \Omega \setminus K$ and some constants $D > 0$, $1 \leq q < \infty$, then

$$C_1 \leq \frac{u}{v} \leq C_2 \quad \text{in } \Omega,$$

where C_1, C_2 are positive constants depending on $\Omega, \delta, K, n, s, p, D, B, M, x_0$ or q.

Up to the knowledge of the author, there are several results for the boundary Harnack theorem for the linear fractional Laplacian, see [1–3, 26, 27], but there is none for the fractional p-Laplacian. In the case of $g_u = 0$, $v = u_{\mathrm{tor}}$ in Theorem 1.3, we derive the following corollary:

Corollary 1.4. *Let $\delta > 0$, $u \in C(\overline{\Omega}_\delta) \cap V^{s,p}_0(\Omega|\mathbb{R}^n)$ satisfy $u > 0$ in Ω and*

$$-2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K u^{p-1}(y)\,\mathrm{d}y \leq (-\Delta_p)^s u \leq 1 \quad \text{in } \Omega,$$

in the locally weak sense, where $K \Subset \Omega$. If either $u(x_0) \geq D$ or $\|u\|_{L^q(\Omega\setminus K)} \geq D$, where $x_0 \in \Omega \setminus K$ is a fixed point and $D > 0$, $1 \leq q < \infty$ are fixed constants, then

$$\sup_B u \leq C \inf_B u,$$

for every subset $B \Subset \Omega$, where C is a positive constant, which depends on $\Omega, \delta, K, n, s, p, D, x_0$ or q.

In the last result, we prove the isolation of the first (s,p)-eigenvalue.

Define the Sobolev exponent

$$p_s^* := \begin{cases} \dfrac{pn}{n-ps} & \text{if } ps < n, \\ \infty & \text{if } ps \geq n, \end{cases}$$

and consider the following minimization problem:

$$\Lambda_{p,q} := \inf_{\phi \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y : \|\phi\|_{L^q(\Omega)} = 1 \right\}, \tag{1.1}$$

for $1 < q < p_s^*$.

Theorem 1.5. *If $1 < q \leq p$ and Ω has regular boundary for the s-fractional p-Laplacian, then there exist no sequences $\lambda_i > \Lambda_{p,q}$, $u_i \in V_0^{s,p}(\Omega|\mathbb{R}^n)$, which satisfy*

$$\begin{aligned} \|u_i\|_{L^q(\Omega)} &= 1, \\ \lim_{i\to\infty} \lambda_i &= \Lambda_{p,q}, \\ (-\Delta_p)^s u_i &= \lambda_i |u_i|^{q-2} u_i \quad \textit{weakly in } \Omega. \end{aligned}$$

This generalizes previous results in [1, 5, 11]. Moreover, we bring a shorter proof for the case of $C^{1,1}$ boundary condition and $p \geq 2$, see Remark 5.5. We remark that the range $1 < q \leq p$ is used in two major parts, which play key roles in the proof of Theorem 1.5. First, we derive that the minimizes of $\Lambda_{p,q}$ are unique, up to a multiplication by a constant, and strictly positive or negative on Ω, see Proposition 5.3. Second, we imply that the only (s,p)-eigenvalue $\lambda > 0$ with non-negative (s,p)-eigenfunction u, i.e., a non-negative function $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ which satisfies $(-\Delta_p)^s u = \lambda |u|^{q-2} u$ weakly in Ω and $\|u\|_{L^q(\Omega)} = 1$, is $\Lambda_{p,q}$, see Proposition 5.4. Notice that in the case of $q = p$ in [5], the isolation of the first (s,p)-eigenvalue is proved without the assumption of Wiener regularity on the boundary of Ω.

The main difficulty in this work is the proof of a similar version of [1, Lemma 5.1] for solutions to fractional p-Laplacian. The nonlinearity of the equation causes the argument in [1, Lemma 5.1] to fall down. To resolve the issue, we apply the method of the proof for the comparison principle for viscosity solutions. Then the argument follows in the same direction as proof of [13, Lemma 3.1]. One of the interesting aspects of this work is the simplicity of proofs, although one works with the weakest notion of solutions, namely the viscosity solutions.

The summary of the whole work is as follows: First, we bring the definitions of spaces and notions of solutions in Section 2. Then, in Section 3, we present the Wiener criterion for the s-fractional p-Laplacian with a nonzero right-hand side. Afterward, the first two results, namely a generalized Hopf's lemma and global boundary Harnack inequality are proved in Section 4. Finally, in Section 5, we prove the isolation of the first (s,p)-eigenvalue.

2 Preliminaries

In the entire work, $p > 1$, $0 < s < 1$, and $\Omega \subset \mathbb{R}^n$ is a bounded open set.

2.1 Spaces

For every $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define $K + x := \{y \in \mathbb{R}^n : y = z + x, z \in K\}$. Let $L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n)$ be the space of measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $\|u\|_{L^{p-1}(K)} < \infty$ for every compact subset $K \subset \mathbb{R}^n$. The space $L^{p-1}_{ps}(\mathbb{R}^n)$

consists of $u \in \mathrm{L}^{p-1}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|u\|^{p-1}_{\mathrm{L}^{p-1}_{ps}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{1+|x|^{n+ps}}\,\mathrm{d}x < \infty.$$

For every measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we define the semi-norm

$$[u]^p_{V^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

For the boundary value of solutions, we consider the space of functions

$$V^{s,p}(\Omega|\mathbb{R}^n) := \Big\{u : \mathbb{R}^n \to \mathbb{R} : u|_\Omega \in \mathrm{L}^p(\Omega),\ \frac{u(x)-u(y)}{|x-y|^{n/p+s}} \in \mathrm{L}^p(\Omega\times\mathbb{R}^n)\Big\},$$

with the norm

$$\|u\|^p_{V^{s,p}(\Omega|\mathbb{R}^n)} := \int_\Omega |u(x)|^p\,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y.$$

We denote $V^{s,p}_{\mathrm{loc}}(\Omega|\mathbb{R}^n)$ as functions $u : \mathbb{R}^n \to \mathbb{R}$ which satisfy

$$\int_K |u(x)|^p\,\mathrm{d}x + \int_{K\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

for every compact set $K \subset \Omega$. Note that by fractional Poincaré–Sobolev inequality, see Theorem 2.1, $[\,\cdot\,]_{V^{s,p}}$ is a norm on $C_0^\infty(\Omega)$. The completion of the space $C_0^\infty(\Omega)$ in $V^{s,p}(\Omega|\mathbb{R}^n)$ with respect to the norm $[\,\cdot\,]_{V^{s,p}(\mathbb{R}^n)}$ is denoted by $V_0^{s,p}(\Omega|\mathbb{R}^n)$, and the space of functions with boundary $g \in V^{s,p}(\Omega|\mathbb{R}^n)$ is defined by

$$V_g^{s,p}(\Omega|\mathbb{R}^n) := \{u \in V^{s,p}(\Omega|\mathbb{R}^n) : u - g \in V_0^{s,p}(\Omega|\mathbb{R}^n)\}.$$

We introduce $(V_0^{s,p}(\Omega|\mathbb{R}^n))^*$ as the dual space of $V_0^{s,p}(\Omega|\mathbb{R}^n)$, and let $\|\cdot\|_{(V_0^{s,p}(\Omega|\mathbb{R}^n))^*}$ denote the natural norm on $(V_0^{s,p}(\Omega|\mathbb{R}^n))^*$. Define the Hölder dual of p by p', i.e., $\frac{1}{p'}+\frac{1}{p}=1$. For every measurable $u : \mathbb{R}^n \to \mathbb{R}$ and $K \subset \mathbb{R}^n$, the norm $\|\cdot\|_{C^\alpha(K)}$ is defined by

$$\|u\|_{C^\alpha(K)} := \sup_{(x,y)\in K\times K} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

The set of critical points of a differentiable function u is denoted by N_u. Let $D \subset \Omega$ be an open subset. For every $\beta \geq 1$, we define $C^2_\beta(D)$ as the space of C^2 functions u on D satisfying

$$\sup_{x\in D\setminus N_u}\Big(\frac{\min\{\mathrm{dist}(x,N_u)^{\beta-1},1\}}{|\nabla u(x)|} + \frac{|D^2u(x)|}{\mathrm{dist}(x,N_u)^{\beta-2}}\Big) < \infty.$$

For example, $u(x) = |x-x_0|^\beta \in C^2_\beta(\Omega)$ for every $x_0 \in \mathbb{R}^n$, $\beta \geq 2$. Note that we need to use the space $C^2_\beta(D)$ to define the notion of viscosity solutions, see [18].

2.2 Fractional Poincaré–Sobolev inequality and fractional Sobolev embedding

We bring the fractional Poincaré–Sobolev inequality.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset. Then, for every $u \in C_0^\infty(\Omega)$, we have*

$$\|u\|_{\mathrm{L}^{p_s^*}(\Omega)} \leq C_1[u]_{V^{s,p}(\mathbb{R}^n)} \quad \text{if } ps < n,$$
$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq C_2[u]_{V^{s,p}(\mathbb{R}^n)} \quad \text{if } ps > n,$$
$$\|u\|_{\mathrm{L}^q(\Omega)} \leq C_3[u]_{V^{s,p}(\mathbb{R}^n)} \quad \text{if } ps = n,$$

for every $1 \leq q < \infty$, where C_1 depends on n, s, p, C_2 depends on n, s, p, Ω, and C_3 depends on n, s, p, q, Ω.

Proof. The proof is essentially contained in [23, Theorems 8.1, 8.2, 9.1]. However, since the norms and spaces are slightly different, we bring the proof for the reader's convenience. First, by [6, Propositions 4.1 and 4.5], we derive that the fractional Sobolev spaces in [23], which are defined via interpolation, are equivalent to

$V^{s,p}(\mathbb{R}^n)$. Now, for the case $ps < n$, we refer to [23, Theorem 8.1]. Moreover, the proof of the case $ps > n$ exists in [4, Propositions 2.9]. For last case $ps = n$, we first use [23, Theorem 9.1] and Hölder's inequality to imply that

$$\|u\|_{L^q(\Omega)} \leq C(\|u\|_{L^p(\mathbb{R}^n)} + [u]_{V^{s,p}(\mathbb{R}^n)}), \tag{2.1}$$

for every $u \in C_0^\infty(\Omega)$ and $1 \leq q < \infty$, where C is a constant depending on n, s, p, q, Ω. Now, by [4, Lemma 2.4], it is obtained that

$$\|u\|_{L^p(\Omega)} \leq C'[u]_{V^{s,p}(\mathbb{R}^n)}, \tag{2.2}$$

for every $u \in C_0^\infty(\Omega)$, where C' is a constant depending on n, s, p, Ω. Hence, by (2.1) and (2.2), we finish the proof for case of $ps = n$. □

Remark 2.2. By Hölder's inequality and Theorem 2.1, we have $L^{(p_s^*)'}(\Omega) \subset (V_0^{s,p}(\Omega|\mathbb{R}^n))^\star$ if $ps \neq n$, where we define the action of $f \in L^{(p_s^*)'}(\Omega)$ on $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ by the paring

$$\int_{\mathbb{R}^n} f\, u \,\mathrm{d}x.$$

Moreover, by Hölder's inequality and Theorem 2.1, $L^q(\Omega) \subset (V_0^{s,p}(\Omega|\mathbb{R}^n))^\star$ if $ps = n$ for every $1 < q < \infty$.

Note that, for every $1 \leq q < p_s^*$, we have

$$\|u\|_{L^q(\Omega)}^p \leq C[u]_{V^{s,p}(\mathbb{R}^n)}^p \quad \text{for } u \in V_0^{s,p}(\Omega|\mathbb{R}^n),$$

where we used Theorem 2.1 and Hölder's inequality. Hence, the eigenvalue $\Lambda_{p,q}$ defined in (1.1) is strictly positive, and the inverse $\Lambda_{p,q}^{-1}$ is the best constant C in the above inequality.

Theorem 2.3. *The space $V_0^{s,p}(\Omega|\mathbb{R}^n)$ is compactly embedded in $L^q(\Omega)$ for bounded open sets $\Omega \subset \mathbb{R}^n$ and $1 < q \leq p$.*

Proof. Similar to [1, Theorem 2.2], we consider a large enough ball $B \subset \mathbb{R}^n$ such that $\Omega \subset B$. Then $V_0^{s,p}(\Omega|\mathbb{R}^n) \subset V_0^{s,p}(B|\mathbb{R}^n)$ and B is an extension domain, see [10, Theorem 5.4]. By [10, Theorem 7.1], $V_0^{s,p}(B|\mathbb{R}^n)$ is compactly embedded in $L^q(B)$ for $1 < q \leq p$. In conclusion, $V_0^{s,p}(\Omega|\mathbb{R}^n)$ is compactly embedded in $L^q(\Omega)$ for $1 < q \leq p$. □

2.3 Notions of solutions

In this subsection, we bring the different notions of solutions for fractional p-Laplacian equations.

The first notion is pointwise solutions.

Definition 2.4. For every $f \in C(\Omega)$, we say that $u \in L_{ps}^{p-1}(\mathbb{R}^n) \cap C(\Omega)$ satisfies $(-\Delta_p)^s u = f$ pointwise in Ω if

$$2\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} \,\mathrm{d}y := 2\lim_{\epsilon\to 0} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} \,\mathrm{d}y = f(x) \quad \text{for } x \in \Omega.$$

For simplicity, we remove the notation P.V. in the rest of the work.

The following is the definition of the locally weak and weak solutions.

Definition 2.5. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. We say that $u \in L_{ps}^{p-1}(\mathbb{R}^n) \cap V_{\mathrm{loc}}^{s,p}(\Omega|\mathbb{R}^n)$ is locally weak subsolution (supersolution) of the equation $(-\Delta_p)^s u = f$ in Ω, or equivalently we say that $(-\Delta_p)^s u \leq (\geq)\, f$ in Ω in the locally weak sense if

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\phi(x)-\phi(y))}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y \leq (\geq) \int_{\Omega} f(x)\,\phi(x)\,\mathrm{d}x, \tag{2.3}$$

for all non-negative $\phi \in C_0^\infty(\Omega)$. Note that, if ϕ is supported in $K \Subset \Omega$, we have

$$\begin{aligned}\int_{K\times K} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\phi(x)-\phi(y))}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y + 2\int_{K\times\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\phi(x)-\phi(y))}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\,(\phi(x)-\phi(y))}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

so the left-hand-side of (2.3) is well-defined for $u \in L^{p-1}_{ps}(\mathbb{R}^n) \cap V^{s,p}_{\mathrm{loc}}(\Omega|\mathbb{R}^n)$. If u is a locally weak subsolution and supersolution of $(-\Delta_p)^s u = f$ in Ω, then we say that $(-\Delta_p)^s u = f$ locally weakly in Ω. In the case that $u \in V^{s,p}_g(\Omega|\mathbb{R}^n)$ for a function $g \in V^{s,p}(\Omega|\mathbb{R}^n)$, $f \in (V^{s,p}_0(\Omega|\mathbb{R}^n))^*$, and $(-\Delta_p)^s u = f$ locally weakly in Ω, we call u a weak solution of $(-\Delta_p)^s u = f$ in Ω.

The following is the definition of viscosity solutions.

Definition 2.6. We say that $u \in L^{p-1}_{ps}(\mathbb{R}^n) \cap C(\Omega)$ is a viscosity subsolution (supersolution) of $(-\Delta_p)^s u = f$ in Ω for $f \in C(\Omega)$, or equivalently we say that $(-\Delta_p)^s u \leq (\geq)\ f$ in Ω in the viscosity sense, if for every $x_0 \in \Omega$, $\overline{B(x_0,r)} \subset \Omega$, and a function $\phi \in C^2(B(x_0,r))$ which touches u from above (below) at x_0, i.e.,

$$\begin{aligned} \phi(x_0) &= u(x_0), \\ \phi > (<)\ u &\quad \text{in } \overline{B(x_0,r)} \setminus \{x_0\}, \end{aligned}$$

and satisfies at least one of the following:
(a) $\nabla\phi(x_0) \neq 0$ or $p > \frac{2}{2-s}$,
(b) $\nabla\phi(x_0) = 0$, x_0 is an isolated critical point of ϕ, $\phi \in C^2_\beta(B(x_0,r))$ for some $\beta > \frac{ps}{p-1}$, and $1 < p \leq \frac{2}{2-s}$,
we have

$$2\int_{\mathbb{R}^n} \frac{|w(x_0)-w(y)|^{p-2}(w(x_0)-w(y))}{|x_0-y|^{n+ps}}\,dy \leq (\geq)\ f(x_0),$$

where

$$\begin{aligned} w &= \phi \quad \text{in } B(x_0,r), \\ w &= u \quad \text{in } \mathbb{R}^n \setminus B(x_0,r). \end{aligned}$$

We define $(-\Delta_p)^s u = f$ in Ω in the viscosity sense if u is a viscosity subsolution and supersolution of $(-\Delta_p)^s u = f$ in Ω.

Now, we prove the weak supersolutions with right-hand sides are viscosity supersolutions, see [18] for zero right-hand side.

Proposition 2.7. *Let $u \in L^{p-1}_{ps}(\mathbb{R}^n) \cap C(\Omega) \cap V^{s,p}_{\mathrm{loc}}(\Omega|\mathbb{R}^n)$ be a locally weak subsolution (supersolution) of $(-\Delta_p)^s u = f$ in Ω, where $f \in C(\Omega)$. Then u is a viscosity subsolution (supersolution) of $(-\Delta_p)^s u = f$ in Ω.*

Proof. We prove the case of subsolutions and the other one follows immediately by replacing u with $-u$. Assume that $x \in \Omega$, $r > 0$ satisfy $\overline{B(x,r)} \subset \Omega$, and $\phi \in C^2(B(x,r))$ touches u from above at the point x, which satisfies either the condition (a) or (b) in Definition 2.6. Define

$$\begin{aligned} w &:= \phi \quad \text{in } B(x,r), \\ w &:= u \quad \text{in } \mathbb{R}^n \setminus B(x,r). \end{aligned}$$

Assume that $(-\Delta_p)^s w(x) > f(x)$. Then, by the continuity of f and $(-\Delta_p)^s w$ in $B(x,r)$, see [18, Lemma 3.8], we have $(-\Delta_p)^s w \geq f + \delta$ pointwise and locally weakly in $B(x,r')$ for some small enough $\delta > 0$, $0 < r' < r$. By [18, Lemma 3.9], for small enough $\epsilon > 0$ and $\eta \in C^2_0(B(x,r'))$ satisfying $\eta(x) = 1$ and $0 \leq \eta \leq 1$, we have $(-\Delta_p)^s(-w+\epsilon\eta) \leq -f$ pointwise in $B(x,r')$, and $-w+\epsilon\eta$ is a locally weak subsolution of $(-\Delta_p)^s(-w+\epsilon\eta) = -f$ in $B(x,r')$. Hence, $(-\Delta_p)^s(w-\epsilon\eta) \geq f$ pointwise, and $w-\epsilon\eta$ is a locally weak supersolution of $(-\Delta_p)^s(w-\epsilon\eta) = f$ in $B(x,r')$. Now, define another function

$$\begin{aligned} w' &:= \phi \quad \text{in } B(x,r'), \\ w' &:= u \quad \text{in } \mathbb{R}^n \setminus B(x,r'). \end{aligned}$$

Then

$$\begin{aligned} &2\int_{\mathbb{R}^n} \frac{|(w'-\epsilon\eta)(z)-(w'-\epsilon\eta)(y)|^{p-2}((w'-\epsilon\eta)(z)-(w'-\epsilon\eta)(y))}{|z-y|^{n+ps}}\,dy \\ &\geq 2\int_{\mathbb{R}^n} \frac{|(w-\epsilon\eta)(z)-(w-\epsilon\eta)(y)|^{p-2}((w-\epsilon\eta)(z)-(w-\epsilon\eta)(y))}{|z-y|^{n+ps}}\,dy \\ &= (-\Delta_p)^s(w-\epsilon\eta) \geq f(z), \end{aligned}$$

pointwise for $z \in B(x, r')$. Hence, $w' - \epsilon\eta = u$ on $\mathbb{R}^n \setminus B(x, r')$ and and $w' - \epsilon\eta$ is a locally weak supersolution of $(-\Delta_p)^s(w' - \epsilon\eta) = f$ in $B(x, r')$. Then, by [15, Propostion 2.10], we have $w' - \epsilon\eta \geq u$ in $\mathbb{R}^n$. This is in contradiction with

$$w'(x) - \epsilon\eta(x) = \phi(x) - \epsilon\eta(x) = u(x) - \epsilon < u(x).$$

In conclusion, $(-\Delta_p)^s w(x) \leq f(x)$ for every $x \in \Omega$, which implies that u is a viscosity subsolution of $(-\Delta_p)^s u = f$ in Ω. □

The following proposition can be proved along the lines of [21, Theorem 8] and [18, Theorem 2.4].

Proposition 2.8. *For every $f \in (V_0^{s,p}(\Omega|\mathbb{R}^n))^\star$, $g \in V^{s,p}(\Omega|\mathbb{R}^n)$, there exists a weak solution $u \in V_g^{s,p}(\Omega|\mathbb{R}^n)$ to*

$$(-\Delta_p)^s u = f \quad \text{in } \Omega.$$

Proposition 2.9. *Let $g \in V^{s,p}(\Omega|\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \setminus \Omega)$ and let $u \in V_g^{s,p}(\Omega|\mathbb{R}^n)$ be a locally weak subsolution of $(-\Delta_p)^s u = 1$ in Ω. Then*

$$u \leq M \quad \text{in } \Omega,$$

for a constant M depending on $n, s, p, \Omega, \|g\|_{L^\infty(\mathbb{R}^n\setminus\Omega)}$.

Proof. By Remark 2.2 and Proposition 2.8, there exists a weak solution $v \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ of $(-\Delta_p)^s v = 1$ in Ω. Then, by the comparison principle, see [15, Proposition 2.10], we have $v \geq 0$ and

$$u \leq v + \|g\|_{L^\infty(\mathbb{R}^n\setminus\Omega)} \quad \text{in } \Omega.$$

This together with [5, Theorem 3.1] concludes the proof. □

Finally, we prove the stability of viscosity solutions.

Proposition 2.10. *Let $u_i \in L_{ps}^{p-1}(\mathbb{R}^n) \cap C(\Omega)$ be a uniformly bounded sequence of viscosity supersolutions of $(-\Delta_p)^s u_i = f_i$ in Ω for $f_i \in C(\Omega)$. Assume that u_i converges locally uniformly in Ω to $u \in L_{ps}^{p-1}(\mathbb{R}^n)$, f_i converges locally uniformly in Ω to f, and u_i converges pointwise a.e. in $\mathbb{R}^n$ to u. Then u is a viscosity supersolutions of $(-\Delta_p)^s u = f$ in Ω.*

Proof. The proof follows the argument in [7, Lemma 4.5]. Let $x \in \Omega$ and $\phi \in C^2(\overline{B(x, r)})$ touch u from below at x and satisfy either (a) or (b) in Definition 2.6, where $\overline{B(x, r)} \subset \Omega$. Take a point $x_i \in \overline{B(x, r)}$ such that

$$u_i(x_i) - \phi(x_i) = \inf_{\overline{B(x,r)}} u_i - \phi.$$

Since x is the minimum point of $u - \phi$ in $\overline{B(x, r)}$ and u_i converges uniformly to u in $\overline{B(x, r)}$, the points x_i converges to x and x_i is a local minimum for $u_i - \phi$ in $B(x, r)$. Define

$$w_i := \begin{cases} \phi + u_i(x_i) - \phi(x_i) & \text{in } B(x, r), \\ u_i & \text{in } \mathbb{R}^n \setminus B(x, r). \end{cases}$$

Then w_i touches u_i from below at x_i and

$$2 \int_{\mathbb{R}^n} \frac{|w_i(x_i) - w_i(y)|^{p-2}(w_i(x_i) - w_i(y))}{|x_i - y|^{n+ps}}\, dy \geq f_i(x_i)$$

Now, letting $i \to \infty$ and using Lebesgue dominated convergence theorem, we obtain

$$2 \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))}{|x - y|^{n+ps}}\, dy \geq f(x),$$

where

$$w := \begin{cases} \phi & \text{in } B(x, r), \\ u & \text{in } \mathbb{R}^n \setminus B(x, r). \end{cases}$$

This completes the proof. □

3 Wiener criterion

In this section, we bring the proof of the Wiener criterion for weak solutions to the s-fractional p-Laplacian with a nonzero right-hand side. We mention that in [17, Theorem 1.1] the Wiener criterion for zero right-hand side is proved, and in [19, A.4.] the sufficiency part of the results in [17] has been extended to include equations with bounded right-hand sides. We derive the same result as [19, A.4.] by using the case of zero right-hand side, [17, Theorem 1.1], and a perturbation argument.

Definition 3.1. Define

$$\operatorname{cap}_{s,p}(\overline{B(\xi_0,r)}\setminus\Omega, B(\xi_0,2r)) := \inf_{v} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

where the infimum is taken over all $v \in C_0^\infty(B(\xi_0,2r))$ such that $v \geq 1$ on $\overline{B(\xi_0,r)}\setminus\Omega$. We say that a point $\xi_0 \in \partial\Omega$ satisfies the Wiener criterion for the s-fractional p-Laplacian if

$$\int_0^1 \left(\frac{\operatorname{cap}_{s,p}(\overline{B(\xi_0,r)}\setminus\Omega, B(\xi_0,2r))}{r^{n-ps}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} = \infty.$$

Now, we define the notion of regular boundaries and the Wiener criterion.

Definition 3.2. A point $\xi_0 \in \partial\Omega$ is said to be regular for the s-fractional p-Laplacian if for every $f \in \mathrm{L}^\infty(\Omega)$, $g \in C(\mathbb{R}^n) \cap V^{s,p}(\Omega|\mathbb{R}^n)$, $u \in V_g^{s,p}(\Omega|\mathbb{R}^n)$ satisfying $(-\Delta_p)^s u = f$ weakly in Ω, we have $\lim_{\xi\to\xi_0} u(\xi) = g(\xi_0)$. We say that Ω has Wiener regular boundary for the s-fractional p-Laplacian if all the points on $\partial\Omega$ are regular for the s-fractional p-Laplacian.

Proposition 3.3. *A point $\xi_0 \in \partial\Omega$ is regular for the s-fractional p-Laplacian if and only if it satisfies the Wiener criterion for the s-fractional p-Laplacian.*

Proof. Our argument is similar to [21, Lemma 29]. By [17, Theorem 1.1], it follows that if ξ_0 is regular, then it satisfies the Wiener criterion for the s-fractional p-Laplacian. Now, assume that $\xi_0 \in \partial\Omega$ satisfies the Wiener criterion for the s-fractional p-Laplacian. Let $f \in \mathrm{L}^\infty(\Omega)$ and $u \in V_g^{s,p}(\Omega|\mathbb{R}^n)$ be a weak solution of $(-\Delta_p)^s u = f$ with boundary value $g \in C(\mathbb{R}^n) \cap V^{s,p}(\Omega|\mathbb{R}^n)$. Assume that B is large ball such that $\overline{\Omega} \subset B$, $2B$ is the ball with the same center as B and double radius, and $\eta \in C_0^\infty(\mathbb{R}^n\setminus B)$ is a non-negative function, where $\eta = 1$ on $\mathbb{R}^n \setminus 2B$. Then, by Lemma [15, Lemma 2.8], $(-\Delta_p)^s(u+M\eta) = f + h_M$ weakly in Ω, where

$$h_M(x) = 2\int_{\mathbb{R}^n\setminus B} \frac{|u(x)-u(y)-M\eta(y)|^{p-2}(u(x)-u(y)-M\eta(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}}\,\mathrm{d}y,$$

for $M \in \mathbb{R}$ and a.e. Lebesgue point $x \in \mathbb{R}^n$ of u. Hence, for $M > 0$ large enough, we have

$$(-\Delta_p)^s(u-M\eta) \geq 0 \geq (-\Delta_p)^s(u+M\eta) \quad \text{locally weakly in } \Omega.$$

Now, by Proposition 2.8, there exist weak solutions $u_1 \in V_{g-M\eta}^{s,p}(\Omega|\mathbb{R}^n)$, $u_2 \in V_{g+M\eta}^{s,p}(\Omega|\mathbb{R}^n)$, such that

$$(-\Delta_p)^s u_1 = (-\Delta_p)^s u_2 = 0 \quad \text{weakly in } \Omega.$$

Then, by the comparison principle [15, Proposition 2.10], we obtain

$$u_1 \leq u - M\eta, \quad u_2 \geq u + M\eta \quad \text{in } \Omega.$$

In conclusion, by the Wiener criterion for the s-fractional p-Laplacian and [17, Theorem 1.1], u_1, u_2 take their boundary values continuously and

$$\begin{aligned}\limsup_{\xi\to\xi_0} u(\xi) = \limsup_{\xi\to\xi_0} u(\xi) + M\eta(\xi) \leq \lim_{\xi\to\xi_0} u_2(\xi) = g(\xi_0) = \lim_{\xi\to\xi_0} u_1(\xi)\\ \leq \liminf_{\xi\to\xi_0} u(\xi) - M\eta(\xi) = \liminf_{\xi\to\xi_0} u(\xi).\end{aligned}$$

This completes the proof. □

4 Hopf's lemma and Global boundary Harnack inequality

In this section, we prove Hopf's lemma and global boundary Harnack inequality for solutions to fractional p-Laplacian equations.

First, we need the following two preliminary lemmas.

Lemma 4.1. *Let $\delta > 0, \beta > 0$ and $u, v \in C(\overline{\Omega}_\delta)$. Assume that $0 \le u \le Bv$ in $\mathbb{R}^n \setminus \Omega$ for a constant $B > 0$ and*

$$\sup_\Omega \frac{u}{C} - v > 0$$

for a constant $C > B$. Then, for every $\epsilon_0 > 0$, there exist $0 < \epsilon < \epsilon_0$, $(x_\epsilon, y_\epsilon) \in \Omega \times \Omega$ such that

$$\sup_{(x,y)\in\Omega_\delta\times\Omega_\delta} \frac{u(x)}{C} - v(y) - \frac{|x-y|^\beta}{\epsilon} = \frac{u(x_\epsilon)}{C} - v(y_\epsilon) - \frac{|x_\epsilon - y_\epsilon|^\beta}{\epsilon}. \tag{4.1}$$

Proof. Let us define (x_ϵ, y_ϵ) for every $\epsilon > 0$, which satisfies (4.1). Then

$$\frac{u(x_\epsilon)}{C} - v(y_\epsilon) - \frac{|x_\epsilon - y_\epsilon|^\beta}{\epsilon} \ge \sup_\Omega \frac{u}{C} - v > 0. \tag{4.2}$$

Hence,

$$|x_\epsilon - y_\epsilon|^\beta \le \epsilon\Big(\frac{u(x_\epsilon)}{C} - v(y_\epsilon)\Big).$$

Letting $\epsilon \to 0$, we arrive at, up to a subsequence, x_ϵ, y_ϵ converge to $z \in \overline{\Omega}_\delta$. Hence, by (4.2), we get

$$\frac{u(z)}{C} - v(z) \ge \sup_\Omega \frac{u}{C} - v > 0.$$

In conclusion, we obtain $z \in \Omega$ since $C > B$ and $0 \le u \le Bv$ in $\mathbb{R}^n \setminus \Omega$. In particular, for small $0 < \epsilon < \epsilon_0$, we have $x_\epsilon, y_\epsilon \in \Omega$. □

Lemma 4.2. *Let $r_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, and $\beta > \max(\frac{ps}{p-1}, 2)$. Then*

$$\Bigg|\int\limits_{B(y_0,r_0)\setminus B(y_0,\epsilon)} \frac{||y-x_0|^\beta - |y_0 - x_0|^\beta|^{p-2}(|y-x_0|^\beta - |y_0-x_0|^\beta)}{|y-y_0|^{n+ps}}\,dy\Bigg| \le C_{r_0},$$

for every $0 < \epsilon < r_0$ such that $r_0 < \frac{|x_0-y_0|}{2}$ if $x_0 \ne y_0$, where C_{r_0} is independent of y_0 and satisfies $\lim_{r_0\to 0} C_{r_0} = 0$.

Proof. If $x_0 = y_0$, then

$$\Bigg|\int\limits_{B(x_0,r_0)\setminus B(x_0,\epsilon)} \frac{||y-x_0|^\beta - |x_0 - x_0|^\beta|^{p-2}(|y-x_0|^\beta - |x_0-x_0|^\beta)}{|y-x_0|^{n+ps}}\,dy\Bigg| = \frac{n|B(0,1)|}{\beta(p-1)-ps} r_0^{\beta(p-1)-ps}.$$

If $x_0 \ne y_0$, then the result follows by [18, Lemma 3.6, Lemma 3.7]. □

The following two lemmas are the main tools for the proof of Hopf's lemma, global boundary Harnack theorem, and the isolation of the first fractional (s,p)-eigenvalue.

Lemma 4.3. *Let $\delta > 0$ and $v_i \in L^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ be a sequence of functions which are viscosity supersolution of $(-\Delta_p)^s v_i = f_i$ in Ω for $f_i \in C(\Omega)$. Assume that v_i converges uniformly to v on compact subsets of Ω, $v > 0$ in Ω, and*

$$\limsup_{i\to\infty} f_i(x_i) \ge -2\int\limits_K \frac{v^{p-1}(y)}{|x-y|^{n+ps}}\,dy \quad \text{if } x_i \in \Omega,\ \lim_{i\to\infty} x_i = x \in \partial\Omega,\ \limsup_{i\to\infty} v_i(x_i) \le 0,$$

where $K \Subset \Omega$. Moreover, $u_i \in L^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ is a sequence of non-negative uniformly bounded functions which satisfy

$$\begin{aligned} 0 \le u_i &\le Bv_i \quad &&\text{in } \mathbb{R}^n \setminus \Omega,\\ (-\Delta_p)^s u_i &\le D &&\text{in } \Omega, \end{aligned}$$

in the viscosity sense for every i, where B, D are positive constants. Then

$$u_i \le Cv_i \quad \text{in } \mathbb{R}^n,$$

for $i > N$, where N, C are positive constants.

Proof. We prove the lemma in several steps. By taking $K \Subset \Omega$ a bit larger and using $v > 0$, we assume that

$$\limsup_{i\to\infty} f_i(x_i) > -2 \int_K \frac{v^{p-1}(y)}{|x-y|^{n+ps}}\,dy \quad \text{if } \lim_{i\to\infty} x_i = x \in \partial\Omega,\ \limsup_{i\to\infty} v_i(x_i) \le 0. \tag{4.3}$$

Assume that by contradiction, there is a sequence of $C_i > B$ increasing to infinity, such that

$$m_i := \sup_{\mathbb{R}^n} \frac{u_i}{C_i} - v_i > 0.$$

Since $C_i > B$ and $0 \le u_i \le Bv_i$ in $\mathbb{R}^n \setminus \Omega$, we have

$$m_i = \sup_{\Omega} \frac{u_i}{C_i} - v_i > 0.$$

Let us choose $\beta > \max(\frac{ps}{p-1}, 2)$. Then, by Lemma 4.1, there exist $0 < \epsilon_i < \frac{1}{i(1+\|v_i\|_{L^\infty(\Omega)})}$ and $(x_i, y_i) \in \Omega \times \Omega$ such that

$$0 < m_i \le \sup_{(x,y)\in\Omega_\delta\times\Omega_\delta} \frac{u_i(x)}{C_i} - v_i(y) - \frac{|x-y|^\beta}{\epsilon_i} = \frac{u_i(x_i)}{C_i} - v_i(y_i) - \frac{|x_i-y_i|^\beta}{\epsilon_i}. \tag{4.4}$$

Without loss of generality, up to a subsequence, we assume $\lim_{i\to\infty} x_i = \tilde{x}$, $\lim_{i\to\infty} y_i = \tilde{y}$. Then, by (4.4), we get

$$|\tilde{x}-\tilde{y}|^\beta = \limsup_{i\to\infty} |x_i - y_i|^\beta \le \limsup_{i\to\infty} \frac{\frac{u_i(x_i)}{C_i} + |v_i(y_i)|}{i(1+\|v_i\|_{L^\infty(\Omega)})} = 0.$$

Hence, $\tilde{x} = \tilde{y} \in \overline{\Omega}$. If $\tilde{x} = \tilde{y} \in \Omega$, then we arrive at the following contradiction:

$$0 \le \liminf_{i\to\infty} m_i \le \liminf_{i\to\infty} \frac{u_i(x_i)}{C_i} - v_i(y_i) = -v(\tilde{y}) < 0.$$

In conclusion, $\tilde{x} = \tilde{y} \in \partial\Omega$. Also, by (4.4) and uniform boundedness of the sequence u_i, we derive

$$0 \le \liminf_{i\to\infty} \frac{u_i(x_i)}{C_i} - v_i(y_i) = \liminf_{i\to\infty} -v_i(y_i) = -\limsup_{i\to\infty} v_i(y_i).$$

Thus,

$$\limsup_{i\to\infty} v_i(y_i) \le 0. \tag{4.5}$$

Taking i large enough, we assume that $|x_i - y_i| \le \frac{\delta}{3}$ and x_i, y_i belong to $\Omega \setminus \overline{K}$. Let $r_i > 0$ be small enough, such that $B(x_i, r_i) \cup B(y_i, r_i) \subset \Omega \setminus \overline{K}$, $r_i < \frac{|x_i - y_i|}{2}$ if $x_i \ne y_i$, and

$$C_{r_i} \le \frac{\epsilon_i^{p-1}}{i}, \tag{4.6}$$

where C_{r_i} is the constant in Lemma 4.2, replacing x_0, y_0 with x_i, y_i, respectively. Then, by (4.4), the following inequalities are deduced:

(i) If we set $x = x_i$, then

$$v_i(y) - v_i(y_i) \ge \frac{|y_i - x_i|^\beta}{\epsilon_i} - \frac{|y - x_i|^\beta}{\epsilon_i} \quad \text{in } \Omega_\delta.$$

(ii) If we set $x = y + x_i - y_i$, then

$$v_i(y) - v_i(y_i) \ge \frac{u_i(y + x_i - y_i)}{C_i} - \frac{u_i(x_i)}{C_i} \quad \text{in } \Omega_{\frac{\delta}{2}}.$$

Notice that we used the fact that $|x_i - y_i| \le \frac{\delta}{3}$ in the above inequality.

(iii) If we set $y = y_i$, then

$$\frac{u_i(x)}{C_i} - \frac{u_i(x_i)}{C_i} \leq \frac{|x - y_i|^\beta}{\epsilon_i} - \frac{|x_i - y_i|^\beta}{\epsilon_i} \quad \text{in } \Omega_\delta.$$

Now, define the functions

$$w_i(y) := \begin{cases} v_i(y_i) + \dfrac{|y_i - x_i|^\beta}{\epsilon_i} - \dfrac{|y - x_i|^\beta}{\epsilon_i} & \text{in } B(y_i, r_i), \\ v_i(y) & \text{in } \mathbb{R}^n \setminus B(y_i, r_i), \end{cases}$$

$$\tilde{w}_i(x) := \begin{cases} \dfrac{u_i(x_i)}{C_i} + \dfrac{|x - y_i|^\beta}{\epsilon_i} - \dfrac{|x_i - y_i|^\beta}{\epsilon_i} & \text{in } B(x_i, r_i), \\ \dfrac{u_i(x)}{C_i} & \text{in } \mathbb{R}^n \setminus B(x_i, r_i). \end{cases}$$

Hence, by (i), (iii), w_i touches v_i from below at y_i, and $\tilde{w}_i$ touches $\frac{u_i}{C_i}$ from above at x_i. In conclusion,

$$\begin{aligned} 2 \int_{\mathbb{R}^n} \frac{|w_i(y) - w_i(y_i)|^{p-2}(w_i(y) - w_i(y_i))}{|y - y_i|^{n+ps}} \, dy &\leq -f_i(y_i), \\ 2 \int_{\mathbb{R}^n} \frac{|\tilde{w}_i(y) - \tilde{w}_i(y_i)|^{p-2}(\tilde{w}_i(y) - \tilde{w}_i(y_i))}{|y - y_i|^{n+ps}} \, dy &\geq -\frac{D}{C_i^{p-1}}. \end{aligned} \tag{4.7}$$

Now, the rest of the proof aims at deriving a contradiction from the inequalities above and (4.3). By the first inequality in (4.7), it is obtained that

$$\begin{aligned} \liminf_{i\to\infty} -f_i(y_i) &\geq 2 \liminf_{i\to\infty} \int_{\mathbb{R}^n} \frac{|w_i(y) - w_i(y_i)|^{p-2}(w_i(y) - w_i(y_i))}{|y - y_i|^{n+ps}} \, dy \\ &\geq 2 \liminf_{i\to\infty} \int_{\mathbb{R}^n \setminus \Omega_{\frac{\delta}{2}}} \frac{|v_i(y) - v_i(y_i)|^{p-2}(v_i(y) - v_i(y_i))}{|y - y_i|^{n+ps}} \, dy \\ &\quad - 2 \limsup_{i\to\infty} \frac{1}{\epsilon_i^{p-1}} \int_{B(y_i,r_i)} \frac{||y - x_i|^\beta - |y_i - x_i|^\beta|^{p-2}(|y - x_i|^\beta - |y_i - x_i|^\beta)}{|y - y_i|^{n+ps}} \, dy \\ &\quad + 2 \liminf_{i\to\infty} \int_K \frac{|v_i(y) - v_i(y_i)|^{p-2}(v_i(y) - v_i(y_i))}{|y - y_i|^{n+ps}} \, dy \\ &\quad + 2 \liminf_{i\to\infty} \int_{\Omega_{\frac{\delta}{2}} \setminus (B(y_i,r_i) \cup K)} \frac{|v_i(y) - v_i(y_i)|^{p-2}(v_i(y) - v_i(y_i))}{|y - y_i|^{n+ps}} \, dy \\ &\geq J_1 + J_2 + J_3 + J_4. \end{aligned}$$

For the left-hand side, by (4.3), we obtain

$$\liminf_{i\to\infty} -f_i(y_i) = -\limsup_{i\to\infty} f_i(y_i) < 2 \int_K \frac{v^{p-1}(y)}{|y - \tilde{y}|^{n+ps}} \, dy.$$

For the term J_1, we use $0 \leq u_i \leq B v_i$ in $\mathbb{R}^n \setminus \Omega$, (4.5), and Lebesgue dominated convergence to obtain

$$J_1 \geq 2 \liminf_{i\to\infty} \int_{\mathbb{R}^n \setminus \Omega_{\frac{\delta}{2}}} \frac{-|v_i(y_i)|^{p-2} v_i(y_i)}{|y - y_i|^{n+ps}} \, dy \geq 0.$$

By Lemma 4.2 and (4.6), we get

$$J_2 = 0.$$

For the term J_3, we use that v_i converges uniformly to v on K, together with (4.5), to derive

$$J_3 \geq 2 \int_K \frac{v^{p-1}(y)}{|y - \tilde{y}|^{n+ps}} \, dy.$$

The last term requires more work, and this is the place where we need to use the assumption $(-\Delta_p)^s u \le D$ and the function $\tilde{w}_i$. By (ii) and integration by substitution, we obtain

$$J_4 \ge 2 \liminf_{i\to\infty} \int\limits_{\Omega_{\frac{\delta}{2}}+x_i-y_i\setminus(B(x_i,r_i)\cup K+x_i-y_i)} \frac{\left|\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right|^{p-2}\left(\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right)}{|y-x_i|^{n+ps}}\,dy.$$

Hence, by the second inequality in (4.7), (4.6), and Lemma 4.2, we imply

$$\begin{aligned}
0 &= \liminf_{i\to\infty} -\frac{D}{C_i^{p-1}}\\
&\le 2\lim_{i\to\infty}\frac{1}{\epsilon_i^{p-1}}\int\limits_{B(x_i,r_i)}\frac{||y-y_i|^\beta-|x_i-y_i|^\beta|^{p-2}(|y-y_i|^\beta-|x_i-y_i|^\beta)}{|y-x_i|^{n+ps}}\,dy,\\
&\quad+2\lim_{i\to\infty}\int\limits_{K+x_i-y_i}\frac{\left|\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right|^{p-2}\left(\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right)}{|y-x_i|^{n+ps}}\,dy\\
&\quad+2\lim_{i\to\infty}\int\limits_{\mathbb{R}^n\setminus\Omega_{\frac{\delta}{2}}+x_i-y_i}\frac{\left|\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right|^{p-2}\left(\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right)}{|y-x_i|^{n+ps}}\,dy\\
&\quad+2\liminf_{i\to\infty}\int\limits_{\Omega_{\frac{\delta}{2}}+x_i-y_i\setminus(B(x_i,r_i)\cup K+x_i-y_i)}\frac{\left|\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right|^{p-2}\left(\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right)}{|y-x_i|^{n+ps}}\,dy\\
&=2\liminf_{i\to\infty}\int\limits_{\Omega_{\frac{\delta}{2}}+x_i-y_i\setminus(B(x_i,r_i)\cup K+x_i-y_i)}\frac{\left|\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right|^{p-2}\left(\frac{u_i(y)}{C_i}-\frac{u_i(x_i)}{C_i}\right)}{|y-x_i|^{n+ps}}\,dy\\
&\le J_4.
\end{aligned}$$

In conclusion, $J_4 \ge 0$ and

$$2\int\limits_K \frac{v^{p-1}(y)}{|\tilde{y}-y|^{n+ps}}\,dy > J_1+J_2+J_3+J_4 \ge 2\int\limits_K \frac{v^{p-1}(y)}{|\tilde{y}-y|^{n+ps}}\,dy,$$

which provides the contradiction. □

Lemma 4.4. *Let $u \in L^{p-1}_{ps}(\mathbb{R}^n)\cap C(\Omega)$ be a viscosity supersolution of $(-\Delta_p)^s u = f$ in Ω for $f \in C(\Omega)$. If there exists a point $x_0 \in \Omega$ such that $u(x_0) = \inf_{\mathbb{R}^n} u$, then*

$$2\int\limits_{\mathbb{R}^n}\frac{|u(x_0)-u(y)|^{p-2}(u(x_0)-u(y))}{|x_0-y|^{n+ps}}\,dy \ge f(x_0).$$

Proof. The proof follows the same argument as [1, Lemma 5.3]. Let $x_0 \in \Omega$ satisfy $u(x_0) = \inf_{\mathbb{R}^n} u$. Since $u(x_0)-u(y) \le 0$ for $y \in \mathbb{R}^n$, the integral above is well-defined without P.V. and might be $-\infty$. Let us fix $\beta > \max(\frac{ps}{p-1}, 2)$. Define $u_\epsilon(y) = u(x_0) - |x_0-y|^\beta$ for $y \in B(x_0,\epsilon)$, and $u_\epsilon = u$ in $\mathbb{R}^n \setminus B(x_0,\epsilon)$ for $\epsilon > 0$ small enough, such that $\overline{B(x_0,\epsilon)} \subset \Omega$. Then u_ϵ touches u from below at x_0 and satisfies either condition (a) or (b) in Definition 2.6. Hence,

$$\begin{aligned}
&\frac{2n|B(0,1)|}{\beta(p-1)-ps}\epsilon^{\beta(p-1)-ps}+2\int\limits_{\mathbb{R}^n\setminus B(x_0,\epsilon)}\frac{|u(x_0)-u(y)|^{p-2}(u(x_0)-u(y))}{|x_0-y|^{n+ps}}\,dy\\
&=2\int\limits_{B(x_0,\epsilon)}\frac{|x_0-y|^{\beta(p-1)}}{|x_0-y|^{n+ps}}\,dy+2\int\limits_{\mathbb{R}^n\setminus B(x_0,\epsilon)}\frac{|u_\epsilon(x_0)-u_\epsilon(y)|^{p-2}(u_\epsilon(x_0)-u_\epsilon(y))}{|x_0-y|^{n+ps}}\,dy\\
&=2\int\limits_{\mathbb{R}^n}\frac{|u_\epsilon(x_0)-u_\epsilon(y)|^{p-2}(u_\epsilon(x_0)-u_\epsilon(y))}{|x_0-y|^{n+ps}}\,dy \ge f(x_0).
\end{aligned}$$

In conclusion,

$$f(x_0) + 2 \int\limits_{\mathbb{R}^n \setminus B(x_0,\epsilon)} \frac{|u(y) - u(x_0)|^{p-2}(u(y) - u(x_0))}{|x_0 - y|^{n+ps}} \, dy \le \frac{2n|B(0,1)|}{\beta(p-1) - ps} \epsilon^{\beta(p-1)-ps}.$$

Now, letting $\epsilon \to 0$ and using the monotone convergence theorem, it is obtained that

$$f(x_0) \le 2 \int\limits_{\mathbb{R}^n} \frac{|u(x_0) - u(y)|^{p-2}(u(x_0) - u(y))}{|x_0 - y|^{n+ps}} \, dy. \qquad \square$$

In particular, in the above lemma, if u is non-negative and

$$f(x_0) > -2 \int\limits_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}} \, dy,$$

for any $x_0 \in \Omega$ satisfying $u(x_0) = 0$, then $u > 0$ in Ω. This is the strong maximum principle for fractional p-Laplacian equations. We bring an example to justify the sharpness of the condition

$$f(x_0) > -2 \int\limits_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}} \, dy.$$

Let $\beta > \frac{ps}{p-1}$ and

$$u(x) := \begin{cases} |x|^\beta, & x \in B(0,1), \\ 1, & x \in \mathbb{R}^n \setminus B(0,1). \end{cases}$$

Then $u(0) = 0$ and

$$(-\Delta_p)^s u(0) = -2 \int\limits_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|y|^{n+ps}} \, dy.$$

Now, we prove Hopf's lemma.

Proof of Lemma 1.1. By Lemma 4.4 and the condition

$$f(x_0) > -2 \int\limits_{\mathbb{R}^n} \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}} \, dy \quad \text{if } x_0 \in \Omega, \ u(x_0) = 0,$$

we obtain $u > 0$ in Ω. Finally, if we set $u_i = u_{\mathrm{tor}}$, $v_i = u$ in Lemma 4.3 and use

$$\limsup_{\Omega \ni x \to x_0} f(x) \ge -2 \int\limits_K \frac{u^{p-1}(y)}{|x_0 - y|^{n+ps}} \, dy \quad \text{if } x_0 \in \partial\Omega, \ u(x_0) = 0,$$

we conclude that $u \ge C u_{\mathrm{tor}}$ for a positive constant C. $\square$

Lemma 4.5. *Let $\delta > 0$, $u \in L^{p-1}_{ps}(\mathbb{R}^n) \cap C(\overline{\Omega}_\delta)$ be a viscosity supersolution of*

$$(-\Delta_p)^s u = -2(\operatorname{diam} \Omega)^{-(n+ps)} \int\limits_K u^{p-1}(y) \, dy \quad \text{in } \Omega,$$

for a subset $K \Subset \Omega$, which satisfies $u \ge 0$ in $\mathbb{R}^n \setminus \Omega$. Then either $u = 0$ a.e. in $\mathbb{R}^n \setminus K$, or $u > 0$ in Ω.

Proof. If $u > 0$ in Ω, then the proof is complete. Now, assume that $u(x_0) = \inf_{\mathbb{R}^n} u \le 0$ for a point $x_0 \in \Omega$. Then, by Lemma 4.4,

$$\begin{aligned} 2 \int\limits_{\mathbb{R}^n} \frac{|u(x_0) - u(y)|^{p-2}(u(x_0) - u(y))}{|x_0 - y|^{n+ps}} \, dy &\ge -2(\operatorname{diam} \Omega)^{-(n+ps)} \int\limits_K u^{p-1}(y) \, dy \\ &\ge 2(\operatorname{diam} \Omega)^{-(n+ps)} \int\limits_K |u(x_0) - u(y)|^{p-2}(u(x_0) - u(y)) \, dy \\ &\ge 2 \int\limits_K \frac{|u(x_0) - u(y)|^{p-2}(u(x_0) - u(y))}{|x_0 - y|^{n+ps}} \, dy. \end{aligned}$$

Hence, $u(y) = u(x_0) \le 0$ a.e. in $\mathbb{R}^n \setminus K$. In conclusion, by the assumption $u \ge 0$ in $\mathbb{R}^n \setminus \Omega$, we arrive at $u = 0$ a.e. in $\mathbb{R}^n \setminus K$. $\square$

Proof of Theorem 1.3. Let $\delta > 0$, $u_i \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_{u_i}}(\Omega|\mathbb{R}^n)$, $v_i \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_{v_i}}(\Omega|\mathbb{R}^n)$ be positive on Ω,

$$0 \leq \frac{1}{B} g_{v_i} \leq g_{u_i} \leq B g_{v_i} \leq M \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

and

$$\begin{aligned} -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K u_i^{p-1}(y)\,dy \leq (-\Delta_p)^s u_i \leq 1 \quad \text{in } \Omega, \\ -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K v_i^{p-1}(y)\,dy \leq (-\Delta_p)^s v_i \leq 1 \quad \text{in } \Omega, \end{aligned} \tag{4.8}$$

in the locally weak sense, where $K \Subset \Omega$. We also assume that

$$u_i(x_0) \geq D, \quad v_i(x_0) \geq D \quad (\|u_i\|_{L^q(\Omega)} \geq D,\ \|v_i\|_{L^q(\Omega)} \geq D),$$

for $D > 0$, $1 < q < \infty$. By Proposition 2.9, $\|u_i\|_{L^\infty(\Omega)} + \|v_i\|_{L^\infty(\Omega)} \leq C$ for a constant C depending on n, s, p, Ω, B, M. Also, by local Hölder regularity, see [15, Theorem 5.4], the Arzéla–Ascoli theorem, and passing to a subsequence, u_i, v_i converge uniformly to u, v, respectively, on compact subsets of Ω. Define $u = v = 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$\tilde{u}_i := \begin{cases} u_i & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

$$\tilde{v}_i := \begin{cases} v_i & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, by (4.8) and Definition 2.6, we have

$$\begin{aligned} -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K \tilde{u}_i^{p-1}(y)\,dy \leq (-\Delta_p)^s \tilde{u}_i \quad \text{in } \Omega, \\ -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K \tilde{v}_i^{p-1}(y)\,dy \leq (-\Delta_p)^s \tilde{v}_i \quad \text{in } \Omega, \end{aligned}$$

in the viscosity sense. Hence, by the Lebesgue dominated convergence, Proposition 2.7, and Proposition 2.10, we have

$$u(x_0) \geq D, \quad v(x_0) \geq D \quad (\|u\|_{L^q(\Omega)} \geq D,\ \|v\|_{L^q(\Omega)} \geq D),$$

and

$$\begin{aligned} -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K u^{p-1}(y)\,dy \leq (-\Delta_p)^s u \quad \text{in } \Omega, \\ -2(\operatorname{diam}\Omega)^{-(n+ps)} \int_K v^{p-1}(y)\,dy \leq (-\Delta_p)^s v \quad \text{in } \Omega, \end{aligned}$$

in the viscosity sense. Hence, by Proposition 2.7 and Lemma 4.5, u_i, v_i satisfy (4.8) in the viscosity sense and u, v are strictly positive in Ω. Finally, by Lemma 4.3, $\frac{u_i}{v_i}$ and $\frac{v_i}{u_i}$ are uniformly bounded from below in Ω. This completes the proof. □

In the case that K is empty, we obtain the following result:

Corollary 4.6. *Let $\delta > 0$, $u \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_u}(\Omega|\mathbb{R}^n)$, $v \in C(\overline{\Omega}_\delta) \cap V^{s,p}_{g_v}(\Omega|\mathbb{R}^n)$ satisfy*

$$u > 0, \quad v > 0 \quad \text{in } \Omega,$$

$$0 \leq \frac{1}{B} g_v \leq g_u \leq B g_v \leq M \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

for $B > 0$, $M \geq 0$, and

$$0 \leq (-\Delta_p)^s u \leq 1 \quad \text{in } \Omega,$$
$$0 \leq (-\Delta_p)^s v \leq 1 \quad \text{in } \Omega,$$

in the locally weak sense. If either $u(x_0) \geq D$, $v(x_0) \geq D$ *for a fixed point* $x_0 \in \Omega$ *or* $\|u\|_{L^q(\Omega)} \geq D$, $\|v\|_{L^q(\Omega)} \geq D$ *for* $D > 0$, $1 \leq q < \infty$, *then*

$$C_1 \leq \frac{u}{v} \leq C_2 \quad \text{in } \Omega,$$

where C_1, C_2 *are positive constants depending on* $\Omega, \delta, n, s, p, D, B, M, x_0$ *or* q.

5 Eigenvalue problem

In this section, $\Omega \subset \mathbb{R}^n$ is a bounded open set with Wiener regular boundary for the s-fractional p-Laplacian. Note that, as mentioned in Section 2.2, $\Lambda_{p,q} > 0$ in (1.1) by fractional Poincaré–Sobolev theorem, see Theorem 2.1. Now, it is aimed to prove Theorem 1.5. We divide the proof into several steps. First, we prove the following global boundedness of weak solutions.

Proposition 5.1. *Let* $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ *be a nonzero weak solution of*

$$(-\Delta_p)^s u = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \quad \text{in } \Omega,$$

where $1 < q < p_s^*$, $\lambda > 0$. *Then*

$$\|u\|_{L^\infty(\Omega)} \leq C\lambda^\theta,$$

where θ *is a positive constant depending on* n, s, p, q *and* C *is a constant depending on* n, s, p, q, Ω.

Proof. By Theorem 2.1 and Hölder's inequality, we have $u \in L^q(\Omega)$. Since the problem is scale-invariant, without loss of generality, we can assume that $\|u\|_{L^q(\Omega)} = 1$. Now, we consider three cases. First, if $ps > n$, we can simply apply Theorem 2.1, together with the equation for u, to derive that

$$\|u\|_{L^\infty(\Omega)}^p \leq C[u]_{V^{s,p}(\mathbb{R}^n)}^p = C\lambda,$$

which concludes the proposition for $\theta = \frac{1}{p}$. Now, for $ps = n$, we have

$$\|u\|_{L^\infty(\Omega)} \leq C\| \lambda |u|^{q-2} u \|_{L^{\frac{q}{q-1}}(\Omega)}^{\frac{1}{p-1}} = C\lambda^{\frac{1}{p-1}},$$

where we used [22, Lemma 2.3]. This completes the proposition for $\theta = \frac{1}{p-1}$. Finally, if $ps < n$, we use [22, Lemma 2.3] multiple times until we get L^∞-estimate for u. We note that by Theorem 2.1 and the equation for u, we derive that

$$\|u\|_{L^{p_s^*}(\Omega)} \leq C[u]_{V^{s,p}(\mathbb{R}^n)} = C\lambda^{\frac{1}{p}}.$$

For the case that $u \in L^r(\Omega)$ where $r > \frac{(q-1)n}{ps}$, we can use [22, Lemma 2.3] once to imply that

$$\|u\|_{L^\infty(\Omega)} \leq C\| \lambda |u|^{q-2} u \|_{L^{\frac{r}{q-1}}(\Omega)}^{\frac{1}{p-1}} = C\lambda^{\frac{1}{p-1}} \|u\|_{L^r(\Omega)}^{\frac{q-1}{p-1}},$$

which concludes the proof. Now, if $u \in L^r(\Omega)$ for $r = (q-1)\frac{n}{ps}$, we get

$$\|u\|_{L^t(\Omega)} \leq |\Omega|^{\frac{1}{t}-\frac{1}{r}} \|u\|_{L^r(\Omega)}$$

by Hölder's inequality, where $q - 1 < t < (q-1)\frac{n}{ps}$. Hence, by [22, Lemma 2.3], we obtain

$$\|u\|_{L^{r'}(\Omega)} \leq C\| \lambda |u|^{q-2} u \|_{L^{\frac{t}{q-1}}(\Omega)}^{\frac{1}{p-1}} \leq C\lambda^{\frac{1}{p-1}} |\Omega|^{\frac{q-1}{p-1}(\frac{1}{t}-\frac{1}{r})} \|u\|_{L^r(\Omega)}^{\frac{q-1}{p-1}},$$

where $r' = \frac{n(p-1)t}{n(q-1)-pst}$. Note that as t converges to $(q-1)\frac{n}{ps}$, r' goes to infinity. Hence, by choosing t close enough to $(q-1)\frac{n}{ps}$, we arrive at $u \in L^{r'}(\Omega)$ for $r' > (q-1)\frac{n}{ps}$. In conclusion, the proof follows from the previous case. Finally, we claim that if $p_s^* < \frac{n(q-1)}{ps}$ and we start with $u \in L^{r_1}(\Omega)$ for some $p_s^* \leq r_1 < \frac{n(q-1)}{ps}$, then $u \in L^{r_2}(\Omega)$ with $\frac{r_2}{r_1} > \alpha > 1$, where α depends on n, s, p, q, and

$$\|u\|_{L^{r_2}(\Omega)} \leq C\lambda^{\frac{1}{p-1}} \|u\|_{L^{r_1}(\Omega)}^{\frac{q-1}{p-1}}, \tag{5.1}$$

where C depends on n, s, p, q, Ω. Note that by using the claim finitely many times, we get that u belongs to $\mathrm{L}^r(\Omega)$ for some $r \geq \frac{n(q-1)}{ps}$ which, together with previous cases, completes the proof. To prove the claim, we use [22, Lemma 2.3] as before to obtain (5.1) for

$$r_2 := \frac{n(p-1)r_1}{n(q-1) - psr_1}.$$

Then, by $r_1 \geq p_s^*$ and $q < p_s^*$, we deduce

$$\frac{r_2}{r_1} \geq \frac{n(p-1)}{n(q-1) - psp_s^*} > \frac{n(p-1)}{n(p_s^*-1) - psp_s^*} = 1,$$

which concludes the above claim for $\alpha = \frac{n(p-1)}{n(q-1)-psp_s^*}$. □

Remark 5.2. The same proof can be applied to the general equation

$$(-\Delta_p)^s u = f(u) \quad \text{in } \Omega,$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(u)| \leq C_1 + C_2|u|^{q-1}$ for $1 < q < p_s^*$ and positive constants C_1, C_2, to derive

$$\|u\|_{\mathrm{L}^\infty(\Omega)} \leq g(\|u\|_{\mathrm{L}^q(\Omega)}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a non-negative function depending on n, s, p, q, Ω. Moreover, we do not need the Wiener regular property of Ω in the proof of Proposition 5.1.

Second, we demonstrate that the first eigenvalue $\Lambda_{p,q}$ is simple for $1 < q \leq p$, and the first eigenfunction does not change sign.

Proposition 5.3. *For every $1 < q \leq p$, the weak solutions $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ of*

$$(-\Delta_p)^s u = \Lambda_{p,q}|u|^{q-2}u \quad \text{in } \Omega$$

satisfying $\|u\|_{\mathrm{L}^q(\Omega)} = 1$, are proportional and strictly positive or negative on Ω.

Proof. Assume that $u, v \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ satisfy $\|u\|_{\mathrm{L}^q(\Omega)} = 1 = \|v\|_{\mathrm{L}^q(\Omega)}$ and are weak solutions of

$$(-\Delta_p)^s u = \Lambda_{p,q}|u|^{q-2}u, \quad (-\Delta_p)^s v = \Lambda_{p,q}|v|^{q-2}v \quad \text{in } \Omega. \tag{5.2}$$

Then, by the triangle inequality,

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big||u(x)| - |u(y)|\big|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

and the equality holds if and only if u does not change sign. Hence, by (1.1), the equality occurs above, and, without loss of generality, we can assume that u is non-negative on Ω. To prove that u is positive in Ω, we note that, by Proposition 3.3 and Proposition 5.1, $u \in C(\overline{\Omega})$. In conclusion, by Corollary 1.2 and Proposition 2.7, we derive that $u > 0$ in Ω. Likewise, up to a multiplication with -1, we can assume that $v > 0$ in Ω.

Now, we show that u and v are proportional. Define $\chi_t := (t^{\frac{1}{q}}u, (1-t)^{\frac{1}{q}}v)$ and $\|\cdot\|_{l^q}$ as the l^q-norm in $\mathbb{R}^2$. Since $t \to t^{\frac{p}{q}}$ is a convex function on $\mathbb{R}^+$, we have

$$\|\chi_t(x) - \chi_t(y)\|_{l^q}^p \leq t|u(x) - u(y)|^p + (1-t)|v(x) - v(y)|^p \quad \text{for } x, y \in \mathbb{R}^n.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big|\|\chi_t(x)\|_{l^q} - \|\chi_t(y)\|_{l^q}\big|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y &\leq t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y + (1-t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \Lambda_{p,q},\end{aligned} \tag{5.3}$$

by the triangle inequality and (5.2). Also, we have

$$\big\|\|\chi_t\|_{l^q}\big\|_{\mathrm{L}^q(\Omega)}^q = t\|u\|_{\mathrm{L}^q(\Omega)}^q + (1-t)\|v\|_{\mathrm{L}^q(\Omega)}^q = 1.$$

Hence, by (1.1), the inequalities in (5.3) are equalities. In conclusion, we have the equality

$$\big|\|\chi_t(x)\|_{l^q}-\|\chi_t(y)\|_{l^q}\big|=\|\chi_t(x)-\chi_t(y)\|_{l^q}\quad\text{for a.e. } x,y\in\mathbb{R}^n$$

in the triangle inequality. It follows that $\chi_t(x)=c(x,y)\chi_t(y)$ for a.e. $x,y\in\mathbb{R}^n$. Hence, we have $\frac{u(x)}{v(x)}=\frac{u(y)}{v(y)}$ for a.e. $x,y\in\Omega$. □

The next proposition shows that only for the first eigenvalue there exists a non-negative eigenfunction.

Proposition 5.4. *Let $u\in V_0^{s,p}(\Omega|\mathbb{R}^n)$ be a weak solution of $(-\Delta_p)^s u=\lambda|u|^{q-1}u$, where $1<q\le p$, $\lambda>0$, and $\|u\|_{\mathrm{L}^q(\Omega)}=1$. If u is non-negative in Ω, then $\lambda=\Lambda_{p,q}$.*

Proof. The argument is the same as [12, Theorem 4.1]. Let $\lambda>0$ and $u\in V_0^{s,p}(\Omega|\mathbb{R}^n)$ be a non-negative weak solution of $(-\Delta_p)^s u=\Lambda_{p,q}u^{q-1}$, where $\|u\|_{\mathrm{L}^q(\Omega)}=1$. Assume that $v\in V_0^{s,p}(\Omega|\mathbb{R}^n)$ is a non-negative weak solution of $(-\Delta_p)^s v=\lambda v^{q-1}$ satisfying $\|v\|_{\mathrm{L}^q(\Omega)}=1$. Define the functions $u_\epsilon:=u+\epsilon$, $v_\epsilon:=v+\epsilon$, and

$$\sigma_t^\epsilon(x):=\big(tu_\epsilon^q(x)+(1-t)v_\epsilon^q(x)\big)^{\frac{1}{q}}\quad\text{for } x\in\Omega,\ t\in[0,1].$$

Since $t\to t^{\frac{p}{q}}$ is a convex function on $\mathbb{R}^+$, by the triangle inequality for $\|\cdot\|_{l^q}$, it is obtained that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}&\frac{|\sigma_t^\epsilon(x)-\sigma_t^\epsilon(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y\\&\le t\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)\\&\le t(\Lambda_{p,q}-\lambda).\end{aligned}\tag{5.4}$$

Now, by the convexity of the map, $t\to t^p$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}&\frac{|\sigma_t^\epsilon(x)-\sigma_t^\epsilon(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}}\,\mathrm{d}x\,\mathrm{d}y\\&\ge p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{n+ps}}\big(\sigma_t^\epsilon(x)-\sigma_t^\epsilon(y)-(v(x)-v(y))\big)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{5.5}$$

Hence, using $(-\Delta_p)^s v=\lambda v^{q-1}$ weakly in Ω and the test function $\sigma_t^\epsilon-v_\epsilon$, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{n+ps}}\big(\sigma_t^\epsilon(x)-\sigma_t^\epsilon(y)-(v_\epsilon(x)-v_\epsilon(y))\big)\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega\lambda v^{q-1}(x)\,(\sigma_t^\epsilon(x)-v_\epsilon(x))\,\mathrm{d}x.\tag{5.6}$$

In conclusion, by (5.4), (5.5), and (5.6), we arrive at

$$\lambda p\int_\Omega v^{q-1}(x)\frac{\sigma_t^\epsilon(x)-v_\epsilon(x)}{t}\,\mathrm{d}x\le\Lambda_{p,q}-\lambda,\tag{5.7}$$

where we used $v_\epsilon(x)-v_\epsilon(y)=v(x)-v(y)$ for all $x,y\in\mathbb{R}^n$. Since $t\to t^{\frac{1}{q}}$ is a concave function of $t\in\mathbb{R}^+$, we have

$$v^{q-1}\frac{\sigma_t^\epsilon-v_\epsilon}{t}\ge v^{q-1}(u-v)\quad\text{in }\Omega,$$

and $v^{q-1}(u-v)\in\mathrm{L}^1(\Omega)$ by Proposition 5.1. Then, by applying Fatou's lemma to (5.7), it is implied that

$$\frac{\lambda p}{q}\int_\Omega\Big(\frac{v(x)}{v_\epsilon(x)}\Big)^{q-1}(u_\epsilon^q(x)-v_\epsilon^q(x))\le\lambda p\liminf_{t\to0}\int_\Omega v^{q-1}(x)\frac{\sigma_t^\epsilon(x)-v_\epsilon(x)}{t}\,\mathrm{d}x\le\Lambda_{p,q}-\lambda,$$

for small enough $\epsilon>0$. Hence, by the assumption $v>0$ in Ω and Lebesgue dominated convergence, we deduce

$$0=\frac{\lambda p}{q}\int_\Omega u^q(x)-v^q(x)\,\mathrm{d}x\le\Lambda_{p,q}-\lambda.$$

Combining the above inequality with (1.1) implies that $\lambda=\Lambda_{p,q}$. □

Proof of Theorem 1.5. Let $\lambda_i > \Lambda_{p,q}$, $v_i \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ be a sequence such that

$$\|v_i\|_{\mathrm{L}^q(\Omega)} = 1,$$
$$\lim_{i\to\infty} \lambda_i = \Lambda_{p,q},$$
$$(-\Delta_p)^s v_i = \lambda_i |v_i|^{q-2} v_i,$$

weakly in Ω. By Proposition 5.1,

$$\|v_i\|_{\mathrm{L}^\infty(\Omega)} \leq C\lambda_i^\theta \tag{5.8}$$

for some positive constants θ, C, which are independent of i. Also, $[v_i]^p_{V^{s,p}(\mathbb{R}^n)} = \lambda_i$. Hence, by Theorem 2.3, [4, Theorem 2.7], and passing to a subsequence, v_i converges strongly in Ω to $u \in V_0^{s,p}(\Omega|\mathbb{R}^n)$ with respect to the L^q-norm and $\|u\|_{\mathrm{L}^q(\Omega)} = 1$. Then

$$(-\Delta_p)^s u = \Lambda_{p,q}|u|^{q-2}u \quad \text{weakly in } \Omega.$$

Now, by Proposition 5.3, $u > 0$ a.e. in Ω up to multiplication by a constant. By (5.8), Hölder's regularity for the s-fractional Laplacian, see [15, Theorem 5.4], and the Arzéla–Ascoli theorem, v_i converges uniformly to u on compact subsets of Ω up to a subsequence and multiplication by a constant. Since Ω has a Wiener regular boundary for the s-fractional p-Laplacian, we derive u, v_i, u_{tor} belong to $C(\mathbb{R}^n)$. Now, we want to apply Lemma 4.3 to sequences $u_i = u, v_i$. Since $u > 0$, we need to only check that

$$\lim_{i\to\infty} \lambda_i |v_i(x_i)|^{q-2} v_i(x_i) = 0$$

for every sequence $x_i \in \Omega$ such that $\lim_{i\to\infty} x_i \in \partial\Omega$. To prove this, define

$$\tilde{v}_i := \big(\lambda_i \|v_i\|^{q-1}_{\mathrm{L}^\infty(\Omega)}\big)^{\frac{1}{p-1}} u_{\mathrm{tor}}.$$

Then $\tilde{v}_i = v_i = 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$(-\Delta_p)^s \tilde{v}_i = \lambda_i \|v_i\|^{q-1}_{\mathrm{L}^\infty(\Omega)},$$

weakly in Ω. Hence, $-\tilde{v}_i \leq v_i \leq \tilde{v}_i$ in Ω by comparison principle, see [15, Proposition 2.10]. In particular, by (5.8) and $u_{\mathrm{tor}} \in C(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\lim_{i\to\infty} \lambda_i |v_i(x_i)|^{q-2} v_i(x_i) &= \lim_{i\to\infty} \lambda_i |\tilde{v}_i(x_i)|^{q-2} \tilde{v}_i(x_i)\\ &= \lim_{i\to\infty} \big(\lambda_i \|v_i\|^{q-1}_{\mathrm{L}^\infty(\Omega)}\big)^{\frac{q-1}{p-1}} |u_{\mathrm{tor}}(x_i)|^{q-2} u_{\mathrm{tor}}(x_i) = 0,\end{aligned}$$

for every sequence $x_i \in \Omega$ satisfying $\lim_{i\to\infty} x_i \in \partial\Omega$. In conclusion, by Lemma 4.3, we obtain $v_i \geq Cu$ in Ω for a constant $C > 0$ and large enough i. In particular, $v_i > 0$ in Ω for large enough i, which implies that $\lambda_i = \Lambda_{p,q}$ by Proposition 5.4. This derives the desired contradiction. □

Remark 5.5. If Ω has a $C^{1,1}$ boundary and $p \geq 2$, then one can simplify the proof of Theorem 1.5 by using [16, Theorem 1.1]. Indeed, taking a sequence of functions v_i as in the above proof, we have, by [16, Theorem 1.1], $\frac{v_i}{\mathrm{dist}(x,\partial\Omega)^s}$ is uniformly bounded, and

$$\left\| \frac{v_i}{\mathrm{dist}(x,\partial\Omega)^s} \right\|_{C^\alpha(\bar{\Omega})} \leq C$$

for some positive constants C, α. By the same argument as proof of Theorem 1.5, up to multiplication by a constant and passing to a subsequence, we derive that v_i is sign-changing on Ω and converges pointwise to a non-negative function u. Hence, by the Arzéla–Ascoli theorem, $\frac{v_i}{\mathrm{dist}(x,\partial\Omega)^s}$ converges uniformly to $\frac{u}{\mathrm{dist}(x,\partial\Omega)^s}$ up to a subsequence. Since $\frac{v_i}{\mathrm{dist}(x,\partial\Omega)^s}$ is also continuous and sign-changing on Ω, we derive that $\frac{u}{\mathrm{dist}(x,\partial\Omega)^s}$ goes to zero at a boundary point. This contradicts Corollary 1.2 and [16, Lemma 2.3], since $u \geq C_1 u_{\mathrm{tor}}(x) \geq C_2 \,\mathrm{dist}(x,\partial\Omega)^s$ for $x \in \Omega$, where C_1, C_2 are positive constants.

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