

A Dynamic Stiffness Approach for Vibration Analysis of a Laminated Composite Beam

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ABSTRACT

An exact dynamic stiffness approach for vibration analysis of laminated composite beams with arbitrary ply orientation is introduced in this paper. The influences of Poisson effect, shear deformation and rotary inertia are accounted for in the formulation. The coupled equations of motion are derived first by using the Hamilton principle and the dynamic stiffness matrix is established based on the analytical solutions of the governing differential equations. The automated Muller root search algorithm is then applied to the developed dynamic stiffness matrix to calculate the natural frequencies and mode shapes of the particular composite beams. The influences due to Poisson effect, shear deformation, anisotropy, slenderness ratio and boundary condition on the natural frequencies of the composite beams are investigated. Numerical results of the present method are validated by comparison with those previously published in literature.

Keywords: Laminated beam; Arbitrary ply; Free vibration; Poisson effect; Shear deformation; Rotary inertia

1. INTRODUCTION

Composite materials are widely used in many branches of aerospace, mechanical and civil engineering in the past few decades. Laminated composite beams made of anisotropic materials are ideal for structural applications because of their high strength-to-weight and stiffness-to-weight ratios as well as their ability to be tailored to meet the design requirements of stiffness and strength and favorable fatigue characteristics. The great promise provided by the fiber-reinforced composite materials can be used to improve the performance of the structures that operate in complex environmental conditions. Since the composite beam members have found increasing applications in various areas of technology, it is important to ensure that their design is reliable and safe. The determination of the natural frequencies of the structures is the first step and is usually critical in the design process because much structural behavior and integrity can be deduced from the knowledge of the natural frequencies. Therefore, it is essential for design engineers to evaluate the dynamic characteristics of the composite beam members accurately.

Due to the practical interest and future potential of the laminated composite beams, particularly in the context of aerospace and mechanical applications, the dynamics of the composite beams is a subject of great interest. The different aspects concerning the free vibration behavior of the composite beams have been investigated by a number of researchers who have developed several theories.

Abarcar and Cunniff /1/ performed the free vibration analysis of a simple laminated composite beam without the effects of shear deformation and rotary inertia included. Miller and Adams /2/ presented the vibration characteristics of orthotropic fixed-free beams without including the shear deformation. Vinson and Sierakowski /3/ obtained the analytical solutions for simply-supported composite beams using the classical lamination theory. Banerjee and Williams /4/ developed the exact dynamic stiffness matrix for a uniform, straight, bending-torsion coupled, composite beam with the effects of shear deformation and rotary inertia ignored. Mahapatra *et al.* /5/ proposed a spectral element for Bernoulli-Euler composite beams.

The classical lamination beam theory, in which the transverse shear strain is neglected, overpredicts the natural frequencies. The high ratio of extensional modulus to shear modulus renders classical theory inadequate for the analysis of laminated beams made of advanced composites. An adequate theory must account for the transverse shear deformation.

Teh and Huang /6/ presented two finite element models based on a first-order theory for the free vibration analysis of fixed-free beams of general orthotropy. Chen and Yang /7/ worked out free vibration analysis of symmetrically laminated beams based on the first-order shear deformation theory using finite elements. Chandrashekhara *et al.* /8/ derived the equations of motion of composite beams using a first-order shear deformation theory and obtained the exact frequencies and mode shapes of the composite beams with several boundary conditions. Bhimaraddi and Chandrashekhara /9/ modeled laminated beams by a systematic reduction of the constitutive relations of the three-dimensional anisotropic body and obtained the basic equations of the beam theory based on the parabolic shear deformation theory. Soldatos and Elishakoff /10/ developed a third-order shear deformation theory for static and dynamic analysis of an orthotropic beam with the effects of transverse shear and transverse normal deformations incorporated. Abramovich /11/ gave the exact solutions, based on the Timoshenko type equations, for symmetrically laminated composite beams with ten different boundary conditions. The effects of the rotary inertia and shear deformation on the natural frequencies were investigated for simply-supported beams with square section. Krishnaswamy *et al.* /12/ used the Hamilton principle to formulate the dynamic equations governing the free vibration of laminated composite beams. The influences of transverse shear deformation and rotary inertia were included, and the analytical solutions for unsymmetric laminated beams were obtained by applying the Lagrange multipliers method. Chandrashekhara and Bangera /13/ studied the free vibration characteristics of laminated composite beams using a higher-order beam theory. Abramovich and Livshits /14/ considered the free bending vibration of non-symmetric cross-ply laminated composite Timoshenko beams. Nabi and Ganesan /15/ solved the free vibration problem of laminated composite beams using the finite element method based on the first-order shear deformation theory in which biaxial bending as well as both torsional and longitudinal oscillations were accounted for. Eisenberger *et al.* /16/ used dynamic stiffness analysis and the first-order shear deformation theory to study the free vibration of laminated beams. Teboub and Hajela /17/ used the first-order shear deformation theory to analyze the free vibrations of generally layered composite beams. The model used in this study accounted for both in-plane and rotary inertias. Banerjee and Williams /18/ presented an exact dynamic stiffness matrix for a composite beam with the effects of shear deformation, rotary inertia and coupling between the bending and the torsional deformations included. Kant *et al.* /19/ presented an analytical method for the dynamic analysis of laminated beams using higher-order refined theory. Shimpi and Ainapure /20/ studied the free vibration of two-layered laminated cross-ply beams using the variationally consistent layerwise trigonometric shear deformation theory. Yildirim and Kiral /21/ studied the out-of-plane free vibration problem of symmetric cross-ply laminated beams by the transfer matrix method based on the first-order shear deformation theory. Yildirim /22/ used the stiffness method for the solution of the purely in-plane free vibration problem of symmetric cross-ply laminated beams with the effects of rotary inertia, axial and transverse shear deformations included by the first-order shear deformation theory. Shimpi and Ainapure /23/ presented a simple one-dimensional beam finite element with two nodes and three degrees of freedom per node, based on the layerwise trigonometric shear deformation theory. Rao *et al.* /24/ proposed a higher-order mixed theory for determining the natural frequencies of a diversity of laminated simply-supported beams. Arya *et al.* /25/ developed a zigzag model for symmetric laminated beam in which a sine term was used to represent the nonlinear displacement field across the thickness as compared to a third-order polynomial term in conventional theories. Zero transverse shear stress boundary conditions at the top and bottom of the beam were satisfied. Chakraborty *et al.* /26/ presented a new refined locking free first-order shear deformable finite element and demonstrated its utility in solving the free vibration and wave propagation problems in the laminated composite beam structures with symmetric as well as asymmetric ply stacking. Mahapatra and Gopalakrishnan /27/ presented a spectral finite element model for analysis of axial-flexural-shear coupled wave propagation in thick laminated composite beams and derived an exact dynamic stiffness matrix. Chen *et al.* /28/ presented a new approach combining the state space method and the differential quadrature method for freely vibrating laminated beams based on two-dimensional theory of elasticity. Ruotolo /29/ developed a spectral element for anisotropic, laminated composite beams. The axial-bending coupled equations of motion were derived under the assumptions of the first-order shear deformation theory and the spectral element matrix was formulated.

A review of the literature indicates that, although a large number of publications address the vibration problem of the composite beams, most of the works on the subject focus on the cross-ply composite beams; only a few researches are available that pertain to the vibration of generally layered composite beams. In order to develop a valid mathematical model of generally laminated composite beams, the shear deformation and rotary inertia, which are of great importance from the existing studies, are considered in the present work. Also, the Poisson effect which is significant when considering the analysis of one-dimensional beams with arbitrary ply orientation is included in the beam constitutive equations. The aim of this paper is to formulate the dynamic stiffness matrix for an anisotropic, generally laminated composite beam based on first-order shear deformation theory. The dynamic stiffness method in vibration analysis of laminated beams has certain advantages over the traditional finite element method, particularly when higher frequencies and better accuracies of results are required. This is because, unlike the traditional finite element and other approximate methods, the dynamic stiffness method is based on the analytical solutions of the governing differential equations of motion, and it allows an infinite number of natural frequencies to be accounted for, without any loss of accuracy. Despite the outstanding features of the dynamic stiffness method, there have been few applications to generally lay-up composite beams. The available dynamic stiffness formulations of the composite beams mostly deal with the cross-ply configurations.

This paper is partly motivated by the earlier works and gives the derivation of the exact dynamic stiffness matrix of a uniform generally laminated composite beam member, starting from its basic governing differential equations. The equations of motion of a laminated beam in free vibration are obtained by means of the Hamilton principle. The Poisson effect, shear deformation and rotary inertia are accounted for in the formulation. The application of the proposed dynamic stiffness matrix is discussed and the numerical results for natural frequencies and mode shapes of the laminated beams are presented. The present results are compared with those available in the literature, whenever possible. A parametric study of the influences of the Poisson effect, shear deformation, anisotropy, slenderness ratio and boundary condition on the natural frequencies of the composite beams is performed.

2. ANALYTICAL FORMULATION

A laminated composite beam is considered, and referred to a system of Cartesian coordinates originating on the mid-plane of the beam, with the x-axis being coincident with the beam axis, as shown in Fig. 1. The laminated beam is made of many plies of orthotropic materials, and the principal material axes of a ply may be oriented at an arbitrary angle with respect to the x-axis. The length, breadth and thickness of the beam are represented by L, b and h, respectively.

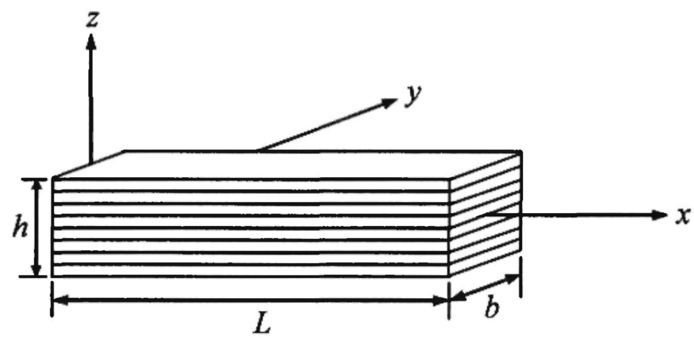


Fig. 1: Geometry of a laminated composite beam.

The displacements of the beam based on first-order shear deformation theory are assumed to be of the forms

$$u(x,z,t) = u_0(x,t) + z\theta(x,t) \tag{1a}$$

$$v(x,z,t) = 0 \tag{1b}$$

$$w(x,z,t) = w_0(x,t) \tag{1c}$$

where $u_0(x,t)$ and $w_0(x,t)$ represent the axial and lateral displacements of a point on the mid-plane in the x and z directions respectively, $\theta(x,t)$ represents the rotation of the normal to the mid-plane about the y axis and t is time.

The constitutive equations of the laminate based on the classical lamination theory can be expressed as

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{Bmatrix} \tag{2}$$

where $\mathbf{N} = \{N_x \quad N_y \quad N_{xy}\}^T$ and $\mathbf{M} = \{M_x \quad M_y \quad M_{xy}\}^T$ are the resultant vectors, $\boldsymbol{\varepsilon} = \{\varepsilon_x^0 \quad \varepsilon_y^0 \quad \gamma_{xy}\}^T$ and $\boldsymbol{\kappa} = \{\kappa_x \quad \kappa_y \quad \kappa_{xy}\}^T$ are the mid-plane vector and the curvature vector, respectively.

For a laminated beam, the membrane forces N_y, N_{xy} and the moments M_y, M_{xy} are zero while the mid-plane strains ε_y^0, γ_{xy} and the curvatures κ_y, κ_{xy} are nonzero. Thus, Eq. (2) can be rewritten as

$$\begin{Bmatrix} N_x \\ M_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \kappa_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix} \begin{Bmatrix} \partial u_0/\partial x \\ \partial \theta/\partial x \end{Bmatrix} \tag{3}$$

where N_x and M_x are the normal force and bending moment per unit length, respectively.

$$\begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} - \begin{bmatrix} A_{12} & A_{16} & B_{12} & B_{16} \\ B_{12} & B_{16} & D_{12} & D_{16} \end{bmatrix} \begin{bmatrix} A_{22} & A_{26} & B_{22} & B_{26} \\ A_{26} & A_{66} & B_{26} & B_{66} \\ B_{22} & B_{26} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{26} & D_{66} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} \\ A_{16} & B_{16} \\ B_{12} & D_{12} \\ B_{16} & D_{16} \end{bmatrix}$$

It is noted that ε_y^0, γ_{xy}, κ_y and κ_{xy} are solved in terms of ε_x^0 and κ_x in order to obtain Eq. (3).

If the effect of transverse shear deformation is taken into account, then

$$Q_{xz} = A_{55}\gamma_{xz} = A_{55}(\partial w_0/\partial x + \theta) \tag{4}$$

where Q_{xz} is the transverse shear force per unit length.

The laminate stiffness coefficients A_{ij}, B_{ij}, D_{ij} ($i, j = 1,2,6$) and the transverse shear stiffness A_{55}, which are functions of laminate ply orientation, material property and stack sequence, are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2)dz \qquad (i, j = 1,2,6) \tag{5a}$$

$$A_{55} = k\int_{-h/2}^{h/2} \bar{Q}_{55}dz \tag{5b}$$

where k is the shear correction factor, which is chosen as 5/6 in order to compare with available results in the literature, although the factor may change with mode number, Poisson's ratio and lamination scheme. The transformed reduced stiffness constants $\bar{Q}_{ij}$ ($i, j = 1,2,6$) and $\bar{Q}_{55}$ can be found in Ref. /30/.

For the present laminated composite beam, the total strain energy V is given by the relationship

$$V = \frac{1}{2}\int_0^L [N_x\,\partial u_0/\partial x + M_x\,\partial\theta/\partial x + Q_{xz}(\partial w_0/\partial x + \theta)]bdx \tag{6}$$

The total kinetic energy T of the laminated beam is given by

$$T = \frac{1}{2}\int_0^L [I_1(\partial u_0/\partial t)^2 + I_3(\partial\theta/\partial t)^2 + 2I_2(\partial u_0/\partial t)(\partial\theta/\partial t) + I_1(\partial w_0/\partial t)^2]bdx \tag{7}$$

where ρ is the mass density of beam material and

$$(I_1, I_2, I_3) = \int_{-h/2}^{h/2} \rho(1, z, z^2)dz$$

The governing equations of motion for the laminated composite beam are derived using the Hamilton principle

$$\int_{t_1}^{t_2}(\delta T - \delta V)dt = 0 \tag{8}$$

$$\delta u_0 = \delta w_0 = \delta\theta = 0 \text{ at } t = t_1, t_2$$

Substituting Eqs. (3) and (4) into Eq. (6), then substituting the resulting equation and Eq. (7) into Eq. (8) and carrying out the variational operations yields the following set of partial differential equations which govern the motion of laminated beam having constant properties along its length

$$-I_1(\partial^2 u_0/\partial t^2) - I_2(\partial^2\theta/\partial t^2) + \bar{A}_{11}(\partial^2 u_0/\partial x^2) + \bar{B}_{11}(\partial^2\theta/\partial x^2) = 0 \tag{9a}$$

$$-I_1(\partial^2 w_0/\partial t^2) + A_{55}(\partial^2 w_0/\partial x^2) + A_{55}(\partial\theta/\partial x) = 0 \tag{9b}$$

$$-I_3(\partial^2\theta/\partial t^2) - I_2(\partial^2 u_0/\partial t^2) + \bar{B}_{11}(\partial^2 u_0/\partial x^2) + \bar{D}_{11}(\partial^2\theta/\partial x^2) - A_{55}(\partial w_0/\partial x) - A_{55}\theta = 0 \tag{9c}$$

3. DYNAMIC STIFFNESS MATRIX

The harmonic solutions of Eqs. (9) for the displacements u_0, w_0 and rotation θ are assumed as

$$u_0(x,t) = U(x)\sin\omega t \qquad w_0(x,t) = W(x)\sin\omega t \qquad \theta(x,t) = \Theta(x)\sin\omega t \tag{10}$$

where ω is the circular frequency, $U(x)$, $W(x)$, and $\Theta(x)$ are the amplitudes of the sinusoidally varying axial displacement, lateral displacement, and normal rotation respectively.

Substitution of Eqs. (10) into Eqs. (9) leads to the following ordinary differential equations

$$\omega^2 I_1 U + \omega^2 I_2 \Theta + \bar{A}_{11} U'' + \bar{B}_{11} \Theta'' = 0 \tag{11a}$$

$$\omega^2 I_1 W + A_{55} W'' + A_{55} \Theta' = 0 \tag{11b}$$

$$\omega^2 I_3 \Theta + \omega^2 I_2 U + \bar{B}_{11} U'' + \bar{D}_{11} \Theta'' - A_{55} W' - A_{55} \Theta = 0 \tag{11c}$$

where the superscript primes refer to the derivatives with respect to the axial coordinate x.

Equations (11) are solved by choosing for U, W and Θ the following dependence on x

$$U(x) = \bar{A} e^{\kappa x} \qquad W(x) = \bar{B} e^{\kappa x} \qquad \Theta(x) = \bar{C} e^{\kappa x} \tag{12}$$

Inserting Eqs. (12) into Eqs. (11), the algebraic eigenvalue equation is obtained and the equation has nontrivial solutions when the determinant of the coefficient matrix of the $\bar{A}$, $\bar{B}$, and $\bar{C}$ vanishes. Setting the determinant equal to zero yields the characteristics equation, which is a sixth-order polynomial equation.

$$\eta_3 \kappa^6 + \eta_2 \kappa^4 + \eta_1 \kappa^2 + \eta_0 = 0 \tag{13}$$

where

$$\eta_3 = -(\bar{B}_{11}^2 - \bar{A}_{11}\bar{D}_{11})A_{55}$$

$$\eta_2 = -\bar{B}_{11}^2 I_1 \omega^2 + (\bar{A}_{11}(\bar{D}_{11} I_1 + I_3 A_{55}) + (\bar{D}_{11} I_1 - 2\bar{B}_{11} I_2) A_{55})\omega^2$$

$$\eta_1 = -A_{55} I_1 \bar{A}_{11} \omega^2 + (\bar{D}_{11} I_1^2 - 2\bar{B}_{11} I_1 I_2 + \bar{A}_{11} I_1 I_3 - (I_2^2 - I_1 I_3) A_{55})\omega^4$$

$$\eta_0 = I_1 \omega^4 (-A_{55} I_1 + (-I_2^2 + I_1 I_3)\omega^2)$$

Equation (13) can be rewritten as

$$\chi^3 + a_1 \chi^2 + a_2 \chi + a_3 = 0 \tag{14}$$

where

$$\chi = \kappa^2 \qquad a_1 = \eta_2/\eta_3 \qquad a_2 = \eta_1/\eta_3 \qquad a_3 = \eta_0/\eta_3$$

The three roots of Eq. (14) can be expressed as

$$\chi_1 = 2\sqrt{q/3}\cos(\vartheta/3) - a_1/3 \quad \chi_2 = 2\sqrt{q/3}\cos\big((2\pi + \vartheta)/3\big) - a_1/3 \quad \chi_3 = 2\sqrt{q/3}\cos\big((4\pi + \vartheta)/3\big) - a_1/3 \tag{15}$$

where

$$q = -a_2 + a_1^2/3 \qquad \bar{\vartheta} = \cos^{-1}\left[-27a_3 + 9a_1a_2 - 2a_1^3 \Big/ 2\sqrt{\left(a_1^2 - 3a_2\right)^3} \right]$$

The general solutions to Eqs. (11) are given by

$$U(x) = A_1e^{\kappa_1 x} + A_2e^{-\kappa_1 x} + A_3e^{\kappa_2 x} + A_4e^{-\kappa_2 x} + A_5e^{\kappa_3 x} + A_6e^{-\kappa_3 x} = \sum_{j=1}^{3}(A_{2j-1}e^{\kappa_j x} + A_{2j}e^{-\kappa_j x}) \tag{16a}$$

$$W(x) = B_1e^{\kappa_1 x} + B_2e^{-\kappa_1 x} + B_3e^{\kappa_2 x} + B_4e^{-\kappa_2 x} + B_5e^{\kappa_3 x} + B_6e^{-\kappa_3 x} = \sum_{j=1}^{3}(B_{2j-1}e^{\kappa_j x} + B_{2j}e^{-\kappa_j x}) \tag{16b}$$

$$\Theta(x) = C_1e^{\kappa_1 x} + C_2e^{-\kappa_1 x} + C_3e^{\kappa_2 x} + C_4e^{-\kappa_2 x} + C_5e^{\kappa_3 x} + C_6e^{-\kappa_3 x} = \sum_{j=1}^{3}(C_{2j-1}e^{\kappa_j x} + C_{2j}e^{-\kappa_j x}) \tag{16c}$$

where $\kappa_1 = \sqrt{\chi_1}$, $\kappa_2 = \sqrt{\chi_2}$, $\kappa_3 = \sqrt{\chi_3}$. If any of the χ_j's are zero or are repeated in the solution of Eq. (14), the solutions (16) have to be modified according to the well-known methods of ordinary differential equations with constant coefficients. From Eqs. (11), only six of the eighteen constants are independent. The relationship among the constants is given by

$$A_{2j-1} = t_jC_{2j-1} \qquad A_{2j} = t_jC_{2j}$$
$$B_{2j-1} = \bar{t}_jC_{2j-1} \qquad B_{2j} = -\bar{t}_jC_{2j}$$

where

$$t_j = -(\bar{B}_{11}\kappa_j^2 + I_2\omega^2)/(\bar{A}_{11}\kappa_j^2 + I_1\omega^2) \qquad \bar{t}_j = -A_{55}\kappa_j/(A_{55}\kappa_j^2 + I_1\omega^2) \quad (j = 1\sim 3)$$

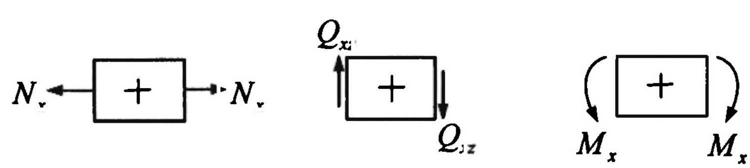


Fig. 2: Sign convention for positive normal force $N_x(x)$, shear force $Q_{xz}(x)$, and bending moment $M_x(x)$.

According to the sign convention shown in Fig. 2, the expressions of normal force $N_x(x)$, shear force $Q_{xz}(x)$, and bending moment $M_x(x)$ can be obtained from Eqs. (3), (4) and (16) as follows

$$N_x(x) = \bar{A}_{11}\, dU/dx + \bar{B}_{11}\, d\Theta/dx = \sum_{j=1}^{3}(\bar{A}_{11}\kappa_j t_j + \bar{B}_{11}\kappa_j)(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}) \tag{17a}$$

$$Q_{xz}(x) = -(A_{55}\, dW/dx + A_{55}\Theta) = \sum_{j=1}^{3} -(A_{55}\kappa_j\bar{t}_j + A_{55})(C_{2j-1}e^{\kappa_j x} + C_{2j}e^{-\kappa_j x}) \tag{17b}$$

$$M_x(x) = -(\bar{B}_{11}\, dU/dx + \bar{D}_{11}\, d\Theta/dx) = \sum_{j=1}^{3} -(\bar{B}_{11}\kappa_j t_j + \bar{D}_{11}\kappa_j)(C_{2j-1}e^{\kappa_j x} - C_{2j}e^{-\kappa_j x}) \tag{17c}$$

Referring to Fig. 3, the boundary conditions for displacements and forces of the laminated composite beam element are, respectively,

$$\begin{aligned} x=0:&\ U=U_1 \quad W=W_1 \quad \Theta=\Theta_1 \\ x=L:&\ U=U_2 \quad W=W_2 \quad \Theta=\Theta_2 \end{aligned} \tag{18a}$$

$$\begin{aligned} x=0:&\ N_x=-N_{x1} \quad Q_{xz}=Q_{xz1} \quad M_x=M_{x1} \\ x=L:&\ N_x=N_{x2} \quad Q_{xz}=-Q_{xz2} \quad M_x=-M_{x2} \end{aligned} \tag{18b}$$

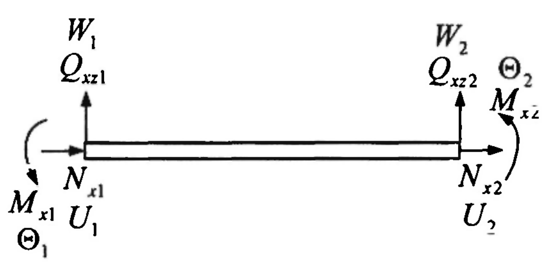


Fig. 3: Boundary conditions for displacements and forces of composite beam element.

Substituting Eqs. (18a) into Eqs. (16), the nodal displacements defined in Fig. 3 can be expressed in terms of C_j's as

$$\{D\} = [R]\{C\} \tag{19}$$

where $\{D\}$ is the nodal degree-of-freedom vector defined by

$$\{D\} = \{U_1 \quad W_1 \quad \Theta_1 \quad U_2 \quad W_2 \quad \Theta_2\}^T$$

$$\{C\} = \{C_1 \quad C_3 \quad C_5 \quad C_2 \quad C_4 \quad C_6\}^T$$

$$[R] = \begin{bmatrix} t_1 & t_2 & t_3 & t_1 & t_2 & t_3 \\ \bar{t}_1 & \bar{t}_2 & \bar{t}_3 & -\bar{t}_1 & -\bar{t}_2 & -\bar{t}_3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ t_1e^{\kappa_1 L} & t_2e^{\kappa_2 L} & t_3e^{\kappa_3 L} & t_1e^{-\kappa_1 L} & t_2e^{-\kappa_2 L} & t_3e^{-\kappa_3 L} \\ \bar{t}_1e^{\kappa_1 L} & \bar{t}_2e^{\kappa_2 L} & \bar{t}_3e^{\kappa_3 L} & -\bar{t}_1e^{-\kappa_1 L} & -\bar{t}_2e^{-\kappa_2 L} & -\bar{t}_3e^{-\kappa_3 L} \\ e^{\kappa_1 L} & e^{\kappa_2 L} & e^{\kappa_3 L} & e^{-\kappa_1 L} & e^{-\kappa_2 L} & e^{-\kappa_3 L} \end{bmatrix}$$

Substituting Eqs. (18b) into Eqs. (17), the nodal forces defined in Fig. 3 can be expressed in terms of C_j's as

$$\{F\} = [H]\{C\} \tag{20}$$

where $\{F\}$ is the nodal force vector defined by

$$\{F\}=\{N_{x1}\ \ Q_{xz1}\ \ M_{x1}\ \ N_{x2}\ \ Q_{xz2}\ \ M_{x2}\}^T$$

$$[H]=\begin{bmatrix} -\hat{t}_1 & -\hat{t}_2 & -\hat{t}_3 & \hat{t}_1 & \hat{t}_2 & \hat{t}_3 \\ \bar{\bar{t}}_1 & \bar{\bar{t}}_2 & \bar{\bar{t}}_3 & \bar{\bar{t}}_1 & \bar{\bar{t}}_2 & \bar{\bar{t}}_3 \\ \hat{\hat{t}}_1 & \hat{\hat{t}}_2 & \hat{\hat{t}}_3 & -\hat{\hat{t}}_1 & -\hat{\hat{t}}_2 & -\hat{\hat{t}}_3 \\ \hat{t}_1 e^{\kappa_1 L} & \hat{t}_2 e^{\kappa_2 L} & \hat{t}_3 e^{\kappa_3 L} & -\hat{t}_1 e^{-\kappa_1 L} & -\hat{t}_2 e^{-\kappa_2 L} & -\hat{t}_3 e^{-\kappa_3 L} \\ -\bar{\bar{t}}_1 e^{\kappa_1 L} & -\bar{\bar{t}}_2 e^{\kappa_2 L} & -\bar{\bar{t}}_3 e^{\kappa_3 L} & -\bar{\bar{t}}_1 e^{-\kappa_1 L} & -\bar{\bar{t}}_2 e^{-\kappa_2 L} & -\bar{\bar{t}}_3 e^{-\kappa_3 L} \\ -\hat{\hat{t}}_1 e^{\kappa_1 L} & -\hat{\hat{t}}_2 e^{\kappa_2 L} & -\hat{\hat{t}}_3 e^{\kappa_3 L} & \hat{\hat{t}}_1 e^{-\kappa_1 L} & \hat{\hat{t}}_2 e^{-\kappa_2 L} & \hat{\hat{t}}_3 e^{-\kappa_3 L} \end{bmatrix}$$

in which

$$\hat{t}_j=\bar{A}_{11}\kappa_j t_j+\bar{B}_{11}\kappa_j \qquad \bar{\bar{t}}_j=-(A_{55}\kappa_j\bar{t}_j+A_{55}) \qquad \hat{\hat{t}}_j=-(\bar{B}_{11}\kappa_j t_j+\bar{D}_{11}\kappa_j)\ \ (j=1\sim 3)$$

Eliminating the coefficients C_j's from Eqs. (19) and (20) gives the following relationship between the nodal forces and nodal displacements

$$\{F\}=[H][R]^{-1}\{D\}=[K]\{D\} \tag{21}$$

where $[K]$ is the frequency dependent dynamic stiffness matrix. The dynamic equilibrium equation (21), from a purely formal point of view, resembles the well-known static equilibrium equation of the laminated beam. It should be mentioned that the explicit analytical expressions for the elements of the dynamic stiffness matrix could be derived using the symbolic manipulator software such as Mathematica /31/, although the expressions are too lengthy to list in the paper.

4. AUTOMATED MULLER ROOT SEARCH METHOD

Once the dynamic stiffness matrix is obtained, the appropriate boundary conditions for the particular laminated beams under consideration are applied to obtain the frequency characteristic equation. It should be noted that the conventional finite element method generally leads to a linear eigenvalue problem whereas the dynamic stiffness method often leads to a transcendental eigenvalue problem. A simple automated Muller root search method /32/ is adopted in the present study to obtain all the natural frequencies in a given frequency band. The mode shapes corresponding to the natural frequencies can be found in the usual way by making an arbitrary assumption about one unknown variable of the laminated beam and then calculating the remaining variables in terms of the arbitrarily chosen one.

Let us denote the frequency characteristic determinant by $f(\omega)$ and consider the solution of $f(\omega)=0$ by Muller's method. Muller's method uses a quadratic approximation to the function f by interpolating a quadratic polynomial through the last three computed points and then determining where this curve crosses the ω axis. The following algorithm is an implementation of Muller's method. This algorithm terminates when a prescribed error tolerance on $|f(\omega)|$ has been achieved or a maximum number of iterations has been performed.

(1) Pick three distinct values of ω, i.e., $\omega_0, \omega_1, \omega_2$. Set $i = 1$ and compute

$$h_i = \omega_i - \omega_{i-1}\,;\ h_{i+1} = \omega_{i+1} - \omega_i\,;\ \delta_i = \left(f(\omega_i) - f(\omega_{i-1})\right)/h_i\,;$$
$$\delta_{i+1} = \left(f(\omega_{i+1}) - f(\omega_i)\right)/h_{i+1}\,;\ d_i = \left(\delta_{i+1} - \delta_i\right)/\left(h_{i+1} + h_i\right).$$

(2) Compute

$$b_i = \delta_{i+1} + h_{i+1} d_i\,;\ D_i = \left(b_i^2 - 4f(\omega_{i+1}) d_i\right)^{1/2}.$$

(3) If $|b_i - D_i| < |b_i + D_i|$ then set $E_i = b_i + D_i$; else set $E_i = b_i - D_i$

(4) Set $h_{i+2} = -2f(\omega_{i+1})/E_i$; $\omega_{i+2} = \omega_{i+1} + h_{i+2}$.

where ω_{i+2} is the new approximation of the root ω_{i+1}. The process is continued until a desired accuracy is obtained. The rate of convergence is high and may be increased if an estimate of the eigenvalue is available. Previously roots may be removed from $f(\omega)$ by dividing it by $\prod_{k=1}^{p}(\omega - \omega_k)$, where ω_k ($k = 1, 2, \cdots, p$) are the first p roots.

5. NUMERICAL RESULTS

Four examples are presented to demonstrate the correctness and accuracy of the dynamic stiffness formulation presented in the preceding sections. For each example, the following boundary conditions: Clamped-Clamped, Clamped-Simply supported, Free-Free, Clamped-Free and Simply-Supported are considered.

Clamped edge: $U = W = \Theta = 0$
Simply supported edge: $U = W = M_x = 0$
Free edge: $N_x = Q_{xz} = M_x = 0$

As a first example, the natural frequencies of a glass-polyester composite beam of rectangular cross-section with lay-up scheme [40°/50°/40°/50°] are calculated and the results are given in Tables 1 and 2. The properties of the laminated beam are given as follows:

$E_1 = 37.41\times10^9$ Pa	$E_2 = 13.67\times10^9$ Pa	$G_{12} = 5.478\times10^9$ Pa	$G_{13} = 6.03\times10^9$ Pa
$G_{23} = 6.666\times10^9$ Pa	$\nu_{12} = 0.3$	$\rho = 1968.9$ kg/m^3	
$L = 0.11179$ m	$b = 12.7\times10^{-3}$ m	$h = 3.38\times10^{-3}$ m	

The natural frequencies of the glass-polyester laminated beam with five different boundary conditions are calculated by using the dynamic stiffness matrix developed in this paper, and the numerical results of the first four frequencies of the case study are summarized in Tables 1 and 2. The results shown in Table 2 are obtained by neglecting the Poisson effect, i.e., assuming the composite beam in cylindrical bending.

Tables 1 and 2 illustrates quite well that Poisson effect can have a strong influence on the natural frequencies of the composite beam. From this point of view, any modeling methods based on the assumption, that the influence of Poisson effect is neglected, may yield considerable inaccuracy for the natural frequency prediction.

Table 1

Natural frequencies (in Hz) of glass-polyester composite beam

Mode No.	Clamped-Clamped	Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
1	770.6	532.4	772.7	121.7	341.5
2	2108.2	1715.1	2117.6	759.8	1360.0
3	4091.2	3547.6	4116.3	2113.8	3038.3
4	6678.5	5999.1	6729.8	4103.6	5348.9

Table 2

Natural frequencies (in Hz) of glass-polyester composite beam with Poisson effect ignored

Mode No.	Clamped-Clamped	Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
1	899.3	621.9	903.0	142.3	399.4
2	2454.5	1999.9	2471.5	887.3	1587.7
3	4749.7	4126.6	4794.7	2464.0	3540.5
4	7727.1	6957.3	7818.6	4772.0	6216.7

The second example is a graphite-epoxy composite beam with lay-up [30°/50°/30°/50°]. This example is selected due to some comparative results being available in the literature /13, 33/. All layers of the composite beam have the same thickness and every orthotropic lamina has the following material properties:

$$E_1 = 144.80\times10^9\,\text{Pa} \qquad E_2 = 9.65\times10^9\,\text{Pa} \qquad G_{12} = G_{13} = 4.14\times10^9\,\text{Pa} \qquad G_{23} = 3.45\times10^9\,\text{Pa}$$

$$\nu_{12} = 0.3 \qquad \rho = 1389.23\,\text{kg/m}^3 \qquad L = 0.381\,\text{m} \qquad b = 25.4\times10^{-3}\,\text{m} \qquad h = 25.4\times10^{-3}\,\text{m}$$

Table 3 lists the first five natural frequencies of the beam under various boundary conditions. A good agreement can be observed with the corresponding reference values given in Refs. /13, 33/.

Considering Table 3, the authors think that the fourth natural frequency of Ref. /13/, obtained by the third-order shear deformation theory, is incorrect. As can be seen from the numerical results shown in Table 3, the fourth frequency of Ref. /13/ for Clamped-Clamped boundary condition is smaller than the one for Simply-Supported boundary condition.

In order to investigate the effect of shear deformation on the natural frequencies, the numerical results of this example with the shear deformation excluded are displayed in Table 4.

It can be observed from Tables 3 and 4 that the shear deformation is seen to have relatively marginal effect on the lower natural frequencies of this particular beam. Indeed, the fourth natural frequency of the Free-Free beam and Clamped-Free beam and the fifth natural frequency of the Simply-Supported beam are virtually unaltered due to these frequencies corresponding to the longitudinal modes, for which shear deformation would not be expected to have any major effect. It can also be seen that the effect of shear deformation increases when the number of modes increases.

Table 3
Natural frequencies (in Hz) of graphite-epoxy composite beam

Mode No.	Clamped-Clamped Present	Clamped-Clamped Ref. /13/	Clamped-Clamped Ref. /33/	Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
1	639.0	640.5	639.1	451.0	660.1	105.4	295.2
2	1656.7	1666.8	1663.5	1391.0	1741.4	638.2	1134.3
3	3028.3	3059.5	3071.2	2724.8	3219.7	1699.0	2419.1
4	4644.0	3397.8	N/A	4338.6	4951.1	2475.5	4022.6
5	4950.7	4712.5	N/A	4949.7	4971.4	3120.7	4949.4

Note: N/A denotes the result not available.

Table 4
Natural frequencies (in Hz) of graphite-epoxy composite beam with shear deformation ignored

Mode No.	Clamped-Clamped	Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
1	673.5	464.5	668.9	106.0	298.5
2	1845.2	1496.0	1824.2	660.9	1182.4
3	3582.7	3091.6	3526.8	1835.4	2637.9
4	4951.1	4942.7	4952.0	2475.6	4612.0
5	5846.2	5228.1	5733.0	3554.6	4967.0

The third example is related to a laminated composite beam with lay-up sequence [30°/-60°/30°/-60°]. The material properties of the beam are taken from Ref. /17/ and are given below

$E_2 = 6.9\times10^9$ Pa $\quad G_{12} = 4.8\times10^9$ Pa $\quad G_{13} = 4.14\times10^9$ Pa $\quad G_{23} = 3.45\times10^9$ Pa

$\nu_{12} = 0.3 \quad \rho = 1550.1$ kg/m^3 $\quad b = 25.4\times10^{-3}$ m $\quad h = 25.4\times10^{-3}$ m

In order to illustrate the effect of material anisotropy on the natural frequencies of the beam, two different values of material constant E_1, i.e. $E_1 = 221\times10^9$ Pa and $E_1 = 303\times10^9$ Pa are investigated. Also two different beam lengths are considered: $L = 0.381$ m and $L = 0.572$ m.

Results for the first six natural frequencies of the laminated beam with $E_1 = 221\times10^9$ Pa and $L = 0.381$ m are shown in Table 5. The comparison indicates that the present results agree very well with the corresponding results from Ref. /17/, with a maximum relative error of 1%.

Table 6 lists the results for the natural frequencies of the laminated beam with $E_1 = 303\times10^9$ Pa and $L = 0.381$ m. Even here, the present results compare well with the results reported in Ref. /17/. As can be observed from Tables 5 and 6, an increase in value of E_1 results in increasing natural frequencies.

Table 7 shows the natural frequencies of the composite beam with $E_1 = 221\times10^9$ Pa and $L = 0.572$ m. Results tabulated in Tables 5 and 7 indicate that the natural frequencies decrease with the increase of beam length.

The normal mode shapes associated with the first six natural frequencies of the Clamped-Free beams in the cases of $E_1 = 221\times10^9$ Pa, $L = 0.381$ m; $E_1 = 303\times10^9$ Pa, $L = 0.381$ m and $E_1 = 221\times10^9$ Pa, $L = 0.572$ m are presented in Figs. 4-6, respectively. It is seen from Figs. 4-6 that the axial displacement is small as compared to the lateral displacement for each of the six modes except for the fourth mode in Figs. 4 and 5 and the fifth mode in Fig. 6.

It is apparent from Figs. 4 and 5 that there is little difference for the first six modes, except for the fourth mode. As can be seen from Figs. 4 and 6, there is noticeable difference for the first six mode shapes.

Table 5

Natural frequencies (in Hz) of composite beam with $E_1 = 221\times10^9$ Pa and $L = 0.381$ m

Mode No.	Clamped-Clamped		Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
	Present	Ref. /17/				
1	636.3	635.8	450.1	660.7	105.6	294.6
2	1639.2	1633.0	1381.6	1735.5	637.0	1131.2
3	2977.9	2957.5	2690.8	3191.9	1686.9	2398.7
4	4541.8	4496.4	4261.2	4900.6	2472.1	3969.4
5	4944.2	N/A	4943.9	4944.3	3080.7	4943.8
6	6257.9	N/A	6004.7	6767.4	4714.2	5739.5

Table 6

Natural frequencies (in Hz) of composite beam with $E_1 = 303\times10^9$ Pa and $L = 0.381$ m

Mode No.	Clamped-Clamped		Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
	Present	Ref. /17/				
1	646.9	629.2	458.0	672.8	107.6	299.9
2	1663.2	1617.4	1403.6	1764.8	648.0	1150.5
3	3016.0	2931.1	2728.6	3240.5	1713.5	2435.6
4	4592.5	4459.7	4313.9	4966.7	2517.5	4023.6
5	5035.1	N/A	5034.9	5035.2	3123.7	5034.8
6	6319.0	N/A	6069.4	6847.6	4772.1	5807.8

Table 7

Natural frequencies (in Hz) of composite beam with $E_1 = 221\times10^9$ Pa and $L = 0.572$ m

Mode No.	Clamped-Clamped	Clamped-Simply supported	Free-Free	Clamped-Free	Simply-Supported
1	291.9	203.6	297.0	47.0	131.7
2	778.2	643.0	800.3	289.7	516.8
3	1464.7	1295.0	1519.1	789.5	1129.4
4	2310.9	2122.2	2412.4	1491.3	1933.0
5	3282.4	3087.6	3293.1	1646.6	2890.0
6	3293.3	3293.2	3442.7	2360.4	3292.5

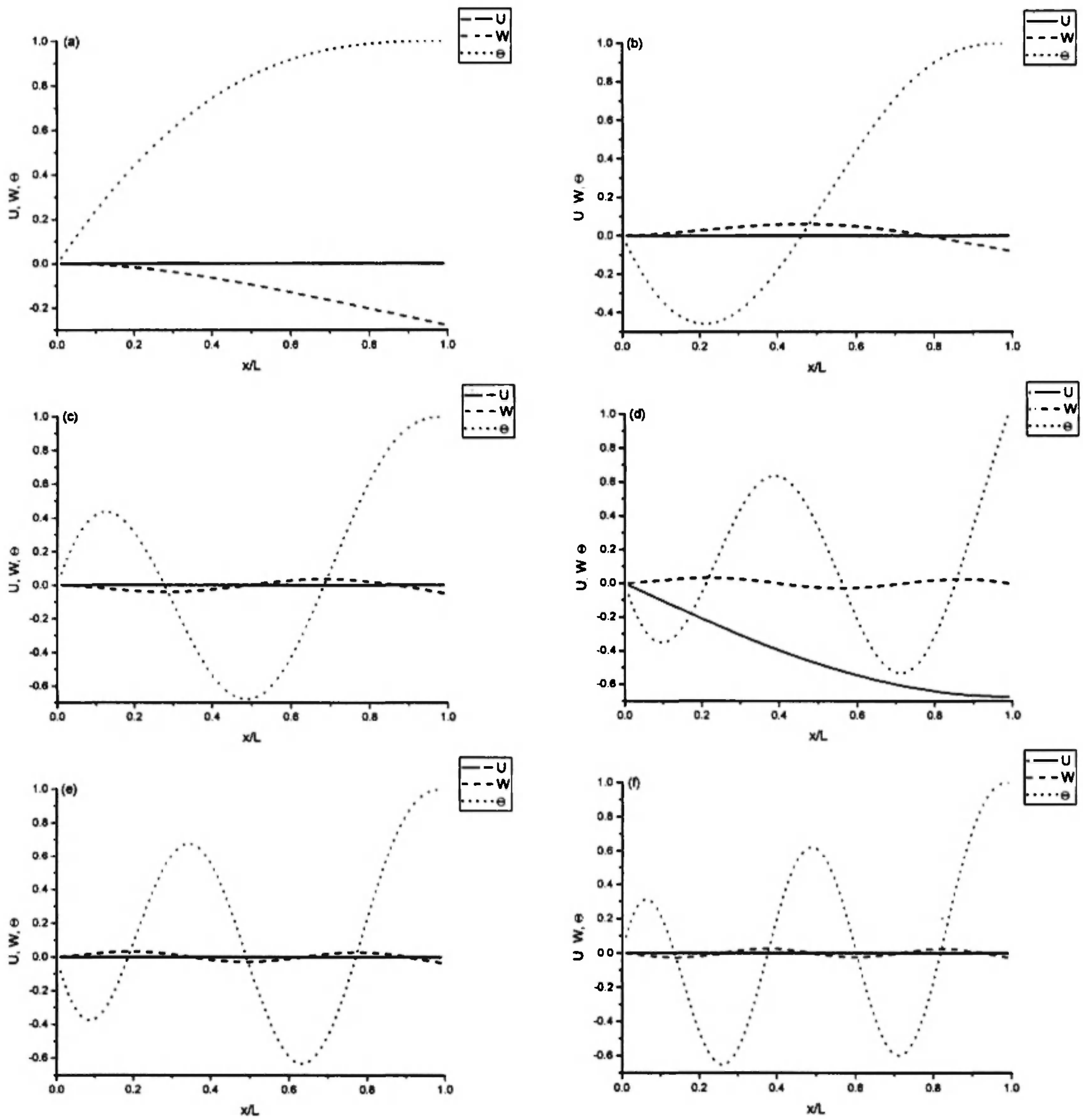
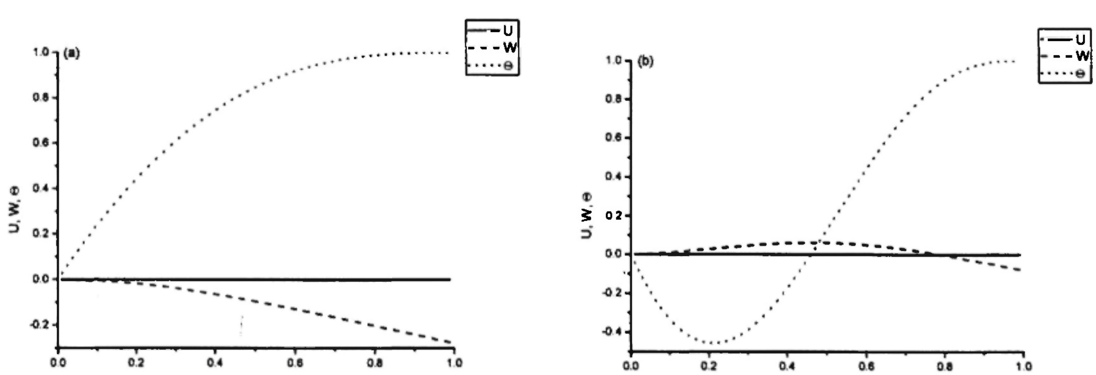


Fig. 4: First six normal mode shapes of Clamped-Free composite beam with $E_1 = 221\times10^9$ Pa and $L = 0.381$ m (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5; (f) mode 6.



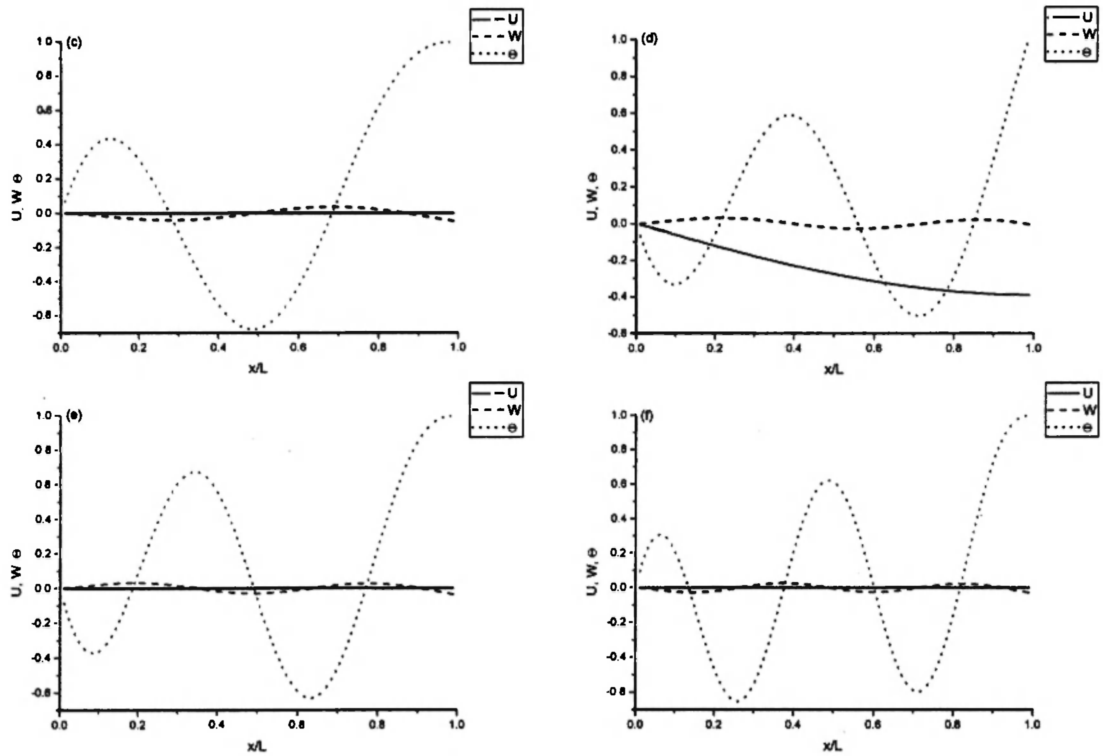
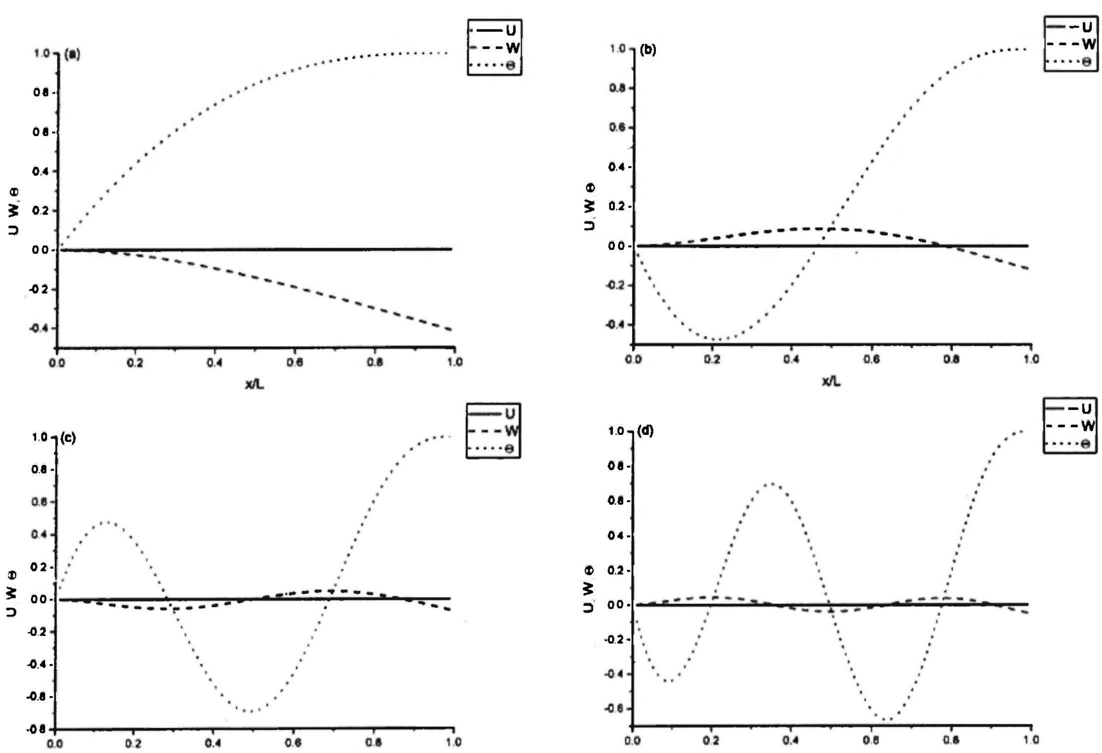


Fig. 5: First six normal mode shapes of Clamped-Free composite beam with $E_1 = 303\times10^9$ Pa and $L = 0.381$ m (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5; (f) mode 6.



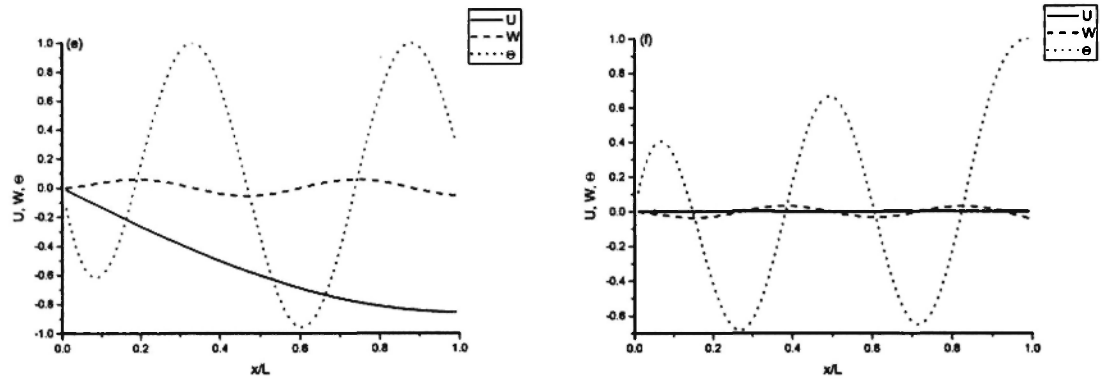


Fig. 6: First six normal mode shapes of Clamped-Free composite beam with $E_1 = 221\times10^9$ Pa and $L = 0.572$ m (a) mode 1; (b) mode 2; (c) mode 3; (d) mode 4; (e) mode 5; (f) mode 6.

The final example is chosen for further verification of the present results compared to the known values. This example is for a [0°/90°] cross-ply composite beam taken from Refs. /14, 16/. The beam is of rectangular cross-section and the data used in the analysis are as follows:

$E_1 = 144.80\times10^9$ Pa $E_2 = 9\ 65\times10^9$ Pa $G_{12} = G_{13} = 4.14\times10^9$ Pa $G_{23} = 3.45\times10^9$ Pa

$\nu_{12} = 0.3$ $\rho = 1389.23$ kg/m^3 $L = 0.381$ m $b = 25.4\times10^{-3}$ m $h = 38.1\times10^{-3}$ m

The numerical results for the first ten natural frequencies of the beam with four different boundary conditions are calculated and shown in Table 8, along with the results from Refs. /14/ and /16/. The axial displacements were restrained in the clamped edge and simply-supported edge. It can be seen that the present results are in very good agreement with the results obtained in Refs. /14/ and /16/ for all the frequencies and boundary conditions.

Table 8

Natural frequencies (in Hz) of cross-ply composite beam

Mode No.	Clamped-Clamped			Clamped - Simply supported	Clamped-Free			Simply-Supported		
	Present	Ref. /14/	Ref. /16/		Present	Ref. /14/	Ref. /16/	Present	Ref. /14/	Ref. /16/
1	1093.6	1094.9	1091.9	865.5	201.9	202.2	202.0	733.3	734.4	733.5
2	2565.7	2567.7	2558.5	2310.9	1125.2	1126.7	1124.1	1950.8	1953.4	1948.6
3	4338.8	4341.3	4323.5	4116.3	2743.9	2746.7	2738.2	3945.3	3948.7	3935.9
4	6260.2	6262.8	6235.1	6062.2	4573.7	4577.9	4564.2	5747.6	5752.2	5732.0
5	8270.8	8273.2	8234.4	8046.5	4926.5	4933.4	4928.9	8029.9	8039.4	8019.3
6	9244.7	9257.7	9249.5	8697.5	6689.0	6692.7	6665.3	8050.5	8054.1	8022.6
7	10334.1	10336.3	10285.8	10313.1	8784.3	8787.8	8748.6	10304.6	10307.4	10259.3
8	12386.7	12388.7	12325.3	12335.3	10841.8	10844.9	10794.4	12250.5	12253.4	12193.7
9	14466.4	14467.3	14391.4	14437.9	12786.6	12793.6	12744.9	14424.4	14425.8	14351.1
10	15817.0	15830.8	15786.6	15515.7	13242.5	13251.7	13226.8	15171.0	15187.9	15150.7

6. CONCLUDING REMARKS

In the present paper the dynamic stiffness method is introduced for the free vibration analysis of laminated composite beams. The shear deformation, rotary inertia, axial deformation and Poisson effects are considered in the formulation. The exact dynamic stiffness matrix is derived by use of the analytical solutions of the governing differential equations of the laminated beam in free vibration. The accuracy of the natural frequencies obtained by the present formulation is demonstrated by comparison with the reported results in the literature. The influences of Poisson effect, shear deformation, anisotropy, slenderness ratio and boundary condition on the natural frequencies of the laminated beams are investigated.

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