

Formation of Shear Stress Equations for Transversely Isotropic Finite Length Bar under Torsion

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ABSTRACT

In this study the derivation of the most general equilibrium and shear stress expressions for transversely isotropic fiber composite bar were theoretically investigated with the coordinate parameters of torsion problem. Nonlinear governing equations were modified in a linear system of differential equations by the application of assumptions. Equilibrium equation was separated into three different term-parts in terms of the uncoupled forms of the stress and warping functions. Derived resultant expressions have shown that the total number of independent elastic constants are reduced from twenty-one to five. Coordinate dependent variances of two shear stresses were presented on the transverse and lateral cross sections of fiber composite cylindrical bar in terms of the five unknown elastic constants.

Keywords: Torsion loading; shear modulus; fiber composite; transversely isotropic, partial differential equation

1. INTRODUCTION

Fiber reinforced structure of the layered composite material can be defined as containing different material properties in their each ply. In order to calculate the

general elasticity matrix of such a complex structure; it is necessary to apply the elasticity or stiffness matrix component to each ply and finally to get the sum of them. Selection of the material type and the way of loading is important in the definition of stiffness matrix. The shear modulus function $G(x,y)$ is defined as the x and y coordinates of the cross sectioned thin tubular composite bar under torsion (Stokes /1-2/). The derived expressions define the average shear moduli. Several authors are reported the resultant equations and experimental results about the stress distributions of fiber composite bars under torsion (Ergüven /3/, Yoshihara /4/, Liu /5/, Kardomateas /6/, Ecsedi /7/, Ting /8/). The distributions under tensile loading of fiber-matrix structure are also studied by Akbarov /9/ and Dong *et al.* /10/ previously.

The objective of this paper is to improve the coordinate dependent shear stresses $\tau_{r\theta}(r,\theta,z)$ and $\tau_{\theta z}(r,\theta,z)$ by using the elliptical stress function $\phi(r,z)$ and finally to show the shear modulus functions $G_{\theta z}$ and $G_{r\theta}$ in terms of 5 elastic constants C_{14}, C_{44}, C_{46} and C_{16}, C_{66} . These constants define the shear stress expressions. The torsion loaded composite material is transversely isotropic bar which has a finite length. The improvement of the coordinate dependent functions $\tau_{r\theta}(r,\theta,z)$ and $\tau_{\theta z}(r,\theta,z)$ that we performed in this study requires a sequence of developing steps. These are defined in the following sections of this paper.

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2. MATHEMATICAL MODEL FOR TRANSVERSELY ISOTROPIC FIBER COMPOSITE CYLINDRICAL BAR UNDER TORSION

In improving the coordinate dependent $\tau_{r\theta}(r,\theta,z)$ and $\tau_{\theta z}(r,\theta,z)$, we followed the main steps as below:

1. Applying the general Hooke's law which is expressed in terms of 21 independent elastic constants (Appendix-(A1-A2));
2. Writing the nonlinear equilibrium equation in z-direction (Appendix (A3-A6)); (we have assumed nonzero displacement components in the radial $u(r,z)$, tangential $v(r,z)$ and axial $w(r,\theta)$ directions in this equation);
3. Reducing the number of independent constants from 21 to 5 by using some assumptions;
4. Using $\phi(r,z)$ stress and $\psi(r,\theta)$ warping function assumptions, the equilibrium equation is contracted into two subgroups;
5. Defining the stress equations $\tau_{r\theta}(r,\theta,z)$ and $\tau_{\theta z}(r,\theta,z)$ in terms of partial differential equations by using the two shear modulus functions. These functions contain 5 independent elastic constants ($C_{14}, C_{44}, C_{46}, C_{16}, C_{66}$).

These procedures are explained in detail and sample distributions of these two shear modulus functions are provided in the later parts of the paper.

2.1 General Shear Modulus Formulation

Transversely isotropic fiber composite material is organized by some layer arrangements which also defines 5 independent elasticity constants settled in it (Fig. 1)). These composite materials are widely used in the industrial applications so that both the already defined elastic constants and developing mathematical models will be important in future material researches and applications.

Displacement functions u, v are the components of displacements in the radial and tangential directions which are defined in terms of cosine hyperbolic function (Ergüven /3/). Displacement in z-direction is defined by w which is declared with warping function ψ . It is a function of radius r and angle θ /11/:

$$\begin{aligned} u(r, z) &= v(r, z) = \phi(r, z) / \cosh(kz) \\ w(r, \theta) &= \theta_0 \psi(r, \theta) \end{aligned} \tag{1}$$

where θ_0 is a constant and named as initial angle of twist per unit length.

In this study, we have used 21 independent constants despite the wide spread use of 5 constants in the literature survey (Appendix-(A1), (A2)). Under the torsion load; the developing stresses on the longitudinal surface, and in the transversely sectioned part are defined in terms of 11 constants. They are defined as below:

$$\begin{aligned} \tau_{\theta z} &= C_{14}(r, z) \epsilon_r + C_{24}(r, z) \epsilon_\theta + C_{34}(r, z) \epsilon_z \\ &+ C_{44}(r, z) \gamma_{z\theta} + C_{45}(r, z) \gamma_{rz} + C_{46}(r, z) \gamma_{r\theta} \end{aligned} \tag{2}$$

$$\begin{aligned} \tau_{r\theta} &= C_{16}(r, z) \epsilon_r + C_{26}(r, z) \epsilon_\theta + C_{36}(r, z) \epsilon_z \\ &+ C_{46}(r, z) \gamma_{z\theta} + C_{56}(r, z) \gamma_{rz} + C_{66}(r, z) \gamma_{r\theta} \end{aligned} \tag{3}$$

The stress components $\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$ vanish in torsion problem /12/. Therefore, the strain expressions $\epsilon_z, \epsilon_r, \epsilon_\theta, \gamma_{rz}$ also have zero value under torsion. Tangential strain ϵ_θ is accepted as zero related to the rigid body rotation, but in our study the other three strain values are accepted as different from zero ($\epsilon_z, \epsilon_r, \gamma_{rz}$). These values are responsible for the complex interaction between fibers and matrix and cause additional displacements in deformation under shear loading. As a consequence of the assumption explained above, strain components $\epsilon_r, \epsilon_z, \epsilon_\theta, \gamma_{r\theta}, \gamma_{z\theta}, \gamma_{rz}$ (Appendix (A6)) are substituted into previously defined two stress equations. In this way these general stress expressions below are brought out:

$$\begin{aligned} \tau_{\theta z} &= C_{14}(r, z) \frac{\partial u}{\partial r} + C_{24}(r, z) \left(\underbrace{\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}}_{=0} \right) + \\ &C_{34}(r, z) \underbrace{\frac{\partial w}{\partial z}}_{=0} + C_{44}(r, z) \left(\underbrace{\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}}_{=0} \right) + \end{aligned} \tag{4}$$

$$\left[C_{45}(r,z) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + C_{46}(r,z) \left(\frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) \right]$$

$$\tau_{r\theta} = C_{16}(r,z) \frac{\partial u}{\partial r} + C_{36}(r,z) \frac{\partial w}{\partial z} + C_{26}(r,z)$$

$$\left(\frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} \right) + C_{46}(r,z) \left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right) +$$

$$C_{56}(r,z) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + C_{66}(r,z) \left(\frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r} \right) \quad (5)$$

Equilibrium equations through r and θ directions are identically satisfied (Appendix (A3-A5)). After substitution of Eqs. (4) and (5) into the z-directed differential equation of equilibrium, Eqs. (6-7) are developed:

$$\frac{\partial \tau_{r\theta}(r,z)}{\partial r} + \frac{\partial \tau_{\theta z}(r,z)}{\partial z} + \frac{2\tau_{r\theta}(r,z)}{r} = 0 \quad (6)$$

$$\phi(r,z) \left[\left(\frac{1}{r} \frac{\partial C_{24}(r,z)}{\partial z} + \frac{1}{r} \frac{\partial C_{26}(r,z)}{\partial r} + \frac{C_{26}(r,z)}{r^2} \right) \frac{1}{\cosh(kz)} + \right.$$

$$\left. \left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} + \frac{C_{24}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} + \right.$$

$$\left. \left(\frac{1}{r} \frac{\partial C_{46}(r,z)}{\partial z} - \frac{1}{r} \frac{\partial C_{66}(r,z)}{\partial r} - \frac{C_{66}(r,z)}{r^2} \right) \frac{1}{\cosh(kz)} + \right.$$

$$\left. \left(\frac{\partial C_{44}(r,z)}{\partial z} + \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} + \right.$$

$$\left. C_{45}(r,z) \frac{(k^2 \tanh^2(kz) - k^2)}{\cosh(kz)} + C_{44}(r,z) \frac{(k^2 \tanh^2(kz) - k^2)}{\cosh(kz)} \right] +$$

$$\frac{\partial \phi(r,z)}{\partial r} \left[\left(\frac{\partial C_{14}(r,z)}{\partial z} + \frac{\partial C_{16}(r,z)}{\partial r} + \frac{2C_{16}(r,z)}{r} + \frac{C_{26}(r,z)}{r} \right) \right.$$

$$\frac{1}{\cosh(kz)} + \left(\frac{\partial C_{46}(r,z)}{\partial z} + \frac{\partial C_{66}(r,z)}{\partial r} + \frac{C_{66}(r,z)}{r} \right)$$

$$\frac{1}{\cosh(kz)} + \left(C_{14}(r,z) + C_{56}(r,z) \right) \frac{(-k \tanh(kz))}{\cosh(kz)} +$$

$$\left. \left(\frac{2C_{46}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} \right] +$$

$$\frac{\partial \phi(r,z)}{\partial z} \left[\left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} + \frac{C_{24}(r,z)}{r} \right) \right.$$

$$\frac{1}{\cosh(kz)} + \left(\frac{\partial C_{44}(r,z)}{\partial z} + \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r} \right) \frac{1}{\cosh(kz)} +$$

$$\left. C_{45}(r,z) \frac{(-2k \tanh(kz))}{\cosh(kz)} + C_{44}(r,z) \frac{(-2k \tanh(kz))}{\cosh(kz)} \right] +$$

$$\frac{\partial^2 \phi(r,z)}{\partial r \partial z} \left[(C_{14}(r,z) + C_{56}(r,z)) \frac{1}{\cosh(kz)} + \left(\frac{2C_{46}(r,z)}{r} \right) \frac{1}{\cosh(kz)} \right] +$$

$$\frac{\partial^2 \phi(r,z)}{\partial r^2} \left[\frac{C_{16}(r,z)}{\cosh(kz)} + \frac{C_{66}(r,z)}{\cosh(kz)} \right] + \frac{\partial^2 \phi(r,z)}{\partial z^2} \left[\frac{C_{45}(r,z)}{\cosh(kz)} + \frac{C_{44}(r,z)}{\cosh(kz)} \right] +$$

$$\theta_0 \frac{\partial \psi(r,\theta)}{\partial r} \left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} \right) +$$

$$\theta_0 \frac{\partial^2 \psi(r,\theta)}{\partial r^2} (C_{56}(r,z))$$

$$\theta_0 \frac{\partial \psi(r,\theta)}{\partial \theta} \left(\frac{1}{r} \frac{\partial C_{44}(r,z)}{\partial z} + \frac{1}{r} \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r^2} \right) +$$

$$\theta_0 \frac{\partial^2 \psi(r,\theta)}{\partial r \partial \theta} \left(\frac{C_{46}(r,z)}{r} \right) = 0 \quad (7)$$

Eqn. (8) which finally has 10 parentheses is obtained by contracting Eqn. (7). All segments are defined by Roman letters as (I), (II), (III), (IV), (V) and (VI), (VII), (VIII), (IX), (X). Expanded forms of parenthesis are given in Appendix-B in detail.

$$\frac{\partial \phi(r,z)}{\partial r} (I) + \frac{\partial^2 \phi(r,z)}{\partial r^2} (II) + \phi(r,z) (III) + \frac{\partial \phi(r,z)}{\partial z} (IV) +$$

$$\frac{\partial^2 \phi(r,z)}{\partial z^2} (V) + \frac{\partial^2 \phi(r,z)}{\partial r \partial z} (VI) + \frac{\partial \psi(r,\theta)}{\partial r} (VII) +$$

$$\frac{\partial^2 \psi(r,\theta)}{\partial r^2} (VIII) + \frac{\partial^2 \psi(r,\theta)}{\partial r \partial \theta} (IX) + \frac{\partial \psi(r,\theta)}{\partial \theta} (X) = 0 \quad (8)$$

As seen from the last contracted expression, the third equilibrium equation can be seen in two groups of uncoupled linear partial differential equations. While the first one includes the stress φ(r,z) function, the second one has the warping function ψ(r,θ) alone.

MATHEMATICA 4.0 program was used in substitutions of all mathematical equations.

2.2 Basic Assumptions Used and Generation of Stress Function

Wood, thermoplastic cables, bone, honey-comb are the examples for transversely isotropic layered structure. In order to generate this type fiber composite bar, it is assumed that thin rectangular strips can be used throughout the transverse cross section. Layered sections are defined by thin strip rectangular areas with different material properties which are rotated 180° or 360° about its main central axis to form a cylindrical layered composite structure (Figs. 1-3).

Layered surfaces and related displacement fields are defined mathematically by suitable parameters. They are considered in the generation of modified stress function $\phi(r, z)$. It is related to the structural formation. In this study this function is chosen as:

$$\phi(r, z) = K_1(r^2 - R^2) + K_2(z^2 - L^2) \tag{9}$$

The related sample plot of this function is represented in Fig. 4. In this figure r and z coordinates are chosen between the ranges $-2 < r < 2$ and $-3 < z < 3$. Total length (L) of bar should be defined along z -axis and stress function distribution is shown through this axis. So that, this formulation can be applicable on every z cross-sectional area of the transversely isotropic composite bar. It is assumed that the selected material constants are fix for the definite $r=R$ at each cross section and on the surface. For example wood has continuous parabolic/elliptical/ circular layered surfaces. The developed 3-D curved ply surface is based on the arrangements of fibers and their accumulated distributions in matrix section of fiber composite. So that in the selection step for stress function $\phi(r, z)$, description of the generation of these multiple layers with the fiber arrangements and the necessity of the satisfaction of boundary conditions are held on (Eqs. (9) and (10)).

$$\begin{aligned} r = \pm R & \quad ; \quad \phi = 0, \phi = c \\ z = \pm L & \quad ; \quad \phi = 0, \phi = c \end{aligned} \tag{10}$$

It is known that the stress function must be equal to zero or a constant number on the transverse and circumferential surfaces of the bar.

2.3 Steps for Reduction of Partial Differential Equations

Some steps are provided to simplify the complex equations in our study /12-13/. One of them is to check out the usability of the $\phi(r, z)$ function. It depends on obtaining an acceptability condition by the application of the bi-harmonic equation ($\nabla^2 \phi = -2G_{r\theta} \theta_0$) (Appendix-(A7)).

The second shows Eqn. (8) in three different linear partial differential equations. This equation is divided into three linear system of equations considering $\phi(r, z)$ and $\psi(r, z)$ and to their partial derivatives with respect to r, θ and z . After this procedure, equations are seen as in the uncoupled form. For example, the function $f(r, \theta, z) = f_1(r) + f_2(\theta) + f_3(z)$ can be reordered in more simplified forms: $f_1(r); f_2(\theta); f_3(z)$ and these forms can be used in the exact formulation of $\phi(r, z)$ and $\psi(r, \theta)$ functions.

The third one is related to the generation of the geometrical view of the cross-sectional shape of the transversely isotropic fiber composite bar. In the elliptical section, the $\phi(r, z)$ function is described as in Eqn.(11) and this function can also be used for circular sections by equating the minor/major axes of ellipse to each other as $a = b$. It is known that elasticity formulations of circular cross-sectional isotropic bars have zero warping function in the elasticity formulations.

$$\phi = c_1 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \tag{11}$$

Substitution of $\phi(r, z)$ into the bi-harmonic equation by considering the equality of minor and major axes on the circular sections as $a = b$ gives constant c_1 as below:

$$c_1 = -G_{r\theta}^{avg} \theta_0 \frac{a^2}{2} \tag{12}$$

Here, the stress function is defined at the circular cross section with constant average shear modulus $G_{r\theta}^{avg}$.

Using polar coordinate transformations in Eqn. (11) as below:

$$\phi(r, z) = -\frac{G_{r\theta}^{avg}}{2} (r^2 - a^2) \tag{13}$$

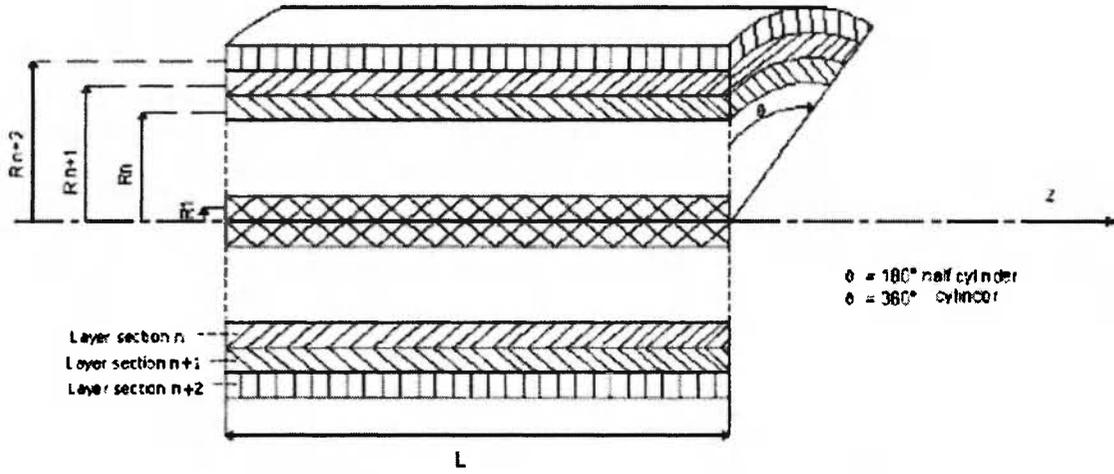


Fig. 1: 3-D development of transversely isotropic fiber composite cylinder

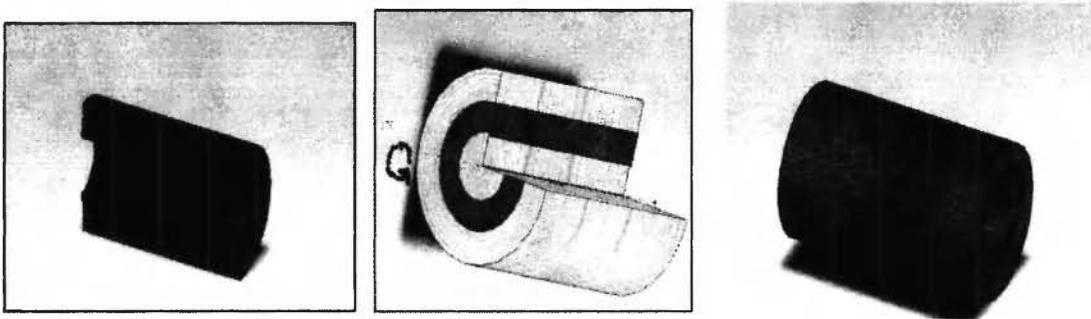


Fig. 2: Representation of transversely isotropic structure with cylindrical wood bars

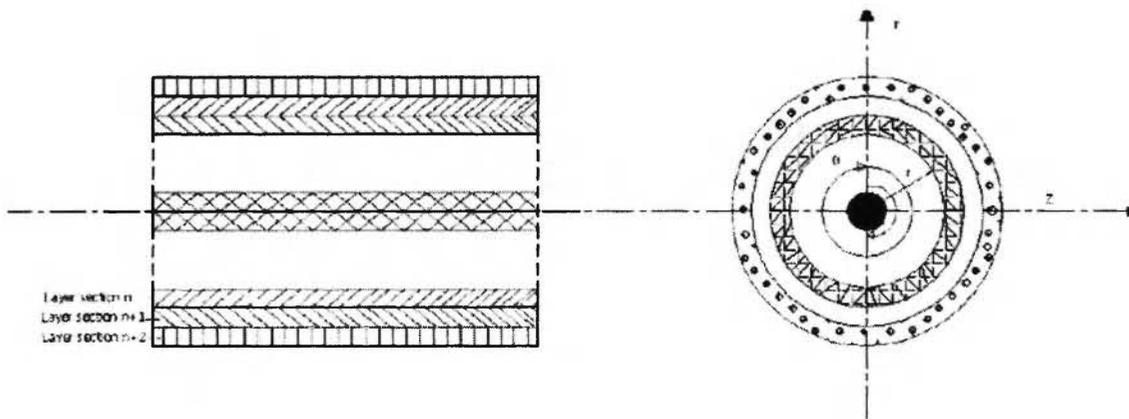


Fig. 3: Side views of transversely isotropic layered composite cylinder

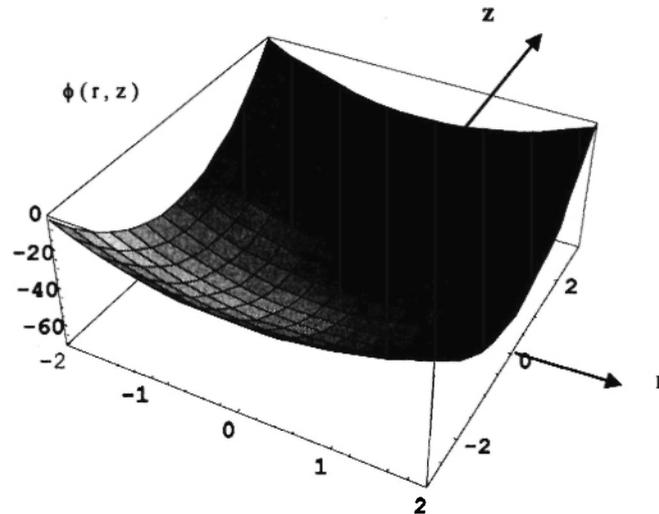


Fig. 4: Stress function $\phi(r, z)$ distribution on r-z coordinate plane

Eqs. (13) and (9) define the same problem. The first part of the modified expression has $K_1 = -G_{z\theta} \theta_0$ on the circular sectional surface and providing that $K_2 = -G_{r\theta} \theta_0$ valid on the cylindrical sectional surface. They are used to complete the definition of torsion problem for two different shear modulus data $G_{r\theta}, G_{z\theta}$. Considering these, the modified stress function $\phi^m(r, z)$ can be written as:

$$\phi^m(r, z) = - \left[G_{r\theta}(r, z)(r^2 - R^2) + G_{z\theta}(r, z)(z^2 - L^2) \right] \theta_0 \tag{14}$$

In order to express the point-wise shear moduli data $G_{r\theta}(r, z)$ and $G_{z\theta}(r, z)$ in terms of the overall average one G^{avg} , the stress function is defined as below:

$$K_1' = K_1 / G\theta_0, K_2' = K_2 / G\theta_0 \tag{15}$$

$$\phi^{rel}(r, z) = K_1'(r^2 - R^2) + K_2'(z^2 - L^2) \tag{16}$$

In Eqns. (15) and (16), $K = G^{avg} = G\theta_0$ is used for the average shear modulus. In the following steps K_1', K_2' and ϕ^{rel} are written as K_1, K_2 and ϕ respectively in order to simplify the notation in the partial differential equations.

Disregarding warping function by taking $\psi(r, z) = 0$, Eqn. (8) is reduced as:

$$\frac{\partial \phi(r, z)}{\partial r} \text{(I)} + \frac{\partial^2 \phi(r, z)}{\partial r^2} \text{(II)} + \phi(r, z) \text{(III)} + \frac{\partial \phi(r, z)}{\partial z} \text{(IV)} + \frac{\partial^2 \phi(r, z)}{\partial z^2} \text{(V)} + \frac{\partial^2 \phi(r, z)}{\partial r \partial z} \text{(VI)} = 0 \tag{17}$$

From this equation two new uncoupled differential equations can be seen as:

$$\frac{\partial \phi(r, z)}{\partial r} \text{(I)} + \frac{\partial^2 \phi(r, z)}{\partial r^2} \text{(II)} + \phi(r, z) \text{(III)} = 0 \tag{18}$$

$$\frac{\partial \phi(r, z)}{\partial z} \text{(IV)} + \frac{\partial^2 \phi(r, z)}{\partial z^2} \text{(V)} + \frac{\partial^2 \phi(r, z)}{\partial r \partial z} \text{(VI)} = 0 \tag{19}$$

The last term in Eqn.(19) is equal to zero for chosen $\phi(r, z)$. These two additional equations will be used to calculate the unknown elastic constants in future work.

In order to plot the variances of the shear moduli data, the third reduced equilibrium equation (Eqn. 17) is used. Coordinate changes through r and z directions are inserted into this equation in order to get two different equations in terms of $\phi(r, z)$. The first expression is obtained at the tip of the bar where $z = \pm L$ constant and

r is a variable one. The second equation is obtained at the surface of the bar where radii data have $r = \pm R$ while z is variable. The following two simplified equations are obtained according to these coordinate changes.

$$(r^2 - R^2)(III) + 2r(I) + 2(II) = K_1' \tag{20}$$

$$(z^2 - L^2)(III) + 2z(IV) + 2(V) = K_2' \tag{21}$$

All of the expressions in common parentheses show us the coordinate dependent values of $\phi(r, z)$, elastic constants and their variances. It is interesting to note that these two equations are also related to functions $G_{r\theta}(r, z), G_{z\theta}(r, z)$ in terms of K_1, K_2 , or K_1', K_2' as described in Eqn. (15). For test data, sample graphics are presented in the following section.

The reduced form of the second differential equation

from Eqn. (8), which includes the warping function, comes to form the following equation with θ -independent symmetrical distribution.

$$\begin{aligned} \frac{\partial \psi(r, \theta)}{\partial r} (VII) + \frac{\partial^2 \psi(r, \theta)}{\partial r^2} (VIII) &= 0 \\ \frac{\partial}{\partial r} \psi(r, \theta) \left((VII) + \frac{\partial}{\partial r} (VIII) \right) &= 0 \\ \psi(r, \theta) \left((VII) + \frac{\partial}{\partial r} (VIII) \right) &= C(r) \end{aligned} \tag{22}$$

$C(r)$ is the r coordinate dependent functional constant. The last three equations (namely Eqns. (20), (21) and (22)) are the final appearance of the corrected Eqn. (8).

In the last step; shear stress equations (Eqn. (4) and (5)) are rewritten by the substitution of Eqn. (1) which includes stress function $\phi(r, z)$.

$$\begin{aligned} \tau_{z\theta} = C_{14} \left(\frac{2rK_1}{\cosh(kz)} \right) + C_{24} \left(\frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right) + \\ C_{44} \left[\left(\frac{1}{\cosh(kz)} \left(2K_2z - k \tanh(kz)(K_1(r^2 - R^2) + K_2(z^2 - L^2)) \right) \right) + \frac{1}{r} \theta_r \frac{\partial \psi(r, \theta)}{\partial \theta} \right] + \\ C_{45} \left[\frac{1}{\cosh(kz)} \left[2K_2z - k \tanh(kz) \left[K_1(r^2 - R^2) + K_2(z^2 - L^2) \right] \right] + \theta_r \frac{\partial \psi(r, \theta)}{\partial r} \right] + \\ C_{46} \left[\frac{2rK_1}{\cosh(kz)} - \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right] \end{aligned} \tag{23}$$

$$\begin{aligned} \tau_{r\theta} = C_{16} \left(\frac{2rK_1}{\cosh(kz)} \right) + C_{26} \left(\frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right) + \\ C_{46} \left[\left(\frac{1}{\cosh(kz)} \left(2K_2z - k \tanh(kz)(K_1(r^2 - R^2) + K_2(z^2 - L^2)) \right) \right) + \frac{1}{r} \theta_r \frac{\partial \psi(r, \theta)}{\partial \theta} \right] + \\ C_{56} \left[\frac{1}{\cosh(kz)} \left[2K_2z - k \tanh(kz) \left[K_1(r^2 - R^2) + K_2(z^2 - L^2) \right] \right] + \theta_r \frac{\partial \psi(r, \theta)}{\partial r} \right] + \\ C_{66} \left[\frac{2rK_1}{\cosh(kz)} - \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right] \end{aligned} \tag{24}$$

This step is based on the cylindrically symmetric deformation case with the corresponding stress components $\sigma_r = 0, \sigma_\theta = 0, \sigma_z = 0, \tau_{rz} = 0$ which simplify the system of differential equations as

described in the preceding section. Transversely the isotropic cylinder which is subjected to torsion has four simplified equations which still contain the unknown elastic constants (C_{ij}).

$$\begin{aligned} \sigma_r : C_{14} \left(\frac{\partial v}{\partial z} \right) + C_{16} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) &= - \left[C_{11} \frac{\partial u}{\partial r} + C_{12} \frac{u}{r} + C_{13} \frac{\partial w}{\partial z} + C_{15} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right] \\ \sigma_\theta : C_{24} \left(\frac{\partial v}{\partial z} \right) + C_{26} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) &= - \left[C_{21} \frac{\partial u}{\partial r} + C_{22} \frac{u}{r} + C_{23} \frac{\partial w}{\partial z} + C_{25} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right] \\ \sigma_z : C_{34} \left(\frac{\partial v}{\partial z} \right) + C_{36} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) &= - \left[C_{31} \frac{\partial u}{\partial r} + C_{32} \frac{u}{r} + C_{33} \frac{\partial w}{\partial z} + C_{35} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right] \\ \tau_{rz} : C_{45} \left(\frac{\partial v}{\partial z} \right) + C_{56} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) &= - \left[C_{15} \frac{\partial u}{\partial r} + C_{25} \frac{u}{r} + C_{35} \frac{\partial w}{\partial z} + C_{55} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right] \end{aligned} \tag{25}$$

By substitution of the displacement components in terms of stress $\phi(r, z)$ and warping $\psi(r, \theta)$ functions

(Eqn. (1), Eqn. (9)), four expressions arise from Eqn. (25) as written below:

$$\begin{aligned} C_{14} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) + C_{16} \left(\frac{2rK_1}{\cosh(kz)} - \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} \right) &= \\ - \left[C_{11} \frac{2rK_1}{\cosh(kz)} + C_{12} \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} + C_{15} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) \right] & \\ C_{24} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) + C_{26} \left(\frac{2rK_1}{\cosh(kz)} - \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} \right) &= \\ - \left[C_{21} \frac{2rK_1}{\cosh(kz)} + C_{22} \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} + C_{25} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) \right] & \\ C_{34} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) + C_{36} \left(\frac{2rK_1}{\cosh(kz)} - \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} \right) &= \\ - \left[C_{31} \frac{2rK_1}{\cosh(kz)} + C_{32} \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} + C_{35} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) \right] & \\ C_{45} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) + C_{56} \left(\frac{2rK_1}{\cosh(kz)} - \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} \right) &= \\ - \left[C_{15} \frac{2rK_1}{\cosh(kz)} + C_{25} \frac{\left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right)}{r \cosh(kz)} + C_{55} \left(\frac{1}{\cosh(kz)} \left[2K_2 z - k \tanh(kz) \left(K_1 (r^2 - R^2) + K_2 (z^2 - L^2) \right) \right] \right) \right] & \end{aligned} \tag{26}$$

By comparing the coefficients of strain terms, elastic constants are written in terms of each other as seen in Eqn. (27). Then simplifications are applicable to them.

$$\begin{aligned}
 C_{14} &= K_3 C_{24} = K_4 C_{34} = K_5 C_{45} \\
 C_{16} &= K_6 C_{26} = K_7 C_{36} = K_8 C_{56} \\
 C_{11} &= K_9 C_{21} = K_{10} C_{31} = K_{11} C_{15} \\
 C_{12} &= K_{12} C_{22} = K_{13} C_{32} = K_{14} C_{25} \\
 C_{13} &= K_{15} C_{23} = K_{16} C_{33} = K_{17} C_{35} \\
 C_{15} &= K_{18} C_{25} = K_{19} C_{35} = K_{20} C_{55}
 \end{aligned}
 \tag{27}$$

Elastic constants in each set are multiplicities of others. If we use the simplifying assumption such that each of them has a constant value one, we can see the most reduced system of equations as below. Otherwise, the new eighteen unknowns will appear in the equations ($K_3 - K_{20}$).

$$\begin{aligned}
 C_{14} &= C_{24} = C_{34} = C_{45} \\
 C_{16} &= C_{26} = C_{36} = C_{56} \\
 C_{11} &= C_{21} = C_{31} = C_{15} \\
 C_{12} &= C_{22} = C_{32} = C_{25} \\
 C_{13} &= C_{23} = C_{33} = C_{35} \\
 C_{15} &= C_{25} = C_{35} = C_{55}
 \end{aligned}
 \tag{28}$$

The number of unknown constants is reduced from twenty one to nine ($C_{14}, C_{44}, C_{46}, C_{16}, C_{66}, C_{11}, C_{12}, C_{13}, C_{15}$) in the general torsion problem definition. Using the first two expressions $C_{14} = C_{24} = C_{45}$ and $C_{16} = C_{36} = C_{56}$ in Eqn. (28), the modified shear stress expressions are given as below:

$$\begin{aligned}
 \tau_{z\theta}(r, z) &= C_{14} \left[\left(\frac{2rK_1}{\cosh(kz)} + \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right) + \right. \\
 &\quad \left. \left[\frac{1}{\cosh(kz)} \left[2K_2z - k \tanh(kz) \left[K_1(r^2 - R^2) + K_2(z^2 - L^2) \right] \right] + \theta_r \frac{\partial \psi(r, \theta)}{\partial r} \right] \right] + \\
 &\quad C_{44} \left[\left(\frac{1}{\cosh(kz)} \left(2K_2z - k \tanh(kz) (K_1(r^2 - R^2) + K_2(z^2 - L^2)) \right) \right) + \frac{1}{r} \theta_r \frac{\partial \psi(r, \theta)}{\partial \theta} \right] + \\
 &\quad + C_{46} \left[\frac{2rK_1}{\cosh(kz)} - \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right]
 \end{aligned}
 \tag{29}$$

$$\begin{aligned}
 \tau_{r\theta}(r, z) &= C_{16} \left[\left(\frac{2rK_1}{\cosh(kz)} + \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right) + \right. \\
 &\quad \left. \left[\frac{1}{\cosh(kz)} \left[2K_2z - k \tanh(kz) \left[K_1(r^2 - R^2) + K_2(z^2 - L^2) \right] \right] + \theta_r \frac{\partial \psi(r, \theta)}{\partial r} \right] \right] + \\
 &\quad C_{46} \left[\left(\frac{1}{\cosh(kz)} \left(2K_2z - k \tanh(kz) (K_1(r^2 - R^2) + K_2(z^2 - L^2)) \right) \right) + \frac{1}{r} \theta_r \frac{\partial \psi(r, \theta)}{\partial \theta} \right] + \\
 &\quad C_{66} \left[\frac{2rK_1}{\cosh(kz)} - \frac{K_1(r^2 - R^2) + K_2(z^2 - L^2)}{r \cosh(kz)} \right]
 \end{aligned}
 \tag{30}$$

In conclusion; with the application of the reduction method, we have found 5 independent unknown constants instead of 11. In the shear stress equations, the

number of unknown independent constants is seen as five.

3. SHEAR MODULUS DISTRIBUTIONS ON COMPOSITE CYLINDERS

Sample graphics for shear modulus functions $f(r) = G_{r\theta}$ and $f(z) = G_{z\theta}$ are obtained and plotted in Fig. 5, Fig. 6, Fig. 7, and Fig. 8 by using Eqs. (20) and (21). Using the suitable sample data for variable coordinates r and z , equations are converted into parabolic relationships. Through the r -axis of the cylinder, parabolic equation $r^2 - 4 + 2r + 2 = r^2 + 2r - 2$ is representing the variance of shear modulus for the r -interval $R = -2$ to $R = 2$ in which they are the end points of the cylinder where $z = \pm L$. The first example is given as $z^2 - 9 + 2z + 2 = z^2 + 2z - 7$ between the tip points of the cylinder $-1.5 \leq L \leq +1.5$ and over the surface of the

cylinder where $-R \leq r \leq +R$ or $-2 \leq r \leq +2$ (Eqn. (21)). All illustrations are shown with $2L$ total length and R outer radius of the cylinders. The cartesian coordinate system is attached to the central axis of the cylinder.

The second example has the parabolic variances $r^2 + 2r + 1$ and $z^2 + 2z - 98$ for shear moduli. The selected dimensions are $-1 \leq r \leq +1$ and $-10 \leq L \leq +10$ (Fig. 6). The third one is given by $r^2 + 2r + 1$ and $z^2 + 2z - 398$. The studied field is defined by $-1 \leq r \leq +1$ and $-20 \leq L \leq +20$ (Fig. 7). The fourth representation is given by $3r^2 + 4r + 5$ and $3z^2 + 10z + 11$. The used dimensions are $-1 \leq r \leq +1$ and $-1 \leq L \leq +1$ (Fig. 8). Graphics are showing the distributions for the sample test data according to the parenthesis (I) = 2, (II) = 4, (III) = 3, (IV) = 5, (V) = 7 (Eqn. (20), (21)).

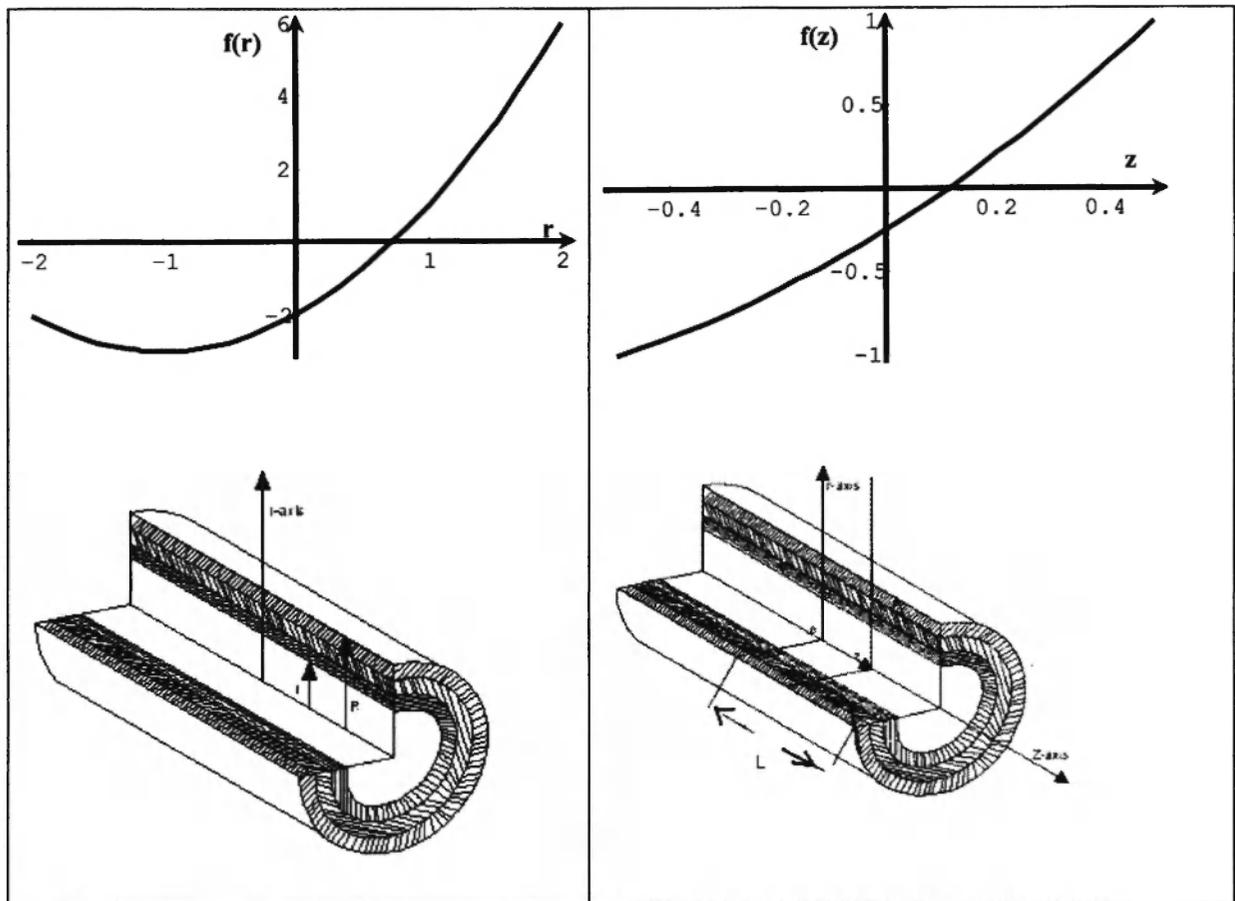


Fig. 5: Graphical representations of shear modulus functions $f(r) = G_{r\theta} = r^2 + 2r - 2$ and $f(z) = G_{z\theta} = z^2 + 2z - 0.25$ for the transversely isotropic cylinder

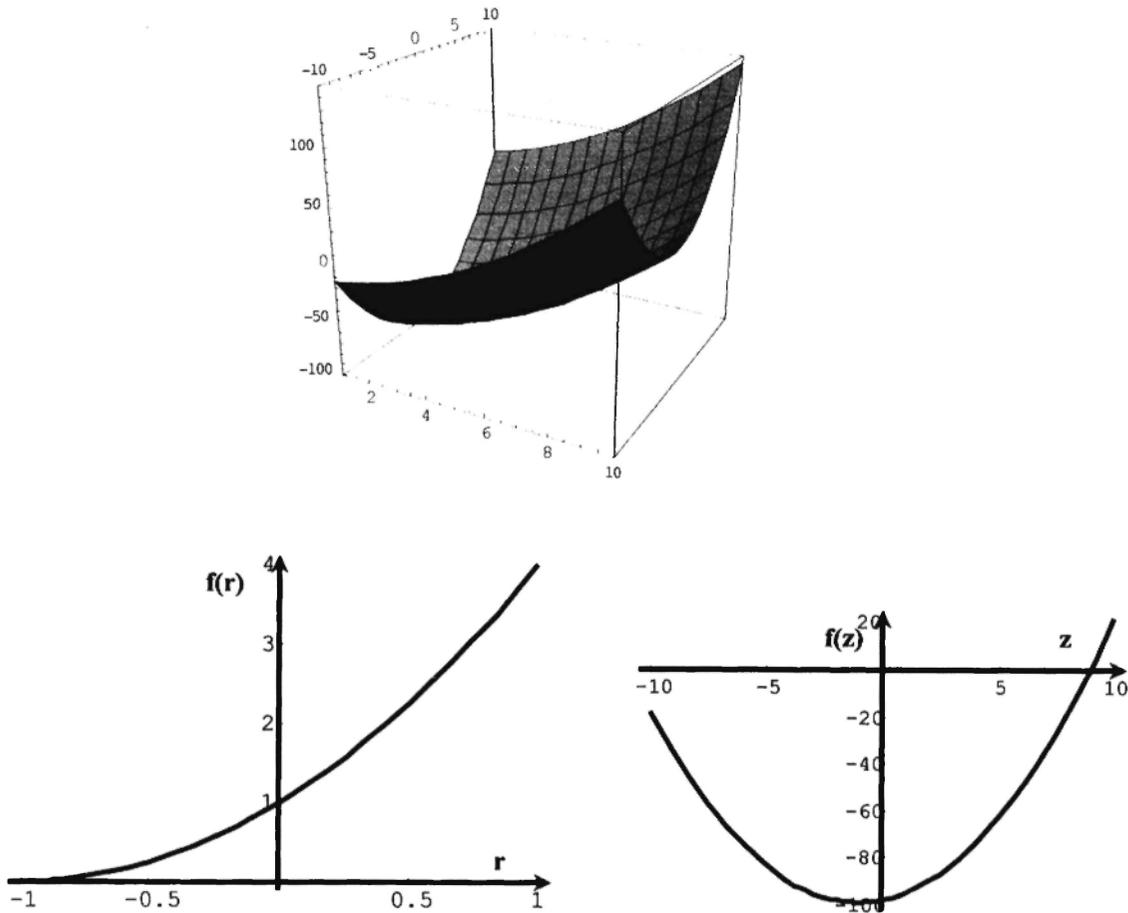


Fig. 6: 3D and 2D illustrations of shear modulus functions $f(r) = G_{r\theta} = r^2 + 2r + 1$, $f(z) = G_{z\theta} = z^2 + 2z - 98$ for the fiber composite cylinder

4. CONCLUSIONS

An elasticity shear stress ($\tau_{r\theta}(r, z), \tau_{z\theta}(r, z)$) formulation for the torsion of transversely isotropic fiber composite cylinder with variable shear modulus descriptions through the thickness and the axial direction was presented. The material has cylindrical symmetry referred to the main axis of the attached coordinate system on the cylindrical body. It is assumed that all planes perpendicular to the main axis have geometrical symmetry due to fiber arrangements. Coordinate dependent nonlinear equilibrium differential and boundary conditions that govern the stress function have been developed. In the formulation, the total

number of degree of freedom (u, v, w) was chosen in terms of displacements as three. They are defined in terms of the stress $\phi(r, z)$ function by hyperbolic cosine. The derived equations have nonzero warping function $\psi(r, \theta)$. The nonlinear equilibrium equation was written as a linear one by selecting suitable stress function and terms were grouped into a set of partial differential equations. In this study, the highest power of partial differentiations is in second order and the stress function is chosen as a quadratic one to obtain a constant value after differentiations. Partial differential equations are obtained in terms of r, θ and z . Equations were constructed from six different non-zero strain expressions; $\epsilon_r, \epsilon_z, \epsilon_\theta, \gamma_{r\theta}, \gamma_{z\theta}, \gamma_{rz}$ and γ_{rz} .

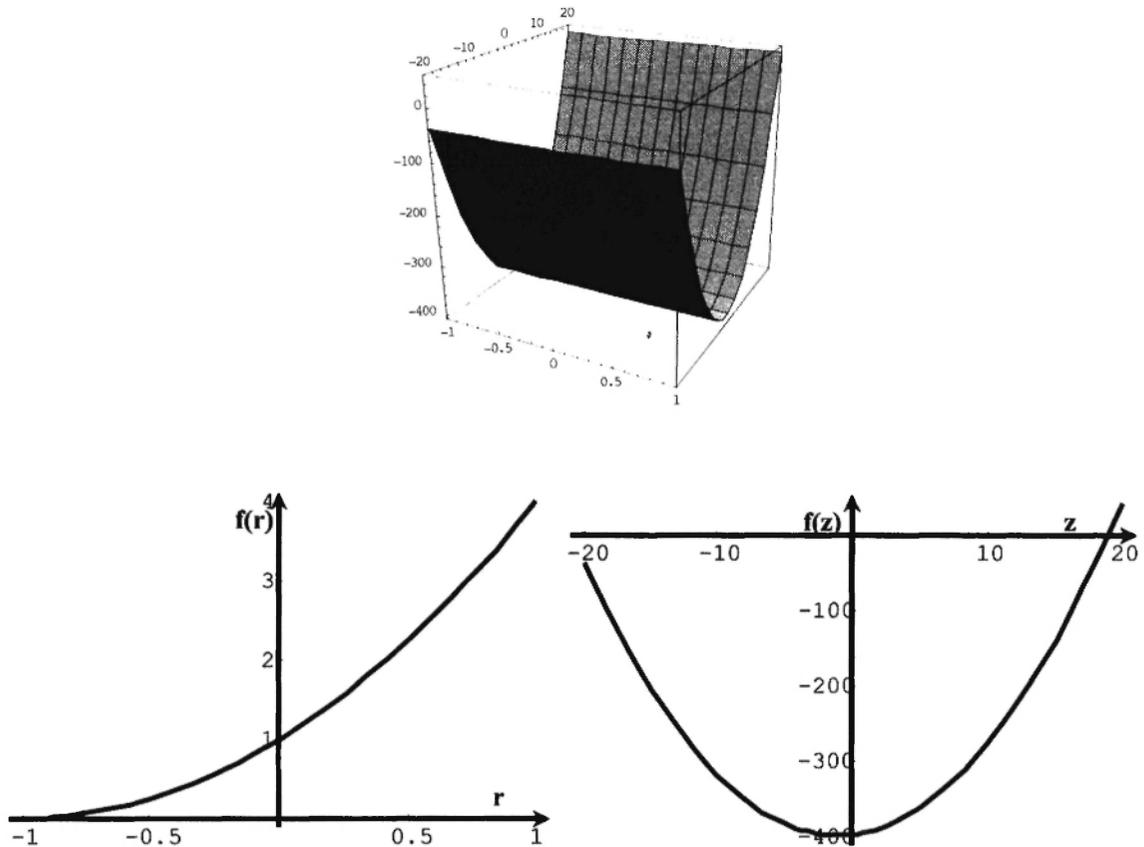


Fig. 7: 3D and 2D illustrations of shear modulus functions $f(r) = G_{r\theta} = r^2 + 2r + 1$, $f(z) = G_{z\theta} = z^2 + 2z - 398$ for the fiber composite cylinder

Strain coupling behaviour was included into the strain-displacement components and then into the Hooke's law. So, all of the related strain terms were included in the shear stress-strain equations. The shear modulus functions $G_{r\theta}(r, \theta, z)$ and $G_{z\theta}(r, \theta, z)$ were expressed with the coordinate dependent constants K'_1 and K'_2 . It was shown that transversely isotropic fiber composite material defined with five independent elastic constants $C_{14}, C_{44}, C_{46}, C_{16}, C_{66}$ in the shear stress components

$\tau_{r\theta}, \tau_{\theta z}$ under torsion. When the general elastic material matrix is considered, 4 constants are introduced ($C_{11}, C_{12}, C_{13}, C_{15}$) into the equations additionally and with the other two unknowns K'_1, K'_2 totally eleven elastic constants are seen. These are dependent ones. Finally; these analytical studies will be modified and solved with the data obtained from future experimental studies.

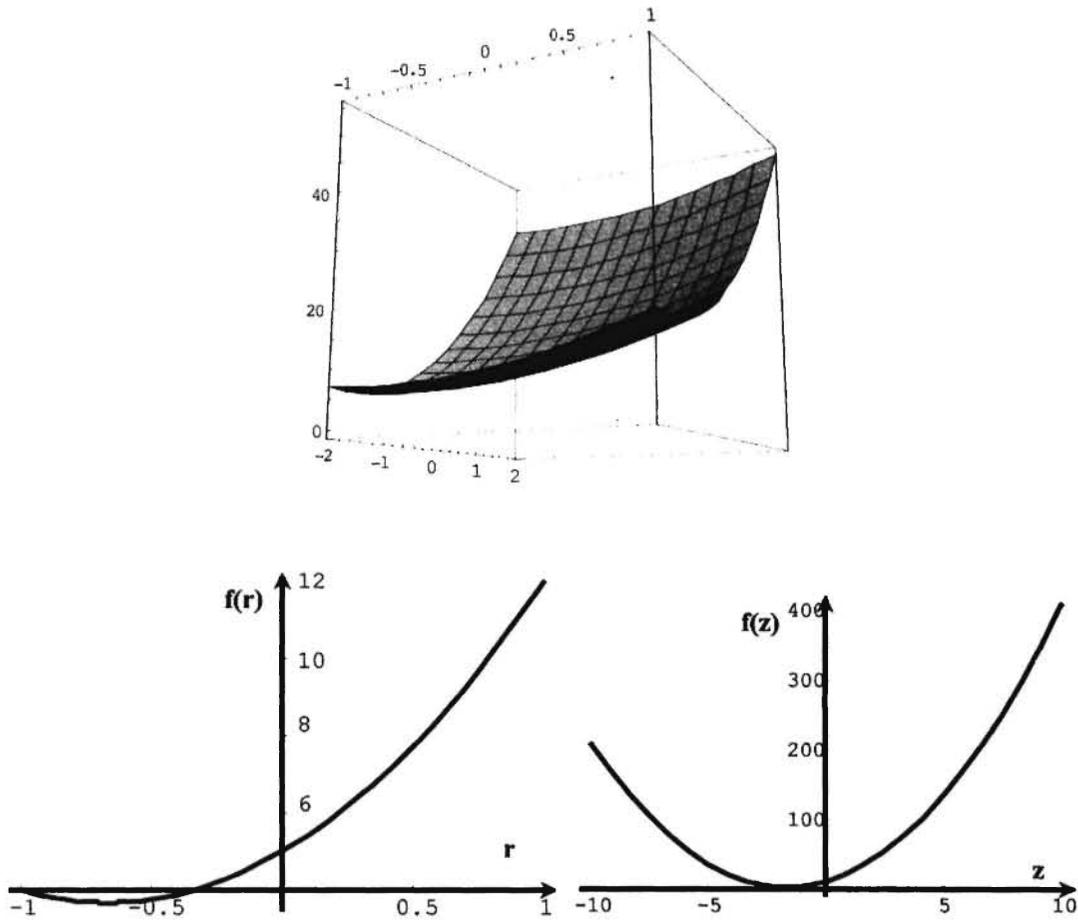


Fig. 8: 3D and 2D illustrations of shear modulus functions $f(r) = G_{r\theta} = 3r^2 + 4r + 5$, $f(z) = G_{z\theta} = 3z^2 + 10z + 11$ for the fiber composite cylinder

APPENDIX-A :

General transversely isotropic fiber composite elasticity matrix with twenty-one independent elastic constants (examined elasticity constants for torsion loading are shown by bold letters):

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ \mathbf{C_{14}} & \mathbf{C_{24}} & \mathbf{C_{34}} & \mathbf{C_{44}} & \mathbf{C_{45}} & \mathbf{C_{46}} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ \mathbf{C_{16}} & \mathbf{C_{26}} & \mathbf{C_{36}} & \mathbf{C_{46}} & \mathbf{C_{56}} & \mathbf{C_{66}} \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{Bmatrix} \tag{A1}$$

Simplified transversely isotropic fiber composite elasticity matrix with five independent elastic constants:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{C_{44}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{C_{44}} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{Bmatrix} \tag{A2}$$

Equilibrium equations in the cylindrical coordinate system:

$$r \text{ direction: } \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (A3)$$

$$z \text{ direction: } \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \quad (A4)$$

$$\theta \text{ direction: } \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \quad (A5)$$

Strain-displacement relationships:

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \quad \epsilon_\theta = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}, \\ \gamma_{r\theta} &= \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}, \quad \gamma_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}, \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \quad (A6)$$

Biharmonic equation:

$$\nabla^2 \phi := \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \quad (A7)$$

APPENDIX-B

(I) :

$$\begin{aligned} \phi(r,z) &\left[\left(\frac{1}{r} \frac{\partial C_{24}(r,z)}{\partial z} + \frac{1}{r} \frac{\partial C_{26}(r,z)}{\partial r} + \frac{C_{26}(r,z)}{r^2} \right) \frac{1}{\cosh(kz)} + \right. \\ &\left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} + \frac{C_{24}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} + \\ &\left(\frac{1}{r} \frac{\partial C_{46}(r,z)}{\partial z} - \frac{1}{r} \frac{\partial C_{66}(r,z)}{\partial r} - \frac{C_{66}(r,z)}{r^2} \right) \frac{1}{\cosh(kz)} + \\ &\left(\frac{\partial C_{44}(r,z)}{\partial z} + \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} + \\ &\left. C_{45}(r,z) \frac{(k^2 \tanh^2(kz) - k^2)}{\cosh(kz)} + C_{44}(r,z) \frac{(k^2 \tanh^2(kz) - k^2)}{\cosh(kz)} \right] \end{aligned} \quad (B1)$$

(II):

$$\begin{aligned} \frac{\partial \phi(r,z)}{\partial r} &\left[\left(\frac{\partial C_{14}(r,z)}{\partial z} + \frac{\partial C_{16}(r,z)}{\partial r} + \frac{2C_{16}(r,z)}{r} + \frac{C_{26}(r,z)}{r} \right) \frac{1}{\cosh(kz)} + \right. \\ &\left(\frac{\partial C_{46}(r,z)}{\partial z} + \frac{\partial C_{66}(r,z)}{\partial r} + \frac{C_{66}(r,z)}{r} \right) \frac{1}{\cosh(kz)} + \end{aligned}$$

$$\left. (C_{14}(r,z) + C_{56}(r,z)) \frac{(-k \tanh(kz))}{\cosh(kz)} + \left(\frac{2C_{46}(r,z)}{r} \right) \frac{(-k \tanh(kz))}{\cosh(kz)} \right] \quad (B2)$$

(III):

$$\begin{aligned} \frac{\partial \phi(r,z)}{\partial z} &\left[\left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} + \frac{C_{24}(r,z)}{r} \right) \frac{1}{\cosh(kz)} + \right. \\ &\left(\frac{\partial C_{44}(r,z)}{\partial z} + \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r} \right) \frac{1}{\cosh(kz)} + \\ &\left. C_{45}(r,z) \frac{(-2k \tanh(kz))}{\cosh(kz)} + C_{44}(r,z) \frac{(-2k \tanh(kz))}{\cosh(kz)} \right] \end{aligned} \quad (B3)$$

(IV):

$$\frac{\partial^2 \phi(r,z)}{\partial r \partial z} \left[(C_{14}(r,z) + C_{56}(r,z)) \frac{1}{\cosh(kz)} + \left(\frac{2C_{46}(r,z)}{r} \right) \frac{1}{\cosh(kz)} \right] \quad (B4)$$

$$(V): \frac{\partial^2 \phi(r,z)}{\partial r^2} \left[\frac{C_{16}(r,z)}{\cosh(kz)} + \frac{C_{66}(r,z)}{\cosh(kz)} \right] \quad (B5)$$

$$(VI): \frac{\partial^2 \phi(r,z)}{\partial z^2} \left[\frac{C_{45}(r,z)}{\cosh(kz)} + \frac{C_{44}(r,z)}{\cosh(kz)} \right] \quad (B6)$$

$$(VII): \theta_0 \frac{\partial \psi(r,\theta)}{\partial r} \left(\frac{\partial C_{45}(r,z)}{\partial z} + \frac{\partial C_{56}(r,z)}{\partial r} + \frac{2C_{56}(r,z)}{r} \right) \quad (B7)$$

$$(VIII): \theta_0 \frac{\partial^2 \psi(r,\theta)}{\partial r^2} (C_{56}(r,z)) \quad (B8)$$

(IX):

$$\theta_0 \frac{\partial \psi(r,\theta)}{\partial \theta} \left(\frac{1}{r} \frac{\partial C_{44}(r,z)}{\partial z} + \frac{1}{r} \frac{\partial C_{46}(r,z)}{\partial r} + \frac{C_{46}(r,z)}{r^2} \right) \quad (B9)$$

$$(X): \theta_0 \frac{\partial^2 \psi(r,\theta)}{\partial r \partial \theta} \left(\frac{C_{46}(r,z)}{r} \right) \quad (B10)$$

REFERENCES

1. V.K. Stokes, Design with nonhomogeneous materials-Part I: Pure bending of prismatic bars, *ASME Journal of Vibration Acoustics Stress and Reliability in Design*, **108**, 82-86 (1987).
2. V.K. Stokes, Design with nonhomogeneous materials-Part II: Torsion of thin-walled prismatic bars, *ASME Journal of Vibration Acoustics Stress and Reliability in Design*, **109**, 87-91 (1987).
3. M.E. Ergüven, Torsional loading of an elastic transversely isotropic nonhomogeneous semi-infinite solid, *Acta Mechanica*, 165-174 (1986).
4. H. Yoshihara, H. Ohsaki, Y. Kubojima and M. Ohta, Comparisons of shear stress/shear strain relations of wood obtained by Iosipescu and torsion tests, *Wood and Fiber Science*, **33** (2), 275-283 (2001).
5. J.Y. Liu, Effects of shear coupling on shear properties of wood, *Wood and Fiber Science*, **32** (4), 458-465 (2000).
6. G.A. Kardomateas, Torsion of a filament wound anisotropic elliptic cylinder with variable moduli of elasticity, *Int. J. Mech. Sci.* **31** (6), 459-469 (1989).
7. I. Ecsedi, Elliptic cross section without warping under torsion, *Mechanics Research Communications*, **31**, 147-150 (2004).
8. T. Ting, Pressuring shearing torsion and extension of circular tube or bar of cylindrically anisotropic material, *Proc. R. Soc. Lond. A.* **452**, 2397-2421 (1996).
9. S.D. Akbarov, R. Kosker and Y. Ucan, Stress with a periodically curved row of fibers, *Mechanics of Composite Materials*, **40** (3), 191-202 (2004).
10. X.N. Dong, X. Zhang, Y.Y. Huang and X.E. Guo, Generalized self-consistent estimate for the effective elastic moduli of fiber reinforced composite materials with multiple transversely isotropic inclusions, *Int. J. Mech. Sci.* **47**, 922-940 (2005).
11. S. Konaklı, Mathematical Modelling for Shear Modulus of Transversely Isotropic Circular Cross Sectional Bars under Torsional Loading, M.Sc. Thesis, The Institute of Science and Technology of Gazi University, Turkey, 2003.
12. A.C. Ugural and S.K. Fenster, *Advanced Strength and Applied Elasticity*, Prentice Hall, New Jersey, 1994.
13. J.T. Oden and E.A. Ripperger, *Mechanics of Elastic Structures*, McGraw-Hill, 1981.

