Bending Analysis of Sandwich Panels Under Point Forces

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ABSTRACT

A refined version of an applied theory is developed that adopts the analytical solution of the bending problem of a three-layer plate of arbitrary asymmetric structure under point loading. Local effects are investigated within the scope of the discrete model considering the specific character of soft filler elastic properties. The boundary problem of a panel supported along the surface of its lower face layer with free ends is reduced to the Cauchy problem. Variants are examined for the limiting transformation of the model parameters leading to a qualitative change in its kinematics and the corresponding simplified bending models. Advantages of the solution are the expressions of bending characteristics in a closed form: layer curvatures, displacements and stresses. These characteristics are shown to be varied several-fold according to the asymmetry parameters of the structure and the relative stiffness of the layers.

Keywords: sandwich composite panel, discrete non-symmetric structure, bending stresses, transversal stresses, concentrated forces, free end effects.

1. INTRODUCTION

The problem of the stress state of structures such as sandwich panels has been studied by applied methods of varying complexity including numerical and analytical methods. Classification of the theories and a detailed discussion of some of them was given in /1,2/. The expanding range of investigations is motivated by the need for consideration of the physical features in the behavior of structural sandwich elements under different loading conditions, concentrated loads included. The local stress fields in the vicinity of point forces and/or fixing points of a sandwich are still not clearly understood. Nevertheless, the general method of calculation of local characteristics, based on the solutions in boundary functions for the semi-infinite layered models, was given in /3/. The solution of the boundary-value problem for the finite length panel /4/ was proposed in the form of a method incommensurably more complex than the design procedure needed in engineering practice. The guideline in selection of a particular version of the theory is the opportunity to provide a clear-cut physical interpretation of all final and intermediate results of its application and the scope of going from a complex structural geometry to an ordinary geometry (e.g. from a layered structure to a uniform one).

Because of the introduction of simplifying hypotheses incident to applied methods, a solution does not always result in expressions acceptable for engineering practice and, consequently, numerical methods are used /1-3/. Obviously, closed-form analytical expressions, as functions of elastic properties and geometric parameters of a panel, are preferred for design analysis. The deduction of these expressions on a discrete layer model raises calculation difficulties. The asymmetry of a panel structure and the loading by point forces introduces additional mathematical difficulties into these calculations. Some design formulas obtained on the discrete model appertain to a symmetrical structure of a panel only /5, 6/

2. PROBLEM STATEMENT

In the present report, an analytical method was used to derive local bending effects for a sandwich panel according to the discrete model of three rigidly connected layers. In order to derive the analytical solution version, the Kirchhoff-Love kinematics of deformation of face layers and the transversal compression combined with shear of a midlayer were used. Under these assumptions, the mechanism of traction transfer between the layers, based on energy estimates, is not considerably simplified, however it can be analyzed more easily than with the use of numerical solutions.

Cylindrical bending by point forces, each of them being invariable along the panel width, is considered. Thus the discrete model possesses four degrees of displacement freedom: deflections w_i and longitudinal displacements u_i of face layers i = 1,2. Introducing the local coordinate system half-way of an either face layer thickness, see Fig. 1, the layer kinematics can be readily deduced from the four functions in the interval $-l \le x \le l$ and, respectively over the intervals $-h_i/2 < z^{(i)} \le h_i/2$, I = 1,2, and the problem equations are derived by a variational method.

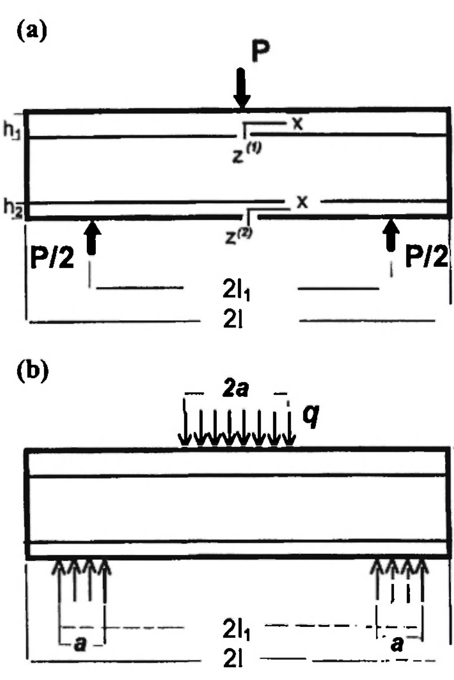

Fig. 1: Loading chart of a sandwich panel.

Omitting the details of this procedure, which is described in /6/, let us lay our emphasis on the analysis of the equations and the solution, separating a symmetric part from a nonsymmetric one and their dependence on the parameters corresponding to a symmetric and asymmetric panel structure. Symbolizing the components of the symmetric solution part by the index "*s*" and the nonsymmetric one, respectively, by "*as*", the four displacement functions of face layer middle planes and the surface loads take the following form:

$$u_{s,as} = (u_2 \pm u_1)/2,$$
$$w_{s,as} = (w_2 \mp w_1)/2, \qquad (1)$$
$$q_{s,as} = (q_2 \mp q_1)/2,$$

where the upper sign at the right part of Eq (1) corresponds to the index "*s*" at the left-hand side and the lower sign, respectively, to "*as*". Eq (1) corresponds to the index "*s*" at the left-hand side and the lower sign, respectively, to "*as*". The set of the Euler equations of the variational problem $\delta V = 0$ for the functional of the total energy V, after taking into account Eq (1) and having introduced the structure asymmetry parameters ϑ and θ (presented below), assume the following dimensionless form

$$\begin{cases} u''_{s,as} - \vartheta_{s,as}(k_1/4)\left[(1+\chi_0)w'_{as} - (1-\chi_0)w'_s + 4u_{as}\right] = 0 \\ w^{IV}_{s,as} - \theta_{s,as}(3k_1/2\mu^2) \\ \left[(1+\chi_0)w''_{as} - (1-\chi_0)w''_s + 4u'_{as}\right] + 12k_2\theta_{as,s}w_s = q_{s,as} \end{cases} \qquad (2)$$

The derivation in Eqs (2) is taken with respect to the dimensionless variable, all functions, physical and geometric parameters are expressed in a dimensionless form:

$$\xi = \frac{x}{h_1 + h_0}, \quad u,w = \frac{\mathrm{u,w}}{h_1 + h_0}, \quad \chi_0 = \frac{h_2 + h_1}{h_1 + h_0}, \quad \chi = \frac{h_2}{h_1}$$

$$n_x = \frac{E_x^{(2)} \cdot (1-\nu_1^2)}{E_x^{(1)}(1-\nu_2^2)}, n_z = \frac{E_z}{E_x^{(1)}}(1-\nu_1^2),$$

$$m_z = \frac{G_{xz}}{E_x^{(1)}}(1-\nu_1^2) \qquad (3)$$

$$\gamma_1 = 1 + \frac{1}{n_x\chi}, \gamma_2 = \frac{1}{n_x\chi^3},$$

continued ...

$$\vartheta_{s,as} = \frac{1}{n_x\chi} \mp 1, \theta_{s,as} = \frac{1}{n_x\chi^3} \mp 1$$

$$\mu = \frac{h_1}{h_1 + h_0},$$

$$k_1 = \frac{m_z}{\mu(1-\mu)}, \quad k_2 = \frac{n_z}{\mu^3(1-\mu)},$$

$$k_3 = \frac{3\chi_0 m_z}{\mu^3(1-\mu)}$$

$$q_0 = \frac{6P(1-\nu_1^2)}{E_x^{(1)} b(h_1 + h_0)\mu^3},$$

$$q_1 = 2q_0\delta(\xi), \quad q_2 = -\gamma_2 q_0 \delta(|\xi| - \xi_1)$$

The last line of Eq (3) including the Dirac function δ should be replaced by a linear combination of the unit functions η of a "lagging" argument, when forces *P* and *P/2* are substituted for a uniform load $q = P/2ab$, applied, respectively, at lengths *2a* and *a* in the vicinity of points $x = 0$ and $x = \pm l_1$ of the upper and lower face layers, see Fig.1 *b*. Thus:

$$\delta(\xi) \Rightarrow (1/2\xi_a)[\eta|\xi| - \eta(|\xi| - \xi_a)],$$
$$\delta(|\xi| - \xi_1) \Rightarrow (1/\xi_a)[\eta(|\xi| - \xi_1 + \xi_a) - \eta(|\xi| - \xi_1)]$$
$$\xi_1 = \frac{l_1}{h_1 + h_0}, \xi_l = \frac{l}{h_1 + h_0}, \xi_a = \frac{a}{h_1 + h_0}, \ 0 \le a \le l_1 \quad (4)$$

Inspection of Eqs (2) shows that, for the structure asymmetry when all $\vartheta_{s,as}, \theta_{s,as} \neq 0$, the equations of the symmetric and nonsymmetric parts of the displacements are not separated and in this situation these equations are devoid of advantages over the set of the equations in /6/. However, for the structure symmetry it follows that $\vartheta_s = \theta_s = 0$, $\chi_0 = 1$ and all equations can be separated into the sets, each defining the symmetric and nonsymmetric component of the solution. Further still, in the set for the symmetric part the equations of the functions u_s, w_s themselves can also be separated.

Boundary conditions for the set of Eqs (2) of the twelfth order in the absence of the longitudinal force at the panel ends $\pm\xi_l$ are the natural homogeneous conditions derived from variation of an energy functional:

$$w'''_{s,as}(\pm\xi_l) - \bar{\theta}_{s,as}(3k_1/2\mu^2) \quad (5)$$
$$[(1+\chi_0)w'_{as} - (1-\chi_0)w'_s + 4u_{as}]_{\pm\xi_l} = 0$$
$$w''_{s,as}(\pm\xi_l) = u'_{s,as}(\pm\xi_l) = 0, (\bar{\theta}_{s,as} = \chi_0/(n_x\chi^3) + 1)$$

By virtue of the fact that the membrane stresses of the face layers in the summarized axial load $N_x = 0$ make the couple $N_1 = -N_2$, the relation $u'_2(\xi) = -(1/n_x\chi)u'_1(\xi)$ is realized. Thus the expression of the moment at the centre of a sandwich panel loaded by point forces (Fig.1a) in relative variables can be deduced as:

$$w''_1(0) + n_x\chi^3 w''_2(0) + (6/\mu^2)(1+\chi_0)u'_1(0) = -q_0\xi_1 \quad (6)$$

and with a piecewise distributed load (Fig.1b) as

$$w''_1(0) + n_x\chi^3 w''_2(0) + (6/\mu^2)(1+\chi_0)u'_1(0) = -q_0(\xi_1 - \xi_a) \quad (7)$$

The Laplace transform was used in solving the set of Eqs. (2) in light of the discontinuous nature of the load upon the action of point forces and piecewise loads, as specified by $q_i(\xi)$ in (3) and (4). All other attendant conditions for symmetric load distribution about the central section $\xi = 0$, as the initial values at that section taken as $w'_i(0) = u_i(0) = 0$, $i = 1,2$ match the applied method. On the other hand, the fundamental functions available from this method give us a convenient way to deduce the kinematic characteristics at the central panel section in terms of the unknown constants in the Cauchy problem.

As a result, the solution of the bending problem for the asymmetric structure panel would be expressible in a closed form in terms of the fundamental functions, whose specific character is defined by the following properties:

1. Eleven linearly independent functions (with $N_x = 0$ the necessity to evaluate the twelfth function is eliminated) are recursively deduced by derivation:

$$p_{n-5}(\xi) = \frac{d^n}{d\xi^n} p_{-5}(\xi), \qquad n = 0,...,10 \quad (8)$$

The expression $p_{-5}(\xi)$ can be written in the form of a sum of terms that are the products of hyperbolic and trigonometric functions of a longitudinal coordinate. The arguments of these functions are rescaled by the factors that are the real and the imaginary parts of complex characteristic numbers with the notations α and β . The isolated terms of the sum are hyperbolic functions dependent on a real characteristic number denoted by a_1. The total notation of the set of functions is quoted in the Appendix. For a material of the midlayer which is absolutely pliable in transverse shear, that is $G_{xz} = 0$ (and $k_1 = 0$), the following expression of the primary fundamental function can be derived:

$$p_{-5}(\xi) = \frac{\xi^6}{6!b_1} - \frac{\xi^2}{2b_1^2} + \frac{1}{b_1^{5/2}} sh\,\alpha\xi \sin\alpha\xi \qquad (9)$$

where right now $\alpha = (\sqrt{2}/2)b_1^{1/4}$.

2. The functions $p_n(\xi)$ are normalized so that:

$$p_n(0) = 0, \text{ when } n \neq 5 \text{ and } p_5(0) = 1 \qquad (10)$$

3. The following auxiliary linear relationship of the fundamental functions that are identically equal to unity was deduced:

$$b_0 p_{-1}(\xi) + b_1 p_1(\xi) + b_2 p_3(\xi) + p_5(\xi) = 1 \qquad (11)$$

where

$$b_0 = -a_1^6 - b_2 a_1^4 - b_1 a_1^2 = -12k_1k_2[\gamma_1(1+\gamma_2) + \gamma_2(3/\mu^2)(1+\chi_0)^2]$$
$$b_1 = (\alpha^2+\beta^2)^2 + 2a_1^2(\alpha^2-\beta^2) = 12k_2(1+\gamma_2) \qquad (12)$$
$$b_2 = -a_1^2 - 2(\alpha^2-\beta^2) = -k_1[\gamma_1 + (3/\mu^2)(1+\gamma_2\chi_0^2)]$$

The displacements and their derivatives at the central panel section are derived from the boundary relationships (5) considering (6) as well as from the fact that a zero value of the deflection of the lower face layer is defined at a point of support, $\xi = \xi_1$, i.e. $w_2(\xi_1) = 0$. Eliminating $w_i(0)$ and $u_1'(0)$ from these equations, a set of two equations for $w_i''(0)$ i = 1,2 is obtainable, which solution provides a basis for the following analysis of the results given below. It should be noted that, as experience shows, the calculation procedure can be incorrect for a panel span $\xi_1 > 3$. This is attendant upon the necessity of regularization of the terms for the sum of products of hyperbolic and trigonometric functions with large arguments. The correcting calculation can be obtained by the substitution of the mentioned products with the real and imaginary parts of these functions now of a complex argument, that is, by isolation of their modulus as $\sqrt{sh^2\alpha\xi_l + \cos^2\beta\xi_l}$ and argument as $arctg(tg\,\alpha\xi_l\, tg\,\beta\xi_l)$.

Determination of the starting displacements and their derivatives at the section $\xi = 0$, in the case the panel is loaded with a uniform piecewise distributed load over the sections of length a (Fig.1b), is carried out analogously to the case of point forces. For this purpose we should replace the functions $p_n(\xi)$ in the particular solution for the case of a point force by the finite differences of their primitives from the lagging argument, divided by the displacement of the argument ξ_a. Also, we should carry out the replacement $\xi_1 \to \xi_1 - \xi_a$ in the case when ξ_1 is a cofactor of q_0.

3. FINITE FORMULAE

At first let us assume a standard curvature value $w_*''(0)$ at the panel center $x = 0$, see Fig. 1, which is derived from the classical Kirchhoff-Love bending model in the context of an asymmetric sandwich structure. For a three-point bending, the classical curvature value at the panel center can be expressed as:

$$w_{*,xx}(0) = -\frac{6Pl_1(1-\nu_{xy}\nu_{yx})}{E_x^{(1)} n_{as}^* b h_1^3} \qquad (13)$$

The ratio $w_*''(0) = -q_0\xi_1/n_{cs}^* = (h_1 + h_0)/R_*$ is decretal for the greatest classical curvature radius R_* in relative variables (3). The factor n_{cs}^* in (13) corrects for structure asymmetry of a sandwich and equals in magnitude the ratio of its cylindrical rigidity to the one of the upper face layer. It can be made available from the classical model with $w_1''(0) = w_2''(0) = w_*''(0)$ and $u_1'(0) = -[(h_2 + h_0 + h_1/2 - h_e)/(h_1 + h_0)]w_*''(0)$.

The distance h_e, counted from the outer surface of the lower face layer to the neutral plane of an asymmetric panel with the soft midlayer unloaded in bending, and the factor n_{cs}^* are defined as

$$h_e = \frac{2h + (n_x\chi^2 - 1)h_1}{2(n_x\chi + 1)},$$

$$n_{as}^* = 1 + n_x\chi^3 + \frac{3n_x\chi(1+\chi_0)[1+\chi+2(1-\mu)/\mu]}{\mu(1+n_x\chi)} \qquad (14)$$

For the symmetry of a sandwich structure we have $n_{as}^* = 2 + 6/\mu^2$, which agrees within designations with the formula from /7/. If $h_0 = 0$, that is $\mu = 1$ and $\chi_0 = \chi$, then the factor for a two-layer panel layered with different layer properties can be derived as $n_{cs}^* = 1 + n_x\chi^3 + 3n_x\chi(1+\chi)^2/(1+n_x\chi)$. In the case of an ideal sliding of the layers along the surface of their contact, we have $n_{as}^* = 1 + n_x\chi^3$.

We can now proceed to the construction of the relationship of deflections and its derivatives according to our discrete model and the discontinuous nature of the load upon the action of point forces. The factor associated with the curvature of the face layers was incorporated here and everywhere. The definition of the term $R_i = w_{i,xx}''$, $i = 1,2$ demands great care in $w_{i,xx}''$ presentation. The exact formulae for evaluation of $w_{i,xx}''$ in detail are clearly set out in /8/. All other attendant conditions for symmetric load distribution about the central section $\xi = 0$, as with the initial values at that section taken as $w_i'(0) = u_i(0) = 0$, $i = 1,2$, match the applied analytical calculations that result from the Laplace method. On the other hand, the fundamental functions available from this method give us a convenient way to deduce the kinematic characteristics at the central panel section in terms of the unknown constants in the Cauchy problem.

For two functions of the deflection, the following expressions were obtained:

$$w_1(\xi) = w_1(0)[1 - s_1(\xi)] + w_2(0)s_1(\xi) - u_1'(0)\phi_3(\xi) + \sum_{i=1}^{2} w_i''(0)\phi_i'(\xi) + q_0[\phi_1(\xi) - \gamma_2\phi_2(\xi - \xi_1)] \qquad (15)$$

$$w_2(\xi) = w_2(0)[1 - s_2(\xi)] + w_1(0)s_2(\xi) - u_1'(0)\varphi_3(\xi) + \sum_{i=1}^{2} w_i''(0)\varphi_i'(\xi) + q_0[\varphi_1(\xi) - \gamma_2\varphi_2(\xi - \xi_1)]$$

in which the definition of new functions is relegated to the Appendix.

The ratio between the maximum bending stress by a discrete model and the corresponding stress by a classical theory can be derived as:

$$\left.\frac{\sigma_x^{(1)}(0)}{\sigma_x^*(0)}\right|_{z^{(1)}=-h_1/2} = \frac{h_1+h_0}{h-h_e}\left\{\left[n_{as}^* - \frac{w_1''(0)+n_x\chi^3 w_2''(0)}{w_*''(0)}\right]\frac{\mu^2}{6(1+\chi_0)} + \frac{\mu}{2}\frac{w_1''(0)}{w_*''(0)}\right\}$$

$$\left.\frac{\sigma_x^{(2)}(0)}{\sigma_x^*(0)}\right|_{z^{(2)}=h_2/2} = \frac{h_1+h_0}{h_e}\left\{\left[n_{as}^* - \frac{w_1''(0)+n_x\chi^3 w_2''(0)}{w_*''(0)}\right]\frac{\mu^2}{6n_x\chi(1+\chi_0)} + \frac{\mu\chi}{2}\frac{w_2''(0)}{w_*''(0)}\right\} \qquad (16)$$

For the transverse normal stress of the midlayer we obtained the following definition:

$$\sigma_z(\xi) = \frac{E_z}{1-\mu}[w_2(\xi) - w_1(\xi)] \qquad (17)$$

The difference of the layer deflections at the center, $\xi = 0$, may be expressed as

$$w_\Delta = w_2(0) - w_1(0) = -c_{10}w_1''(0) - c_{20}w_2''(0) + c_{00} \qquad (18)$$

where the final expressions of the coefficients c_{00}, c_{10}, c_{20} for a sandwich panel with an asymmetric structure ($h_1 \neq h_2, E_x^{(1)} \neq E_x^{(2)}$) are specified in the Appendix.

4. RESULTS AND DISCUSSION

The analysis of the bending characteristics of the asymmetric panel was carried out in different directions from the symmetric structure, which is an intermediate variant with equal layer thickness $h_1 = h_2 = h_0 = 4$ mm and $E_x^{(i)} = E_x = 40 GPa$ for a material of the HEXCEL/Al type. The thickness of the face layers was varied between the limits $(1/10)h \le h_i \le (2/3)h$ at a constant thickness of a panel h and the midlayer $h_0 = 0.24\ h$, and their elasticity moduli were varied in the range $E_x \le E_x^{(i)} \le (20/3)E_x$. The variation of the structural parameters was inversely affected, that is, the initial values of parameters χ and n_x were, respectively, the same as the values $1/\chi$ and $1/n_x$ at the end of a variation interval $0.15 \le \chi, n_x \le 6.66$. To make the variation intervals symmetrical about the values

$\chi = 1$ and $n_x = 1$ for these relative parameters, a logarithmic scale was introduced.

As one can see from (16), the ratio between the curvatures by a discrete and by a classical bending model is not equivalent to the ratio of the maximum bending stresses by these models, other things being equal. It does not ensue that two summands prior to the last one in (16) could not be equal to zero. Indeed for ideal slipping by a discrete model with $G_{xz} = 0$, the ratio $\left[\sigma_x^{(i)\,slip}/\sigma_x^{*\,slip}\right]_{z^{(i)}=\mp h_i/2} = w_i^{''slip}/w_*^{''slip}$, i = 1,2 regards only a filler transverse elasticity with ideal sliding of face layers by a classical model. In this case, the derivative of the longitudinal displacement in the midplane of the upper layer is equal to zero: $u_1'(0) = 0$ and, accordingly, (6), (13) the expression in square parentheses in (16) equals zero. The remnant coefficients $(h_1/2)/(h-h_e), (h_2/2h_e)$ ahead of the ratios $w_i''(0)/w_*'(0)$, i = 1,2 have been substituted for the unit, because the denominators of those coefficients in ideal sliding by a classical model should be equal to $h_1/2$ and $h_2/2$, respectively.

The curves obtained, accordingly, (16) and given in Fig. 2 *a, b,* show the combined influence of transverse shear and compression rigidity of a midlayer on the bending stresses. Note in order that the curves in Fig. 2 (*a*) intersect the ordinate axis above the curves in Fig. 2 (*b*), as with the same index n_x = 1 in Fig. 2 (*a*), (*b*) the values m_z and n_z are different, as per the first case of symmetry $E_x^{(i)} = 3.83E_x$ and per the other $E_x^{(i)} = E_x$ at the same elastic characteristics of the midlayer $E_z = 0.310\ GPa$, $G_{xz} = 0.138\ GPa$.

For a symmetric structure $(E_x^{(1)} = E_x^{(2)}, h_1 = h_2)$, the key characteristics defining local deformation in bending of a sandwich panel are the relative transverse compression of a central section $\hat{w}_\Delta = [w_1(0) - w_2(0)]/q_0$ and the relative increment of a local curvature at the panel center κ_α. Both of these parameters are dependent on merely the relative rigidity in compression $a = (6K_2)^{1/4}$. For the local parameters, the following formulae can be found for the case of loading by three point forces (Fig. 1 *a*):

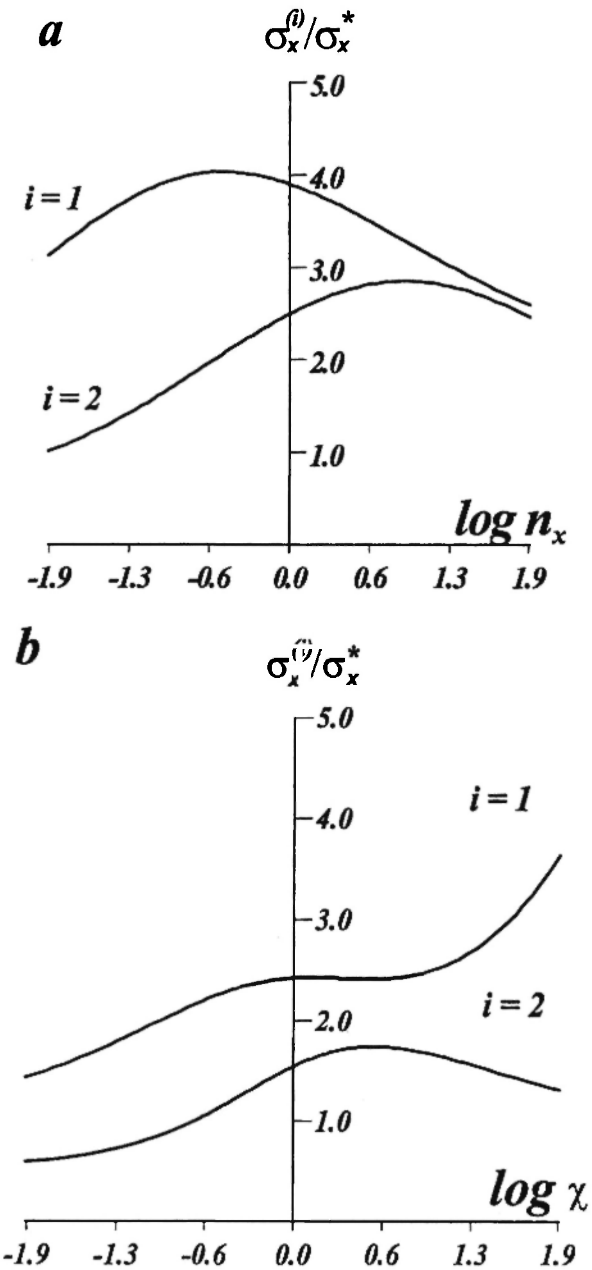

Fig. 2: Effect of the elastic (*a*) and geometric (*b*) asymmetry factors of a layered structure on the ratio of the largest bending stresses.

$$\hat{w}_\Delta = \frac{s''(\alpha\xi_l) + s''[\alpha(\xi_l - \xi_1)] - 2\kappa_\alpha s'''(\alpha\xi_l)}{4\alpha^4 s'(\alpha\xi_l)} \quad (19)$$

$$\kappa_\alpha = \frac{2}{\alpha}\,\frac{s'(\alpha\xi_l)[s'(\alpha\xi_l) + s'[\alpha(\xi_l - \xi_1)]] - s(\alpha\xi_l)[s''(\alpha\xi_l) + s''[\alpha(\xi_l - \xi_1)]]}{s'^2(2\alpha\xi_l) - s(2\alpha\xi_l)s''(2\alpha\xi_l)} \quad (20)$$

where $s(\xi) = sh\ a\xi \sin a\xi$, and the operation of derivation with respect to relative variable ξ is designated by a prime. The relative values of face layer curvatures $w_1''(0) = -q_0(\hat{\xi}_1/2 + \kappa_\alpha)$ and $w_2''(0) = -q_0(\hat{\xi}_1/2 - \kappa_\alpha)$ are also functions of the shear parameter $a_1 = [2\mu^2 k_1/(3+\mu^2)]^{1/2}$, entering in the expression for a double value of the mean curvature:

$$\hat{\xi}_l = \frac{\xi_l}{1+3/\mu^2} + \frac{sh a_1\xi_l - sh[a_1(\xi_l - \xi_1)]}{a_1(1+\mu^2/3)ch a_1\xi} \quad (21)$$

Then the maximum value of the bending stresses in the central section, at a point reaching the outer surface of the upper face layer, equals

$$\sigma_x^{mom}(0) = -E_x q_0 \mu(\hat{\xi}_1/2 + \kappa_\alpha)/[2(1-\nu_1^2)] \quad (22)$$

which differs from the membrane stress

$$\sigma_x^{memb}(0) = -E_x q_0 \mu^2(\xi_1 - \hat{\xi}_1)[12(1-\nu_1^2)] \quad (23)$$

approximately by a factor of two. The latter stress, defined as the average stress throughout the face layer thickness, is comparable to the value of flexural stress σ_x^{cl}, determined according to the classical Kirchhoff-Love hypotheses for the whole stack of layers /9/.

Notice that for $G_{xz} = 0$, the doubled value of the relative mean curvature equals $\hat{\xi}_1 = \xi_1$, while in the case of $G_{xz} = \infty$ (the bending of a uniform panel according to Kirchhoff-Love) and for a thin filling interlayer, when $\mu_V \to 1$, we have $\hat{\xi}_1 = \xi_1/4$, see Fig. 3. This is also evident from a comparison of the sum of two cylindrical rigidities of plates with a cylindrical rigidity of one plate, which is twice as thick as each of them.

Recurring to the case of asymmetric structure under three-point bending, one might find monotonous variation of the midlayer compression stresses with geometric and elastic asymmetry parameters, see Fig. 4 (*a*), (*b*). Curve behavior is different for the central and the support sections. A distinction between the ordinates of the curves at a zero abscissa in Fig. 4 (*a*), (*b*) is generated by a certain discrepancy between the pairs of

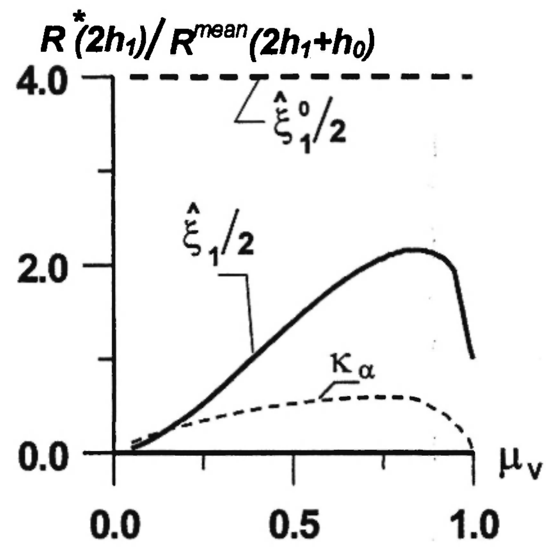

Fig. 3: Summands of relative curvature at the central section of a symmetric sandwich panel versus face layers fraction: $\mu_V = 2h_1/h$, $2l/h = 5$, ($\hat{\xi}_1^0/2$ at $G_{xz} = 0$)

moduli, when $n_x = 1$, i. e., like the situation in Fig. 2 (*a*), (*b*). The ratio (1/2):1:2 that is characteristic of the ordinates of the curves at $n_x = 1$ and $\chi = 1$ is explained by the substantial difference in the length of the panel ends overhanging beyond the supports. This characteristic property pertinent to the ratio in hand was fully examined in /5,6/ for the case of a symmetric structure panel.

In the case of an asymmetric structure the ratio h_2/h_1 together with $\Delta l = l - l_1$ can essentially affect a positive value $\sigma_z(l)$ at the panel end. A bit of parametric surface is isolated by faded colours along the coordinate lines for this case, as shown in Fig. 5.

It should be emphasized that in the case of a symmetric structure the transversal stresses σ_z in the midlayer are identical both for bending combined with slippage ($G_{xz} = 0$) and for bending without slippage. At the center of the panel loaded according to the three point chart shown in Fig. 1 *a*, the stress $\sigma_z(0)$ is evaluated by the formula:

$$\sigma_z(0) = (-1/4)\sigma_z^* \quad (24)$$

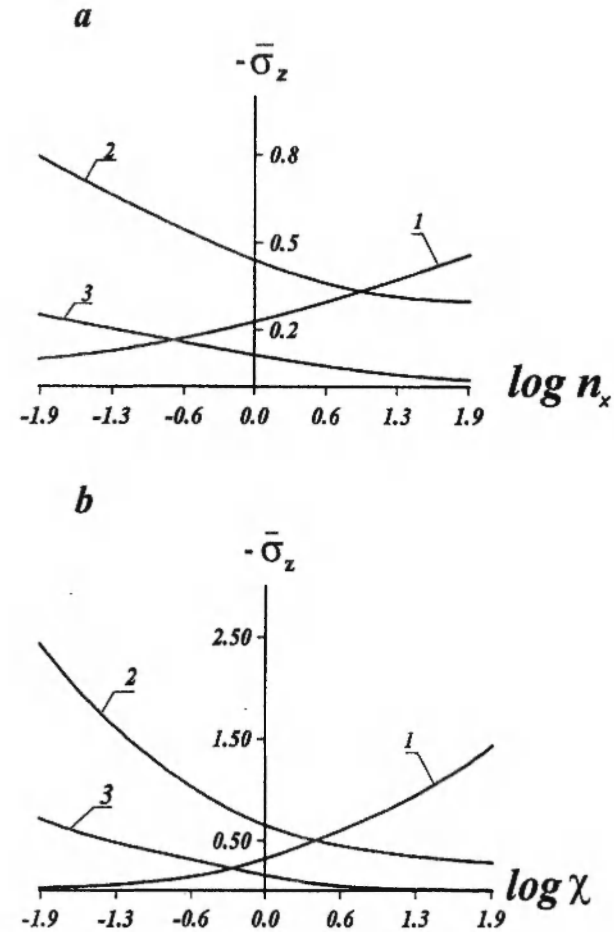

Fig. 4: The effect of elastic (*a*) and geometric (*b*) structure asymmetry parameters under three point panel bending on the value of the midlayer transverse stress $\bar{\sigma}_z = \sigma_z / \sigma_0$, $\sigma_0 = P/bh$ in the following sections: 1 - $\xi = 0$, 2 - $\xi = \xi_1$ for $l = l_1 = 3h$ and 3 - $\xi = \xi_1$ for $l = 4.5h$, $l_1 = 3h$.

where

$$\sigma_z^* = \frac{P}{bh_0}\sqrt[4]{\frac{6(1-\nu_1^2)E_z h_0^3}{E_x h_1^3}} \tag{25}$$

For the support section $\xi = \xi_1$ the asymptotic formula was deduced when $\xi_1 >> 1$ and the following estimate was found as $\sigma_z(\xi_1) = -(1/8)\sigma_z^*$. However the cases of $\xi_1 \neq \xi_l$ and $\xi_1 = \xi_l$ have called for careful analysis. A fundamental change in the transfer mechanism of pressure on a midlayer occurs when the overhang of panel length beyond the supports is varied between 0 and *(1/4)h*, see Fig. 6. The distribution of $\sigma_z(\xi)$ with superposition of

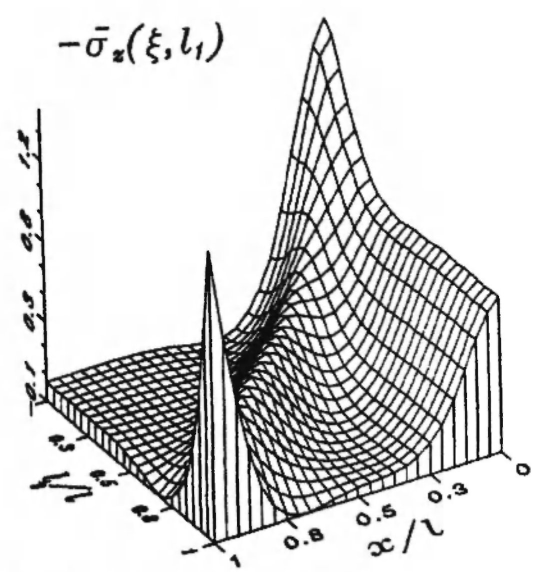

Fig. 6: Transverse normal stress diagrams along the panel length and against the extent of a span between the supports, ($\bar{\sigma}_z = \sigma_z / \sigma_0$, $\sigma_0 = P/bh$).

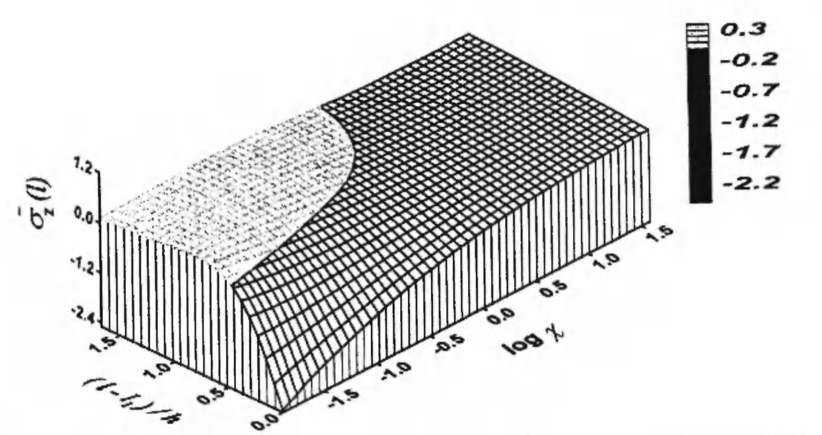

Fig. 5: Variation of the relative transverse stress at the end of a panel $x = l$ according to relative length of the section overhanging beyond a support and the ratio of face layer thickness. (The data are identical with Figs. 2 and 4).

local effects caused by point forces at the support and the center of a panel is shown for the moderate panel length $2l = 5h$.

At the end section $\xi=\xi_l$ of the panel loaded as shown in Fig. 1 *a*, the relative value of the stresses $\sigma_z(\xi_l)/\sigma_z^*$ can be found as $-(1/2)e^{-\alpha(\xi_l-\xi_1)}\cos\alpha(\xi_l-\xi_1)$. The maximum *positive* value of the transversal stress σ_z at the end section $\xi=\xi_l$ occurs when $\xi_l-\xi_1=3\pi/4\alpha$.

Let us give a description of a piecewise homogeneous load statically equivalent to point forces "spread" over small regions, representing the three-point bending of a panel (see Fig. 1 *b*). We obtain, for the central section of the panel, the transverse compression strains w_Δ^{p-w}, the relative curvature $\hat{\xi}_1/2=-(h_1+h_0)/(R_{mid}^{p-w}q_0)$ and the local additive for the curvature of a separate layer, k_a^{p-w} /10/. The ratios of the examined characteristics of bending from a point force and a piecewise constant load are:

In Eqs. (27), (28) the parameters a_1 and α depend on the elastic modulus of the filler in shear, G_{xz}, and in compression, E_z, separately. The parameter q_0 is the same as in Eq. (3), but, for a uniform load q, an equivalent value of the force, $P = 2abq$ must be assumed. In the case of layer slippage in bending, expression (27) becomes indeterminate and, using L'Hospital's rule, a simple linear ratio for the average curvature is found for the two considered variants of loading with respect to the relative length a/l_l:

$$\left.\frac{w_\Delta^{piecewise}}{w_\Delta^{point}}\right|_{\xi=0}=\frac{\dfrac{1-e^{-\alpha\xi_a}\cos\alpha\xi_a-e^{-\alpha\xi_1}\cos\alpha\xi_1+e^{-\alpha(\xi_1-\xi_a)}\cos\left[\alpha(\xi_1-\xi_a)\right]}{4\alpha^4\xi_a}}{\dfrac{1+e^{-\alpha\xi_1}(\cos\alpha\xi_1+\sin\alpha\xi_1)}{4\alpha^3}}, \quad (26)$$

$$\left.\frac{\hat{\xi}_1^{piecewise}}{\hat{\xi}_1^{point}}\right|_{\xi=0}=\frac{\left(\dfrac{\mu^2}{3+\mu^2}\right)\left(\xi_1-\xi_a+\dfrac{3}{\mu^2a_1^2}\cdot\dfrac{1-e^{-a_1\xi_a}-e^{-a_1(\xi_1-\xi_a)}+e^{-a_1\xi_1}}{\xi_a}\right)}{\dfrac{\xi_1}{1+3/\mu^2}+\dfrac{1-e^{-a_1\xi_1}}{a_1(1+\mu^2/3)}}, \quad (27)$$

$$\left.\frac{k_a^{piecewise}}{k_a^{point}}\right|_{\xi=0}=\frac{\dfrac{e^{-\alpha\xi_1}\sin\alpha\xi_1+e^{-\alpha\xi_a}\sin\alpha\xi_a-e^{-\alpha(\xi_1-\xi_a)}\sin\left[\alpha(\xi_1-\xi_a)\right]}{4\alpha^2\xi_a}}{\dfrac{1+e^{-\alpha\xi_1}(\cos\alpha\xi_1-\sin\alpha\xi_1)}{4\alpha}}. \quad (28)$$

$$\left.\frac{R_{mid}^{point}}{R_{mid}^{p-w}}\right|_{\xi=0}=1-\frac{\xi_a}{\xi_1} \quad (29)$$

According to the classical theory, relation (29), which also follows from Eq. (27) at $G_{xx} \to \infty$, can be applied to both face layers in the center of the panel. We should note, however, that, according to the discrete model, the curvature of face layers upon layer slippage and upon absolute shear stiffness of the midlayer is different. The calculated curves, illustrating relations (26)-(28) for finite values of elastic constants, are shown in Fig. 7.

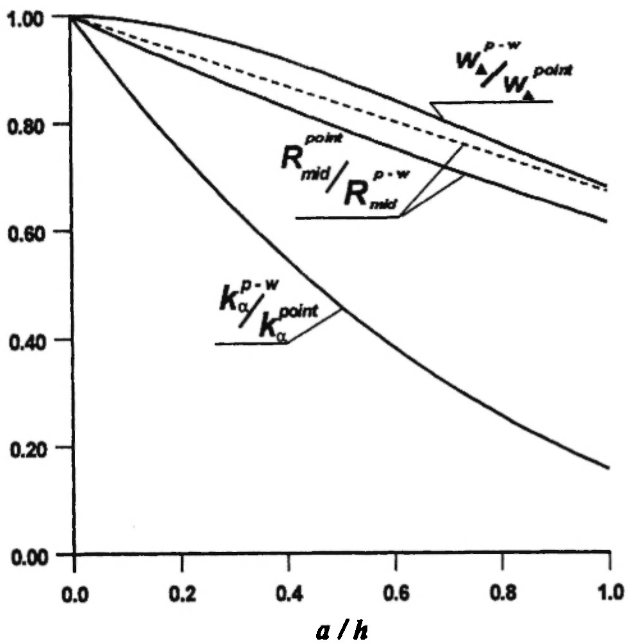

Fig. 7: Variation in the curvature characteristics and transverse deformation at $\xi = 0$ with the point forces replaced by a piecewise distributed pressure (see Fig. 1 *b*). The values of elastic constants as in Fig. 2; $h = 0.012m$, and $h_1 = h_2 = h_0$. The dashed line at $G_{xz} = 0$.

A difference in the values of the average curvature and/or transverse deformation, not exceeding the 5% correction because of interchanging the calculation formulas corresponding to point forces and distributed pressure, as seen from Fig. 7, will take place if the length of the region of "spreading" the point force $2a$ does not exceed 20% of the panel thickness h. In the estimate for the curvature of face layers, the acceptable length of this region for the interchangeability of the formulas is smaller.

5. CONCLUSIONS

An analytical method devised for analysis of the sandwich panel in bending allows us to obtain refined characteristics for the inplane and transverse normal stresses. Taking into account the important aspects neglected in the classical theory the following conclusions were drawn:

- The ratio between the greatest bending stresses and their values by a classical theory varies nonmonotonously with the asymmetry parameters of a panel structure.
- Structure modification of a sandwich panel of the HEXCEL/A1 - type material, varying inversely the elasticity moduli of the upper and the lower face layer as much as six times, gives rise to a change of 20%-40% in the ratio as compared with a symmetric structure. A similar comparative deviation of the stress ratio is deduced also in the redistribution inversely of a face layer thickness. For the layer sliding in bending this ratio is greater everywhere over the range. Face layer curvatures at the places of point force application differ from the classical value to a greater extent than bending stresses.
- For the panel of symmetric structure with respect to the middle plane, it was shown that the local curvatures of layers and the bending stresses can be determined in terms of the mean curvature depending on transverse shear and the additional curvature from transverse compression. The final formulae allow us to analyze the stresses over the whole range of shear rigidity of the midlayer including, $G_{xx} \to \infty$, (the case of slippage over the interlayer surface).
- Squeezing stresses of the midlayer in three-point bending essentially depend on the asymmetry of face layer characteristics. The trends to variation of these stresses at the central and support sections with the inversion modification of the asymmetry parameters are of an opposite nature. As an elasticity modulus or thickness of the face layer, contacting supports, is reduced by a factor of six, the stress increases two to three - fold as compared with its value for the structure symmetry. This stress is the greatest in magnitude at a support section when the length of a free section extending beyond the support is zero.
- The panel parts extending beyond the supports may

be the cause of the advent of a positive value $\sigma_z(l)$ at the free end, which by an order of magnitude less than the value $\sigma_z(0)$ at the panel center.

Local bending characteristics amplify the applied theory of layered plates by the analytical description of local effects, and in the special cases of boundary conditions.

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APPENDIX

The designation of the fundamental function in (8) is taken as

$$p_{-5}(\xi) = \frac{\xi^4}{24b_0} - \frac{\xi^2}{2d_1}\left(\frac{1}{a_1^4} + \frac{\alpha_2+\beta_2\rho_0}{r^8}\right) + \frac{ch\, a_1\xi - 1}{a_1^6 d_1} +$$

$$+\frac{1}{d_1 r^{12}}\left[(\alpha_4+\beta_4\rho_0)(ch\,\alpha\xi\cos\beta\xi - 1) - (\beta_4 - \alpha_4\rho_0)sh\,\alpha\xi\sin\beta\xi\right]$$

The arguments of hyperbolic and trigonometric functions of a longitudinal coordinate are rescaled by the factors, which are the real and the imaginary parts of complex characteristic numbers with the notations α and β. The isolated term of the sum is the hyperbolic function dependent on real the characteristic number denoted by a_1:

$$r^2 = (a_1^4 + b_2 a_1^2 + b_1)^{1/2}, \quad d_1 = 3a_1^4 + 2b_2 a_1^2 + b_1,$$

$$\rho_0 = \sqrt{3}(\rho_1 - \rho_2)/(\rho_1 + \rho_2), \quad \rho_{1\,2} = (\sqrt{\Delta} \mp c_0/2)^{1/3}$$

$$\Delta = (c_1/3)^3 + (c_0/2)^2, \; c_1 = b_1 - b_2^2/3,$$

$$c_0 = (2/27)b_2^3 - (1/3)b_1 b_2 + b_0$$

$$\alpha_1 = \alpha(3\beta^2 - \alpha^2), \quad \beta_1 = \beta(3\alpha^2 - \beta^2)$$

$$\alpha_{k+1} = \alpha\alpha_k + \beta\beta_k, \quad \beta_{k+1} = \alpha\beta_k - \beta\alpha_k, \quad k = 1,2,3$$

$$a_{1,2} = \pm\sqrt{\rho_1 - \rho_2 - b_2/3}, \text{ when } \rho_1 - \rho_2 \ge b_2/3,$$

$$a_{1,2} = \pm i\sqrt{\rho_1 - \rho_2 - b_2/3}, \text{ when } \rho_1 - \rho_2 < b_2/3.$$

$$a_k = \pm(\alpha \pm i\beta), \quad k = 3,4,5,6$$

$$\alpha = (1/2)\sqrt{2r^2 + \rho_2 - \rho_1 - (2/3)b_2},$$

$$\beta = (1/2)\sqrt{2r^2 - \rho_2 + \rho_1 + (2/3)b_2}$$

$$r^2 = \sqrt{\rho_1^2 + \rho_1\rho_2 + \rho_2^2 + (1/9)b_2^2 - (1/3)(\rho_2 - \rho_1)b_2}$$

Here $i = \sqrt{-1}$. The total notation of the set of functions is quoted in /8/.

Accessorial functions take the form

$$s_1(\xi) = 12k_2[p_1(\xi) - k_1\gamma_5 p_{-1}(\xi)],$$
$$s_2(\xi) = 12k_2\gamma_2[p_1(\xi) - k_1\gamma_6 p_{-1}(\xi)]$$

$$\phi_1(\xi) = p_2(\xi) - k_1\gamma_4 p_0(\xi) + \gamma_2\omega(\xi),$$
$$\phi_2(\xi) = k_3 p_0(\xi) + \omega(\xi)$$

$$\phi_3(\xi) = k_4 p_{-3}(\xi) + (2k_3\gamma_1 / \chi_0)p_1(\xi),$$
$$\omega(\xi) = 12k_2[p_{-2}(\xi) - k_1\gamma_1 p_{-4}(\xi)]$$

$$\varphi_1(\xi) = \gamma_2\phi_2(\xi),$$
$$\varphi_2(\xi) = p_2(\xi) - k_1\gamma_3 p_0(\xi) + \omega(\xi)$$

$$\varphi_3(\xi) = k_4 p_{-3}(\xi) + 2k_3\gamma_1\gamma_2 p_1(\xi)$$

$$k_3 = (3/\mu^2)\chi_0 k_1, \qquad k_4 = 24\gamma_1\gamma_2(1/\chi_0 + 1)k_2k_3$$
$$\gamma_3 = \gamma_1 + 3/\mu^2, \qquad \gamma_4 = \gamma_1 + (3/\mu^2)\gamma_2\chi_3^2$$

Second derivatives of the central deflections are expressible in simple equations:

$$w_1''(0) = [(c_{00} + \varphi_{00})(\phi_{20} + \varphi_{20}) - (\phi_{00} + \varphi_{00})(c_{20} + \varphi_{20})]/\Omega_0$$

$$w_2''(0) = [(c_{10} + \varphi_{10})(\phi_{00} + \varphi_{00}) - (\phi_{10} + \varphi_{10})(c_{00} + \varphi_{00})]/\Omega_0$$

$$\Omega_0 = (c_{10} + \varphi_{10})(\phi_{20} + \varphi_{20}) - (\phi_{10} + \varphi_{10})(c_{20} + \varphi_{20})$$

The case in point $\xi = \xi_l$ yields the constants:

$$c_{10} = [(k_0\mu_0 + k_5)p_0(\xi_l) - p_2(\xi_l)]/s_0(\xi_l)$$

$$c_{20} = [(k_0\mu_0/\gamma_2 - k_6)p_0(\xi_l) + p_2(\xi_l)]/s_0(\xi_l)$$

$$c_{00} = -[q_0/s_0(\xi_l)]$$
$$[1/(12k_2) + \xi_1\mu_0k_0p_0(\xi_l) - p_1(\xi_l) - \gamma_2p_1(\xi_l - \xi_1)$$
$$+ k_5p_{-1}(\xi_l) + \gamma_2k_6p_{-1}(\xi_l - \xi_1)]$$

$$s_0(\xi_l) = \xi_l - b_1p_0(\xi_l) - b_0p_{-2}(\xi_l), \quad \mu_0 = \mu^2/[6(1 + \chi_0)]$$

$$k_0 = 2k_3\gamma_1(\gamma_2 - 1/\chi_0),$$
$$k_5 = k_1\gamma_4 + k_3\gamma_2, \quad k_6 - k_3 + k_1\gamma_3$$

$$\phi_{10} = [\phi_1'''(\xi_l) + \mu_0\phi_3''(\xi_l)]/s_1''(\xi_l),$$
$$\phi_{20} = [\phi_2''(\xi_l) + (\mu_0/\gamma_2)\phi_3''(\xi_l)]/s_1''(\xi_l)$$
$$\phi_{00} = -[q_0/s_1''(\xi_l)][\xi_1\mu_0\phi_3''(\xi_l) + \phi_1''(\xi_l) - \gamma_2\phi_2''(\xi_l - \xi_1)]$$

In order of designation $\varphi_{i0}, i = 1,2, \varphi_{00}$ we replace ϕ by φ and $s_1 \rightarrow s_2$ in the last two rows.