A New Approach for the Mechanical Behaviour of Materials Having Different Moduli in Tension and Compression

M. Karama*, S.K. Afaq** and S. Mistou***

Laboratoire Génie de Production, Equipe CMAO, Groupe M^2SF, ENIT, BP 1629, 65016 Tarbes Cedex, France.

ABSTRACT

In this article, an analytical method is presented for predicting the mechanical behaviour of materials with elastic properties in traction which are different from those under compression. The study was carried out on square, simply supported, non-symmetrical multi-layer orthotropic laminates under uniformly distributed sinosoidal load. Transverse shearing stresses were taken into consideration through the use of Touratier's refined theory. The present analytical results of deformation, strains and stresses in flexure and by numerical analysis, are compared to the results of Papazoglou and Tsouvalis /1/ (based on the theory of Reddy /4/).

NOTATION

a,b	*dimensions of plate in x,y directions*
Aij, Bij, Dij, Eij, Fij, Gij	*laminate stiffness*
Et, Ec	*tensile and compressive Young's moduli, respectively*
G	*shear modulus*
h	*plate thickness*
kp, kq, kx	*weighting factors*
Mi, Pi	*stress results*
q	*lateral load*
Q	*layer stiffness matrix*
S	*compliance matrix*
T	*transformation matrix*
u, v, w	*plate displacement in the x, y, z directions*
S	*compliance matrix*
T	*transformation matrix*
u, v, w	*plate displacement in the x, y, z directions*
znx, zny	*natural surface location in x and y directions respectively*
j	*rotation*
p, q, x	*subscripts for principal stress coordinate system*
1,2,3	*subscripts for principal material coordinate system*
g	*superscript for geometrical direction indicator*
m	*superscript for principal material direction indicator*
pr	*superscript for principal stress direction indicator*

INTRODUCTION

Some composite materials exhibit elastic moduli which are different when subjected to tension loading than when the loading is compressive, i.e., the elastic properties are load dependent. The characteristic behaviour is actually curvilinear, and is often approximated by two straight lines with a slope discontinuity at the origin (Fig. 1). Thus these materials are called bi-modulus .

* Professor, corresponding author moussa@enit.fr, Fax +33562442708,

** Phd student,

*** Assistant Professor

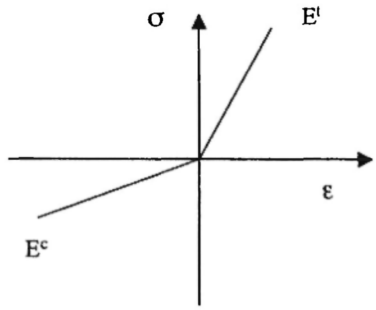

Fig. 1: Stress-strain relation of linearized different modulus material

For example, differences of up to 60% have been reported between tensile and compressive moduli for orthotropic composites /2,3/, or even up to 100% for some rubber-matrix ones /4/.

Initial research on the bimodulus materials studied by Jones /2,5/, uses the Ambartsumyan material model for bimodulus materials in the buckling of circular cylindrical shells. The same author developed a new; more accurate, material model /3/. Subsequently Jones and Morgan /6/ studied the cylindrical bending of bimodulus cross-ply laminates using classical lamination theory. Kamiya /7/, first incorporated the transverse shear deformation theory (SDT) in the analysis of cylindrical bending of bimodulus material, while Fung and Doong /8/, using a higher-order SDT, evaluated results for the bending of cross ply laminates.

All the above studies were restricted to laminates consisting at most of two layers. This restriction is overcome by Papazoglou and Tsouvalis /1/ in studying the multilayered cross-ply laminates by the Reddy higher-order SDT .

In this article, the same method has been adopted for multilayered cross-ply bimodulus laminates, as defined by Papazoglou and Tsouvalis /1/ with a more accurate definition of weighting factors k_p, k_q, k_ξ for calculating the stiffnesses of 'stress zone', as well as well-known shear deformation laminated-plate theory of Touratier /9/ is used to incorporate the transverse shear stresses effect on each interface of the layers in a laminated structure.

THREE-DIMENSIONAL STRESS MATERIAL MODEL

For bi-modulus material, the different properties in tension and compression cause a shift in the neutral surface away from the geometric midplane, and symmetry about the midplane no longer holds. The result of this is that the bending-stretching coupling of an orthotropic type is exhibited, i.e. analogous to a two-layer cross-ply plate (one layer at 0° and the other at 90°) of ordinary orthotropic material. The governing equations of composite materials could be used for bi-modulus materials except that the stress-strain relation must be of the bimodular form.

The term "Material Model", used here, incorporates a way of defining the values of the stiffness or the compliance matrix members. For the sake of simplicity these relations are usually defined in the principal stress coordinates *(p, q, ξ)*, rather than in the geometrical or principal material axes.

The material model which will be used as a basis for the present extension is the Weighted Compliance Matrix material model (WCM Model), presented by Jones /3/. The WCM model introduces weighting factor which depend on the principal stress state and which determine the percentage of the tensile and compressive properties that form the final compliance matrix. In this article, weighting factors are introduced in a little bit different way as defined by Jones.

In this analysis, a shear deformation theory of Touratier /9/ is used and thus the need for a material model valid for the three-dimensional stress state becomes obvious.

In principle stress coordinates, ($\sigma_{pq} = \sigma_{q\xi} = \sigma_{\xi p} = 0$), stress-strain relation:

$$\begin{Bmatrix} \varepsilon_p \\ \varepsilon_q \\ \varepsilon_\xi \\ \varepsilon_{q\xi} \\ \varepsilon_{\xi p} \\ \varepsilon_{pq} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \\ S_{14} & S_{24} & S_{34} \\ S_{15} & S_{25} & S_{35} \\ S_{16} & S_{26} & S_{36} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_p \\ \sigma_q \\ \sigma_\xi \end{Bmatrix} \qquad (1)$$

Here, $\varepsilon_{q\xi}$ $\varepsilon_{\xi p}$, ε_{pq} are not equal to zero, since the principal strain directions do not coincide with principal stress directions in an orthotropic material.

Using Jones's WCM model /1/ to the three-dimensional stress state, with the compliance are assigned as follows, according to the sign of the principal stress *(t: tensile, c: compressive)*

if $\sigma_p > 0, \sigma_q > 0, \sigma_\xi > 0: S_{ij} = S_{ij}^t$

if $\sigma_p > 0, \sigma_q > 0, \sigma_\xi < 0: \left.\begin{matrix} S_{1j} = S_{1j}^t \\ S_{2j} = S_{2j}^t \end{matrix}\right\} \quad j = 1,2,4,5,6$

$S_{3j} = S_{3j}^c \quad j = 3,4,5,6$

$S_{3j} = k_p S_{3j}^t + k_q S_{3j}^t + k_\xi S_{3j}^c \quad j = 1,2$

if $\sigma_p > 0, \sigma_q < 0, \sigma_\xi > 0: \left.\begin{matrix} S_{1j} = S_{1j}^t \\ S_{3j} = S_{2j}^t \end{matrix}\right\} \quad j = 1,3,4,5,6$

$S_{2j} = S_{2j}^c \quad j = 2,4,5,6$

$S_{2j} = k_p S_{2j}^t + k_q S_{2j}^c + k_\xi S_{2j}^t \quad j = 1,$

if $\sigma_p > 0, \sigma_q < 0, \sigma_\xi < 0: \left.\begin{matrix} S_{2j} = S_{2j}^c \\ S_{3j} = S_{3j}^c \end{matrix}\right\} \quad j = 2,3,4,5,6$

$S_{1j} = S_{1j}^t \quad j = 1,4,5,6$

$S_{1j} = k_p S_{1j}^t + k_q S_{1j}^c + k_\xi S_{1j}^c \quad j = 2,3$

if $\sigma_p < 0, \sigma_q > 0, \sigma_\xi > 0: \left.\begin{matrix} S_{2j} = S_{2j}^t \\ S_{3j} = S_{3j}^t \end{matrix}\right\} \quad j = 2,3,4,5,6$

$S_{1j} = S_{1j}^c \quad j = 1,4,5,6$

$S_{1j} = k_p S_{1j}^c + k_q S_{1j}^t + k_\xi S_{1j}^t \quad j = 2,3$

if $\sigma_p < 0, \sigma_q > 0, \sigma_\xi < 0: \left.\begin{matrix} S_{1j} = S_{1j}^c \\ S_{3j} = S_{3j}^c \end{matrix}\right\} \quad j = 1,3,4,5,6$

$S_{2j} = S_{2j}^t \quad j = 2,4,5,6$

$S_{2j} = k_p S_{2j}^c + k_q S_{2j}^t + k_\xi S_{2j}^c \quad j = 1,$

if $\sigma_p < 0, \sigma_q < 0, \sigma_\xi > 0: \left.\begin{matrix} S_{1j} = S_{1j}^c \\ S_{2j} = S_{2j}^c \end{matrix}\right\} \quad j = 1,2,4,5,6$

$S_{3j} = S_{3j}^t \quad j = 3,4,5,6$

$S_{3j} = k_p S_{3j}^c + k_q S_{3j}^c + k_\xi S_{3j}^t \quad j = 1,2$

if $\sigma_p < 0, \sigma_q < 0, \sigma_\zeta < 0: S_{ij} = S_{ij}^c$

(2)

where,

$$k_p = \frac{|\sigma_p|}{\sqrt{|\sigma_p|^2 + |\sigma_q|^2 + |\sigma_\xi|^2}}, \quad k_q = \frac{|\sigma_q|}{\sqrt{|\sigma_p|^2 + |\sigma_q|^2 + |\sigma_\xi|^2}} \text{ and } \quad k_\xi = \frac{|\sigma_\xi|}{\sqrt{|\sigma_p|^2 + |\sigma_q|^2 + |\sigma_\xi|^2}} \tag{3}$$

The weighting factor k_p, k_q, k_ξ is unique for each stress state, hence this result is a unique compliance matrix. The compliances S_{ij}^t *and* S_{ij}^c are those which would have been calculated if the material properties were only the tensile or compressive ones.

GOVERNING EQUATIONS

To determine the bending behaviour of simply supported unsymmetrical specially orthotropic laminates, the well-known shear deformation theory of Touratier will be used for small deflections. The most general form of the displacement field can be written as:

$$U_\alpha(x_1,x_2,x_3) = u_\alpha^o - z w_{,\alpha} + f(z)(w_{,\alpha} + \varphi_\alpha) \quad \alpha = 1,2$$
$$U_3(x_1,x_2,z) = w^o(x_1,x_2); \; x_3 = z \tag{4}$$

Where,

u_α^0 = is the membrane displacement component along the x_α axes of a point on midplane

$w_{,\alpha} = \partial w / \partial x_\alpha$.

f(z) = shear deformation function. For the Touratier kinematics: $f(z) = (h/\pi)\sin(\pi z/h)$

φ_α = rotation of the midplane normal about x_α axes.

By neglecting the non-linear terms of Von Karman, the strain-displacement relation can be written as:

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \tag{5}$$

Now, by the above strain-displacement relation, the field of strains from equation (4) can be written in the vector form as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + f(z) \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix},$$
$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + f'(z) \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \tag{6}$$

where,

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} u_{o,x} \\ v_{o,y} \\ u_{o,y} + v_{o,x} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = -\begin{Bmatrix} w_{o,xx} \\ w_{o,yy} \\ 2w_{o,xy} \end{Bmatrix},$$
$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \varphi_{x,x} + w_{o,xx} \\ \varphi_{y,y} + w_{o,yy} \\ \varphi_{x,y} + \varphi_{y,x} + 2w_{o,xy} \end{Bmatrix}, \tag{7}$$
$$\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \varphi_y + w_{o,y} \\ \varphi_x + w_{o,x} \end{Bmatrix}$$

By the static version of the principle of virtual displacements;

$$0 = \int_{\Omega_n} (\delta U + \delta V) d\Omega \tag{8}$$

The strain energy and potential energy of a plate in bending under lateral load $q(x,y)$, is given by:

$$0 = \iint_{A\,-h/2}^{\;\;h/2} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz})dz dA - \int_A a(x,y)w(x,y)dA \tag{9}$$

where A is the plate area in the undeformed position.

Now, by substituting equation (6) into equation (9):

$$0 = \int_A \Big\{ \int_{-h/2}^{h/2} \Big[\sigma_{xx}(\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)} + f(z)\delta\varepsilon_{xx}^{(3)})$$
$$+ \sigma_{yy}(\delta\varepsilon_{yy}^{(0)} + z\delta\varepsilon_{yy}^{(1)} + f(z)\delta\varepsilon_{yy}^{(3)}) + \sigma_{xy}(\delta\gamma_{xy}^{(0)} + z\delta\gamma_{xy}^{(1)} + f(z)\delta\gamma_{xy}^{(3)})$$
$$+ \sigma_{xz}(\delta\gamma_{xz}^{(0)} + f'(z)\delta\gamma_{xz}^{(2)}) + \sigma_{yz}(\delta\gamma_{yz}^{(0)} + f'(z)\delta\gamma_{yz}^{(2)}) \Big] dz - q(x,y)\delta w \Big\} dxd \tag{10}$$

Stress results can be defined as for the Touratier theory:

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ f(z) \end{Bmatrix} dz, \quad \begin{Bmatrix} Q_\alpha \\ R_\alpha \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ f'(z) \end{Bmatrix} dz \tag{11}$$

So, equation (10) becomes;

$$0 = \int_A \{ N_{xx}\delta u_{o,x} - M_{xx}\delta w_{o,xx} + P_{xx}(\delta\varphi_{x,x} + \delta w_{o,xx})$$
$$+ N_{yy}\delta v_{o,y} - M_{yy}\delta w_{o,yy} + P_{yy}(\delta\varphi_{y,y} + \delta w_{o,yy})$$
$$+ N_{xy}(\delta u_{o,y} + \delta v_{o,x}) - 2M_{xy}(\delta w_{o,xy}) + P_{xy}(\delta\varphi_{x,y} + \delta\varphi_{y,x} + 2\delta w_{o,xy})$$
$$+ Q_x(0) + R_x(\delta\varphi_x + \delta w_{o,x}) + Q_y(0) + R_y(\delta\varphi_y + \delta w_{o,y})$$
$$- q(x,y)\delta w_o \} dxdy \tag{12}$$

Now, by integration by parts:

$$0 = \int_A \{ -N_{xx,x}\delta u_o - M_{xx,xx}\delta w_o - P_{xx,x}\delta\varphi_x + P_{xx,xx}\delta w_o)$$
$$- N_{yy,y}\delta v_o - M_{yy,yy}\delta w_o - P_{yy,y}\delta\varphi_y + P_{yy,yy}\delta w_o$$
$$- N_{xy,y}\delta u_o - N_{xy,x}\delta v_o - 2M_{xy,xy}\delta w_o - P_{xy,y}\delta\varphi_x - P_{xy,x}\delta\varphi_y + 2P_{xy,xy}\delta w_o$$
$$+ R_x\delta\varphi_x - R_{x,x}\delta w_o + R_y\delta\varphi_y - R_{y,y}\delta w_o - q(x,y)\delta w_o \} dxdy \tag{13}$$

which gives the governing equations for $\forall\, \delta u_0, \delta v_0, \delta w_0, \delta\varphi_x, \delta\varphi_y$

$$\begin{aligned} & N_{xx,x} + N_{xy,y} = 0 \\ & N_{yy,y} + N_{xy,x} = 0 \\ & M_{xx,xx} + M_{yy,yy} + 2M_{xy,xy} + R_{x,x} + R_{y,y} - \\ & \qquad (P_{xx,xx} + P_{yy,yy} + 2P_{xy,xy}) + q(x,y) = 0 \\ & P_{xx,x} + P_{xy,y} - R_x = 0 \\ & P_{yy,y} + P_{xy,x} - R_y = 0 \end{aligned} \tag{14}$$

For the laminate under consideration, which has simply supported boundary conditions on all four edges, and which can be expressed mathematically as:

$$\begin{aligned} &u_o(x,0) = u_o(x,b) = v_o(0,y) = v_o(a,y) = 0 \\ &w_o(x,0) = w_o(x,b) = w_o(0,y) = w_o(a,y) = 0 \\ &\varphi_x(x,0) = \varphi_x(x,b) = \varphi_y(0,y) = \varphi_y(a,y) = 0 \\ &N_{yy}(x,0) = N_{yy}(x,b) = N_{xx}(0,y) = N_{xx}(a,y) = 0 \\ &M_{yy}(x,0) = M_{yy}(x,b) = M_{xx}(0,y) = M_{xx}(a,y) = 0 \end{aligned} \quad ,(15)$$

the assumed solution forms for u, v, w, φ_x *and* φ_y that satisfy the above boundary conditions are:

$$u = \sum_{m,n} U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad , \quad v = \sum_{m,n} V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$w = \sum_{m,n} W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad , \quad \varphi_x = \sum_{m,n} X_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\varphi_y = \sum_{m,n} Y_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (16)$$

The strain-displacement relation of equation (7), and the above assumed field of displacement, gives the strain relation as *(for m=n=1)*:

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} -U\alpha\sin\alpha x\sin\beta y \\ -V\beta\sin\alpha x\sin\beta y \\ U\beta\cos\alpha x\cos\beta y + V\alpha\cos\alpha x\cos\beta y \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = -\begin{Bmatrix} -W\alpha^2\sin\alpha x\sin\beta y \\ -W\beta^2\sin\alpha x\sin\beta y \\ 2W\alpha\beta\cos\alpha x\cos\beta \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} -X\alpha\sin\alpha x\sin\beta y - W\alpha^2\sin\alpha x\sin\beta \\ -Y\beta\sin\alpha x\sin\beta y - W\beta^2\sin\alpha x\sin\beta \\ X\beta\cos\alpha x\cos\beta y + Y\alpha\cos\alpha x\cos\beta y + 2W\alpha\beta\cos\alpha x\cos\beta \end{Bmatrix}$$

and,

$$\begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} Y\sin\alpha x\cos\beta y + W\beta\sin\alpha x\cos\beta \\ X\cos\alpha x\sin\beta y + W\alpha\cos\alpha x\sin\beta \end{Bmatrix}$$

(17)

where α = (π/a) and β = (π/b). The stress results $N_{\alpha\beta}$, $M_{\alpha\beta}$ and $P_{\alpha\beta}$ are related to strains by the relations;

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [E] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(3)}\} \end{Bmatrix} , \qquad (18)$$

$$\{R\} = [T]\{\gamma^{(2)}\}$$

where,

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{ij}^{(k)} \left(1, z, z^2, f(z), zf(z), f(z)^2\right) dz \qquad (19)$$

and $T_{ij} = \sum_{k=1}^{n} Q_{ij}^{k} \int_{z_{k-1}}^{z_k} (f'(z))^2 dz$

Now, by equations (17 to 19), the governing equations (14) can be written in matrix form as:

$$\begin{bmatrix} G_{11} & G_{21} & G_{31} & G_{41} & G_{51} \\ G_{12} & G_{22} & G_{32} & G_{42} & G_{52} \\ G_{13} & G_{23} & G_{33} & G_{43} & G_{53} \\ G_{14} & G_{24} & G_{34} & G_{44} & G_{54} \\ G_{15} & G_{25} & G_{35} & G_{45} & G_{55} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \\ X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q \\ 0 \\ 0 \end{Bmatrix} \qquad (20)$$

In the above equation matrix G is symmetric ($G_{ij}=G_{ji}$), whereas its members are functions of the plate dimensions a, b and the plate stiffness $A_{ij}, B_{ij}, \ldots etc., ij \neq 16, 26$, (*Appendix A*). On the right-hand side, Q is the coefficient of the assumed double-Fourier series expansion of the applied transverse load q.

$$q(x,y) = Q\sin\alpha x\sin\beta y ,$$

$$\text{where} \quad Q(z) = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\alpha x\sin\beta y\,dx\,dy \qquad (21)$$

Now, by solving the system of equations (20), the coefficient U, V, W, X *and* Y at any point (x,y) can be evaluated. The strains at point z, in the thickness direction can be evaluated by strain-displacement relations, and corresponding stress can be calculated by the law of the orthotropic materials ($\sigma_{33} = 0$).

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad (22)$$

COMPUTER IMPLEMENTATION

To reduce troublesome calculations, a computer programme has been developed for the proposed method to incorporate the "Material Model" into the computer program for the calculation of the bending behaviour of a simply supported, unsymmetrical, specially orthotropic laminate. It is obvious from equation (1), that the compliance matrix in principal stress coordinates S is needed, and has to be calculated twice, once by tensile properties S^t, and once by compressive properties S^c. The quantities that finally characterize a laminate are its laminate stiffnesses A_{ij}, B_{ij}, etc.

Algorithm /1/

To find the bending behaviour of a bimodulus laminated plate using the "Material Model" and the shear deformation theory of Touratier /9/ previously presented, a computer programme on Mathematica was developed.

The algorithm starts by assuming specific properties (tensile or compressive) for each layer. A parametric investigation showed that by considering in the first instance the upper half of the layers (from the loading side) to have compressive properties and the lower half to have tensile ones, faster convergence can be attained than in the case of all layers having tensile or compressive properties only. This is followed by the computer code. With this in mind, the layer stiffness Q_{ij} and the laminate stiffness A_{ij}, B_{ij} *...etc.*, are calculated, resulting in displacements u, v, w, and rotations φ_x φ_y . With these quantities known, the through-thickness variation of the stresses is calculated and the various "stress zones" according to the principal stresses and equation (2). At this point it is assumed that every "stress zone" constitutes a single effective layer, with layer stiffness Q_{ij} equal to the mean value of the stiffnesses calculated at the various points through its thickness. This is the only major assumption of the proposed method /1/. Considering the new number of these effective layers, new laminate stiffnesses are calculated and the calculations started again from the beginning. The iterations stop when the deflection w and the z-points where the strains ε_x and ε_y become zero, differ by less than 1% from the corresponding values calculated in the previous iteration. The code produced as an output result of the deflection and through-thickness variation of the strains and stresses at a specific point *(x,y)*, has given us an input.

Finite Element Analysis

Since no exact 3D solution exists for unsymmetrical laminated plates, finite element has been taken as a reference for the comparison of different models of laminated plate. Finite element analysis is done using numerical analysis software (Ansys 5.5).

The 3D approximation of the behaviour is carried out by element type "SOLID45" (Brick 8 node). In this finite element analysis, it is assumed that the upper half (from loading side) of the laminate has compressive orthotropic properties, and the lower half has tensile ones. Due to symmetry, the fourth part of the laminated plate has been studied. To validate the finite element results, firstly it is necessary to find out the convergence of laminate meshing. So, within the available resources and limitations of Ansys, we found the following convergence for different numbers of layers (Table 1):

Table 1
Convergence of meshing in finite element analysis

Layers	Numbers of Elements (h, a, b)
2	10800 (12,30,30)
4	10800 (12,30,30)
6	11250 (18,25,25)
8	14400 (16,30,30)

ANALYTICAL RESULTS

The numerical evaluation of the proposed method has been carried out in three stages. First the results are evaluated for different numbers of layers by the computer programme for the proposed analytical method, developed in "Mathematica". In the second stage, the results of the first stage are verified by finite element analysis, i.e. a numerical model developed on "Ansys 5.5". In the final stage, results are compared with Papazoglou and Tsouvalis's /1/ bi-modulus results.

The laminate under consideration is simply supported, unsymmetrical, specially orthotropic. Parametric study carried out for a square *(a/b = 1)*, thin *(a/h = 50)*, cross-ply laminate *(0°/90°/0/90°/.....)*, under uniformly distributed lateral load, with $n = 2,4,6,8$ layers, all from the same material and with the same thickness.

The deflections, strains, and stresses are calculated at the laminate midpoint $(x = a/2,\ y = b/2)$ for three material cases.

Case 1 : The material was considered to be bi-modulus for $(E_2^t/E_2^c = 0.6)$,

Case 2 : The material was considered to be single-modulus, with the tensile properties only,

Case 3 : The material was considered single-modulus again, with the upper half of the layers (the ones from the loading side) having compressive properties and the lower half ones having tensile properties (this case is used for finite element analysis). For the sake of comparison, the material properties used are the same as those used by Papazoglou and Tsouvalis/1/, listed in Table 2:

The following parameters are introduced for the non-dimensionalization of deflections and stresses: $m_1 = (100\ E_2^c\ h^3)/\ (q\ a^4)$ for the deflections and $m_2 = h^2/\ (q\ a^2)$ for the stresses, while the strains are given with their absolute values.

Table 2
Elastic Properties of an Orthotropic Material

Properties (MPa)	Tensile	Compressive
E_1	25 E_2^t	25 E_2^c
E_1	E_2^t/E_2^c	1000
E_1	E_2^t	E_2^c
G_{12}	0.5 E_2^t	0.5 E_2^c
G_{23}	0.2 E_2^t	0.2 E_2^c
ν_{12}	0.25	0.25
$G_{13} = G_{12}$		t: tensile
$\nu_{23} = \nu_{13} = \nu_{12}$		c: compressive

In Figure 2, the variation of the non-dimensional deflections, *w*, is shown as a function of the number of layers for the three material cases. It can be observed that the first material case gives an accurate prediction of bi-modulus behaviour compared to the finite element results. Touratier's theory gives more or less the same prediction of central deflection over the theory of Reddy. The second material case introduces a considerable error as compared to the first material case.

In Figures 3-6, the through-thickness variations of strain ε_x is plotted for the 2,4,6, and 8 -layer laminates.

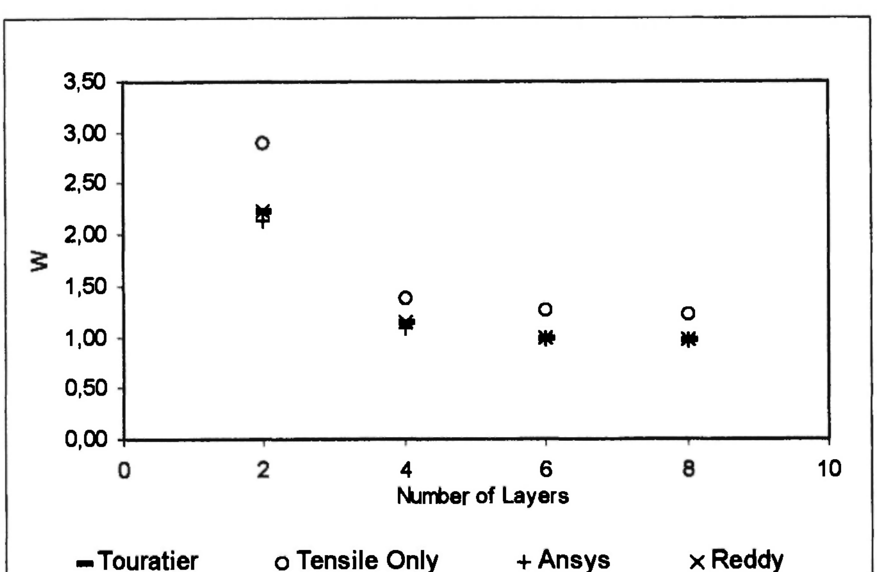

Fig. 2: Non-dimensional centre deflection versus number of layers

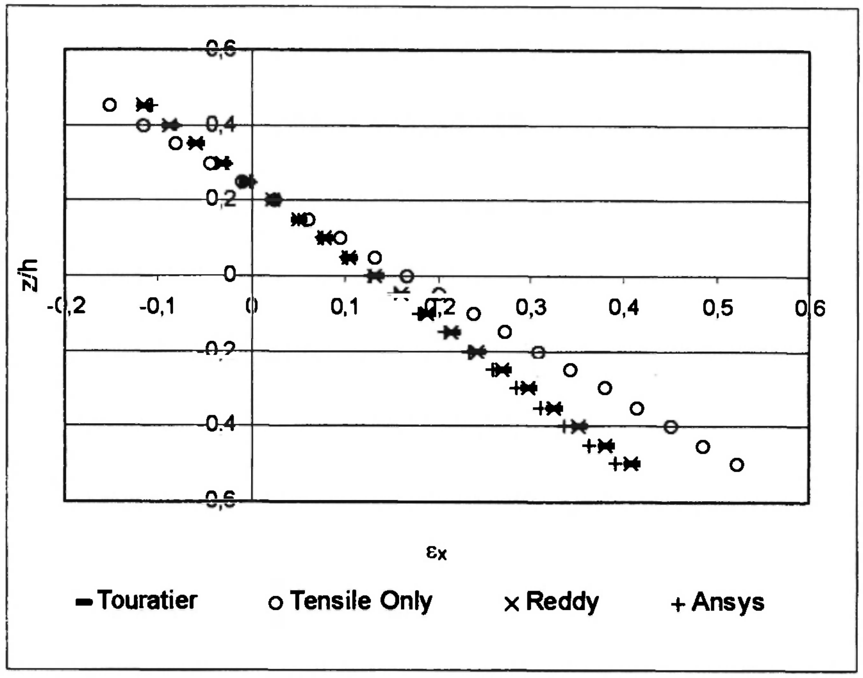

Fig. 3: Through-thickness variation of strain (ε_x), for two layer cross-ply (0°/90°)

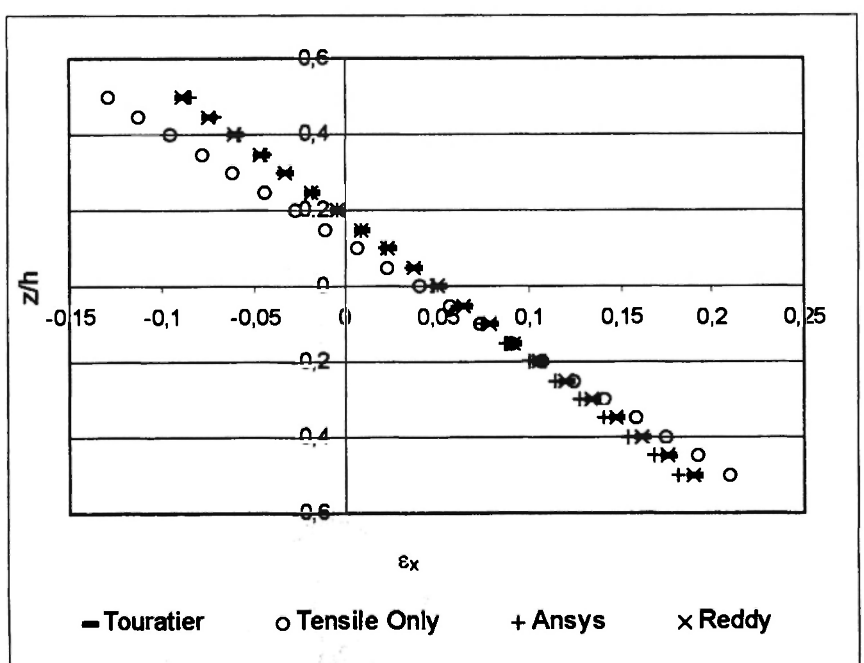

Fig. 4: Through-thickness variation of strain (ε_x), for four layer cross-ply $(0°/90°)_2$

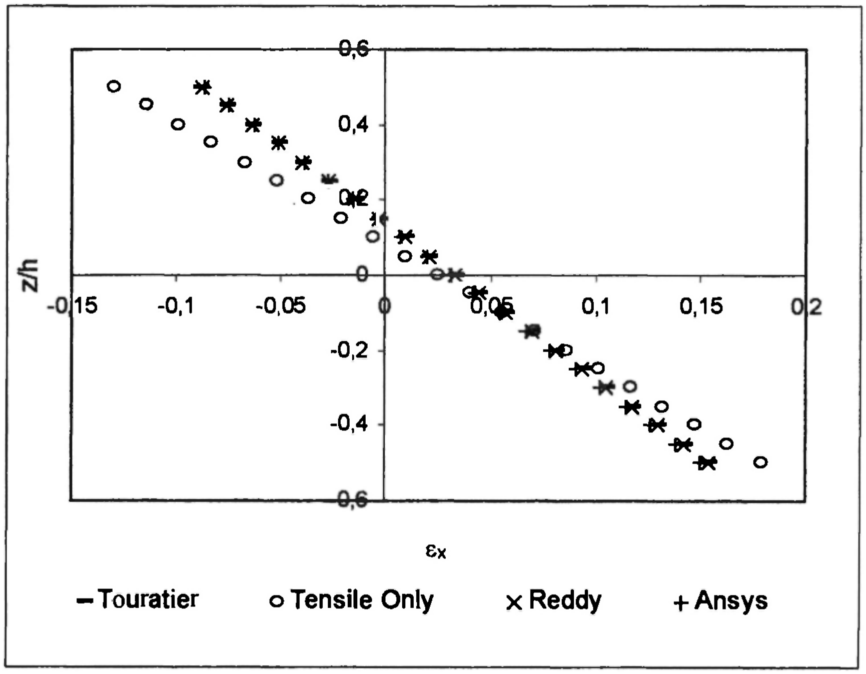

Fig. 5: Through-thickness variation of strain (ε_x), for six layer cross-ply $(0°/90°)_3$

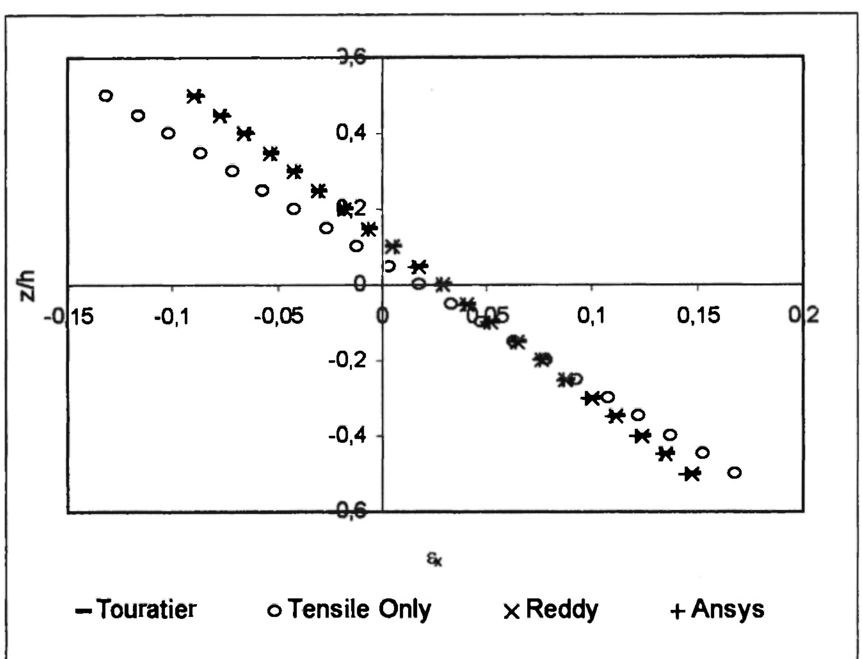

Fig. 6: Through-thickness variation of strain (ε_x), for eight layer cross-ply $(0°/90°)_4$

Again the first material case with Touratier's theory gives quite accurate predictions with the finite element results. Analytical results of strains in the first material case are a little better than using Reddy's theory, and these differences will be clearer in thick laminates. The second material case again shows considerable errors as against the first material case. It can also be seen from these plots that the maximum of strain becomes smaller and converges to a single value, as the number of layers increases, similar to the plot of deflection (Figure 2). This improvement is rapid between laminates with two and four layers, while it becomes smoother for larger numbers of layers. Another point of interest is the fact that *z-point* where ε_x becomes zero, moves towards the geometric midplane as the number of layers increases. This coincides with the generally accepted idea that cross-ply laminates behave like symmetric, specially orthotropic ones for large numbers of layers.

In Figures 7-10, the through-thickness variations of stress σ_x are plotted for the 2,4,6,8 layers. Again, it is clear that the first material case gives an accurate prediction of bi-modulus behaviour compared to the finite element results except for the laminate with two layers. As previously, Touratier's theory again predicts more accurately the stresses of bi-modulus laminates than Reddy's theory as compared to finite element results, and this difference will be more obvious in thick laminates. The second material case again has the same response as previously compared to the first material case.

CONCLUSION

A new approach has been used to predict the bending behaviour of a simply supported, unsymmetrical, specially orthotropic laminate, based on the Touratier shear deformation theory and with a different method of defining the weighting factors for "stress zone". Using this method, the deflections and the through-thickness variations of strains and stresses at any point *(x,y)* of a multilayered laminate can be calculated. The validity of this method is established by comparing it with Papazoglou and Tsouvalis' results, and with numerical analysis on "Ansys". A parametric study showed that the assumption of a single-modulus material for a bimodulus one can lead to significant errors. Bimodulus behaviour of a laminate by

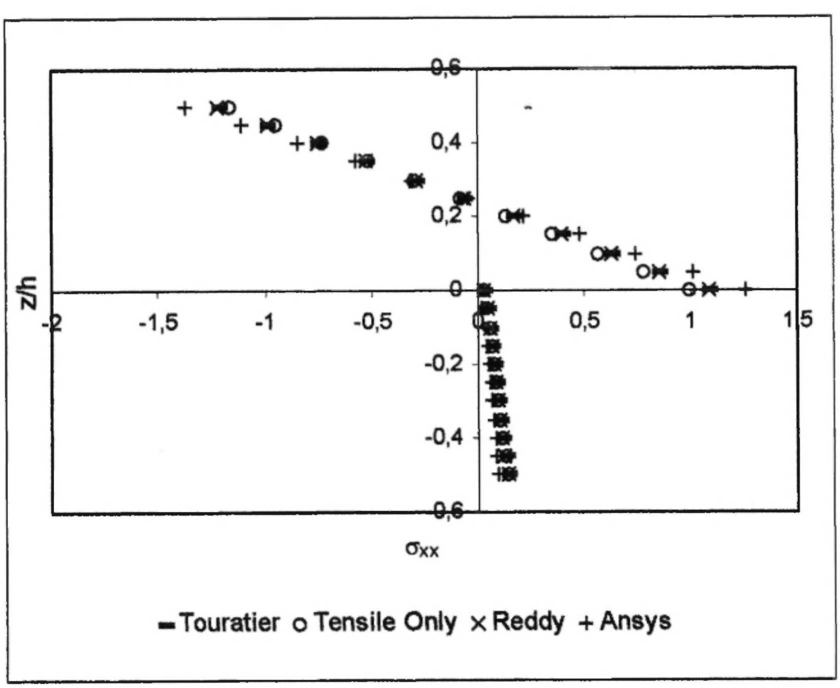

Fig. 7: Through-thickness variation of strain (σ_x), for two layer cross-ply (0°/90°)

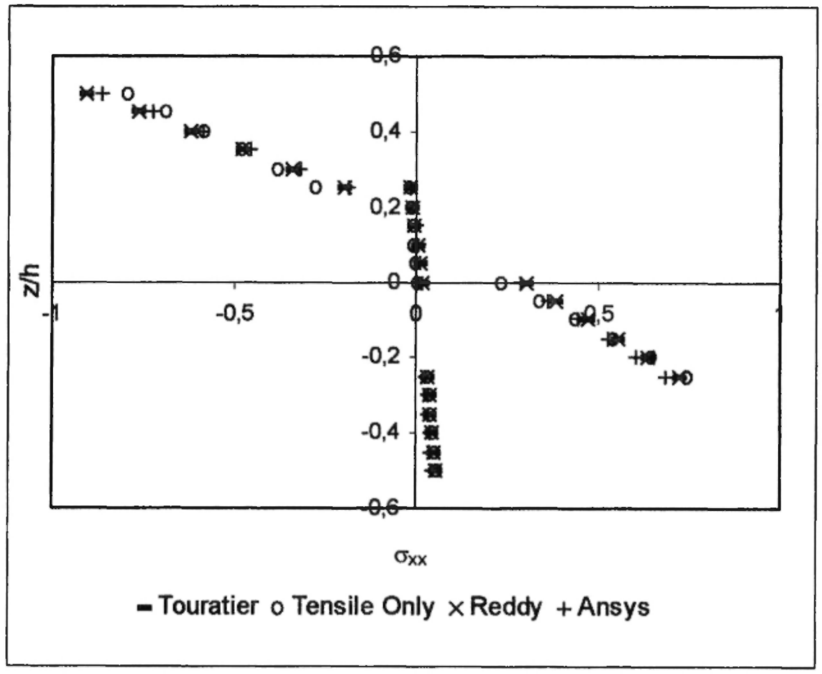

Fig. 8: Through-thickness variation of strain (σ_x), for four layer cross-ply (0°/90°)$_2$

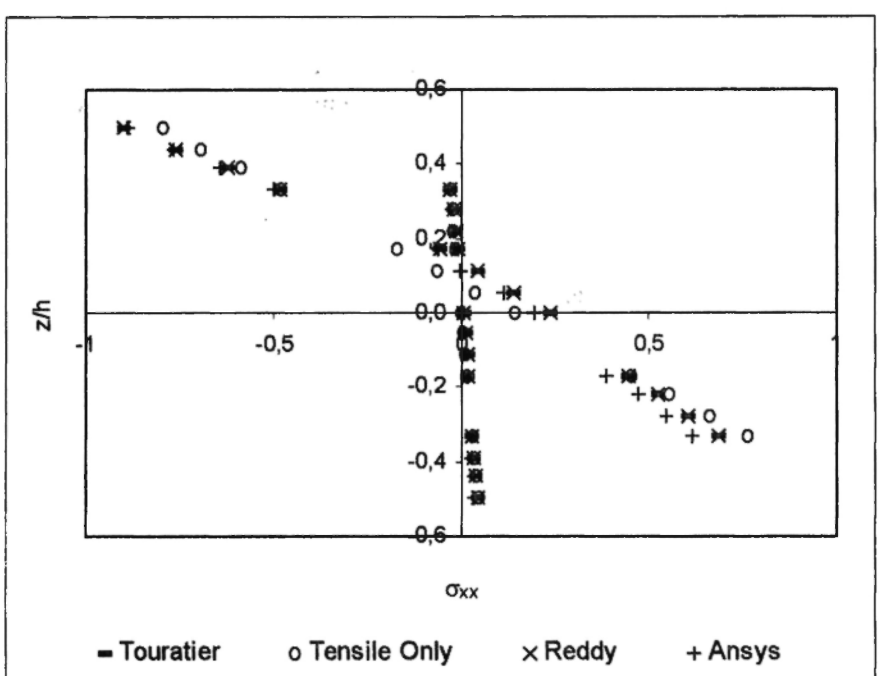

Fig. 9: Through-thickness variation of strain (σ_x), for six layer cross-ply (0°/90°)$_3$

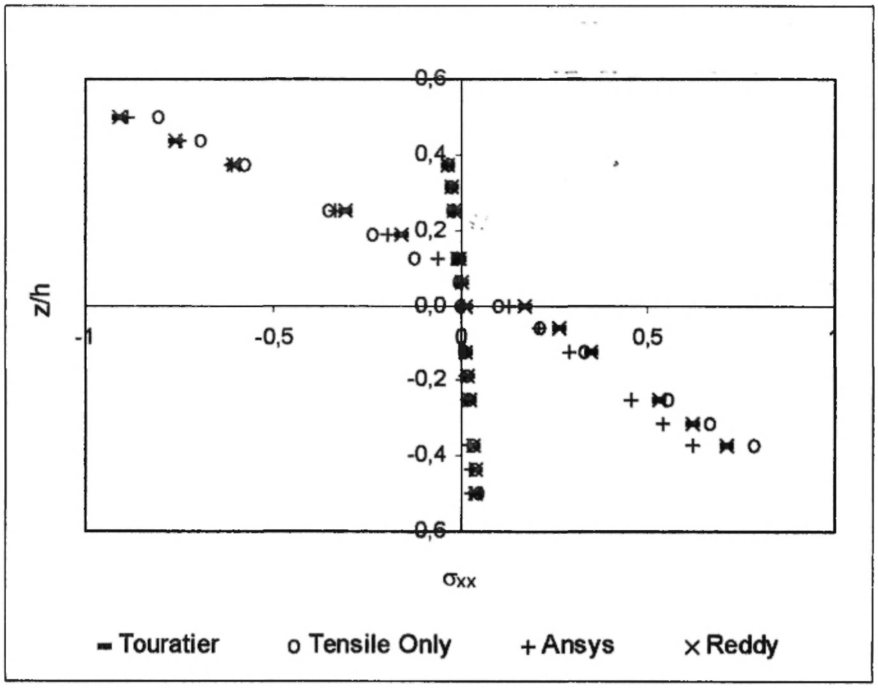

Fig. 10: Through-thickness variation of strain (σ_x), for eight layer cross-ply $(0°/90°)_4$

Touratier's theory is more accurate than by that of Reddy as compared to the finite element results. Accuracy of the bi-modulus behaviour by Touratier's theory can be seen more clearly in thick laminates and this difference is more significant in transverse shear stresses. Future developments of this method include: a more accurate definition of the weighing factors k_p, k_q, k_ξ , a better way of calculating the stiffnesses of a "stress zone", by incorporation of a more accurate shear deformation theory and also by considering the continuity of transverse shear stresses on the layers interfaces in a laminate.

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APPENDIX-A

The terms of the Matrix G in equation (13):

$G_{11} = A_{11}\,\alpha^2 + A_{66}\,\beta^2$, $G_{12} = (A_{12} + A_{66})\,\alpha\,\beta$

$G_{13} =$

$(E_{11}-B_{11})\,\alpha^3 + /E_{12} - B_{12} - 2\,(E_{66} - B_{66})\,/\,\alpha\,\beta^2$

$G_{14} = E_{11}\alpha^2 + E_{66}\,\beta^2$,

$G_{15} = (E_{12} + E_{66})\,\alpha\,\beta$

$G_{21} = G_{12}$, $G_{31} = G_{13}$, $G_{32} = G_{23}$,

$G_{22} = A_{66}\,\alpha^2 + A_{22}\,\beta^2$

$G_{23} =$

$[-B_{12} + E_{12} - 2\,(-B_{66} + E_{66})]\,\beta\,\alpha^2 + (-B_{22} + E_{22})\,\beta^3$

$G_{24} = G_{15}$, $G_{25} = E_{22}\,\beta^2 + E_{66}\,\alpha^2$

$G_{33} =$

$(D_{11} + H_{11} - 2\,F_{11})\,\alpha^4 + (2\,D_{12} + 2\,H_{12} - 4\,F_{12} + 4D_{66} + 4H_{66} - 8\,F_{66})\,\alpha^2\,\beta^2 + (D_{22} + H_{22} - 2\,F_{22})\,\beta^4 + T_{55}\,\alpha^2 + T_{44}\,\beta^2$

$G_{34} -$

$[H_{12} - F_{12} - 2\,(-H_{66} + F_{66})]\,\alpha\,\beta^2 + (H_{11} - F_{11})\,\alpha^3 + \alpha\,T_{55}$

$G_{35} =$

$[H_{12} - F_{12} - 2\,(-H_{66} + F_{66})]\,\alpha^2\,\beta + (H_{22} - F_{22})\,\beta^3 + \beta\,T_{44}$

$G_{41} =$

G_{14}, $G_{42} = G_{24}$, $G_{43} = G_{34}$,

$G_{44} =$

$H_{11}\,\alpha^2 + H_{66}\,\beta^2 + T_{55}$,

$G_{45} =$

$(H_{12} + H_{66})\,\alpha\,\beta$,

$G_{51} =$

G_{15}, $G_{52} = G_{25}$, $G_{53} = G_{35}$, $G_{54} = G_{45}$,

$G_{55} =$

$H_{22}\,\beta^2 + H_{66}\,\alpha^2 + T_{44}$