Dispersion and Nonlinearity of Plane Longitudinal Wave Propagation in Porous Materials

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1. SUMMARY

A mathematical model of an isotropic viscously elastic porous medium is studied in this paper. Cavities are spherical and are equally spaced in the medium. The distance between the cavities is much greater than the radius of the cavity and in turn is a lot smaller than the length of the wave that propagates in the medium. A dependence of velocity of the wave on the porosity of the material is established and corresponding analyses are carried out. The phenomenon of non-linear stationary wave of displacement propagation has been studied for the material also and the corresponding dependencies on porosity have been established.

2. INTRODUCTION

Using inner degrees of freedom (rotational and oscillatory) for modeling material behavior has attracted considerable interest in the past several years, leading to an increase in the number of scientific works in this area /1-3/. One typical example of such materials with oscillatory degrees of freedom is porous materials. Models developed that capture the mechanical response of fluids with gas bubbles are usually used to describe dynamic processes in porous media. A one-dimensional problem of the dynamics for an elastic porous medium was studied in papers by D.M. Donskoy and A.M. Sutin /4/, and in papers by L.A. Ostrovsky /5, 6/. Three-dimensional problems were studied by A.G. Bagdoev and A.V. Shekhoyan /7/.

3. PRELIMINARY ASSUMPTIONS

A mathematical model for the dynamics of a porous medium has been proposed in /7/. The following assumptions were made: the medium was fully isotropic and viscously elastic; the pores were spherical and equally spaced; the propagating wave has a finite amplitude, which means that geometrical, physical and porous nonlinearities were taken into account; the distance between cavities L is a lot larger than the radius of a cavity R_0 $(L >> R_0)$ and in turn a lot smaller than the length of a wave Λ $(L << \Lambda)$, so there can be no interaction between the cavities. We consider that the propagating wave is quasi-longitudinal, so we can state that pressure on a cavity is caused by the longitudinal stress $\sigma_{33} = (\lambda+2\mu)\frac{\partial u_3}{\partial x_3} - (\lambda+2\mu)z$. In this expression $z = NV$, where N is the number of cavities in the volume, V is the volume of a cavity; in particular $V = V_0 + V'$, where V_0 is the starting volume of a cavity, V' is the volume of a cavity indignant by a wave, λ and μ are Lame coefficients, considering $\mu < \lambda$. The pressure inside the cavities is neglected.

4. MATHEMATICAL MODEL

Longitudinal wave propagation in porous media along the x_3 axis can be described using the following combined nonlinear equations (since the one-dimensional problem is studied in this paper, the x_3 coordinate can be denoted by x, and the longitudinal constituent of the displacement vector u_3 is denoted by u):

$$\begin{cases} \rho_0 \dfrac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\dfrac{\partial^2 u}{\partial x^2} - N(\lambda+2\mu)\dfrac{\partial v}{\partial x} + b\dfrac{\partial^3 u}{\partial t\, \partial x^2} + \dfrac{P}{2}\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)^2 \\ \ddot{v} + \omega_0^2 v - \dfrac{R_0}{c_l}\dddot{v} - Gv^2 - \beta_1\left(2v\ddot{v} + \dot{v}^2\right) = (\lambda+2\mu)\dfrac{4\pi R_0}{\rho_0}\left(\dfrac{\partial u}{\partial x} - Nv\right) \end{cases} \tag{1}$$

The first equation describes the propagation of a plane longitudinal wave in a medium with pores taking into account that each pore volume is changing. The second one describes the oscillatory process of cavity volume changes caused by the material deformation.

In these equations ρ_0 is a starting density of the material, $\omega_0^2 = \dfrac{4\mu}{\rho_0 R_0^2}$ is the square of the resonant frequency of cavity volume oscillations and $c_l^2 = \dfrac{(\lambda+2\mu)}{\rho_0}$ is the square of longitudinal velocity. The following are some additional notations:

$G = \dfrac{11\omega_0^2}{16\pi R_0^3}$, $\beta_1 = \dfrac{1}{8\pi R_0^3}$ and $P = (4\mu + 3\lambda + 2A + 6B + 2C)$, where P is a coefficient, caused by geometrical and physical nonlinearities; A, B, C are Landau constants of third order.

Suppose that the viscosity can be neglected. Suppose also that the influence of nonlinearities of order greater than two is a lot smaller than the influence of second order nonlinearities. Taking these assumptions into account and using the notations described above, (1) can be rewritten as:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c_l^2 \dfrac{\partial^2 u}{\partial x^2} - c_l^2 N \dfrac{\partial v}{\partial x} + \dfrac{P}{2\rho_0} \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)^2 \\ \ddot{v} + \omega_0^2 v - G v^2 = 4\pi R_0 c_l^2 \left(\dfrac{\partial u}{\partial x} - N v\right) \end{cases} \tag{2}$$

In order to obtain an equation which describes wave propagation in a porous medium, the first equation of (2) should be differentiated with respect to the x coordinate and the following expression should be evaluated from the second one:

$$\frac{\partial u}{\partial x} = \frac{1}{4\pi R_0 c_l^2}\ddot{v} + \left(N + \frac{\omega_0^2}{4\pi R_0 c_l^2}\right) v - \frac{G}{4\pi R_0 c_l^2} v^2 \tag{3}$$

Differentiating (3) by the x coordinate and time and substituting it into the first equation of (2) we arrive at the equation which describes dynamics of two studied medium in terms of cavity volume.

The equation with all variables written in non-dimensional form (after transforms: $U = \dfrac{v}{v_0}$, $y = \dfrac{x}{\Lambda}$, $\tau = \dfrac{t c_l^2}{\Lambda}$) reads:

$$\begin{aligned} &\left(N + \frac{\omega_0^2}{4\pi R_0 c_l^2}\right)\frac{\partial^2 U}{\partial \tau^2} = \frac{\omega_0^2}{4\pi R_0 c_l^2}\frac{\partial^2 U}{\partial y^2} - \frac{1}{4\pi R_0 \Lambda^2}\left(\frac{\partial^4 U}{\partial \tau^4} - \frac{\partial^4 U}{\partial \tau^2 \partial y^2}\right) + \\ &+ \frac{G U_0}{4\pi R_0 c_l^2}\left(\frac{\partial^2 (U^2)}{\partial \tau^2} - \frac{\partial^2 (U^2)}{\partial y^2}\right) + \frac{P}{2\rho_0}\frac{\partial^2}{\partial y^2}\left\{\frac{U_0}{c_l^2}\left(N + \frac{\omega_0^2}{4\pi R_0 c_l^2}\right)^2 U^2 + f\right\} \end{aligned} \tag{4}$$

where f is a sum of five nonlinear components the expression for f is given below:

$$\begin{aligned} f = &\frac{U_0}{\Lambda^4}\frac{1}{16\pi^2 R_0^2 c_l^2}\left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 + \frac{G^2 U_0^3}{16\pi^2 R_0^2 c_l^6} U^4 + \frac{U_0}{2\pi R_0 c_l^2 \Lambda^2} * \\ * &\left(N + \frac{\omega_0^2}{4\pi R_0 c_l^2}\right) U \frac{\partial^2 U}{\partial \tau^2} - \frac{G U_0^2}{8\pi^2 R_0^2 c_l^4 \Lambda^2} U^2 \frac{\partial^2 U}{\partial \tau^2} - \frac{G U_0^2}{2\pi R_0 c_l^4}\left(N + \frac{\omega_0^2}{4\pi R_0 c_l^2}\right) U^3 \end{aligned} \tag{5}$$

From equations (4) and (5) it can be seen that the consequences of the existence of cavities are the dispersion of a wave (frequency-dependent wave propagation velocity) and an additional nonlinear effect (so-called cavitary nonlinearity). These factors manifest in different regimes.

5. DISPERSIVE PROPERTIES

The dispersive properties of a wave propagating in a porous material can be described using the equation:

$$\omega^4 - \frac{\left(\omega_0^2\Lambda^2 + 4\pi R_0 c_l^2\Lambda^2 N + c_l^2 k^2\right)}{c_l^2}\omega^2 + \frac{\omega_0^2 k^2\Lambda^2}{c_l^2} = 0 \tag{6}$$

from which it follows that the dependence of the wave frequency on the wave number is defined analytically by the expression:

$$\omega = \pm\sqrt{\frac{\left(\omega_0^2\Lambda^2 + 4\pi R_0 c_l^2\Lambda^2 N + c_l^2 k^2\right)}{2c_l^2} \pm \sqrt{\frac{\left(\omega_0^2\Lambda^2 + 4\pi R_0 c_l^2\Lambda^2 N + c_l^2 k^2\right)^2}{4c_l^4} - \frac{\omega_0^2 k^2\Lambda^2}{c_l^2}}};$$

The character of the dependency of the wave frequency on the wave number is shown graphically in Figure 1.

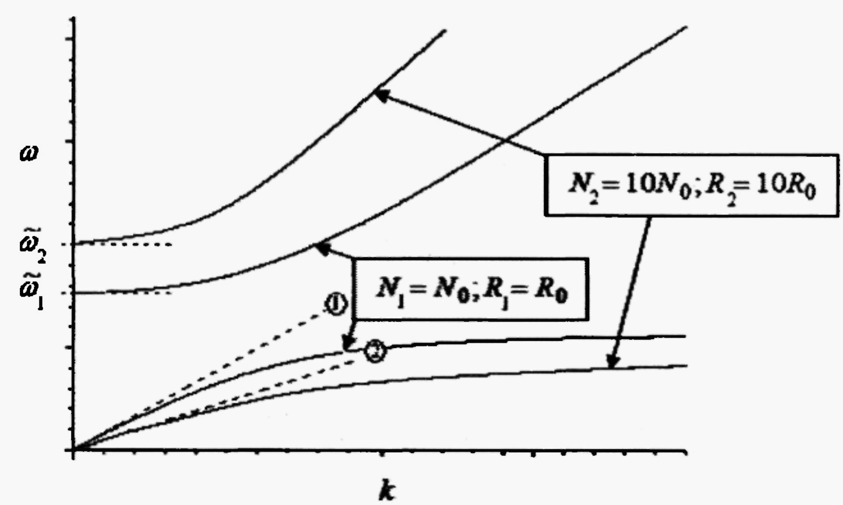

Fig. 1: Dependency of the wave frequency on the wave number

It can be seen that the wave is characterized by two dispersive branches. The analysis shows that an increase of the porosity (which really is an increase of the number of cavities in the unit volume or an increase of the radius of cavities or both) causes (i) an increase of the value of phase velocity, if the propagating wave can be described by the upper dispersive branch, and (ii) it causes a decrease of the value of the phase velocity for the lower dispersive branch. This is shown graphically in Figure 2.

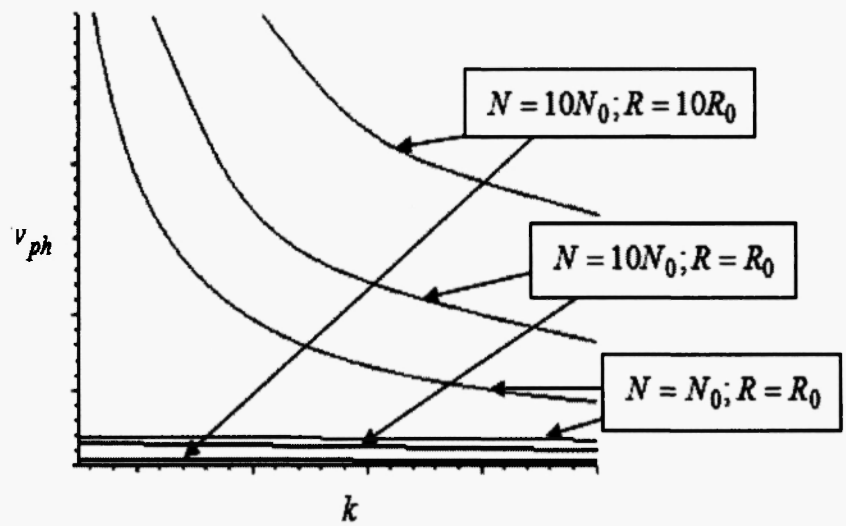

Fig. 2: Dependency of the phase velocity on the wave number

6. SOLITON OF DISPLACEMENT

From (4) it follows that the propagating wave, in terms of this model, is influenced by both dispersion and nonlinearity. Non linearity causes generation of additional harmonics to which the energy is being swapped, which mainly causes drops in the profile of the wave, while dispersion causes smoothing of the drops because of the difference in wave velocities. The combined effect of these two factors can be a reason for forming nonlinear stationary waves. These are the waves that propagate with constant velocity without changing their form.

We shall be looking for a solution of (4) in the form of stationary wave of displacement $W(\xi = x - Vt) = U$, which can be described by the following equation:

$$\frac{\partial^2 W}{\partial \xi^2} + aW + bW^2 = 0 \tag{7}$$

here V is the velocity of the stationary wave and ξ is a coordinate while the coefficients a and b are given by the expressions :

$$a = \frac{4\pi R_0 \Lambda^2}{V^2\left(V^2-1\right)}\left\{V^2\left(N+\frac{\omega_0^2}{4\pi R_0 c_l^2}\right)-\frac{\omega_0^2}{4\pi R_0 c_l^2}\right\} \tag{8}$$

$$b = \frac{4\pi R_0 \Lambda^2}{V^2\left(V^2-1\right)}\left\{V^2\frac{GU_0}{4\pi R_0 c_l^2}+\frac{U_0}{c_l^2}\left(\frac{P}{2\rho_0}\left(N+\frac{\omega_0^2}{4\pi R_0 c_l^2}\right)-\frac{G}{4\pi R_0}\right)\right\} \tag{9}$$

The signs of the coefficients a and b show whether nonlinear stationary waves can be formed or not. Physically realizable are only the cases for which the wave form has no constant constituents. In this particular case it is possible only if wave velocity changes in the ranges $\sqrt{\frac{\omega_0^2}{4\pi R_0 c_l^2 N + \omega_0^2}} < V_i < 1$, which is that $a < 0, b > 0$. Then the solution of the displacement will have the following form:

$$W(\xi) = \frac{A}{ch^2\left(\frac{x-Vt}{\Delta}\right)} \tag{10}$$

where $A = -\frac{3a}{2b}$, $\Delta = \frac{2}{\sqrt{-a}}$.

Most character dependencies for the soliton of displacement are shown in Figure 3.

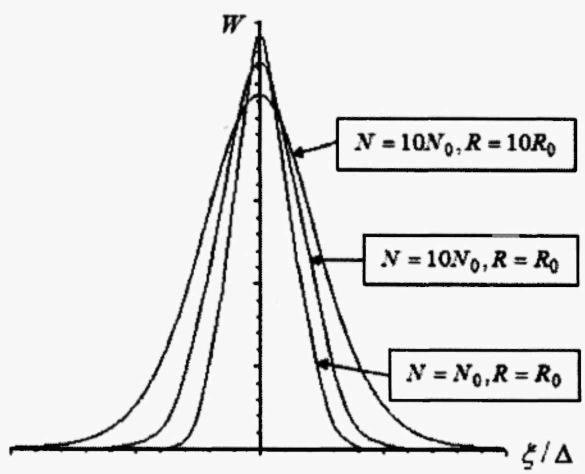

Fig. 3: Form of the soliton of displacement

Analysis of these dependencies lead to the conclusion that the increase of the porosity causes increase of the amplitude of the soliton and a decrease of the width of the soliton.

7. CONCLUSIONS

The main conclusions of analyses performed in this work are: a) The dependence of the velocity of the wave on the porosity of the material has been established and analysed, b) The phenomenon of non-linear stationary wave of displacement propagation has been studied and the respective dependencies on the material porosity have been established.

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