Generalized Solutions of Boundary Value Problems of Dynamics of Anisotropic Elastic Media

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1. SUMMARY

The method of boundary integral equations (BIE) for the solution of non-stationary boundary value problems (BVP) of dynamics of anisotropic elastic mediums is elaborated. The central moment of this method is constructing the fundamental solutions of equations system, kernels of BIE. Here the fundamental solutions in two- and three-dimensional cases (N, M=2,3) are considered and their properties are studied. In the space of generalized functions the solutions of initial BVP are obtained and their integral representations, regular inside a range of definition are given. Generalizing the Green and the Gauss formulas for the generalized solutions of these equations, singular integral equations for the solution of non-stationary BVP are constructed. The uniqueness theorem of the solutions, including for the class of shockwaves, is presented.

2. STATEMENT OF NONSTATIONARY BOUNDARY VALUE PROBLEMS

Let $u(x,t)$ be the solution of the system of hyperbolic equations which describes dynamics of anisotropic elastic mediums. We consider it in the cases of plane (N=2) and space (N=3) deformation:

$$L_{ij}(\partial_x,\partial_t)\, u_j(x,t)+G_i(x,t)=0, \qquad (x,t)\in R^{N+1} \tag{2.1}$$

$$L_{ij}(\partial_x,\partial_t)=C_{ij}^{ml}\partial_m\partial_l-\delta_{ij}\partial_t^2, \quad i,j=\overline{1,M}, \quad m,l=\overline{1,N}$$

$$C_{ij}^{ml}=C_{ij}^{lm}=C_{ji}^{ml}. \tag{2.2}$$

Here C_{ij}^{ml} is the matrix of elastic constants satisfying to the condition of strong hyperbolicity

$$W(n,v) = C_{ij}^{ml} n_m n_l v^i v^j > 0 \ \forall \ n \neq 0, \ v \neq 0$$

$x = (x_1, \ldots, x_N)$, $x \in S^-$, S is the boundary of elastic body $S^- \in R^N$ and belongs to the class of Lyapunov's surfaces with continuous exterior normal n, $\|n\| = 1$, $(x,t) \in D^-$, $D^- = S^- \times (0, \infty)$, $D_t^- = S^- \times (0, t)$, D= $S^- \times (0, \infty)$, $D_t = S \times (0, t)$. Everywhere summation is carried over like indexes in the indicated limits. It is supposed that $u \in C(D^- \cup D)$, $G \in C(D^- \cup D)$ and $G \to 0$, $t \to +\infty$, $\forall x \in S^-$.

Further u is twice differentiated vector function almost everywhere by exception characteristic surfaces - wavefronts F_t, on which following conditions on gaps are executed /1,2/:

$$[u_i(x,t)]_{F_t} = 0 \tag{2.3}$$

$$[u_{i,t} m_l + c u_{i,l}]_{F_t} = 0 \tag{2.4}$$

$$[\sigma_i^l m_l + c u_{i,t}]_{F_t} = 0 \qquad \sigma_i^l = C_{ij}^{ml} u_{j,m} \tag{2.5}$$

Here "c" is the speed of a wavefront motion, which instituted from the solution of the characteristic equation of the system (2.1):

$$\det\{C_{ij}^{ml} \nu_m \nu_l - \nu_t^2 \delta_{ij}\} = 0$$

where $\nu = (\nu_1, \ldots, \nu_N, \nu_t)$ is the vector of characteristic normal:

$$c = -\nu_t / (\nu_l \nu_l)^{1/2}, \ c = \pm c_k(\nu), 0 < c_k \leq c_{k+1}, \ k = 1, \ldots, M-1$$

Wave vector $m = (m_1, \ldots, m_N)$ is the normal to a wave front F_t in R^N : $m = -\nu / (\nu_l \nu_l)^{1/2}$

Problem I. To define the solution of the system (2.1), if next conditions are known:

Initial values

$$u_i(x,0) = u_i^0(x), \quad x \in S^- \cup S, \tag{2.6}$$

$$u_{i,t}(x,0) = u_i^1(x), \quad x \in S^-. \tag{2.7}$$

Dirichlet's conditions

$$u_i(x,t)=w_i(x,t) \quad x\in S, \quad t>0\,; \tag{2.8}$$

or

generalizing Neumann's conditions

$$\sigma_i^l(x,t)=n_l(x)g_i(x,t), \qquad x\in S, \quad t>0, \quad i=1,...N. \tag{2.9}$$

Problem II. To construct boundary integral equations for the solution of following boundary value problems.
First BVP. To find the solution of the Eq. (2.1), satisfying the conditions (2.6)-(2.8) and (2.3)-(2.5).
Second BVP. To find the solution of Eq.(2.1), satisfying the conditions (2.6), (2.7), (2.9) and (2.3)-(2.5).
We will call such solutions *classical* ones.

3. CONSERVATION LAWS AND UNIQUENESS OF THE SOLUTIONS OF BVP

Consider the following functions: $W(u)=0,5\,C_{ij}^{ml}\,u_{i,m}u_{j,l}$, $K(u)=0,5\left\|u_{,t}\right\|^2$, $E(u)=K(u)+W(u)$, $L(u)=K(u)-W(u)$. Functions W, K, E are densities of internal, kinetic and full energy of a field.

Theorem 3.1. If vector-function u is the classical solution of the first (or second) BVP, then

$$\int\limits_{D_t^-} L(u(x,t))dV(x,t)=\int\limits_{D_t^-} G_i(x,t)u_i(x,t)dV(x,t)+$$

$$+\int\limits_{D_t} g_i(x,t)w_i(x,t)dS(x,t)-\int\limits_{S^-}(u_i(x,t)u_{i,t}(x,t)-u_i^0(x)u_i^1(x))dV(x),$$

$$\int\limits_{S^-}(E(u,t)-E(u,0)dV(x)=\int\limits_{D_t^-} G_i(x,t)u_{i,t}(x,t)dV(x,t)+\int\limits_{D_t} g_i(x,t)w_{i,t}(x,t)dS(x,t).$$

If $u_i(x,0)=0$, $u_{i,t}(x,0)=0$, $\lim\limits_{t\to+\infty} u_{i,l}\to 0$, $\lim\limits_{t\to+\infty} u_{i,t}\to 0$, $x\in S^-$ then

$$\int\limits_{D^-} L(u(x,t))dV(x,t)=\int\limits_{D^-} G_i(x,t)u_i(x,t)dV(x,t)+\int\limits_{D} g_i(x,t)w_{i,t}(x,t)dS(x,t)$$

From this theorem follows

Theorem 3.2. If classical solution of first (or second) BVP exists and satisfies the conditions:

$\lim\limits_{t\to+\infty} u_{i,l}\to 0$, $\lim\limits_{t\to+\infty} u_{i,t}\to 0$, $\forall x\in S^-$ then it is unique.

4. THE GENERALIZED SOLUTIONS OF BOUNDARY VALUE PROBLEMS. ANALOGY OF THE KIRCHHOFF AND GREEN FORMULAS

Here $D'_M(R^{N+1})$ is the space of generalized vector functions $\hat{f}(x,t)=(\hat{f}_1,...,\hat{f}_M)$ on the space $D_M(R^{N+1})$ of base vector-functions $\varphi(x,t)=\{\varphi_1,...,\varphi_M\}$, $\forall\varphi_j \in D\ (R^N)$ /3/. For regular $\hat{f}$ we have

$$(\hat{f}(x,t),\varphi(x,t))=\int_{-\infty}^{\infty} d\tau \int_{R^N} f_i(x,\tau)\varphi_i(x,\tau)dV(x),\quad \forall\varphi\in D_M(R^{N+1})$$

For u which is determined on D^- we introduce a generalized function

$$\hat{u}(x,t)=H(t)\,H_S^-(x)\ u(x,t) \tag{4.1}$$

Here $H(t)$ is Heaviside's function, $H_S^-(x)$ is the characteristic function of the set S^- which is equal to 1 for $x\in S^-$, to $0{,}5$ for $x\in S$ and to 0 for $x\in R^N \setminus (S \cup S^-)$. Analogously to (4.1) we have $\hat{G}_k(x,t)=H(t)\,H_S^-(x)\,G_k(x,t)$.

Green's matrix $U_i^k(x,t)$ is the fundamental solution of Eq. (2.1), corresponding to $\hat{G}(x,t)-\delta_{ik}\delta(x,t)$ and

$$U_i^k(x,0)=0,\qquad U_{i,t}^k(x,0)=0\quad x\neq 0$$

Regarding the construction for U see /4/. Also we use the fundamental solution of Eq. (2.1) by $\hat{G}(x,t)=\delta_{ik}\delta(x)H(t)$ as a convolution

$$V_i^k(x,t)=U_i^k(x,t)\underset{t}{*}H(t).$$

Theorem 4.1. If the function u is the classical solution of first (or second) BVP then $\hat{u}(x,t)$ can be represented over convolutions:

$$\begin{aligned}\hat{u}_i(x,t)&=U_i^k(x,t)*\hat{G}_k(x,t)+U_i^k(x,t)\underset{t}{*}u_k^1(x)H_S^-(x)+\\&+(U_{i,t}^k(x,t)\underset{x}{*}u_k^0H_S^-(x)+U_i^k(x,t)*g_k(x,t)\delta_S(x)H(t)-\\&-C_{kj}^{ml}V_{i,l}^k(x,t)*u_{j,t}n_m(x)\delta_S(x)H(t)-C_{kj}^{ml}V_{i,l}^k(x,t)\underset{x}{*}u_j^0(x)n_m(x)\delta_S(x)\end{aligned} \tag{4.2}$$

Here $g_k(x,t)\delta_S(x)H(t)$ is the simple layer on cylinder D; sign "*" means full convolution on (x,t); sign "x" or "t" under an asterisk corresponds to convolution only over x or t accordingly.

The formula of this theorem is the generalized solution of Problem I.

5. MATRICES OF THE FUNDAMENTAL SOLUTIONS V, T, W

New matrices are considered:

$$S_{ik}^{m}(x,t) = C_{ij}^{ml} U_{j,l}^{k}, \qquad \Gamma_i^k(x,t,n) = S_{ik}^{m} n_m$$

$$T_k^i(x,t,n) = -\Gamma_i^k(x,t,n) = -C_{ij}^{ml} n_m U_{j,l}^k(x,t)$$

$$V_j^k(x,t) = U_j^m(x,t) * \delta_{mk}\delta(x)H(t) = U_j^k(x,t) \underset{t}{*} H(t)$$

$$W_j^k(x,t,n) = T_j^m(x,t,n) * \delta_{mk}\delta(x)H(t) = T_j^k(x,t,n) \underset{t}{*} H(t)$$

Some properties of symmetry of these matrices are

$$U_i^k(x,t) = U_i^k(-x,t), \qquad U_i^k(x,t) = U_k^i(x,t),$$

$$V_i^k(x,t) = V_i^k(-x,t), \quad V_i^k(x,t) = V_k^i(x,t)$$

$$S_{ik}^m(x,t) = -S_{ik}^m(-x,t), T_i^k(x,t,n) = -T_i^k(-x,t,n) = -T_i^k(x,t,-n)$$

$$W_i^k(x,t,n) = -W_i^k(-x,t,n) = -W_i^k(x,t,-n)$$

Theorem 5.1. Multipole matrix $T_i^k(x,t,n)$ at fixed "k" is the fundamental solution of the system (2.1), applicable

$$G_i(x,t) = n_m C_{ik}^{ml} \delta_{,l}(x,t).$$

Let us introduce $U_i^{k(s)}(x)$ as the Green matrix of static equations (2.1) (when $\partial_t u = 0$):

$$L_{ij}(\partial_x, 0)\, U_j^{k(s)}(x) + \delta_{ik}\delta(x) = 0, \qquad U_i^{k(s)}(x) \to 0,\ \|x\| \to \infty,$$

$$T_i^{k(s)}(x,n) = -C_{kj}^{ml} n_m U_{j,l}^{i(s)}, \quad T_i^{k(s)}(x,n) = -T_i^{k(s)}(-x,n) = -T_i^{k(s)}(x,-n)$$

Lemma 5.1. $T_i^{k(s)}$ is the fundamental solution of static equations

$$L_{ij}(\partial_x, 0)\mathrm{T}_j^{k(s)}(x,n) - n_m C_{ik}^{ml}\delta_{,l}(x) = 0.$$

It is easy to see that this system is of an elliptical type. The following representations are valid.

Theorem 5.2.

$$V_i^k(x,t) = U_i^{k(s)}(x)H(t) + V_i^{k(d)}(x,t)$$
$$W_i^k(x,t) = T_i^{k(s)}(x)H(t) + W_i^{k(d)}(x,t),$$

where $U_i^{k(s)}(x)\, H(t)$, $T_i^{k(s)}\, H(t)$ are regular functions at $x\neq0$. At $\|x\| \to 0$

$$U_i^{k(s)}(x) \sim ln\|x\| A_{ik}^N(e_x) \qquad T_i^{k(s)}(x,n) \sim \|x\|^{-1} B_{ik}^N(e_x), \quad N=2$$
$$U_i^{k(s)}(x) \sim \|x\|^{-N+2} A_{ik}^N(e_x) \qquad T_i^{k(s)}(x,n) \sim \|x\|^{-N+1} B_{ik}^N(e_x), \quad N>2$$

Here $e_x = x/\|x\|$, $A_{ik}^N(e_x)$, $B_{ik}^{ij}(e_x)$ are continuous and restricted on the sphere $\|e\|=1$ functions; $V_i^{k(d)}$, $W_i^{k(d)}$ are regular functions, continuous at $x=0$, $t>0$. For anyone N: $V_i^{k(d)}(x,t)=0$ and $W_i^{k(d)}(x,t)=0$ by $\|x\|>\max_{\|e\|=1} c_M(e)t$, and for uneven N these equalities are performed and for $\|x\|<\min_{\|e\|=1} c_1(e)t$.

6. BOUNDARY INTEGRAL EQUATIONS

Lemma 6.1 (analogue of Gauss formula). If S is any closed Lyapunov's surface in R^N, then we have

$$\int_S T_i^{k(s)}(y-x, n(y))dS(y) = \delta_{ki}H_S^-(x) \tag{6.1}$$

By $x\in S$ integral is singular, it is calculated in the sense of Value Principle.

When $M=1$ and $L_{1j}(\partial_x, 0) = \partial_j\partial_j = \Delta$, this formula complies with the Gauss formula for potential of double layer of Laplace equations /3/.

Notice that formula (4.2) formally can be presented in the form:

$$\hat{u}_k(x,t) = \int_D (T_k^i(x-y, n(y), t-\tau)u_i(x,t) + U_k^i(x-y, t-\tau)g_i(y,\tau)dD(y,\tau)) +$$
$$+ \int_{S^-} (U_{k,t}^i(x-y,t)u_i^0(y) + U_k^i(x-y,t)u_i^1(y))dS^-(y) + U_k^i * \hat{G}_i$$

Under zero initial conditions this complies with generalizing Green's formula for elliptical systems. However particularities of Green's matrices for wave equations do not allow using this formula for building solutions of boundary value problems, because there are strong singularities on fronts. But the matrices introduced in §5 allow building integral representations of formula (4.2).

Theorem 6.1 If u is the classical solution of boundary value problems then

$$\hat{u}_k(x,t) = U_k^l(x,t) * \hat{G}_l(x,t) + U_k^l(x,t) * g_i(x,t)\delta_S(x)H(t) -$$

$$-V.P.\int_S (T_k^{i(s)}(x-y,n(y))u_i(y,t)dS(y) - \int_S dS(y)\int_0^t W_k^{i(d)}(x-y,n(y),t-\tau)u_{i,t}(y,\tau)d\tau -$$

$$-\int_S (W_k^{i(d)}(x-y,n(y),t)u_i^0(y)dS(y) + (U_k^l(x,t) \underset{t}{*} u_i^0(x)H_S^-(x)),_t$$

Proof.

Integral presentation of formula (4.2) for even N has the form

$$\hat{u}_k(x,t) = \int_S dS(y)\int_0^t (U_k^i(x-y,t-\tau)g_i(y,\tau) - W_k^i(x-y,n(y),t-\tau)u_{i,t}(y,\tau))d\tau -$$

$$-\int_S (W_k^i(x-y,n(y),t)u_i^0(y)dS(y) + \int_{S^-} U_{k,t}^i(x-y,t)u_i^0(y)dS^-(y)$$

$$+\int_{S^-} U_k^i(x-y,t)u_i^1(y)dS^-(y) + \int_{D^-} U_k^i(x-y,t-\tau)G_i(y,\tau)dD^-(y,\tau)$$

If use theorem 5.2 that second summand possible to present so

$$\int_D W_k^i(x-y,n(y),t-\tau)u_{i,t}(y,\tau)dD(y,\tau) = \int_S dS(y)\int_0^t W_k^i(x-y,n(y),t-\tau)du_{i,t}(y,\tau) =$$

$$= \int_S T_k^{i(s)}(x-y,n(y))(u_i(y,t) - u_i^0(y))dS(y) + \int_S dS(y)\int_0^t W_k^{i(d)}(x-y,n(y),t-\tau)w_{i,t}(y,\tau)d\tau$$

According to determination W_k^i

$$\int_S (W_k^i(x-y,n(y),t)-T_k^{i(s)}(x-y,n(y)))u_i^0(y)dS(y)=\int_S W_k^{i(d)}(x-y,n(y),t)u_i^0(y)dS(y)$$

After summation we obtain the formula of the theorem for internal points. If $x^* \in S$, $x \in S^-$ and $x \to x^*$, then

$$\lim_{x\to x^*} u_k(x,t)=u_k(x^*,t)=\lim_{x\to x^*}\int_S T_k^{i(s)}(x-y,n(y))u_i(y,t)dS(y)+$$
$$+\int_S W_k^{i(d)}(x^*-y,n(y),t)u_i^0(y)dS(y)-$$
$$-\int_S dS(y)\int_0^t (U_k^i(x^*-y,t-\tau)g_i(y,\tau)+W_k^{i(d)}(x^*-y,n(y),t-\tau)u_{i,t}(y,\tau))+$$
$$+\int_{S^-} U_k^i(x^*-y,t)u_i^1(y)dS^-(y)+\int_{S^-}(U_k^i(x^*-y,t)u_i^0(y)),_t\, dS^-(y)+$$
$$+\int_{D^-} U_k^i(x^*-y,t-\tau)G_i(y,\tau)dD^-(y,\tau)$$

The limit on the right part can be, by means of lemmas 6.1, converted to type

$$\int_S T_k^{i(s)}(x^*-y,n(y))(u_i(y,t)-u_i(x^*,t))dS(y)+u_i(x^*,t)\delta_k^i=$$

$$=V.P.\int_S T_k^{i(s)}(x^*-y,n(y))u_i(y,t)dS(y)-u_i(x^*,t)\;V.P.\int_S T_k^{i(s)}(x^*-y,n(y))dS(y)+u_i(x^*,t)\delta_k^i=$$

$$=V.P.\int_S T_k^{i(s)}(x^*-y,n(y))dS(y)+0{,}5u_i(x^*,t)\delta_k^i$$

After summation we obtain the formula of the theorem for boundary points, which is the boundary integral equation for solving BVP. Please see paper /5/, where this method was applied for solving wave diffraction problems in elastic media with cavities of different forms.

7. REFERENCES

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