On the Lie Symmetry Groups with Application to a Non Linear Viscoelastic Behaviour

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1. SUMMARY

The continuous symmetry groups of a viscoelastic constitutive behaviour are calculated. The considered constitutive law is expressed within a thermodynamic approach of irreversible processes, called the Distribution of Non-linear Relaxations (acronym DNLR). Interpreting the material behaviour as a partial differential equation satisfied by the stress as a function of the time and of a set of continuously varying parameters, its continuous symmetries are determined, from the generator vector fields associated to each infinitesimal symmetry. The expression of the symmetry groups makes it possible to find a master curve and to underline the well–known time temperature equivalence principle. The symmetry analysis provides a rational and systematic way of construction of the master curves, that appears more economical in terms of experiments compared to the traditional heuristic approach.

2. INTRODUCTION

The Lie group theory (from the mathematician Sophus Lie) is known to be of great importance as a guide for the integration of ordinary or partial differential equations, and to obtain the continuous symmetry groups of a given problem, or the group – invariant solutions of physical systems (see e.g. Olver /1/ and the references therein). It is of great help in various physical fields such as quantum mechanics, electromagnetism, or mechanical engineering (Ibragimov, /2/). For example, Ozer /3/ studied the Lie symmetry groups of nonlocal elasticity equation to find its solution, and Vassilev /4/ used Lie groups to study the linear theory of rods and plates. More recently and focusing on the thermomechanics of dissipative processes in continuous media, Anthony /5/ expressed the field equations of heat conduction from a lagrangian framework, that evidences the so-called gauge invariance, while Mielke /6/ related finite elastoplasticity to Lie groups and geodesics. Following similar lines of thoughts, we calculate in the present work the Lie symmetry groups of the viscoelastic behaviour equation, which is written in the DNLR formalism (Cunat, /7/).

3. DNLR APPROACH AND VISCOELASTIC BEHAVIOUR

The DNLR approach relies on the assumptions that the Gibbs relationship (1902) can be extended to transformations outside equilibrium (see Cunat, /7/ and the references therein that give an historical perspective of the framework), by completing the set of thermodynamic state variables with a set of suitable internal variables. Starting from the specific internal energy $e(\vec{y},\vec{z})$, where $\vec{y}$ is the vector of extensive variables (e.g. the strain ε or the entropy s) and $\vec{z}$ the vector of internal variables (describing the microstructure), we can obtain the rate form of the internal energy as:

$$\dot{e} = e_{,\vec{y}} \cdot \dot{\vec{y}} + e_{,\vec{z}} \cdot \dot{\vec{z}} = \vec{Y}(\vec{y},\vec{z}) \cdot \dot{\vec{y}} - \vec{A}(\vec{y},\vec{z}) \cdot \dot{\vec{z}} \qquad (1)$$

(the subscript denotes the partial derivative), which allows us to define the set of intensive dual variables $\vec{Y}(\vec{y},\vec{z}) = e_{,\vec{y}}$ such as the stress σ or the temperature T conjugated to the strain and entropy respectively, and $\vec{A}(\vec{y},\vec{z}) = -e_{,\vec{z}}$ the generalized affinities (or non equilibrium thermodynamic forces). The evolution of these intensive variables is then given in rate form by the following system:

$$\begin{pmatrix} \dot{\vec{Y}} \\ \dot{\vec{A}} \end{pmatrix} = \begin{pmatrix} e_{,yy} & e_{,zy} \\ e_{,\vec{y}z} & e_{,zz} \end{pmatrix} \begin{pmatrix} \dot{\vec{y}} \\ \dot{\vec{z}} \end{pmatrix} \qquad (2)$$

where the matrix of second order derivatives is, for the sake of simplicity, assumed to be constant. These equations that define the state laws have to be completed by the evolution laws of the internal variables. Thus, for a first order kinetic equation, viz $\dot{\vec{z}} = \bar{\bar{L}}\vec{A}$, with the condition that the affinities rate vanish at relaxed state: $\dot{\vec{A}}' = 0$, one obtains $\dot{\vec{z}} = -\bar{\bar{\tau}}^{-1}\left(\vec{z} - \vec{z}\right)$ with $\bar{\bar{\tau}}^{-1} = \bar{\bar{L}}.e_{,\vec{z}\vec{z}}$ the inverse of relaxation times, the internal variables at the relaxed state being given by $\dot{\vec{z}}^{r} = -\left(e_{,\vec{z}\vec{z}}\right)^{-1} e_{\vec{z}\vec{y}}.\dot{\vec{y}}$. Identifying then, as a peculiar case, $\vec{y}$ with the strain ε, and $\vec{Y}$ with the stress σ, for the simple case of a single dissipative mode ($\vec{z} \equiv z$), it can be shown that the one dimensional equation of viscoelasticity written within the DNLR formalism takes the form:

$$\Delta_{visco}(t,\tau,\dot{\varepsilon},\dot{\sigma},\sigma) = \dot{\sigma}(t) - E_u\dot{\varepsilon}(t) + \frac{\sigma(t) - \sigma^r(t)}{\tau} = 0 \qquad (3)$$

where $\sigma(t)$ and $\varepsilon(t)$ are respectively the stress and the strain at time t. $E_u(=e_{.\varepsilon\varepsilon})$ is the instantaneous Young modulus, $\sigma^r(t)$ the relaxed stress at time t and τ the relaxation time. In a first approach, $\sigma^r(t)$ is assumed to be equal to $E_r\varepsilon(t)$, where $E_r(=e_{.z\varepsilon})$ is the relaxed modulus. Both the relaxed and instantaneous modulus E_r and E_u are supposed to be constant. The chosen loading path is such that $\varepsilon(t)$ is linear in time: $\varepsilon(t)=\dot{\varepsilon}t$ where $\dot{\varepsilon}$ is constant. The relaxation time τ may be expressed for simplicity as a function of the temperature T, according to:

$$\tau(T)=\left(\frac{h}{kT}\right)\exp\left[\left(\frac{\Delta H^{+}-T\Delta S^{+}}{RT}\right)\right] \tag{4}$$

where h is the Planck constant, k the Boltzmann constant, R the ideal gas constant. The parameters ΔH^{+} and ΔS^{+} are supposed to be constant and correspond respectively to the activation enthalpy and entropy. To find the Lie groups of Equ. (3) whith the varying 3 parameters of the problem (i.e. t, τ, $\dot{\varepsilon}$) in a consistent way with the differential geometry method, we have to reconsider the stress $\sigma(t)$ as a new function $\sigma(t,\tau,\dot{\varepsilon})$ of these independent variables. Accordingly, Eq. (3) can be rewritten:

$$\Delta_{visco}(t,\tau,\dot{\varepsilon},\dot{\sigma},\sigma)=\dot{\sigma}(t,\tau,\dot{\varepsilon})-E_u\dot{\varepsilon}+\frac{\sigma(t,\tau,\dot{\varepsilon})-E_r\dot{\varepsilon}t}{\tau}=0 \tag{5}$$

where $\tau=\tau(T)$, with the initial condition $\sigma(t=0,\tau,\dot{\varepsilon})=0$. Let us call $\sigma_{sol}(t,\tau,\dot{\varepsilon})$ the solution of this problem.

4. A COMPUTATIONAL PROCEDURE FOR FINDING SYMMETRY GROUPS

Using Olver notations /1/, one can consider the smooth manifold $M = X \times U$, where X is the set of independent variables $(t,\tau,\dot{\varepsilon})$ and U the set of dependent variables (i.e. the stress $\sigma(t,\tau,\dot{\varepsilon})$). Let $\vec{v}$ be a vector field on M, expressed as an infinitesimal operator by its components:

$$\vec{v}=\xi(t,\tau,\dot{\varepsilon},\sigma)\frac{\partial}{\partial t}+\alpha(t,\tau,\dot{\varepsilon},\sigma)\frac{\partial}{\partial\tau}+\beta(t,\tau,\dot{\varepsilon},\sigma)\frac{\partial}{\partial\dot{\varepsilon}}+\phi(t,\tau,\dot{\varepsilon},\sigma)\frac{\partial}{\partial\sigma} \tag{6}$$

This vector field $\vec{v}$ generates a symmetry group for a first order differential equation $\Delta(t,\tau,\dot{\varepsilon},\sigma,\dot{\sigma})=0$ if and only if it transforms a solution of Δ into another one. A necessary and sufficient condition for $\vec{v}$ to generate a symmetry group for Δ is given by $pr^{(1)}\vec{v}\,\Delta=\vec{0}$, whenever Δ = 0, where $pr^{(1)}\vec{v}$ is the first prolongation of the vector field $\vec{v}$ defined by:

$$pr^{(1)}\vec{v} = \vec{v} + \left(\frac{d\phi}{dt} - \frac{d\xi}{dt}\frac{\partial\sigma}{\partial t} - \frac{d\alpha}{dt}\frac{\partial\sigma}{\partial\tau} - \frac{d\beta}{dt}\frac{\partial\sigma}{\partial\dot{\varepsilon}}\right)\frac{\partial}{\partial\dot{\sigma}} \tag{7}$$

(see /1/ for the general prolongation formula). The explicit expression of a one-parameter group G is given by the vector exponentiation of its generator $\vec{v}$, which consists in solving the differential system:

$$\begin{array}{llll} \dfrac{d\bar{t}}{d\mu} = \xi(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}},\bar{\sigma}) & \dfrac{d\bar{\tau}}{d\mu} = \alpha(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}},\bar{\sigma}) & \dfrac{d\bar{\dot{\varepsilon}}}{d\mu} = \beta(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}},\bar{\sigma}) & \dfrac{d\bar{\sigma}}{d\mu} = \phi(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}},\bar{\sigma}) \\ \bar{t}(\mu=0) = t & \bar{\tau}(\mu=0) = \tau & \bar{\dot{\varepsilon}}(\mu=0) = \dot{\varepsilon} & \bar{\sigma}(\mu=0) = \sigma \end{array} \tag{8}$$

where μ is the parameter of the group. For example, the vector field for which $\xi = t$ and $\alpha = \tau$ gives after exponentiation the finite symmetry group:

$$G \ : \ \left\{ \ \bar{t} = e^{\mu}t \qquad \bar{\tau} = e^{\mu}\tau \qquad \bar{\dot{\varepsilon}} = \dot{\varepsilon} \qquad \bar{\sigma} = \sigma \ \right\} \tag{9}$$

which is the group of dilatation for the time t and the relaxation time τ. As $\bar{\sigma}$ is a function of $(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}})$ one obtains the equality: $\overline{\sigma(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}})} = \sigma(t,\tau,\dot{\varepsilon}) = \sigma(e^{-\mu}\bar{t},e^{-\mu}\bar{\tau},\bar{\dot{\varepsilon}})$. The necessary and sufficient condition $pr^{(1)}\vec{v}\,\Delta = 0$ provides a computational procedure for finding the Lie symmetry group of Δ_{visco}. Starting from an unknown vector field $\vec{v}$ as defined by relation (6), one has to expand the term $pr^{(1)}\vec{v}\,\Delta_{visco}$. This expression, solved with a Matlab program, leads to a polynomial in σ and its time derivative, with coefficients involving the unknown functions (ξ,α,β,ϕ). The term $\dot{\sigma}$ may be substituted using Δ_{visco}, and all polynomial coefficients are then equated to zero. This calculation leads us to a system of PDE for the functions (ξ,α,β,ϕ), whose solution takes the following form (Maple program):

$$\xi = -F_3(\tau,\dot{\varepsilon})\,\tau\, e^{-\frac{t}{\tau}} + F_2(\tau,\dot{\varepsilon})\frac{t}{\tau} + F_4(\tau,\dot{\varepsilon}) \tag{10}$$

$$\alpha = F_2(\tau,\dot{\varepsilon}) \tag{11}$$

$$\beta = F_1(\tau,\dot{\varepsilon}) \tag{12}$$

$$\begin{aligned} \phi = &\left((\sigma - E_r\dot{\varepsilon}\,t)F_3(\tau,\dot{\varepsilon}) + F_5(\tau,\dot{\varepsilon})\right) e^{-\frac{t}{\tau}} + \left((E_u - E_r)\tau + E_r t\right)\frac{\dot{\varepsilon}}{\tau}F_2(\tau,\dot{\varepsilon}) \\ &+ \left((E_u - E_r)\tau + E_r t\right) F_1(\tau,\dot{\varepsilon}) + E_r\dot{\varepsilon}\,F_4(\tau,\dot{\varepsilon}) \end{aligned} \tag{13}$$

where $F_1(\tau,\dot{\varepsilon})$, $F_2(\tau,\dot{\varepsilon})$, $F_3(\tau,\dot{\varepsilon})$, $F_4(\tau,\dot{\varepsilon})$, $F_5(\tau,\dot{\varepsilon})$ are arbitrary functions. Among all these groups, let us now consider the particular groups G for which the following condition is fulfilled:

$$\bar{\sigma}(\bar{t}=0,\bar{\tau},\bar{\dot{\varepsilon}})=0 \tag{14}$$

which means that every function transformed by G, $\bar{\sigma}$, satisfies the same initial condition as $\sigma_{sol}(t,\tau,\dot{\varepsilon})$. Because of the uniqueness of the solution, one can write the equality:

$$\bar{\sigma}(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}})=\sigma_{sol}(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}}) \tag{15}$$

for each value of the parameter μ. The condition (14) is mathematically equivalent to the invariance of algebraic conditions $(t=0,\sigma=0)$. Hence, the conditions $pr^{(1)}\vec{v}\; t=0$ and $pr^{(1)}\vec{v}\; \sigma=0$, whenever $(t,\sigma)=(0,0)$, lead to the new equations:

$$\begin{cases} \xi(t=0,\tau,\varepsilon,\sigma=0)=0 \\ \phi(t=0,\tau,\varepsilon,\sigma=0)=0 \end{cases} \tag{16}$$

Using these two equations, one can express $F_4(\tau,\dot{\varepsilon})$ and $F_5(\tau,\varepsilon)$ in terms of $F_1(\tau,\dot{\varepsilon})$, $F_2(\tau,\dot{\varepsilon})$, and $F_3(\tau,\dot{\varepsilon})$. The solution then takes the form:

$$\xi=\tau\, F_3(\tau,\dot{\varepsilon})\,(1-e^{-\frac{t}{\tau}})+F_2(\tau,\dot{\varepsilon})\frac{t}{\tau} \tag{17}$$

$$\alpha=F_2(\tau,\dot{\varepsilon}) \tag{18}$$

$$\beta=F_1(\tau,\dot{\varepsilon}) \tag{19}$$

$$\begin{aligned}\phi=&F_1(\tau,\dot{\varepsilon})(\tau(E_u-E_r)(1-e^{-\frac{t}{\tau}})+E_r t)+F_2(\tau,\dot{\varepsilon})(\tau(E_u-E_r)(1-e^{-\frac{t}{\tau}})+E_r t)\frac{\dot{\varepsilon}}{\tau}\\ &+F_3(\tau,\dot{\varepsilon})(E_r\,\dot{\varepsilon}\,\tau(1-e^{-\frac{t}{\tau}})+e^{-\frac{t}{\tau}}(\sigma-E_r\dot{\varepsilon}\,t))\end{aligned} \tag{20}$$

To emphasize the significance of this last results, let us examine the particular case where $F_1(\tau,\dot{\varepsilon})=-\dot{\varepsilon}$, $F_2(\tau,\dot{\varepsilon})=\tau$, and $F_3(\tau,\dot{\varepsilon})=0$, leading to the generator:

$$\vec{v}=t\frac{\partial}{\partial t}+\tau\frac{\partial}{\partial \tau}-\dot{\varepsilon}\frac{\partial}{\partial \dot{\varepsilon}} \tag{21}$$

Its exponentiation gives the following group of transformation:

$$G:\left\{\quad \bar{t}=e^{\mu}t \qquad \bar{\tau}=e^{\mu}\tau \qquad \bar{\dot{\varepsilon}}=e^{-\mu}\dot{\varepsilon} \qquad \bar{\sigma}=\sigma_{sol}\quad\right\} \tag{22}$$

Keeping in mind that $\bar{\sigma}$ is a function of $(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}})$ and σ_{sol} a function of $(t,\tau,\dot{\varepsilon})$ yields:

$$\bar{\sigma}(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}}) = \sigma_{sol}(e^{-\mu}\bar{t},e^{-\mu}\bar{\tau},e^{\mu}\bar{\dot{\varepsilon}}) \tag{23}$$

and:

$$\bar{\sigma}(\bar{t}=0,\bar{\tau},\bar{\dot{\varepsilon}}) = \sigma_{sol}(0,e^{-\mu}\bar{\tau},e^{\mu}\bar{\dot{\varepsilon}}) = 0, \quad \forall\,\mu \tag{24}$$

Thus, according to Eq. (15), one obtains :

$$\bar{\sigma}(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}}) = \sigma_{sol}(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}}) \tag{25}$$

leading to a relation (replace $(\bar{t},\bar{\tau},\bar{\dot{\varepsilon}})$ by $(t,\tau,\dot{\varepsilon})$ and $e^{-\mu}=\alpha$ for simplicity) which defines the first "equivalence principle":

$$EP_1: \quad \sigma_{sol}(\alpha\, t,\alpha\,\tau,\frac{\dot{\varepsilon}}{\alpha}) = \sigma_{sol}(t,\tau,\dot{\varepsilon}) \tag{26}$$

In the same way, we consider the two other cases for which $F_1(\tau,\dot{\varepsilon})=\dot{\varepsilon}$, $F_2(\tau,\dot{\varepsilon})=0$, $F_3(\tau,\dot{\varepsilon})=0$ and $F_1(\tau,\dot{\varepsilon})=0$, $F_2(\tau,\dot{\varepsilon})=\tau$, $F_3(\tau,\dot{\varepsilon})=0$, leading to the equalities (where $e^{-\mu}=\alpha$):

$$EP_2: \quad \sigma_{sol}(t,\tau,\alpha\dot{\varepsilon}) = \sigma_{sol}(t,\tau,\dot{\varepsilon}) + (\alpha-1)\left[\tau\,\dot{\varepsilon}\,(E_u-E_r)\,(1-e^{-\frac{t}{\tau}}) + E_r t\dot{\varepsilon}\right] \tag{27}$$

$$EP_3: \quad \sigma_{sol}(\alpha t,\alpha\,\tau,\dot{\varepsilon}) = \sigma_{sol}(t,\tau,\dot{\varepsilon}) + (\alpha-1)\left[\tau\,\dot{\varepsilon}\,(E_u-E_r)\,(1-e^{-\frac{t}{\tau}}) + E_r t\dot{\varepsilon}\right] \tag{28}$$

5. DISCUSSION AND PHYSICAL INTERPRETATION

A significant result of the previous calculation is that the exact solution of the problem (5) explicitly appears in Eqs. (27) and (28):

$$\sigma_{sol}(t,\tau,\dot{\varepsilon}) = \tau\,\dot{\varepsilon}\,(E_u-E_r)\,(1-e^{-\frac{t}{\tau}}) + E_r t\dot{\varepsilon} \tag{29}$$

Thus, using the solution (29), Eqs. (27) and (28) can be rewritten respectively as:

$$\sigma_{sol}(t,\tau,\alpha\dot{\varepsilon}) = \alpha\,\sigma_{sol}(t,\tau,\dot{\varepsilon}) \tag{30}$$

$$\sigma_{sol}(\alpha t,\alpha\tau,\dot{\varepsilon}) = \alpha\sigma_{sol}(t,\tau,\dot{\varepsilon}) \tag{31}$$

Remembering that the relaxation time is a function of temperature, Eq. (31) takes the equivalent form:

$$\sigma_{sol}\left(\alpha t,\alpha\left(\frac{h}{kT}\right)\exp\left[\left(\frac{\Delta H^{+}-T\Delta S^{+}}{RT}\right)\right],\dot{\varepsilon}\right) = \alpha\,\sigma_{sol}\left(t,\left(\frac{h}{kT}\right)\exp\left[\left(\frac{\Delta H^{+}-T\Delta S^{+}}{RT}\right)\right],\dot{\varepsilon}\right) \tag{32}$$

This equality may be regarded as the well–known time temperature equivalence "principle". For instance, performing a loading experiment between t_1 and t_2 at constant T_1 and constant $\dot{\varepsilon}$ will lead, at the same $\dot{\varepsilon}$, between αt_1 and αt_2, at a temperature T_2, the same stress curve multiplied by α. The temperatures T_1 and T_2 are related by the following relationship:

$$\left(\frac{h}{kT_2}\right)\exp\left[\left(\frac{\Delta H - T_2\Delta S}{RT_2}\right)\right] = \alpha\left(\frac{h}{kT_1}\right)\exp\left[\left(\frac{\Delta H - T_1\Delta S}{RT_1}\right)\right] \tag{33}$$

It is also interesting to note that the solution $\sigma_{sol}(t,\tau,\dot{\varepsilon})$ appearing explicitly in the symmetry groups allows us to define a master curve (where E_{eq} is an equivalent modulus):

$$E_{eq} = \frac{\sigma}{t\dot{\varepsilon}} = n_D(E_u - E_r)\left(1 - e^{-\frac{1}{n_D}}\right) + E_r \tag{34}$$

(where $n_D = \frac{\tau}{t}$ denotes the well-known Deborah number). This equation satisfies Eqs. (26), (30) and (31). An example of this master curve for a polymer is given in Figure 1.

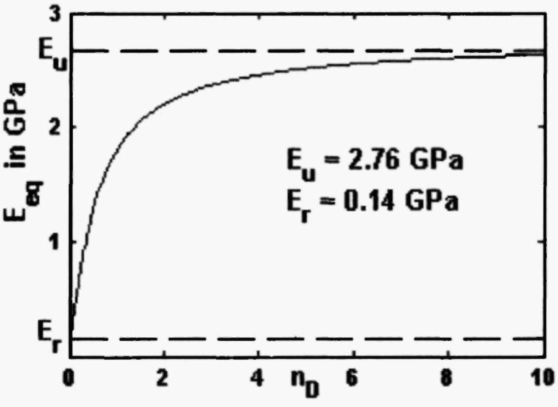

Fig. 1: The master curve corresponding to the linear viscoelastic behaviour of a polymer.

(the properties E_u and E_r denote the instantaneous and relaxed moduli respectively). Further calculations of symmetry groups are in progress.

6. CONCLUSION

The infinitesimal Lie symmetries for a viscoelastic behaviour have been generally written within the DNLR framework. Especially, one has underlined those leaving the initial condition invariant. Three particular cases have been emphasized and one of them can be interpreted as the time temperature equivalence principle, that allows the construction of the master curve in a deductive manner. An extension of this methodology to the exploration of more general dissipative behaviours is in progress (e.g. non linear viscoelastic behaviour), considering variational symmetries as well /8/.

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