Fundamental Solution in the Theory of Consolidation with Double Porosity

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1. SUMMARY

Aifantis' theory of consolidation with double porosity is considered. The fundamental solution of the system of linear partial differential equations of the steady oscillations is constructed in terms of elementary functions and its basic properties are determined.

2. INTRODUCTION

At the beginning of the 1960s, Trusdell and Toupin presented the modern formulation of continuum theories of mixtures /1/. Two decades later, Aifantis /2/ introduced the coupling of multi-porosity theory to a deformable porous medium by employing the theory of mixtures, and, in addition, Wilson and Aifantis /3/ presented the theory of consolidation with double porosity. Furthermore, a multi-porosity thermodynamics continua model was considered by Aifantis /4,6,7/, Aifantis and Beskos /5/, Bai and Roegiers /8/, Masters *et al.* /9/, and the main results obtained in this area were presented by Aifantis and Hill /10/, Hill and Aifantis /11/, Khaled *et al.* /12/, Lewallen and Wang /13/, and Berryman /14/. The basic results and the historical information on the theory of porous media were summarized by de Boer /15/. In order to formulate boundary value problems in classical theory of elasticity and thermoelasticity, by using the boundary integral equation method (potential method), it is necessary to construct fundamental solutions of systems of partial differential equations and establish their basic properties. Several pertinent methods have been developed and can be found in the literature (see Kupradze *et al.* /16/, and Nowacki /17/). Moreover, useful information on fundamental solutions of differential equations is given from Hörmander in /18/.

The fundamental solutions in the linear theory of porous-elasticity (Biot theory) were established in the works of Cleary /19/, Cheng and Liggett /20/ and Rudnicki /21/. The fundamental solutions of equations of the linear theory of binary mixtures for elastic and thermoelastic solids are constructed in the middle of nineties by Svanadze /22-24/. In this article Aifantis' theory of consolidation with double porosity is considered and the fundamental solution of the system of linear partial differential equations of the steady oscillations is constructed in terms of elementary functions and the basic properties are established.

3. BASIC EQUATIONS

Let $x=(x_1,x_2,x_3)$ be the points of the Euclidean three-dimensional space E^3, $|x|=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$, $D_x=(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3})$. The system of the steady oscillations of the Aifantis theory of consolidation with double porosity can be written as (see Wilson and Aifantis /3/)

$$\begin{aligned}
&\mu\Delta u+(\lambda+\mu)graddivu-\beta_1 gradp_1-\beta_2 gradp_2=0,\\
&m_1\Delta p_1+i\omega\alpha_1 p_1-k(p_1-p_2)+i\omega\beta_1 divu=0,\\
&m_2\Delta p_2+i\omega\alpha_2 p_2+k(p_1-p_2)+i\omega\beta_2 divu=0,
\end{aligned}\tag{1}$$

where u is the displacement vector of the solid, p_1 is the pressure in the fissures, p_2 is the pressure in the pores and the various phenomenological coefficients $\lambda,\mu,m_1,m_2,\alpha_1,\alpha_2,\beta_1,\beta_2,k$ are constants, ω is the oscillation frequency ($\omega>0$), Δ is the Laplacian, $i=\sqrt{-1}$.

We introduce the matrix differential operator $A(D_x)=\|A_{lj}(D_x)\|_{5\times5}$, where $A_{lj}(D_x)=\mu\Delta\delta_{lj}+(\lambda+\mu)\frac{\partial^2}{\partial x_l\partial x_j}$, $A_{l,q+3}(D_x)=-\beta_q\frac{\partial}{\partial x_l}$, $A_{q+3,l}(D_x)=i\omega\beta_q\frac{\partial}{\partial x_l}$, $A_{q+3,q+3}(D_x)=m_q\Delta-k+i\omega\alpha_q$, $A_{45}(D_x)=A_{54}(D_x)=k$, $l,j=1,2,3,$ $q=1,2$, and δ_{lj} is the Kronecker delta. The system (1) can be written as $A(D_x)U(x)=0$, where $U=(u,p_1,p_2)$. We assume that

$$\mu(\lambda+2\mu)m_1m_2\neq0. \tag{2}$$

Obviously, if condition (2) is satisfied, then $A(D_x)$ is the elliptic differential operator (see Hörmander /25/).

Definition. The fundamental solution of system (1) is matrix $\Gamma(x)=\|\Gamma_{lj}(x)\|_{5\times5}$ satisfying condition (see Hörmander /25/)

$$A(D_x)\Gamma(x)=\delta(x)J\ ,\ x\in E^3, \tag{3}$$

where δ is the Dirac delta and $J=\|\delta_{lj}\|_{5\times5}$ is the unit matrix.

In this article the matrix $\Gamma(x)$ is constructed in terms of elementary functions and basic properties are established.

4. FUNDAMENTAL SOLUTION

We consider the system of equations

$$\begin{aligned}
&\mu\Delta u + (\lambda+\mu) grad div u + i\omega\beta_1 grad p_1 + i\omega\beta_2 grad p_2 = f_0 ,\\
&-\beta_1 div u + (m_1\Delta + \alpha_3) p_1 + k p_2 = f_2 ,\\
&-\beta_2 div u + k p_1 + (m_2\Delta + \alpha_4) p_2 = f_3 ,
\end{aligned} \tag{4}$$

where $\alpha_q = i\omega a_{q-2} - k$ $(q = 3,4)$; f_0 is a three-component vector function on E^3, f_2 and f_3 are scalar function on E^3. As one may easily verify, the system (4) may be written in the form

$$A^T(D_x)U(x) = F(x) , \tag{5}$$

where A^T is the transpose of matrix A, $F = (f_0, f_2, f_3)$ and $x \in E^3$.

Applying the operator *div* Eq. $(4)_1$ from (4) we have

$$\begin{aligned}
&a\Delta div u + i\omega\beta_1\Delta p_1 + i\omega\beta_2\Delta p_2 = f_1 ,\\
&-\beta_1 div u + (m_1\Delta + \alpha_3) p_1 + k p_2 = f_2 ,\\
&-\beta_2 div u + k p_1 + (m_2\Delta + \alpha_4) p_2 = f_3 ,
\end{aligned}$$

where $a = \lambda + 2\mu$, $f_1 = div f_0$. Its matrix form is

$$B(\Delta)V = \tilde{F} , \tag{6}$$

where $V = (div u, p_1, p_2)$, $\tilde{F} = (f_1, f_2, f_3)$ and

$$B(\Delta) = \| B_{lj}(\Delta) \|_{3\times3} = \begin{Vmatrix} a\Delta & i\omega\beta_1\Delta & i\omega\beta_2\Delta \\ -\beta_1 & m_1\Delta + \alpha_3 & k \\ -\beta_2 & k & m_2\Delta + \alpha_4 \end{Vmatrix}_{3\times3}$$

The system (6) implies

$$\Lambda_1(\Delta)V = \Phi , \tag{7}$$

where

$$\Phi=(\Phi_1,\Phi_2,\Phi_3),\ \Phi_j=d\sum_{l=1}^{3}B_{lj}^{*}f_l,\ \Lambda_1(\Delta)=d\det B(\Delta),\ d=\frac{1}{am_1m_2}, \tag{8}$$

and B_{lj}^{*} is the cofactor of the element B_{lj} of the matrix B. It is easily seen that $\Lambda_1(\Delta)=\Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)$, where λ_1^2 and λ_2^2 are the roots of equation

$$am_1m_2\lambda^2-[a(m_1\alpha_4+m_2\alpha_3)+i\omega(\beta_1^2m_2+\beta_2^2m_1)]\lambda$$
$$+[a(\alpha_3\alpha_4-k^2)+i\omega(\beta_1^2\alpha_4-2\beta_1\beta_2k+\beta_2^2\alpha_3)]=0.$$

Applying the operator $\Lambda_1(\Delta)$ to Eq. $(4)_1$ and taking into account Eq. (7), we obtain

$$\Lambda_1(\Delta)\Delta u=\Phi', \tag{9}$$

where

$$\Phi'=\frac{1}{\mu}[\Lambda_1f_0-(\lambda+\mu)grad\Phi_1-i\omega\beta_1grad\Phi_2-i\omega\beta_2grad\Phi_3], \tag{10}$$

On the basis of Eqs. (7) and (9) we get

$$\Lambda(\Delta)U(x)=\tilde{\Phi}(x), \tag{11}$$

where

$$\tilde{\Phi}=(\Phi',\Phi_2,\Phi_3),\ \Lambda(\Delta)=\|\Lambda_{lj}(\Delta)\|_{5\times5},\ \Lambda_{qq}(\Delta)=\Delta^2(\Delta+\lambda_1^2)(\Delta+\lambda_2^2),$$
$$\Lambda_{44}=\Lambda_{55}=\Lambda_1,\ \Lambda_{lj}(\Delta)=0,\ l,j=1,2,\cdots,5,\ l\neq j,\ q=1,2,3.$$

In what follows we use the notation

$$n_{l1}(\Delta)=-\frac{d}{\mu}[(\lambda+\mu)B_{l1}^{*}(\Delta)+i\omega\beta_1B_{l2}^{*}(\Delta)+i\omega\beta_2B_{l3}^{*}(\Delta)],$$
$$n_{lj}(\Delta)=dB_{lj}^{*}(\Delta),\ l=1,2,3,\ j=2,3. \tag{12}$$

In view of Eq. (12) from Eqs. (8), (10) we have

$$\Phi' = [\frac{1}{\mu}\Lambda_1 I + n_{11} grad div] f_0 + n_{21} grad f_2 + n_{31} grad f_3 ,$$

$$\Phi_2 = n_{12} div f_0 + n_{22} f_2 + n_{32} f_3 , \ \Phi_3 = n_{13} div f_0 + n_{23} f_2 + n_{33} f_3 . \tag{13}$$

Thus, from Eq. (13) we have

$$\tilde{\Phi}(x) = L^T (D_x) F(x) , \tag{14}$$

where

$$L = \| L_{lj} \|_{5\times 5} , \ L_{lj}(D_x) = \frac{1}{\mu}\Lambda_1(\Delta)\delta_{lj} + n_{11}\frac{\partial^2}{\partial x_l \partial x_j} , \ L_{lp}(D_x) = n_{1,p-2}\frac{\partial}{\partial x_l} ,$$

$$L_{pl}(D_x) = n_{p-2,1}\frac{\partial}{\partial x_l} , \quad L_{pq}(D_x) = n_{p-2,q-2} , \ l, j = 1,2,3 , \ p,q = 4,5 .$$

By virtue of Eqs. (4), (14), from (11) it follows that $\Lambda U = L^T A^T U$. It is obvious that $L^T A^T = \Lambda$ and

$$A(D_x) L(D_x) = \Lambda(\Delta) . \tag{15}$$

We assume that $\lambda_1^2 \neq \lambda_2^2$. Let

$$\gamma(x) = \| \gamma_{lj}(x) \|_{5\times 5} , \ \gamma_{qq}(x) = \sum_{j=1}^{4} \eta_{1j}\gamma_j(x) , \ \gamma_{44}(x) = \gamma_{55}(x) = \sum_{j=1}^{3} \eta_{2j}\gamma_j(x) ,$$

$$\gamma_{lj}(x) = 0 , \ l, j = 1,2,\cdots,5 , \ i \neq j , \ q = 1,2,3 , \tag{16}$$

where

$$\gamma_j(x) = -\frac{e^{i\lambda_j |x|}}{4\pi |x|} , \ \gamma_3(x) = -\frac{1}{4\pi |x|} , \ \gamma_4(x) = -\frac{|x|}{8\pi} , \ j = 1,2 ,$$

and

$$\eta_{11} = \frac{1}{\lambda_1^4(\lambda_2^2 - \lambda_1^2)} , \ \eta_{12} = \frac{1}{\lambda_2^4(\lambda_1^2 - \lambda_2^2)} , \ \eta_{13} = -\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^4 \lambda_2^4} , \ \eta_{14} = \frac{1}{\lambda_1^2 \lambda_2^2} ,$$

$$\eta_{21} = \frac{1}{\lambda_1^2(\lambda_1^2 - \lambda_2^2)} , \ \eta_{22} = \frac{1}{\lambda_2^2(\lambda_2^2 - \lambda_1^2)} , \ \eta_{23} = \eta_{14} .$$

It is easily seen that

$$(\Delta+\lambda_j^2)\gamma_j(x)=\delta(x),\ \Delta\gamma_3(x)=\delta(x),\ \Delta^2\gamma_4(x)=\delta(x),\ \Delta\gamma_4(x)=\gamma_3(x),\ x\in E^3,\ j=1,2,$$
$$\eta_{1}+\eta_{2}+\eta_{3}=0,\ \lambda_1^2\eta_{1}+\lambda_2^2\eta_{2}-\eta_{4}=0,\ \lambda_1^4\eta_{1}+\lambda_2^4\eta_{2}=0,\ \lambda_1^6\eta_{1}+\lambda_2^6\eta_{2}=1. \tag{17}$$

Lemma 1. The matrix γ is the fundamental matrix of operator $\Lambda(\Delta)$, that is,

$$\Lambda(\Delta)\gamma(x)=\delta(x)J. \tag{18}$$

Proof. It is sufficient to show

$$\Delta^2(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\gamma_{11}(x)=\delta(x),\ \Delta(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\gamma_{44}(x)=\delta(x),\ x\in E^3. \tag{19}$$

Taking into account the equalities (16) and (17) we have

$$\Delta\Lambda_1\gamma_{11}=\Lambda_1[\sum_{j=1}^{2}\eta_j(\delta-\lambda_j^2\gamma_j)+\eta_3\delta+\eta_4\gamma_3]=\Lambda_1[-\sum_{j=1}^{2}\lambda_j^2\eta_j\gamma_j+\eta_4\gamma_3]$$
$$=(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)[-\sum_{j=1}^{2}\lambda_j^2\eta_j(\delta-\lambda_j^2\gamma_j)+\eta_4\delta]=(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)\sum_{j=1}^{2}\lambda_j^4\eta_j\gamma_j$$
$$=(\Delta+\lambda_1^2)\{\lambda_2^4\eta_2\delta+\lambda_1^4\eta_1[\delta+(\lambda_2^2-\lambda_1^2)\gamma_1]\}=\lambda_1^4(\lambda_2^2-\lambda_1^2)\eta_1(\Delta+\lambda_1^2)\gamma_1=\delta.$$

Equation $(19)_2$ is proved quite similarly.

We introduce the matrix

$$\Gamma(x)=L(D_x)\gamma(x). \tag{20}$$

Using the identity (20) from Eqs. (15), (18) we get

$$A(D_x)\Gamma(x)=A(D_x)L(D_x)\gamma(x)=\Lambda(\Delta)\gamma(x)=\delta(x)J.$$

Hence $\Gamma(x)$ is a solution to Eq. (3). We have thereby proved the following theorem.

Theorem 1. The matrix $\Gamma(x)$ defined by Eq. (20) is the fundamental solution of system (1).

5. BASIC PROPERTIES

Theorem 1 leads to the following results.

Corollary 1. Each column of the matrix $\Gamma(x)$ is the solution to the system (1) at every point $x\in E^3$ except the origin.

Corollary 2. If condition (3) is satisfied, then the fundamental solution of the system

$$\mu \Delta u + (\lambda + \mu) grad div u = 0, \ m_1 \Delta p_1 = 0, \ m_2 \Delta p_2 = 0$$

is the matrix $\Psi(x) = \| \Psi_{lj}(x) \|_{5\times 5}$, where

$$\Psi^{(1)}(x) = (\frac{1}{a} grad div - \frac{1}{\mu} curl curl) \gamma_4(x), \ \Psi^{(1)}(x) = \| \Psi_{lj}(x) \|_{3\times 3},$$

$$\Psi_{qq}(x) = \frac{1}{m_{q-3}} \gamma_3(x), \ \Psi_{ql}(x) = \Psi_{lq}(x) = \Psi_{45}(x) = \Psi_{54}(x) = 0, \ l = 1,2,3, \ q = 4,5.$$

Theorem 2. The relations

a) $|\Gamma_{pq}(x)| < const |x|^{-1}$, $|\Gamma_{44}(x)| < const |x|^{-1}$, $|\Gamma_{55}(x)| < const |x|^{-1}$;
b) $|\Gamma_{4p}(x)| < const$, $|\Gamma_{p4}(x)| < const$, $|\Gamma_{5l}(x)| < const$, $|\Gamma_{l5}(x)| < const$;
c) $\Gamma_{lj}(x) - \Psi_{lj}(x) = const + O(|x|)$;
d) $\dfrac{\partial^s}{\partial x_1^{s_1} \partial x_2^{s_2} \partial x_3^{s_3}} [\Gamma_{lj}(x) - \Psi_{lj}(x)] = O(|x|^{1-s})$,

hold in the neighborhood of the origin, where $s = s_1 + s_2 + s_3$, $s > 1$, $l, j = 1,2,3,4$, $p, q = 1,2,3$.

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