# Static Shakedown Theorem Accounting for Material Damage and Creep

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## 1. SUMMARY

Conditions for shakedown of structures (solids) of elastic plastic creeping materials with anisotropic plastic damage, strain hardening and softening, as subjected to thermo mechanical loading, are in question. The current yield surfaces are assumed to enclose the one of the minimal diameter (*m*-surface). This is valid at the stage of strain hardening, and also at the part of the softening stage. The shakedown conditions are of the classical (Melan) type; however the residual stress may depend on time. The sufficient condition is formulated relative to the *m*-surface.

## 2. INTRODUCTION

In this paper the model of elastic plastic creeping *damage* material is accepted. The argumentation is based on the assumption that the yield surface in the effective stress space is closed, and on the dissipative inequality $\boldsymbol{\sigma}:\mathbf{L}:\boldsymbol{\sigma}>0$ where $\boldsymbol{\sigma}$ denotes the current nominal stress, and $\mathbf{L}$ does the current value of the elastic compliance tensor. It is also assumed that the current yield surfaces enclose the one with minimal diameter (*m*-surface). This assumption is valid both at the stage of material strain hardening, and also at the part of material softening stage.

The paper provides necessary and sufficient conditions for shakedown. The conditions are of the classic static (Melan) condition type; however, unlike it, the virtual residual stress may be time dependent, and also the sufficient shakedown condition relates to the *m*-surface.

*Notations*:
$\mathbf{a}:\mathbf{b}=a_{ij}b_{ij}$, $(\mathbf{A}:\mathbf{B})_{ijkl}=A_{ijmn}B_{mnkl}$, the fourth rank unit tensor is denoted $\mathbf{I}$.

## 3. MATERIAL MODEL

A model of elastic plastic creeping material with limited strain hardening and anisotropic plastic damage was taken. It assimilated some features of the models by Lemaitre /1/ and Ju /2/. The consideration is restricted by the mechanical theory. Inertia forces, thermal fluxes, and changes in temperature due to thermal emission are neglected. The temperature $\theta(\mathbf{x},t)$ is considered as a given function of time and position.

The model is not specified in detail, because only its general features are needed for the subsequent argumentation.

The deformation is assumed small, so that the total strain tensor can be decomposed into elastic $\boldsymbol{\varepsilon}^e$, plastic $\boldsymbol{\varepsilon}^p$, creep $\boldsymbol{\varepsilon}^c$, and thermal (prescribed) $\boldsymbol{\varepsilon}^\theta$ parts.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^c + \boldsymbol{\varepsilon}^\theta \tag{1}$$

Consequently the total strain rate is represented as

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p + \dot{\boldsymbol{\varepsilon}}^c + \dot{\boldsymbol{\varepsilon}}^\theta . \tag{2}$$

### Damage

It is assumed that the damage process is coupled with the process of plastic deformation: it can develop only, if the plastic deformation process is in progress, i.e. the nature of damage is assumed ductile. Due to this assumption, the growth of damage ceases simultaneously with the cessation of plastic deformation. Consequently, damage variables are bounded, if the plastic deformation ceases at some time.

Let $\mathbf{L}$ denote the current (damaged) value of the elastic compliance tensor with its ordinary symmetry, and $\mathbf{C}=\mathbf{L}^{-1}$ denote the corresponding stiffness tensor. Either of these tensors can be taken as the damage variable.

No residual strain and stress are assumed induced by damage.

### Plasticity

The current (damaged) value of the elastic stiffness tensor $\mathbf{C}$ is defined through the fourth rank transformation tensor $\mathbf{M}(\mathbf{x},t)$, as $\mathbf{C}=\mathbf{M}:\mathbf{C}_o$, where $\mathbf{C}_o$ denotes the initial value of $\mathbf{C}$ corresponding to the initial time point of deformation process $t_o$. Analogously $\mathbf{L}=\mathbf{M}^{-1}:\mathbf{L}_o$. The transformation tensor can be also taken for damage variable.

The solid could experience a plastic deformation and damaging before the beginning of deformation process. Therefore it is assumed that $\chi=\chi_o(\mathbf{x})$ and $\mathbf{M}=\mathbf{M}_o(\mathbf{x})$ at $t=t_o$.

The effective stress tensor is defined as $\bar{\boldsymbol{\sigma}} = \mathbf{M}^{-1}:\boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ denotes the nominal stress tensor.

The yield condition is formulated in terms of the nominal stress tensor

$$\Phi(\boldsymbol{\sigma},\mathbf{M},\chi,\theta)=0 \tag{3}$$

where the components of the transformation tensor **M** should be considered as internal variables similar to the strain hardening parameter. This approach is in line with the formulation of some elastic plastic damage material models. See, for example, /1/.

Hardening is assumed limited: $0<\chi<\chi^*$ where $\chi^*$ is the material parameter corresponding to the hardening saturation state of undamaged material.

The yield function $\Phi$ is assumed strictly convex in the argument $\boldsymbol{\sigma}$. It is chosen in such a way that the inequality $\Phi<0$ should correspond to the interior of the yield surface in the stress space $\boldsymbol{\sigma}$, and the point $\boldsymbol{\sigma}=0$ be in the interior. Consequently, if $\Phi(\boldsymbol{\sigma},\mathbf{M},\chi,\theta)=0$, and $\Phi(\hat{\boldsymbol{\sigma}},\mathbf{M},\chi,\theta)<0$, then

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\dot{\boldsymbol{\varepsilon}}^{p} \geq 0 \tag{4}$$

where equality holds only, if $\dot{\boldsymbol{\varepsilon}}^{p}=0$. It is important that (4) is valid, if $\hat{\boldsymbol{\sigma}}$ is in the interior of the initial yield surface.

The mechanical unloading and subsequent reloading process are assumed purely elastic. Therefore during these processes no damaging and hardening occur, and the elastic module and the transformation tensor save their current (damaged) values, which they had at the start of unloading. Consequently the stress tensor can be decomposed as

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^{E}+\boldsymbol{\sigma}^{r} \tag{5}$$

where $\boldsymbol{\sigma}^{E}(\mathbf{x},t)$ represents the pure elastic response of the body under consideration to the current mechanical boundary conditions, and $\boldsymbol{\sigma}^{r}(\mathbf{x},t)$ does the current tensor of residual stress. Thus, the following decomposition of the actual strain tensor is valid

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^{E}+\boldsymbol{\varepsilon}^{r}+\boldsymbol{\varepsilon}^{\theta}=\boldsymbol{\varepsilon}^{E}+\boldsymbol{\varepsilon}^{re}+\boldsymbol{\varepsilon}^{c}+\boldsymbol{\varepsilon}^{p}+\boldsymbol{\varepsilon}^{\theta} \tag{6}$$

where $\boldsymbol{\varepsilon}^{E}$ is the elastic strain corresponding to $\boldsymbol{\sigma}^{E}$, namely, $\boldsymbol{\varepsilon}^{E}=\mathbf{L}:\boldsymbol{\sigma}^{E}$, and $\boldsymbol{\varepsilon}^{re}$ is the elastic part of the residual strain tensor: $\boldsymbol{\varepsilon}^{re}=\mathbf{L}:\boldsymbol{\sigma}^{r}$. Symbol $\boldsymbol{\varepsilon}^{r}$ denotes the total residual strain: $\boldsymbol{\varepsilon}^{r}=\boldsymbol{\varepsilon}^{re}+\boldsymbol{\varepsilon}^{c}+\boldsymbol{\varepsilon}^{p}$.

## Creep

Creep strain rate is defined by the creep flow rule:

$$\dot{\varepsilon}^{c} = \frac{\partial\, F(\mathbf{M},\sigma,\chi,\theta)}{\partial\, \sigma} \tag{7}$$

where the potential function $F(\mathbf{M}, \sigma,\chi,\theta)$ is assumed even, homogeneous and convex in $\boldsymbol{\sigma}$.

## Thermal (prescribed) strain $\boldsymbol{\varepsilon}^{\theta}$.

This could be an initial strain, otherwise it can be originated by thermal stress. Because thermal fluxes, and emission of heat due to deformation are neglected, the thermal strain is defined by the temperature field $\theta(\mathbf{x},t)$, which is considered as given.

## Thermodynamics

The accumulated plastic deformation $\chi$ is taken for the isotropic strain hardening parameter: $\dot{\chi} = (\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}})^{1/2}$.

The local damaged Helmholtz free energy function is formulated as

$$\Psi(\boldsymbol{\varepsilon}^{e},\chi,\mathbf{C}) \equiv \frac{1}{2}(\boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{\theta}) : \mathbf{C} : (\boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{\theta}) + \Psi_{p}(\chi) \tag{8}$$

where the first term represents the reversible part of the free energy, and the term $\Psi_p(\chi)$ does the free energy stored at the micro level due to strain hardening. As in Lemaitre /1/, the effect of damaging on this part of free energy is neglected. The kinematic strain hardening is here not considered. However it can be taken into account by the method proposed in /3/.

The Clausius-Duhem inequality $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} > 0$ should be valid for any mechanical process. Employing the Coleman & Gurtin arguments /4/, we arrive at the elastic strain-stress relation

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{\theta}) \tag{9}$$

and the dissipative inequalities

$$\boldsymbol{\sigma} : (\dot{\varepsilon}^{p} + \dot{\varepsilon}^{c}) - \frac{\partial \Psi_{p}}{\partial \chi} \dot{\chi} \geq 0 \tag{10}$$

and

$$(\boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{\theta}) : \dot{\mathbf{C}} : (\boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{\theta}) < 0. \tag{11}$$

Inequality (11) is valid, if the quadratic function $(\varepsilon^e+\varepsilon^\theta):\dot{\mathbf{C}}:(\varepsilon^e+\varepsilon^\theta)$ is not positive. Thus, all eigenvalues of the matrix $\dot{\mathbf{C}}$ are to be not positive.

As $\mathbf{L}:\mathbf{C}=\mathbf{I}$, then $\dot{\mathbf{C}} = -\mathbf{C}:\dot{\mathbf{L}}:\mathbf{C}$, and $(\varepsilon^e+\varepsilon^\theta):\ \dot{\mathbf{C}}:(\varepsilon^e+\varepsilon^\theta) = -\sigma:\dot{\mathbf{L}}:\sigma$. Consequently

$$\sigma:\dot{\mathbf{L}}:\sigma > 0, \tag{12}$$

i.e. the eigenvalues of the quadratic function $\sigma:\dot{\mathbf{L}}:\sigma$ have to be not negative. Inequalities (11), (12) are transformed into equalities only in the absence of damaging. In the case of isotropic damage inequality (12) results in the known conclusion that the rate of damage parameter is non-negative /1/.

## 4. PRELIMINARIES

### Supposition

During deformation process the yield surface at a point of the solid changes its diameter. Strain hardening increases the diameter; on the contrary damage process decreases it. If $\theta$=const through the solid, the current yield surfaces enclose the initial one. This assertion is valid until the ultimate stress point (USP) is reached. Moreover it is also fulfilled for some part of the unstable stage adjacent to USP.

The length of the stage where the assertion is valid depends on the deformation process, and can be obtained by detailed computing. In the case of isotropic damage it can be estimated directly /3/. This method can be extended to anisotropic damage.

However, increase in temperature reduces the diameter. Therefore it is possible that there exist yield surfaces with diameters less than the initial one.

According to our assumption, the temperature does not depend on the deformation process. Therefore the maximal temperature at a solid point and also the yield surface with the minimal diameter can be found in advance. It could be either the initial yield surface, or the yield surface corresponding to the maximal temperature. This surface will be referred to as the *minimal* yield surface (*m*-yield surface).

The proof of the theorem is based on a Supposition that the current yield surfaces enclose the *min* yield surface. Due to the Supposition, the inequality $\Phi(\sigma(\mathbf{x},t),\mathbf{M}(\mathbf{x},t),\chi(\mathbf{x},t),\theta(\mathbf{x},t))<0$ is valid for any $t>t_o$, if it is valid for the *min* yield surface: $\Phi(\sigma(\mathbf{x},t),\mathbf{M}(\mathbf{x},t_m),\chi(\mathbf{x},t_m),\theta(\mathbf{x},t_m))<0$ for $t>t_o$ where $t_m$, denotes the time point corresponding to *m*-yield surface. That is, if a stress $\hat{\sigma}$ is safe with respect to the *min* yield surface, then it is safe with respect to any current yield one, until the Supposition is valid.

Obviously the notion of *m*-yield surface is similar to the notion of sanctuary /5/.

### Assumption of the closeness the yield surfaces

It is assumed that the yield surface $\bar{\Phi}$ in the effective stress space $\bar{\sigma}$ is closed for any admissible values of its arguments, so that the values of effective stress tensor components are uniformly bounded by a constant B: $|\bar{\sigma}_{ij}| \le B$. Obviously the above assumption does not diminish the generality of the present consideration.

## The notion of shakedown

An elastic plastic creeping damaged structure (solid) subjected to cyclic loading is considered. The term of shakedown (adaptation to cyclic loading), as applied to the taken material model, implies that after a transient (adaptation) period of deformation, the structure achieves a stationary state, during which the processes of plastic deformation and damage are absent, the elastic modules are constant, and the stress state becomes cyclic. This state is defined as the steady cyclic creep state /6/.

## Necessary condition for shakedown

The opposite is true if an elastic plastic creeping damage solid subjected to cyclic loading shakes down, then eventually, after some transient period of deformation, the solid achieves a stationary stage of the deformation process, during which it experiences only elastic and creep deformations. In this case there exists a residual stress $\hat{\boldsymbol{\sigma}}^r(x,t)$ such that for any $t>t_s$, the stress $\hat{\sigma}(\mathbf{x},t) = \sigma^E(\mathbf{x},t) + \hat{\sigma}^r(\mathbf{x},t)$ satisfies the inequality $\Phi(\hat{\boldsymbol{\sigma}}(\mathbf{x},t), \mathbf{M}(\mathbf{x},t), \chi(\mathbf{x},t), \theta(\mathbf{x},t)) < 0$, where $t_s$ is the time corresponding to the beginning of the stationary stage. Thus, the necessary condition for shakedown can be formulated as: if the structure shakes down, then there exist a residual stress $\hat{\boldsymbol{\sigma}}^r(\mathbf{x},t)$ and a yield condition, for which the stress $\hat{\boldsymbol{\sigma}}$ is safe.

# 5. STATIC SHAKEDOWN THEOREM

## Formulation of the theorem

The proposed shakedown theorem can be formulated as follows: if there exists a field of a virtual residual stress $\hat{\boldsymbol{\sigma}}^r(\mathbf{x},t)$, such that the stress $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^E(\mathbf{x},t) + \hat{\boldsymbol{\sigma}}^r(\mathbf{x},t)$ is safe with respect to the *min* yield condition, i.e. satisfies the inequality $\Phi(\hat{\boldsymbol{\sigma}}(\mathbf{x},t), \mathbf{M}(\mathbf{x},t_0), \chi(\mathbf{x},t_0), \theta(\mathbf{x},t)) < 0$ starting from some time $t_o$ on, then the structure under consideration will shake down.

As distinct from the classic shakedown theorem, the proposed one admits time dependence of the virtual residual stress.

The theorem provides a sufficient condition for shakedown.

## Proof of the theorem

To prove the theorem one needs to consider the potential energy corresponding to the difference in actual and virtual residual stresses:

$$W=\frac{1}{2}\int_Q\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right):\mathbf{L}:\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right)dQ \tag{13}$$

where $\boldsymbol{\sigma}^r$ is the actual residual stress, and Q is the volume occupied by the solid.

The derivative of W with respect to time is equal to

$$\dot{W}=\int_Q\left(\dot{\boldsymbol{\sigma}}^r-\dot{\hat{\boldsymbol{\sigma}}}^r\right):\mathbf{L}:\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right)dQ+\frac{1}{2}\int_Q\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right):\dot{\mathbf{L}}:\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right)dQ\,. \tag{14}$$

As $\boldsymbol{\varepsilon}^{re}=\mathbf{L}:\boldsymbol{\sigma}^r$, then $\mathbf{L}:\dot{\boldsymbol{\sigma}}^r=\dot{\boldsymbol{\varepsilon}}^{re}-\dot{\mathbf{L}}:\boldsymbol{\sigma}^r$. Analogously $\mathbf{L}:\dot{\hat{\boldsymbol{\sigma}}}^r=\dot{\hat{\boldsymbol{\varepsilon}}}^{re}-\dot{\mathbf{L}}:\hat{\boldsymbol{\sigma}}^r$ where $\hat{\boldsymbol{\varepsilon}}^{re}=\mathbf{L}:\hat{\boldsymbol{\sigma}}^r$. Having these equalities in hand it is possible to shape (14) into the form

$$\dot{W}=\int_Q\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right):(\dot{\boldsymbol{\varepsilon}}^{re}-\dot{\hat{\boldsymbol{\varepsilon}}}^{re})dQ-\frac{1}{2}\int_Q\boldsymbol{\sigma}^r:\dot{\mathbf{L}}:\boldsymbol{\sigma}^r\,dQ+\frac{1}{2}\int_Q\hat{\boldsymbol{\sigma}}^r:\dot{\mathbf{L}}:\hat{\boldsymbol{\sigma}}^r dQ\,. \tag{15}$$

It results from (11) that $\dot{\boldsymbol{\varepsilon}}=\dot{\boldsymbol{\varepsilon}}^E+\dot{\boldsymbol{\varepsilon}}^{re}+\dot{\boldsymbol{\varepsilon}}^P+\dot{\boldsymbol{\varepsilon}}^c+\dot{\boldsymbol{\varepsilon}}^\theta$. As the stress $\hat{\boldsymbol{\sigma}}=\boldsymbol{\sigma}^E+\hat{\boldsymbol{\sigma}}^r$ is safe, then $\dot{\hat{\boldsymbol{\varepsilon}}}^P=0$, and $\dot{\hat{\boldsymbol{\varepsilon}}}=\dot{\boldsymbol{\varepsilon}}^E+\dot{\hat{\boldsymbol{\varepsilon}}}^{re}+\dot{\hat{\boldsymbol{\varepsilon}}}^c+\dot{\boldsymbol{\varepsilon}}^\theta$. Hence $\dot{\boldsymbol{\varepsilon}}^{re}-\dot{\hat{\boldsymbol{\varepsilon}}}^{re}=(\dot{\boldsymbol{\varepsilon}}-\dot{\hat{\boldsymbol{\varepsilon}}})-\dot{\boldsymbol{\varepsilon}}^P-(\dot{\boldsymbol{\varepsilon}}^c-\dot{\hat{\boldsymbol{\varepsilon}}}^c)$. Furthermore $\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r=\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}$

Because the strains $\boldsymbol{\varepsilon}$, $\hat{\boldsymbol{\varepsilon}}$ and, consequently, the strain rates $\dot{\boldsymbol{\varepsilon}}$, $\dot{\hat{\boldsymbol{\varepsilon}}}$ are kinematically compatible, and the difference $\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}$ is self-equilibrated, and meets zero boundary conditions, $\dot{W}$ can be reshaped as

$$\dot{W}=-\int_Q(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\left(\dot{\boldsymbol{\varepsilon}}^P+(\dot{\boldsymbol{\varepsilon}}^c-\dot{\hat{\boldsymbol{\varepsilon}}}^c)\right)dQ-\frac{1}{2}\int_Q\boldsymbol{\sigma}^r:\dot{\mathbf{L}}:\boldsymbol{\sigma}^r\,dQ+\frac{1}{2}\int_Q\hat{\boldsymbol{\sigma}}^r:\dot{\mathbf{L}}:\hat{\boldsymbol{\sigma}}^r dQ\,. \tag{16}$$

Due to the assumed convexity of the function F($\boldsymbol{\sigma}$,**M**,θ) in $\boldsymbol{\sigma}$, the following inequality is valid

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}):\left(\dot{\boldsymbol{\varepsilon}}^c-\dot{\hat{\boldsymbol{\varepsilon}}}^c\right)\geq 0 \tag{17}$$

Consequently

$$\dot{V}=\dot{W}-\frac{1}{2}\int_Q\hat{\boldsymbol{\sigma}}^r:\dot{\mathbf{L}}:\hat{\boldsymbol{\sigma}}^r dQ<-\int_Q\left(\boldsymbol{\sigma}^r-\hat{\boldsymbol{\sigma}}^r\right):\dot{\boldsymbol{\varepsilon}}^P\,dQ-\frac{1}{2}\int_Q\boldsymbol{\sigma}^r:\dot{\mathbf{L}}:\boldsymbol{\sigma}^r\,dQ \tag{18}$$

where

$$V = W - \frac{1}{2}\int_{t_o}^{T} dt \int_Q \hat{\sigma}^r : \dot{L} : \hat{\sigma}^r dQ = W - \frac{1}{2}\int_Q \hat{\sigma}_*^r : (L - L_o) : \hat{\sigma}_*^r dQ. \tag{19}$$

Above $\hat{\sigma}_*^r$ denotes an average value of $\hat{\sigma}^r$ in the interval $[t_o, T]$, and $\mathbf{L}_o$ denotes the value of $\mathbf{L}$ at $t=t_o$.

According to (4), the right part of (18) is non positive. Hence the left part is non positive as well.

Suppose the structure does not shake down, i.e. $|\dot{\varepsilon}^p_{ij}|>a>0$ for any t, where $a$ is a positive number. In this case inequalities (4) and (12) are strict. Therefore the right side of inequality (18) is strictly negative, and $\dot{V} < -b < 0$ where $b$ is a positive number. Hence $|V(t)| \to \infty$ for $T \to \infty$.

However the magnitude of V(t) is bounded. Actually as $\mathbf{L}=\mathbf{M}^{-1}:\mathbf{L}_o$, this quantity can be estimated as

$$|V(t)| \le \frac{1}{2}\int_Q \left(\sigma^r - \hat{\sigma}^r\right) : \mathbf{M}^{-1} : \mathbf{L}_o : \left(\sigma^r - \hat{\sigma}^r\right) dQ + \frac{1}{2}\int_Q \hat{\sigma}_*^r : (\mathbf{M}^{-1} - \mathbf{I}) : \mathbf{L}_o : \hat{\sigma}_*^r dQ. \tag{20}$$

If the transformations $\sigma = \mathbf{M}^{-1} : \sigma$ and its inverse are non-degenerate, then components of $\mathbf{M}^{-1}$ are bounded. According to the assumption, the yield surface in the effective stress space, $\Phi(\sigma, \chi) = 0$, is closed. Therefore the yield surface in the nominal stress space, $\Phi(\sigma, \mathbf{M}, \chi) = 0$, is closed as well, and the nominal stress tensor components are bounded. As $\hat{\sigma}^r$ is safe, then $\hat{\sigma}_*^r$ is safe also. Hence the components of $\hat{\sigma}_*^r$ are also bounded by the constant B irrespective of value T; and the magnitude of V(t) is bounded for any value of T. Thus, if the transformations $\bar{\sigma} = \mathbf{M}^{-1} : \sigma$ and its inverse are non-degenerate, the supposition $|\dot{\varepsilon}^p_{ij}|>a>0$ leads to contradiction, and should be declined. Consequently $|\dot{\varepsilon}^p_{ij}| \to 0$, which proves the theorem.

Now consider the case where a component of $\mathbf{M}^{-1}(t)$ grows unlimitedly in its magnitude for $T \to \infty$. Then, as $\mathbf{L}=\mathbf{M}^{-1}:\mathbf{L}_o$, some of the $\mathbf{L}$ components tend to infinity. This means that eventually the structure cannot bear the applied loads. This situation should be interpreted as the damage collapse.

The objective of this investigation is to formulate the conditions for shakedown, i.e. for plastic safety of the structure. The problem of damage safety is out of the framework of the paper.

The question of the failure due to damage accumulation in the case of isotropic damage was considered in /3/. The method developed in there can be applied to the model under consideration.

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