On Axisymmetric Multiphase Deformations for Incompressible Elastic Bodies

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1. SUMMARY

In this paper we consider two-phase deformations for incompressible, isotropic elastic bodies in the presence of homogeneous tractions on the boundary. With reference to a trilinear form of the energy function, we discuss in detail the case of pairwise deformations corresponding to an axisymmetric distribution of boundary tractions.

2. INTRODUCTION

The present note deals with the analysis of multiphase deformations for incompressible, isotropic elastic bodies subjected to a homogeneous distribution of dead-load tractions. Within the framework of the non-linear theory of elasticity, the description of multiphase solutions, characterized by deformations with gradient fields discontinuous across a number of surfaces, has been obtained by removing the classical constitutive restrictions based on convexity assumptions for the elastic energy function.

With reference to the case of incompressible elastic materials, equilibrium problems with non-convex energy functions have been widely studied by Fosdick and other authors in a number of papers. Here, we recall the analyses of multiphase deformations in the helical shear /1/ and in the anti-plane shear problems for an elastic tube /2/ and the occurrence of microstructures in the torsional deformations of an elastic cylinder /3-4/. Moreover, in the case of plane deformations, existence conditions for multiphase deformations have been obtained by Knowles & Sternberg /5/ and Abeyaratne & Knowles /6/. Finally, in our preceding paper /7/ stable multiphase plane deformations of an incompressible isotropic body held in equilibrium by a biaxial dead-loading have been determined.

In this paper, in order to characterize multiphase deformations in the presence of homogeneous tractions on the boundary, we consider pairwise deformations for which the Piola stress tensors in the two phases are identical. In particular, we show that these deformations are possible only if the load tensor has at least two opposite forces. Moreover, with reference to a "trilinear" form of the energy function, we discuss in detail an explicit case of pairwise deformations which correspond to an axisymmetric load tensor.

3. PAIRWISE DEFORMATIONS UNDER DEAD-LOAD TRACTIONS

In this section we briefly describe the case of pairwise deformations for a three-dimensional elastic isotropic incompressible body Ω. A deformation of Ω is a function $\mathbf{f}:\Omega \to \mathbb{R}^3$ whose gradient field $\mathbf{F}: = \nabla\mathbf{f}$ must satisfy the incompressibility condition

$$\det \mathbf{F} = 1 \text{ a.e. in } \Omega. \tag{1}$$

For an isotropic incompressible elastic material, the stored energy density σ depends on the first and second invariants $I_{\mathbf{B}}$, $II_{\mathbf{B}}$ of the left Cauchy-Green strain tensor $\mathbf{B} := \mathbf{F}\mathbf{F}^{T}$ (cf., e.g., /8, Section 49/):

$$\sigma = \hat{\sigma}\left(I_{\mathbf{B}}, II_{\mathbf{B}}\right), \tag{2}$$

and the Cauchy stress $\mathbf{T}$ is determined by $\mathbf{B}$ within an arbitrary pressure p:

$$\mathbf{T} = -\,\mathrm{p}\,\mathbf{I} + \alpha\,\mathbf{B} + \beta\,\mathbf{B}^{-1}, \tag{3}$$

where the response coefficients α and β are defined by

$$\alpha: = 2\frac{\partial\hat{\sigma}}{\partial I_{\mathbf{B}}}, \; \beta: = -2\frac{\partial\hat{\sigma}}{\partial II_{\mathbf{B}}}. \tag{4}$$

The constitutive equation (3) can be also expressed in terms of the principal stretches $\lambda_i > 0$ and the principal Cauchy stresses t_i:

$$t_i = -\,p + \alpha\,\lambda_i^2 + \beta\,\lambda_i^{-2}, \; i = 1, 2, 3. \tag{5}$$

We assume the material obeys the Baker-Ericksen inequalities (see, e.g., /8, Section 53/):

$$\begin{aligned} &\alpha - \lambda_i^2\,\beta > 0 \qquad \text{if } \lambda_j \neq \lambda_k \\ &\alpha - \lambda_i^2\,\beta \geq 0 \qquad \text{if } \lambda_j = \lambda_k \;, \end{aligned} \tag{6}$$

where (i, j, k) are any permutation of $(1, 2, 3)$.

In the present paper we consider two-phase homogeneous deformations for which **f** is continuous on Ω whereas **F** suffers a jump discontinuity across a (flat) surface $\mathcal{S} \subset \Omega$ and is constant on each of the two parts of Ω separated by $\mathcal{S}$. Because of the constraint of incompressibility (1), the Hadamard's compatibility condition takes the form (see, e.g., /9/)

$$\mathbf{F}_2 = (\mathbf{I} + \gamma\, \mathbf{b} \otimes \mathbf{n})\, \mathbf{F}_1, \; |\mathbf{n}| = |\mathbf{b}| = 1, \; \mathbf{b} \cdot \mathbf{n} = 0, \; \gamma \in \mathbb{R}, \tag{7}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the two values of **F** on Ω, **n** is the unit normal to the deformed interface $\mathbf{f}(\mathcal{S})$ and $\gamma\,\mathbf{b}$ is the amplitude of the jump. Notice that the phase $\mathbf{F}_2$ can be obtained by superimposing upon $\mathbf{F}_1$ a simple shear of amount γ and axis **b**, which is tangent to the discontinuity surface $\mathbf{f}(\mathcal{S})$.

Here, we focus our attention on the special case of pairwise homogeneous deformations $\mathbf{F}_1$, $\mathbf{F}_2$ which satisfy the condition

$$\mathbf{S}_1 = \mathbf{S}_2 = :\tilde{\mathbf{S}}, \tag{8}$$

where $\mathbf{S}_i = \mathbf{T}_i\mathbf{F}_i^{-T}$ (i = 1, 2) denote the Piola stress tensors corresponding to the phases $\mathbf{F}_1, \mathbf{F}_2$. In the following, a pair of unimodular tensors $\mathbf{F}_1, \mathbf{F}_2$ satisfying the conditions (7) and (8) is called a pairwise deformation with continuous Piola stress.

Remark 1. Pairwise homogeneous deformations satisfying (7), (8) play a fundamental role in various situations of physical interest. In particular, the analysis of such deformations arises in the determination of multiphase deformations for bodies subjected on the boundary $\partial\Omega$ to a homogeneous distribution of surface tractions

$$\hat{\mathbf{s}}(\mathbf{X}) = \tilde{\mathbf{S}}\, \mathbf{n}_0(\mathbf{X}) \quad \forall\; \mathbf{X} \in \partial\Omega, \tag{9}$$

where $\mathbf{n}_0(\mathbf{X})$ is the unit normal to $\partial\Omega$ at $\mathbf{X}$ (see, e.g., /7/).

A general analysis of pairwise deformations with continuous Piola stress is contained in our previous work /10/, where necessary conditions for the existence of this type of deformations are obtained. In particular, in order to state such results, we consider the polar decomposition $\mathbf{F}_1 = \mathbf{V}_1\mathbf{R}_1$, with $\mathbf{V}_1 = \mathbf{B}_1^{1/2}$ and $\mathbf{R}_1 \in \mathrm{Rot}$, and denote by $\bar{\mathbf{S}}$ the Piola stress in the configuration rotated by $\mathbf{R}_1$ with respect to the reference configuration:

$$\bar{\mathbf{S}} = \tilde{\mathbf{S}}\mathbf{R}_1^T = \mathbf{T}_1\mathbf{V}_1^{-1}. \tag{10}$$

Then, the following propositions hold (cf. /10/).

Proposition 1. *Pairwise deformations with continuous Piola stress are possible only if the symmetric tensor* $\bar{\mathbf{S}}$ *has at least two opposite principal values; furthermore, the normal* $\boldsymbol{n}$ *to the discontinuity surface is contained in a plane spanned by two principal directions of* $\bar{\mathbf{S}}$ *to which correspond opposite forces.*

Proposition 2. *For a material whose stored energy density depends only on the first invariant* $I_{\mathbf{B}}$ *(i. e.,* $\sigma = \hat{\sigma}\,(I_{\mathbf{B}})$*), pairwise deformations with continuous Piola stress are possible only if the tensors* $\bar{\mathbf{S}}$ *and* $\mathbf{B}_1$ *correspond to stress and strain states contained in the plane* $(\mathbf{b}, \mathbf{n})$.

4. AXISYMMETRIC PAIRWISE DEFORMATIONS

In this section we illustrate the preceding results by considering an explicit example of pairwise deformations which correspond to an axisymmetric Piola stress. In particular, we assume that the tensor $\bar{\mathbf{S}}$ is defined by loads which are uniform in the plane perpendicular to the unit vector $\mathbf{e}_3$ and opposite to the load in the direction $\mathbf{e}_3$:

$$\bar{\mathbf{S}} = \mathrm{s}\,\left(2\mathbf{e}_3 \otimes \mathbf{e}_3 - \mathbf{I}\right), \tag{11}$$

where s is a constant load parameter. Furthermore, we choose the following special form for the strain energy density σ:

$$\sigma(\mathbf{F}) = \bar{\sigma}\,(\kappa) + \mathrm{c}\,\left(II_{\mathbf{B}} - 3\right), \tag{12}$$

where $\mathrm{c} \geq 0$ is a material parameter, $\kappa := \sqrt{I_{\mathbf{B}} - 3}$ and $\bar{\sigma}\colon [0,\infty[\, \ni \kappa \mapsto \bar{\sigma}(\kappa)$ is the non-convex and strictly increasing function whose derivative $\bar{\sigma}'$ takes the following *trilinear* form (see Fig. 1):

$$\bar{\sigma}'(\kappa) = \begin{cases} \alpha_1\,\kappa & 0 < \kappa \leq \kappa_1 \\ \dfrac{\alpha_2\kappa_2 - \alpha_1\kappa_1}{\kappa_2 - \kappa_1}\,\kappa + \dfrac{(\alpha_1 - \alpha_2)\,\kappa_1\kappa_2}{\kappa_2 - \kappa_1} & \kappa_1 \leq \kappa \leq \kappa_2 \\ \alpha_2\,\kappa & \kappa \geq \kappa_2 \end{cases}. \tag{13}$$

Notice that the density function (12) is consistent with a class of constitutive models used in biomechanics literature to describe the large elastic deformations of biological tissues (see /11/).

Remark 2. It is possible to show that, under the above constitutive assumptions, the strain energy σ defined by (12) is not rank-one convex as a function of the deformation gradient **F** if the material coefficient $c \in \left[0, \dfrac{\alpha_1 \kappa_1 - \alpha_2 \kappa_2}{2\,(\kappa_2 - \kappa_1)}\right]$.

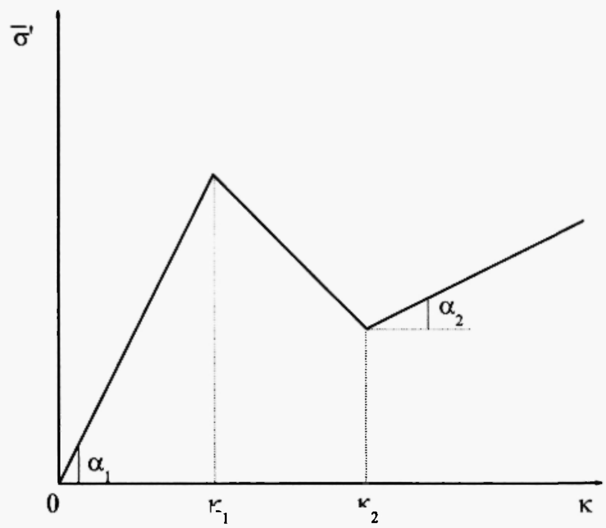

Fig. 1: The trilinear function $\bar{\sigma}'$

As a consequence of (12) and (13), the response coefficients α and β defined by (4) take the form

$$\alpha = \frac{\bar{\sigma}'(\kappa)}{\kappa}\ , \ \beta = -2\,c\,. \tag{14}$$

In particular, the response coefficient α takes the values α_1 and α_2 for each increasing branch of $\bar{\sigma}'$.

We assume that the phase $\mathbf{F}_1$ is axisymmetric and corresponds to the left increasing branch of $\bar{\sigma}'$ (see Fig. 1):

$$\mathbf{F}_1 = \lambda\ e_3 \otimes e_3 + \lambda^{-1/2}\left(\mathbf{I} - e_3 \otimes e_3\right), \tag{15}$$

where, in view of (3), (4), (10), the principal stretch $\lambda(>0)$ is the unique solution of the following equation:

$$s = \left(\alpha_1 \lambda + 2\mathbf{C}\right)\left(1 - \lambda^{-3/2}\right)\ . \tag{16}$$

Notice that we have set the rotation $\mathbf{R}_1 = \mathbf{I}$, so that $\bar{\mathbf{S}} = \tilde{\mathbf{S}}$ and the necessary condition for the existence of pairwise deformations given by Proposition 1 is satisfied.

Now, the values of γ and **n** which completely define the phase $\mathbf{F}_2$ (see (7)) have to be determined by solving the tensorial equation (8). Referring the reader to /12/ for details about the calculation of γ and **n**, here we give their expressions in terms of the material parameters α_1, α_2, c:

$$\lambda\ (\mathbf{n}\cdot\mathbf{e}_3)^2 = \lambda^{-1/2}\left[1-(\mathbf{n}\cdot\mathbf{e}_3)^2\right],$$
$$\gamma = \lambda^{1/4}\left(1-\lambda^{3/2}\right)(\alpha_2-\alpha_1)\ (\alpha_2\lambda+2c)^{-1}, \tag{17}$$

with $\mathbf{F}_2$ corresponding to the right increasing branch of $\bar{\sigma}'$ (see Fig. 1). It is worth noting that the orientation of the normal $\mathbf{n}$ to the deformed interface $\mathbf{f}(\mathcal{S})$ (see $(17)_1$) is determined only by the principal stretch λ. Finally, we have that pairwise deformations $\mathbf{F}_1 / \mathbf{F}_2$ are possible if the following condition is also satisfied:

$$(\alpha_2\lambda+2c)\left(\alpha_2\lambda+2c\lambda^{3/2}\right)+2c\left(\lambda^{3/2}-1\right)(\alpha_1\lambda+2c)=0. \tag{18}$$

In order to discuss some features of pairwise deformations under examination, we consider now a family of hyperelastic materials depending on the material parameter c and characterized by $\alpha_1 = 2$, $\alpha_2 = 1/2$. Then, by (18) and $(17)_2$, we can determine the stretch λ and the amount γ of the superimposed simple shear deformations among the phases as functions of c (see Figures 2, 3).

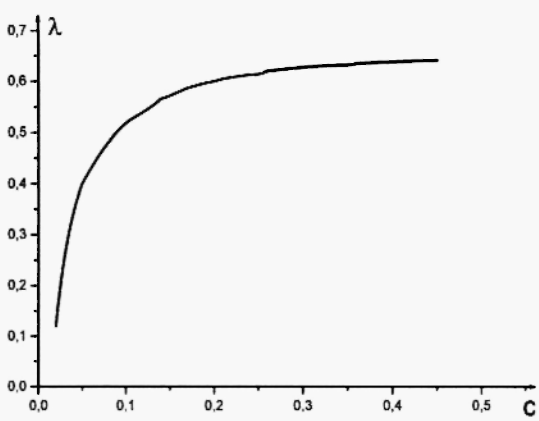

Fig. 2: The stretch λ as a function of c

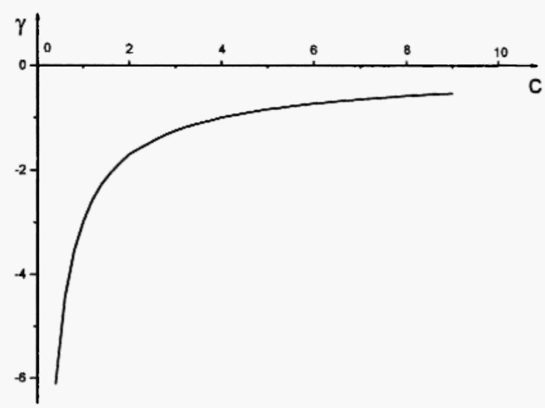

Fig. 3: The shear amount γ as a function of c

Interestingly, Figure 2 shows that if c goes to zero (so that σ becomes independent on $II_{\mathbf{B}}$) the stretch λ also goes to zero; that is, pairwise deformations are not possible. This situation agrees with Proposition 2 which shows that for materials whose energy density depends only on $I_{\mathbf{B}}$, the strain and stress states are necessarily plane. With reference to Figure 3, we see that, if $c \to \infty$, $\gamma \to 0$ and consequently pairwise axisymmetric deformations become trivial (i.e., $\mathbf{F}_2 \to \mathbf{F}_1$). A qualitative "justification" of this result can be easily obtained by noting that, when $c \to \infty$, the energy function σ defined by (12) "converges" to a rank-one convex function: a situation which is not compatible with the occurrence of multiphase solutions.

ACKNOWLEDGEMENTS:

This work has been supported by COFIN-MURST 2002 "Modelli Matematici per la Scienza dei Materiali" and by "Fondi di Ricerca di Ateneo".

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