MECHANICAL MODEL OF A DC MOTOR

Maxime Sadre

Départment de Génie Industriel et Maintenance, Universite de Versailles Saint-Quentin-En-Yveline, IUT, 7 rue Jean Hoet 78200, Mantes La Jolie, France

ABSTRACT

In numerous systems (photovoltaic, wind turbine, electromechanical) for extracting the maximum available power of the source, it is necessary to match the load and source impedances. This paper aims to present a mechanical model of a dc motor based on mechanical impedance. This model permits the study of transient and steady state responses.

1. INTRODUCTION

When a source with non negligible impedance is coupled to a load with variable impedance, the maximum available power is transferred between two systems if the source internal impedance equals the load impedance. We generalize this principle to an electromechanical converter (dc machine) and elaborate a static and dynamic model based on mechanical impedance. This approach of dc machine modeling could be interesting in variable speed applications.

2. MATCHED ELECTRICAL CIRCUITS

Before proceeding with a study of the dc motor, it is desirable to introduce the elementary electrical circuit. Let us consider a voltage source with electromotive force E and internal resistance r, coupled to a resistive element characterized by its current-voltage curve, Fig. 1. In the case of linear elements, we have:

Source: $I = f_1(U) = -U/r + E/r$

Load: $I = f_2(U) = U/R$

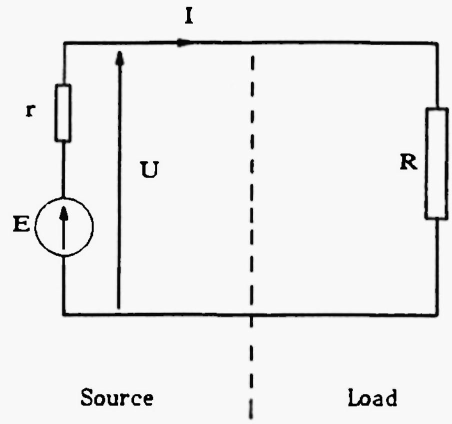


Fig. 1: Elementary electric circuit.

It is interesting to consider the power-voltage curve of the elements with the purpose of analyzing the power transfer procedure from the source to the load.

Source: $P = g_1(U) = U\,I = -U^2/r + E\,U/r$

Load: $P = g_2(U) = U^2/R$

The operating point is obtained by intersection of source and load curves, Fig. 2. The optimal point coordinates are determined by deriving the power with respect to voltage:

$$dP/dU = -2U/r + E/r = 0$$

Thus we obtain:

$$U_{op} = E/2 \quad ; \quad Pop = E^2/4\,r$$

We are now going to examine the power transferred into the load when this one varies and seek the load resistance value which ensures the maximum power transfer. In this case, the load would be characterized by a family of curves in the plane (U, P), in which R is considered as a parameter.

The operating point displaces along the source characteristic from the point A to 0 when R varies from ∞ (open-circuit) up to 0 (short-circuit), Fig. 3; the relative variation of current intensity would be from 0 to E/R (current limitation by the source internal resistance). The current and power delivered to the load would be given by the following relations:

$$I = E/(r+R) \quad ; \quad P = RI^2 = R\,E^2/(r+R)^2$$

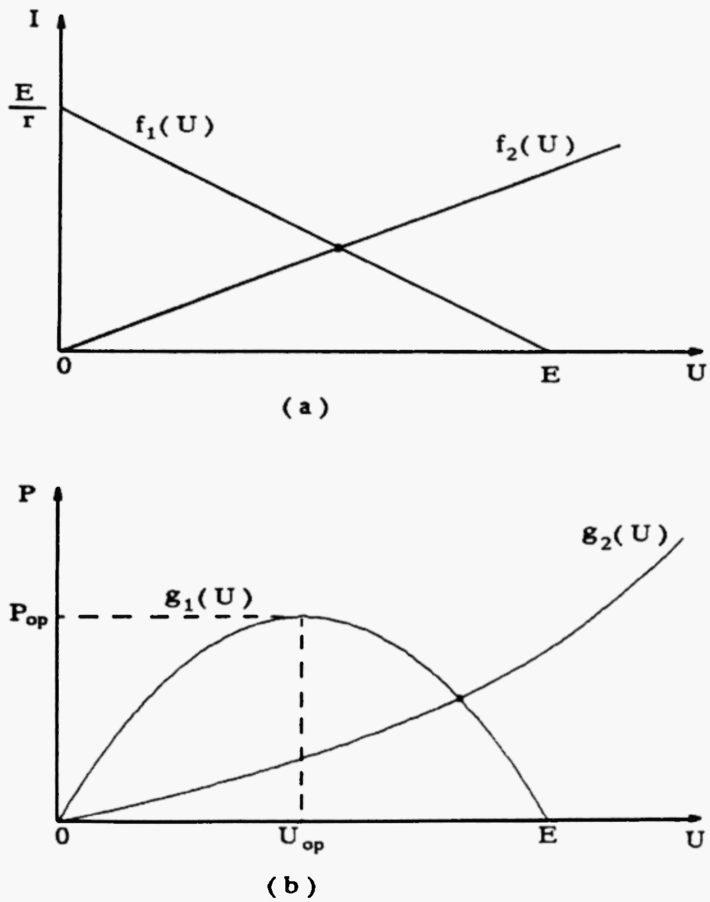
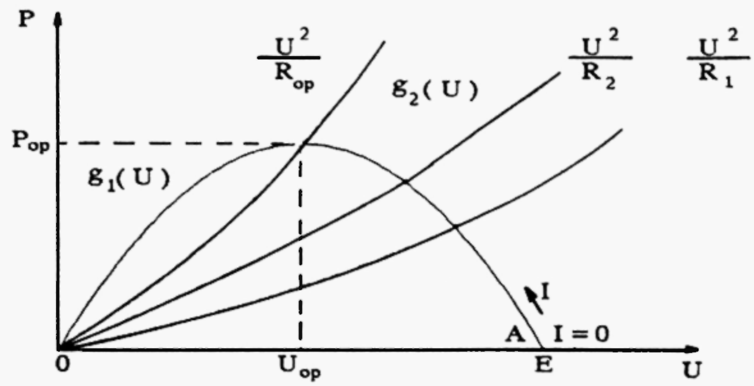


Fig. 2: (a) Current-voltage characteristics; (b) power-voltage characteristics.



Source characteristic : $P = g_1(U)$ at constant E
Load characteristic : $P = g_2(U)$ at variable R

Fig. 3: Source and load characteristics.

The optimal power captured by the load could be determined:

$$dP/dR = E^2\,[(r+R)^2 - 2(r+R)\,r]/(r+R)^4 = 0$$

Thus we obtain: R = r.

The power transfer is optimally performed, if the load resistance equals the source internal resistance (impedance matching), Fig. 3.

We now consider a source with varying electromotive force, Fig. 4. From the above discussion we find that the optimum points would locate on a parabola which has the equation:

$$P_{op} = U_{op}^2/r = E^2/4r$$

The circuit studied here corresponds to the Thevenin model; we can replace it by its Norton model, Fig. 5.

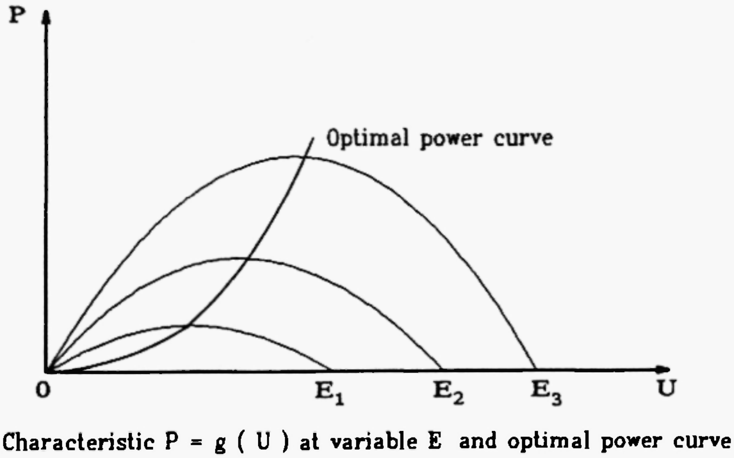


Fig. 4: Optimal power curve.

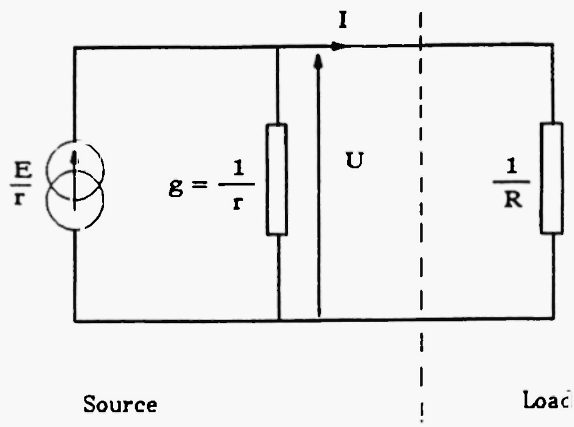


Fig. 5: Equivalent Norton model.

3. MATCHED ELECTROMECHANICAL SYSTEMS

Let the source be a dc motor rotating a mechanical or an electromechanical load.
The counter electromotive force relation is given by:

$$E' = k_m \phi \Omega$$

where ϕ is the magnetic flux, Ω the angular speed and k_m the counter electromotive force constant. In the case of a permanent magnet dc motor, the field is constant:

$$E' = K_m \Omega \quad \text{with } K_m = k_m \phi = \text{constant}$$

The electric equation in steady state could be written:

$$V = R_a I_a + E' \tag{1}$$

Where R_a designates the armature resistance.

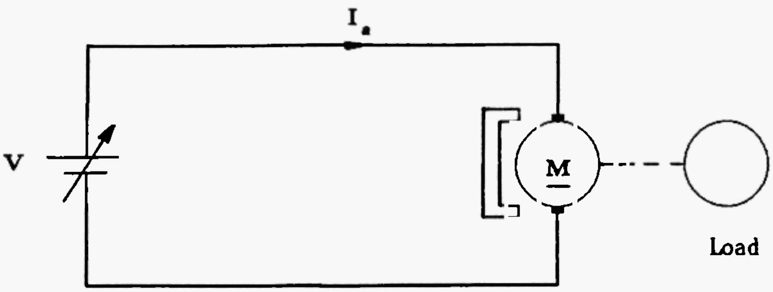


Fig. 6: System scheme

Relations of torque and power

The relation of the electromagnetic torque T_{em} could be elaborated by equating the electromagnetic power P_{em} and the product $E'I_a$:

$$P_{em} = E' I_a$$

$$T_{em} \Omega = E' I_a$$

Thus : $T_{em} = K_m I_a$

The electromagnetic torque is proportional to the current absorbed by the armature.

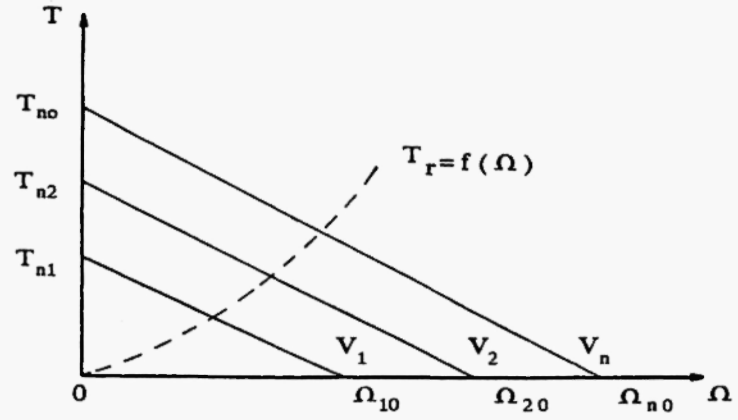


Fig. 7: Torque-angular speed for different armature voltage.

We now take into consideration the magnetic and mechanical losses; these loses could be expressed by a second order relation on Ω:

$$p_m = (T_{fm} + k_{dm}\Omega)\Omega$$

where T_{fm} and k_{dm} respectively designate the torque of dry friction and the coefficient of viscous friction. The output torque and power are hence evaluated by:

$$T_O = T_{em} - (T_{fm} + k_{dm}\Omega)$$
$$P_O = P_{em} - (T_{fm} + k_{dm}\Omega)\,\Omega$$

From (1) we have:

$$I_a = (V - E')/R_a$$

Furthermore: $T_{em} = K_m I_a$

$$T_O = K_m V/R_a - K_m^2\Omega/R_a - (T_{fm} + k_{dm}\Omega) \quad (2)$$

$$T_O = -A\Omega + B$$

with $A = K_m^2/R_a + k_{dm} = \text{Constant}$; $B = K_m V/R_a - T_{fm}$

If the armature voltage is constant, the parameter B would then be constant. In this case the relation (2) would be linear. The load develops a resistant torque $T_r = f_r(\Omega)$, Fig. 7. For a given armature voltage V_n, the no-load angular speed is:

the no-load angular speed is:

$$\Omega_{no} = (K_m V_n / R_a - T_{fm}) / (K_m{}^2 / R_a + k_{dm})$$

The torque for $\Omega = 0$ equals:

$$T_{no} = (K_m V_n / R_a - T_{fm})$$

The output power is expressed by:

$$P_o = (- A\,\Omega + B)\,\Omega \qquad (3)$$

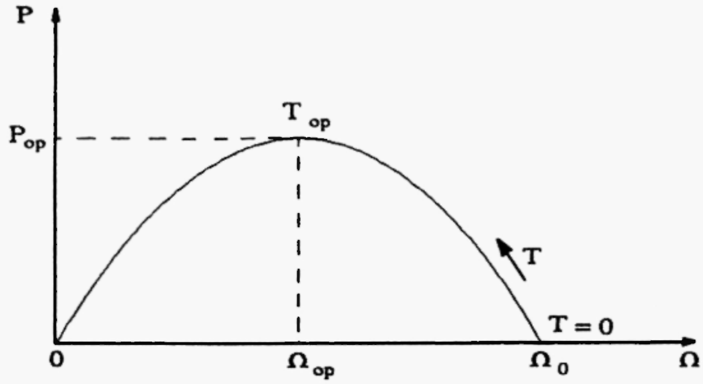


Fig. 8: Output power versus angular revolution speed.

The curve of $P_o = f(\Omega)$ at constant voltage operation possesses an inverted parabolic shape, Fig. 8. We now calculate the optimal operating point coordinates:

$$dP_o/d\Omega = - 2(K_m{}^2/R_a + k_{dm})\,\Omega + K_m V / R_a - T_{fm} = 0$$

$$\Omega_{op} = 0.5\,(K_m V / R_a - T_{fm}) / (K_m{}^2 / R_a + k_{dm})$$

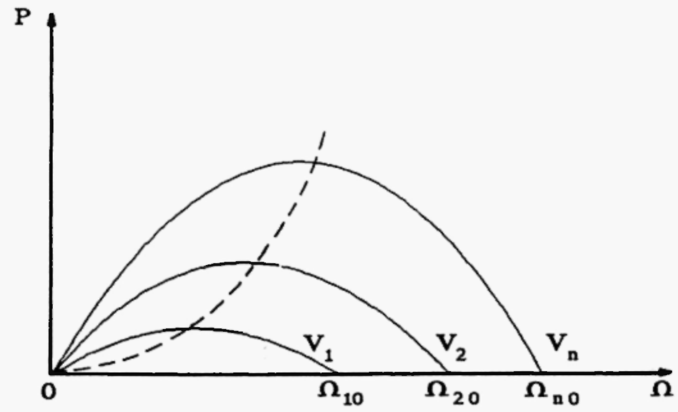
Substituting Ω_{op} in the relations (2) and (3), we obtain the output optimal torque and power:

$$T_{op} = 0.5(K_m V / R_a - T_{fm})$$

$$P_{op} = .25(K_m V / R_a - T_{fm})^2 / (K_m{}^2 / R_a + k_{dm})$$

The curve of maxima powers joining the summits of constant voltage curves is a parabolic function of Ω (or of V), Fig. 9:

$$P_{op} = (K_m{}^2 / R_a + k_{dm})\,\Omega^2$$



Power versus angular speed of a dc motor
—— : curves of power at constant V
– – : curve of optimal power

Fig. 9: Output power versus angular speed of a dc motor.

The notion of mechanical impedance appears here, defined as the quotient of torque to angular speed. The relation (2) may be written:

$$T_0 = -\, r_{mec}\ \Omega\ + E_m - E_i \tag{4}$$

with

with $r_{mec} = K_m{}^2/R_a$; $E_m = K_m V / R_a$; $E_i = T_{fm} + k_{dm}\,\Omega$

where r_{mec} represents the mechanical resistance; E_m is due to the external source (electrical source) and E_i is recognized as the internal source in opposition to the external source, Fig. 10. The quantities E_m

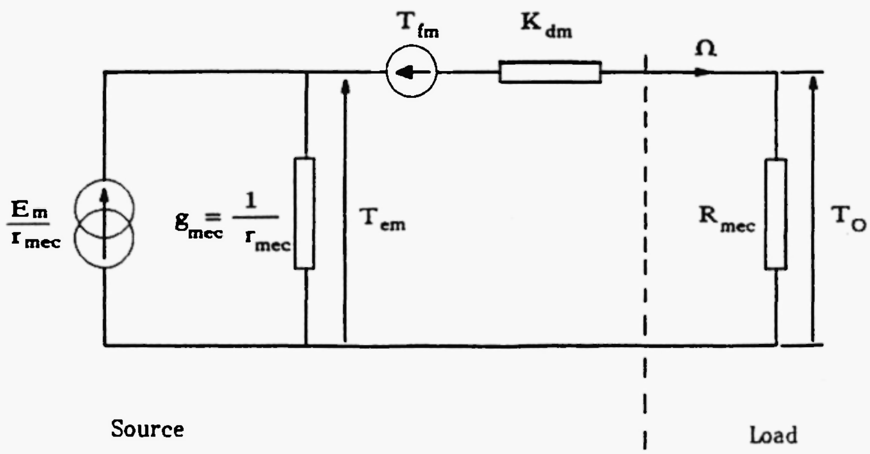


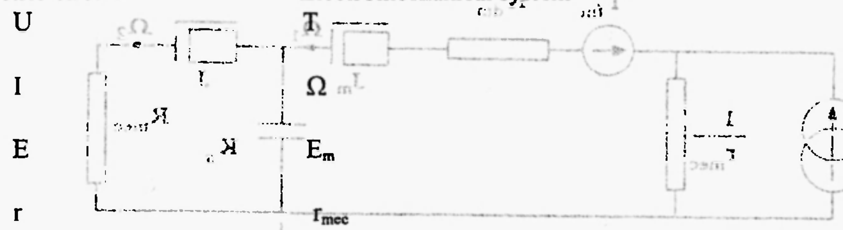
Fig. 10: Electromechanical system.

and r_{mec} play the same role here as do E and r in an electrical circuit. The total mechanical impedance is:

$$r_{tmec} = K_m^2/R_a + k_{dm}$$

We can establish an analogy between this electromechanical system and electrical circuits. The load would be characterized as well by its mechanical resistance.

Electric circuit Electromechanical system



g : Conductance $g_{mec} = 1/r_{mec}$: Mechanical conductance

It should be noted that the developed torque must not exceed the maximum machine rating under pain of winding excessive heating resulting in machine destruction.

We shall now discuss the losses produced in a dc motor. First, as regards no-load operation (R_{mec} = 0), the output torque is null. The mechanical and core losses could be taken into account by T_{fm} and k_{dm} elements ($p_o = T_{fm}\,\Omega + k_{dm}\,\Omega^2$), whereas the copper loss is negligible. Now, considering on-load operation, in addition to the losses mentioned above, copper loss, p_c, would be produced as well. In fact, the electromagnetic torque T_{em} is applied across r_{mec}:

$p_c \;=\; g_{mec}\,T_{em}^2$

As : $T_{em} = K_m\,I_a$

Hence : $p_c = R_a\,I_a^2$

This approach of dc machine modeling could be interesting in variable speed applications, particularly when the extraction of maximum available power is fixed as an objective.

4. DYNAMIC MODEL

Suppose a dc motor rotates a mechanical load via an elastic coupling with stiffness K_s. In the same manner as inductance and capacitance characterize the reactive energy storage elements, the moment of inertia and elastic stiffness represent the energy storage elements in a mechanical system. So we can obtain the equivalence between this mechanical system and associated electric circuit, Fig. 11.

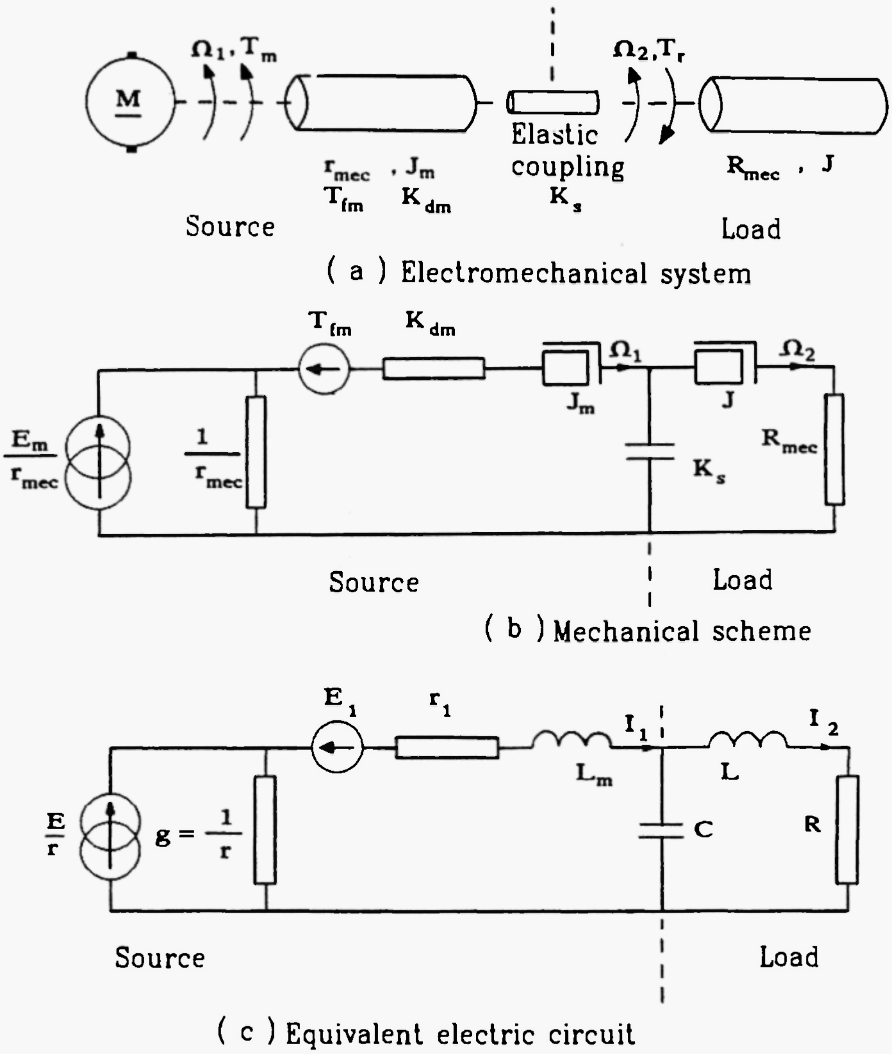


Fig. 11: Electric and mechanical equivalent circuits.

Furthermore, the torque components applied across different elements are:

Element	Torque
r_{mec}	$r_{mec}\,(\Omega_{no} - \Omega_1)$
k_{dm}	$k_{dm}\,\Omega_1$
J_m	$J_m d\Omega_1/dt$
K_s	$K_s \int (\Omega_1 - \Omega_2)dt$
J	$J\,d\Omega_2/dt$
R_{mec}	$R_{mec}\,\Omega_2$

At this stage, we take into account the inductance of the dc motor. The model obtained in this manner represents the dynamics of both electrical and mechanical elements.

The electrical equation of a dc motor in transient state could be written:

$$V = R_a I_a + E' + L\, dI_a/dt$$

In Laplace transform representation we have:

$$V(s) = (R_a + Ls) I_a(s) + K_m \Omega(s)$$

Finally, after calculation we obtain for the torque equation:

$$T_0 = -z_{mec}\ \Omega + E_m - E_f$$

with $z_{mec}(s) = K_m^{\ 2}/(R_a+Ls)$; $E_m = K_m V(s)/(R_a+Ls)$; $E_i = T_{fm} + k_{dm}\,\Omega$

$$T_0(s) = -\,[K_m^{\ 2}/(R_a+Ls)+k_{dm}]\ \Omega(s) + K_m V(s)/(R_a+Ls)\ - T_{fn}$$

The torque equation would be the same as equation (4), provided that R_a is substituted by R_a+Ls in Laplace transform representation.

Figure 12 illustrates the dynamic model.

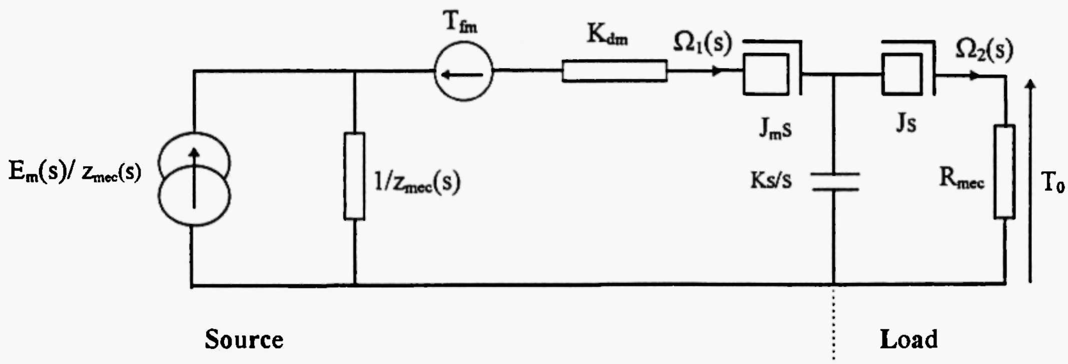


Fig. 12: Dynamic model.

It is obvious that the model could be used to determine electrical quantities such as V, I, electrical power if the mechanical quantities (torque and speed) are known. One should proceed as follows:

$$(T, \Omega) \Rightarrow E_m \Rightarrow V \Rightarrow I$$

4. EXPERIMENTAL RESULTS

We used a dc motor as the mechanical energy source; the load was constituted of a three-phase synchronous generator. The shaft of transmission was infinitely rigid. The dc motor ratings were:

$T_n = 3.8$ Nm ; $U_n = 24$ V ; $I_n = 65$ A

The motor parameters were found by laboratory measurements:

$R_a = 0.116\ \Omega$; $K_m = 0.067$ V rad^{-1} s ;

$T_{fm} = 0.207$ Nm ; $K_{dm} = 2.48.10^{-5}$ Nm rad^{-1} s

The three-phase synchronous generator charged a battery by means of a simple rectifier. It had the following ratings:

$U_n = 14$ V ; $I_n = 25$ A

Table 1

Comparison of theoretical and experimental results

DC motor voltage (V)	Theoretical values			Experimental values			
	Ω_0 (rad/s)	Ω_{op} (rad/s)	P_{op} (W)	Ω_0 (rad/s)	Ω_{op} (rad/s)	P_{op} (W)	r_{mec} (Nmrad^{-1}s)
6	82.7	41.3	68.6	88	44	70	3.61×10^{-2}
8	114.1	57	126	110	62.1	136.7	3.54×10^{-2}
10	144	72	200 .6	150	75	193.3	3.43×10^{-2}
12	173.8	86.9	292.5	180	90	290	$3,58\times10^{-2}$
14	203.7	101.9	402.4	200	109	410	$3,45\times10^{-2}$

6. CONCLUSION

The mechanical model of a dc motor, proposed in this paper, could be used to determine easily both mechanical quantities and electrical ones versus mechanical load. The model also makes it possible to perform impedance matching to extract the maximum available power when speed or source characteristics are variable. The dynamic model allows study of the transient response. The mechanical equivalent circuit illustrates the analogies with electrical circuits; the properties of the former could be obtained from those of the latter.

REFERENCES

1. M. Sadre (Sadrzadeh) and R. Dupuy, "Control of a wind turbine by micro-computer", *European Wind Energy Conference*, Amsterdam, October 1991.

2. M. Sadre (Sadrzadeh), "Etude de la régulation d'un système electromécanique par poursuite d'optimum", Thèse de l'Université Pierre et Marie Curie, juin 1992.
3. A.E. Fitzgerald and C. Kingsley, *Electrical Machinery*, McGraw-Hill, 1971.
4. M. Jufer, *Traite d'électricité, Vol. 9, Transducteur Électromécanique*, Edition Georgie, 1980.