ACHIEVING OOOD OUALITY SURFACE STRESS PREDICTION BY FINITE ELEMENT ANALYSIS

by

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1 Introduction

In assessing the performance of engineered products, considerations of structural integrity or mechanical strength figure importantly. Depending on the application, one may be involved with the static loading of parts made of materials which on the one hand are ductile or, on the other, have low fracture toughness or it may be necessary to consider the possibility of fatigue failure in dynamical situations. Whether, as in the former, local yielding at stress concentrations alleviate the effects of overload or in the latter, where a knowledge of local stress profiles provide a basis for life assessment, in critical applications, good quality predictions of the stress levels at the surface of solids is crucial. To attempt an analytical determination of the stress field in typical engineering components, using the formal notions of solid continuum mechanics, leads to insuperable mathematical difficulties in all but the most trivial cases. Accordingly, to make useful progress, it is necessary to adopt some approximation method. Thus, one possible approach is to form rather gross idealisations into very simple models to which may be applied the concepts of elementary mechanics of materials; local predictions may then be enhanced on the basis of such published data as the stress concentration factors associated with certain geometric discontinuities(1,2,3) or stress intensity factors at cracks(4) However, most machine or structural elements are not readily identifiable with catalogued cases and a more reasoned approach is necessary to obtain stress predictions of good accuracy. In this connection, the finite element process(5) has transformed the engineer's capability for design analysis in general and in the description of structural behaviour or the assessment of mechanical strength in particular. Notwithstanding the rational basis for approximation embodied in finite element analysis, however, assessing the quality of its predictions still requires the exercise of insight and

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judgement by the practitioner if the capability of the process is not to be overstretched and abused leading to faulty conclusions regarding the quality of a design.

Unless specifically stated to the contrary, finite element analysis is an approximation process based upon piecewise assumed displacement fields. Since strains, and hence stresses, have to be determined by differentiation of displacement functions, when the latter are approximate, the corresponding predicted stress levels are inferior in quality to the displacements themselves. Then, to recover, by conventional finite element methods, stress values of adequate quality requires that the displacements be determined to an accuracy which is greater than may really be required and this can incur an unacceptable computational overhead. This being particularly the case in the analysis of fully three-dimensional situations. Consequently, a number of investigators have addressed the problem of enhancing the quality of finite element stress prediction and various methods have been proposed.

In this paper, we briefly review some of these approaches and more fully discuss a very effective procedure, developed by the authors(6,7), for locally enhancing the quality of finite element surface stress prediction at a modest computational cost.

2. Finite Element Prediction of Stress Levels

The finite element method has been thoroughly presented in a number of texts(5,8,9) so that only the briefest outline is necessary here for convenient reference in the discussion on element performance and surface stress prediction presented below.

The formulation of any stressing problem requires a consideration of three essential ingredients;

- i) satisfying the requirements of equilibrium,
- ii) satisfying the requirements of continuity of displacements,
- iii) describing the mechanical behaviour of the material in some constitutive law.

Within the formal framework of solid continuum mechanics, manipulating these ingredients leads to an "exact" formulation in which the mechanical behaviour of an object under load is described in terms of partial differential equations and associated boundary conditions. Closed form solutions to these equations are only available for a few idealised situations which bear little relation to most actual engineering components with their complex shapes and loading patterns. Accordingly, as pointed out above, for useful progress to be made it is necessary to resort to some approximation process or other. When consciously seeking an approximate solution, it is necessary to consider what quality of performance prediction is deemed to be adequate for the purpose. Thus, in the early stages of the design process, say, a rough and ready estimate of stress levels may be all that is required whereas in arriving at a final refined product design a high level of precision may be called for. Such considerations influence the choice of approach to be used in a particular situation.

Being a rational approximation process, finite element analysis yields results having a quality which is very largely under the control of the practitioner, albeit at some consideration of cost; this is in contrast to the often gross idealisations involved in modelling situations on the basis of elementary mechanics of materials and the use of various handbooks. Effectively, to use the method, it is not necessary to be familiar with the detailed construction of the various commercial programmes which are available. However, it is essential to have a sound appreciation of the foundations of the process and especially of the approximations implicit in it as an accompaniment to a good "feel" for structural behaviour. This has a particular bearing on the accurate determination of the surface stresses which are of interest here and why we now briefly outline the essentials of the method.

Without further qualification, finite element stress analysis is usually understood to be an approximation process based on piecewise approximations to displacement fields and the exploitation of a variational principle. That is, the region occupied by a solid object is imagined to be divided into a finite number of relatively simply shaped sub-regions, or elements, comprising lines, surfaces or volumes established by meshes of curves which intersect to define various nodal points. Each element is assumed to deform in a relatively simple admissible manner, such that continuity between adjacent elements is assured. On the basis of such an assumption, usually expressed in terms of polynomials involving a finite number of parameters which comprise generalised co-ordinates, the infinite degrees of freedom characterising a continuum are replaced by a finite number. Strain values may be inferred and, in the case of in linear elastic situations, an expression for strain energy density determined so that integration over the volume of the element leads to the total strain energy stored in the form $U^e = 1/2 \{u\}_0^{et} [k]^e \{u\}_0^e$ which is characteristic of a finite degree of freedom system(10). The elements in $[k]^e$, called the element stiffness matrix, usually require numerical integration, commonly achieved by Gaussian quadrature. On the same basis, the work done by forces acting on the element may be computed. Taking into account the requirements of continuity of displacements, the total potential energy in the assemblage of elements representing the original solid may be computed and the application of the principle of virtual work leads to

$$[K]\{q\} = \{Q\}$$

This is a stiffness equilibrium statement characteristic of any discrete, linear elastic system. Here $[K]$, corresponds to the assemblage stiffness matrix, $\{Q\}$ is the vector of generalised forces corresponding to the vector of generalised co-ordinates $\{q\}$.

The finite element process thus generates a finite degree of freedom system the mechanical characteristics of which approximate those of the original continuum. In relation to the three essential ingredients for formulation referred to above, it can be seen that the finite element model:-

i) aims to provide continuity of displacements*
ii) a proper constitutive law is implicit in the expression for strain energy density
iii) equilibrium can only be satisfied in some average or weighted fashion consequent upon the variational principle being used in conjunction with approximations to the displacement fields

This last factor is a crucial one and provides a measure of the quality of the finite element prediction of stresses. It is also important to notice that the primary unknowns in the formulation are displacement parameters and that the strains (and hence stresses) are obtained by differentiating approximating displacement functions in which these parameters figure; the stresses are thus inevitably less accurate than the displacements from which they are derived, and one is led to the considerations which underlie the main theme of this paper.

3. The Quality of Finite Element Stress Predictions

Various element designs have been created utilising "low" or "high" order polynomials to provide approximations to the element displacement fields and rules have been established regarding what is admissible(5). Conventional extraction of strain, and hence stress, values is achieved by applying the appropriate strain-displacement-stress relations to these approximate displacement fields, the amplitudes of which are given by the nodal displacements of the direct finite element solution. Broadly, higher order polynomials allow fewer elements to be used to achieve a given accuracy, and experience has shown that a good compromise between accuracy and computational

* *For certain problems, as in plates and shells, matters are not so straightforward. See ref (5).*

costs is achieved by quadratic elements. In particular, the so-called isoparametric elements have become widely used in the various commercial computer codes available, but they too are not free of problems.

Difficulties associated with obtaining the "best" set of stresses, particularly those at the surface of a component, from a given analysis, can be appreciated by considering first the simplest linear displacement triangle element for plane stress/strain situations. Such a displacement model implies a constant state of stress within a typical element with step changes between adjacent elements. Intuitively, one might attribute the stress values to the centroid of the element., on the other hand, one may attribute, to a node of interest, some weighted average value from the elements meeting at the mode, or a least squares smoothing process adopted(11). The benefits of such approaches are least when stresses at the surface of a component or at the interface between two phases in a nonhomogeneous object are required. To improve matters, local surface equilibrium conditions may be imposed as constraints as exemplified by Hollaway(12) and Allison and Soh(13).

Higher order elements usually employ incomplete polynomials so that the behaviour is non-isotropic in the sense that a parabolic displacement field can imply a linear strain variation with respect to one co-ordinate whilst there is quadratic variation with respect to another. Such loops, or ripples, are not smoothed out if one chooses to utilise nodal average stresses from elements meeting at a node in an attempt to refine stress prediction; serious errors can be encountered even with fine meshes. It has been shown(14) that for conventional extraction of the stress levels, the optimum sampling points correspond to the Gauss integration points. Then, in pursuit of further improvements, various least squares smoothing schemes have been proposed(15,16) to allow extrapolation from the Gauss points to the nodes; the net effect is not necessarily worthwhile, however. Commercial finite element software do not incorporate any of the various stress refinement processes described above. However, the present authors have proposed a scheme for isoparametric elements(6,7) specifically designed to be interfaced, as an optional routine, to a commercially available package. An outline of this approach together with a description of some test results are presented below.

Other factors, relating to mesh design and geometric distortion, also have an important influence on element performance and certain of these are recognised in some computer codes which issue warnings when selected criteria of good mesh design are not satisfied. In broad terms, performance deteriorates the further an element is distorted from its basic shape. Thus, a long thin triangle is not as satisfactory as an equilaterial one and a thin rectangle is inferior to a square. The formulation of isoparametric elements involves a mapping process to transform generic triangles and squares into curvilinear triangles and rectangles. Various studies have shown that the performance of elements is seriously impaired when they are highly curved(17,18) and, in fact, ref (19) advised that straight sides should always be used unless good matching to a boundary contour makes curved sides essential. Other studies have shown that geometric distortion can imply a singularity in the stress field in regions outside the element(20); this can be exploited to advantage, of course, when such a singularity is known to exist as at the tip of a crack(21).

The remainder of this paper is concerned with enhancing the prediction of stresses at the surface of solid objects using quadratic isoparametric element meshes which are designed to be satisfactory in the senses implied above.

4. The Efficient Enhancement of Surface Stress Prediction

The process for enhancing finite element surface stress prediction proceeds essentially as follows. Accepting the nodal displacements provided by conventional finite element analysis, utilise a least squares process, constrained by satisfaction of the boundary traction conditions, to generate new "smoothed" displacement functions local to elements of interest adjacent to the surface of a given component. In the case of isoparametric elements, the mapping processes involved in formulating the element characteristics introduce complexities into the process which are not met in the simpler elements addressed in, say, ref (13). The analytic basis of the process is outlined below.

Expressed in terms of the local curvilinear co-ordinate system, the displacement field for a typical eight node (plane) isoparametric quadrilateral element is described by

$$u = [P]\{\beta\}_{1-8}, \qquad v = [P]\{\beta\}_{9-16} \tag{1}$$

where

$$[P] = [1 \ r \ s \ rs \ r^2 \ s^2 \ r^2s \ rs^2] \tag{2}$$

and the generic element occupies the region $-1 \leq r \leq 1$, $-1 \leq s \leq 1$.

For our purposes here, $\beta_1, \beta_2, \ldots \beta_{16}$ will, in effect, be known for a typical element from the results of a finite element analysis already carried out.

In the stress refinement process, we define new displacement functions

$$u' = [P']\{\beta'\}_{1-9}, \qquad v' = [P']\{\beta'\}_{10-18} \qquad (3)$$

$$\text{where } [P'] = [1 \; r \; s \; rs \; r^2 \; s^2 \; r^2s \; rs^2 \; r^2s^2] \qquad (4)$$

and $\beta'_1, \beta'_2 \ldots \beta'_{18}$ are to be found by the constrained least squares procedure.

For a typical boundary element, as shown in Fig. 1, the traction boundary conditions will be

$$_\ell\sigma_{rr} = {}_\ell\bar{\sigma}_{rr} = \text{prescribed} \qquad (5)$$

$$_\ell\sigma_{r\theta} = {}_\ell\bar{\sigma}_{r\theta} = \text{prescribed} \qquad (6)$$

for ℓ = 1,5,2 corresponding to the element nodes actually on the solid surface. Then, the constraint conditions to be introduced into the least squares procedure are

$$_\ell(\sigma'_{rr} - \bar{\sigma}_{rr}) = 0 \qquad (7)$$

$$_\ell(\sigma'_{r\theta} - \bar{\sigma}_{r\theta}) = 0 \qquad (8)$$

for ℓ = 1,5,2.

Constraints (7) and (8) may be introduced into an auxilliary function $\phi(\beta, \lambda)$ by means of Lagrange multipliers λ_j. That is,

$$\phi(\beta,\lambda) = \sum_{k=1}^{3}\left\{(u'_k - u_k)^2 + (v'_k - v_k)^2\right\} + \sum_{j=1}^{3}\lambda_j\,{}_{j\ell}(\sigma'_{rr} - \bar{\sigma}_{rr}) + \sum_{j=4}^{6}\lambda_j\,{}_{j\ell}(\sigma'_{r\theta} - \bar{\sigma}_{r\theta}) \qquad (9)$$

Then, the constrained least squares process requires that

$$\delta \phi(\beta, \lambda) = 0 \tag{10}$$

Actually, to carry out the variation in equation (10) requires first that the u'_k, v'_k, σ'_{rr}, $\sigma'_{r\theta}$ be expressed in terms of β'_1, β'_2 ------ β'_{18}. This is very complicated in the case of the stress components requiring a consideration of the co-ordinate transformations and associated jacobian involved and the appropriate form of Hooke's law (for plane stress, plane strain or axisymmetric situations). Such particulars are presented elsewhere (6) and it is sufficient to state here that 24 linear simultaneous algebraic equations are generated from which the β'_1.....β'_{18} and λ_1 ...... λ_6 may be calculated.

In this way, the refined displacement functions u' and v' are determined from which the refined stresses are obtained by means of the appropriate strain-displacement and stress-strain relations for each element midside node of interest at the surface of the component. For corner nodes, the procedure is similar except that a fictitious element straddling two adjacent normal elements is introduced such that the fictitious midside node on the surface coincides with the actual common corner node. Interpolation is used where necessary to locate new fictitious nodes and the "actual" displacements associated with them. The procedure is then as before.

The process has been described in relation to plane problems. The semi-analytic generalisations necessary to treat axisymmetric solids, loaded in a non axisymmetric manner, have also been detailed elsewhere (7) and an application to three dimensional situations is presented here.

5 Interfacing the Scheme with Existing Finite Element Software

The procedures described above can be coded as self-contained routines which can be called as desired by a control statement in the stress recovery phase of a main finite element programme. There is thus no question of modifying existing well proven codes.

The extra computing cost associated with carrying out the stress refinement process in a typical problem is very modest, being of the order of 10% - 15% of the total cost of the analysis.

6. Some Applications of the Stress Refinement Process

The elliptic plate with an elliptic hole subjected to a uniform external pressure has been adopted by the National Agency for Finite Element Methods and Standards (NAFEMS) as providing a bench mark for plane elements. Fig. 2 shows the very coarse mesh used for the trial and Fig. 3 the excellent results provided by the refinement process.

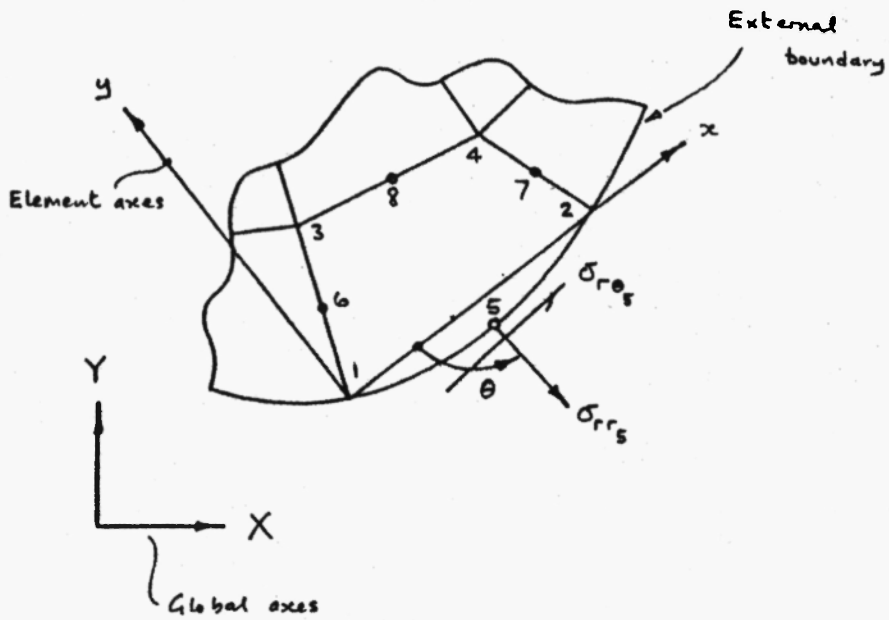

Fig. 1. A typical "boundary element"

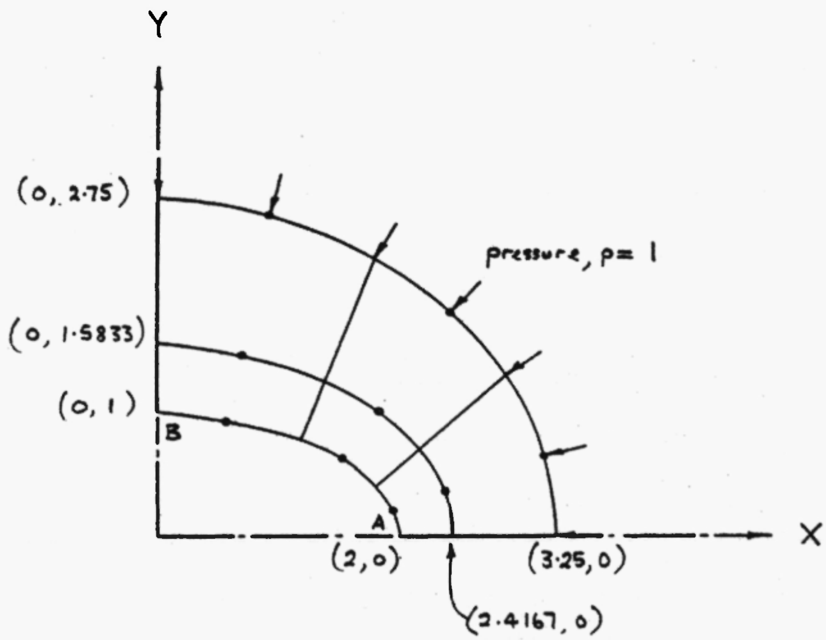

Fig. 2. Coarse mesh for elliptical plate.

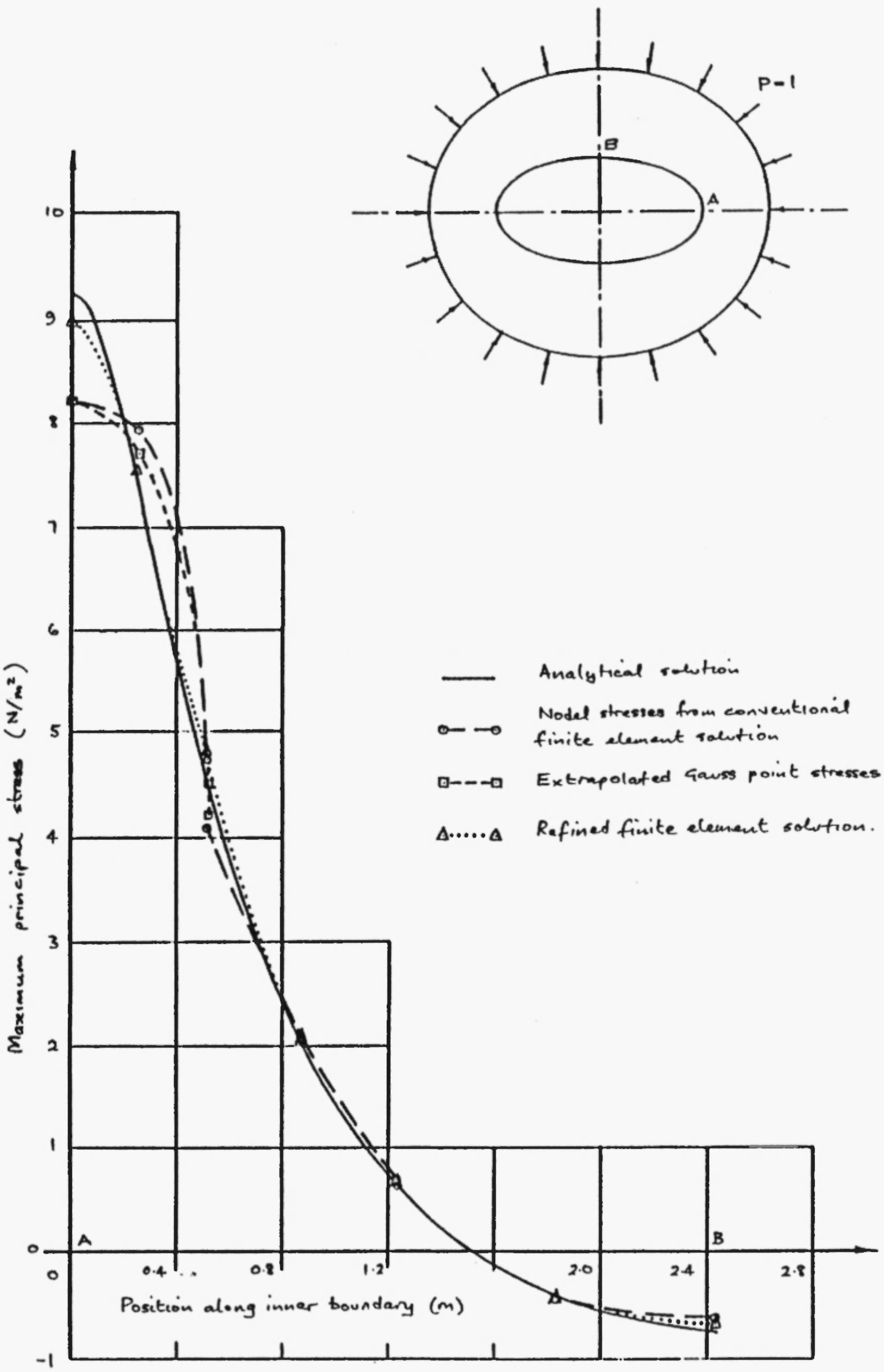

Fig. 3. Inner boundary maximum principal stress for elliptical plate subjected to unit external pressure, obtained by conventional and refined finite element solutions to coarse mesh.

A commonly utilized component is an axially loaded round bar having a circumferential semi-circular groove as shown in Fig. 4. Stress concentration factors (SCFs) for such components have been published previously (1,2) and provide useful comparisons. Fig. 5 shows

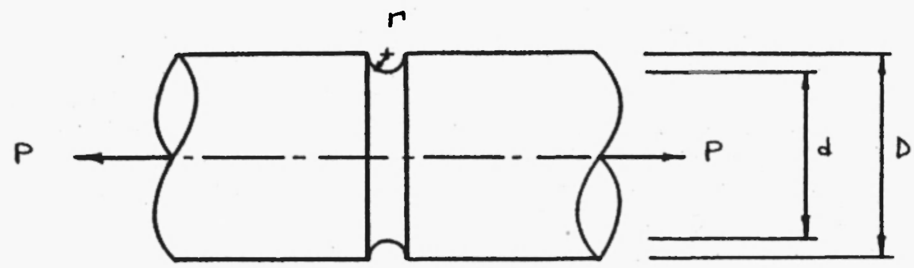

Fig. 4. Semi-circular groove in a tension rod.

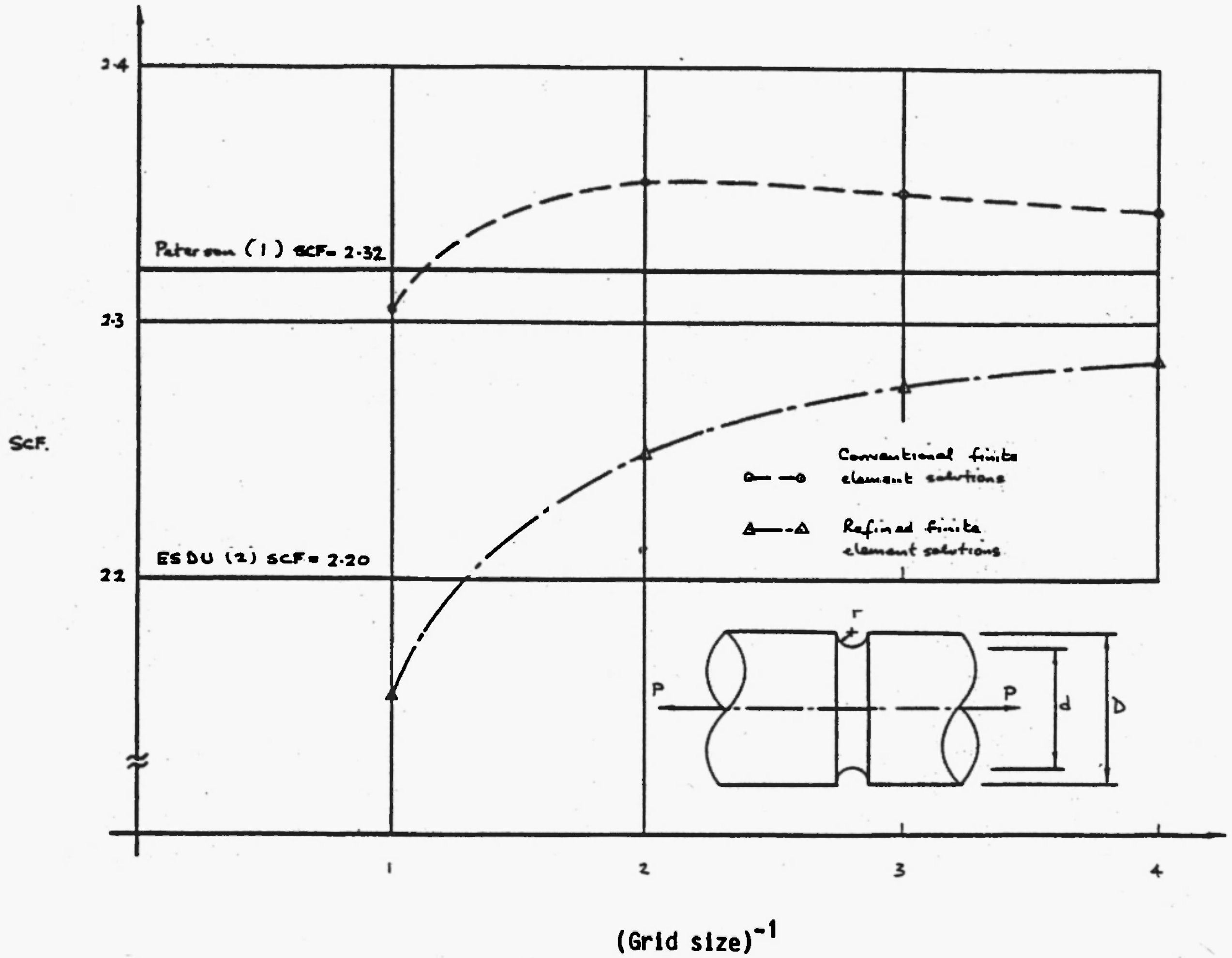

Fig. 5. Maximum stress concentration factor (SCF) for a tension rod with a U-groove (r/d = 0.1 and D/d = 1.2) obtained by conventional and refined finite element solutions.

the effect of refining the mesh by factors of 2, 3, 4 successively on the SCF predicted here, the coarsest mesh being shown in Fig. 6. For the bar, r/d = 0.1 and D/d = 1.2.

Fig. 6. Coarse, 10-element, 85-dof. mesh (grid size = 1)

If the grooved bar of Fig. 4 is subjected to bending or torsion, the situation corresponds to an axisymmetric solid loaded in a non-axisymmetric manner. A finite element formulation for such cases can be achieved by using ring type elements originally proposed by Wilson (22) and subsequently documented in various texts eg.(5,9). Then, the displacement components are assumed to vary according to equations (1) and (2) in a diametral plane, but to vary in a harmonic fashion circumferentially. Since sines and cosines provide orthogonal sets, the harmonic contributions uncouple in finite element formulations leading to very efficient analyses for this class of three dimensional problems. A semi-analytic stress refinement process can thus be formulated with respect to each uncoupled harmonic contribution which is completed analogous to that described above(7). Figs. 7 and 8 show the effect of mesh refinement on the quality of SCF prediction, with and without the refinement process described above, for the torsion and bending cases respectively.

As a final example, again we consider the case of the axially loaded grooved bar described above, but now analysed as a solid model. Fig. 9 shows the effect of refining the mesh by successive sub-division (in all three directions), the coarsest mesh being shown in Figure 10. Here the 20 node isoparametric brick element was used and, whilst the stress refinement process was conceptually the same as described above, the implementation required a considerable amount of care and meticulous attention to detail.

7. Conclusions

The method of enhancing the finite element prediction of surface stresses for a variety of classes of problems described here has proved to be accurate and cost effective. It may be appended to

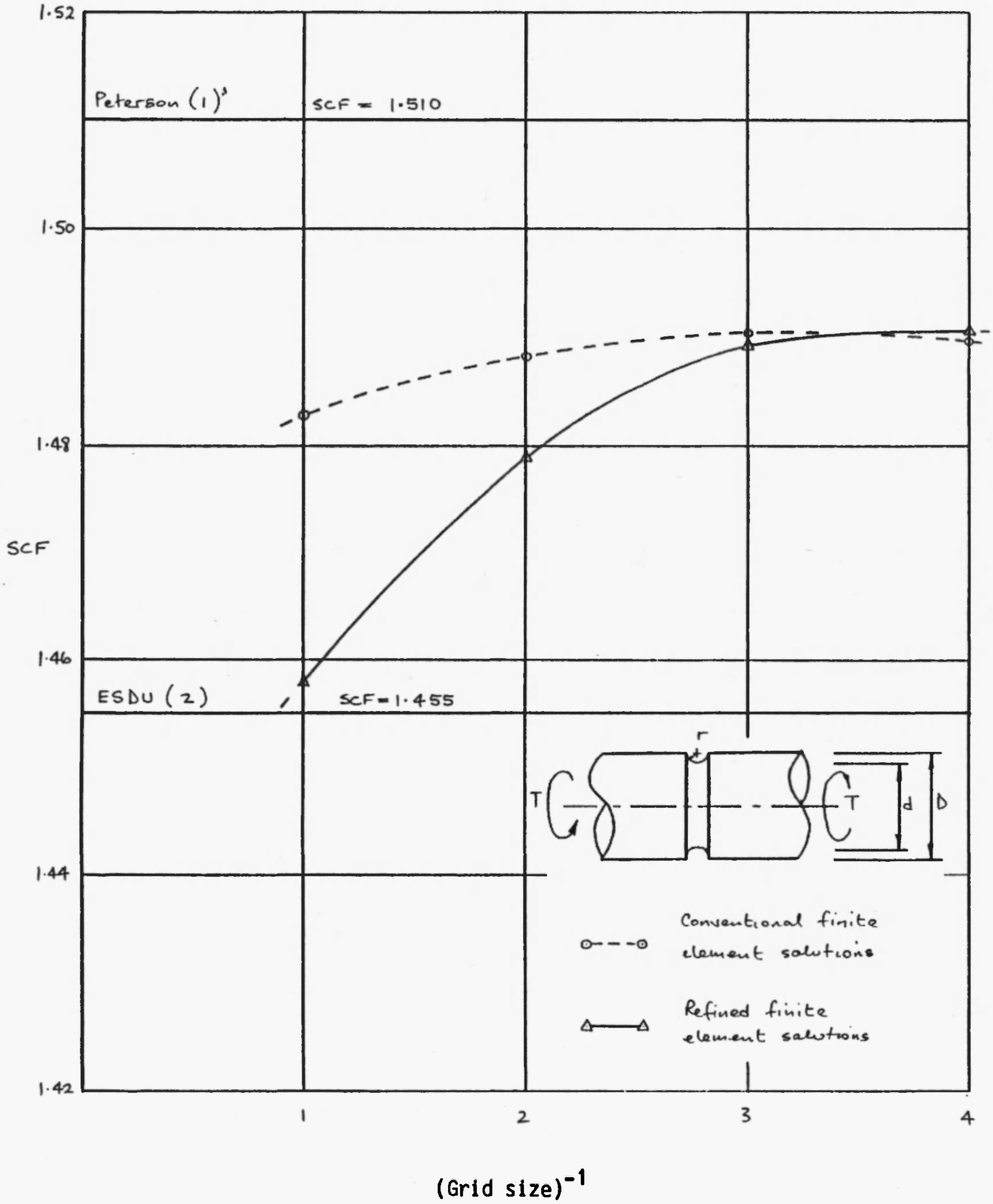

Fig. 7. Maximum stress concentration factor (SCF) for rod with a U-groove ($r/d = 0.1$ and $D/d = 1.2$) subjected to pur torsion obtained by conventional and refined finite element solutions.

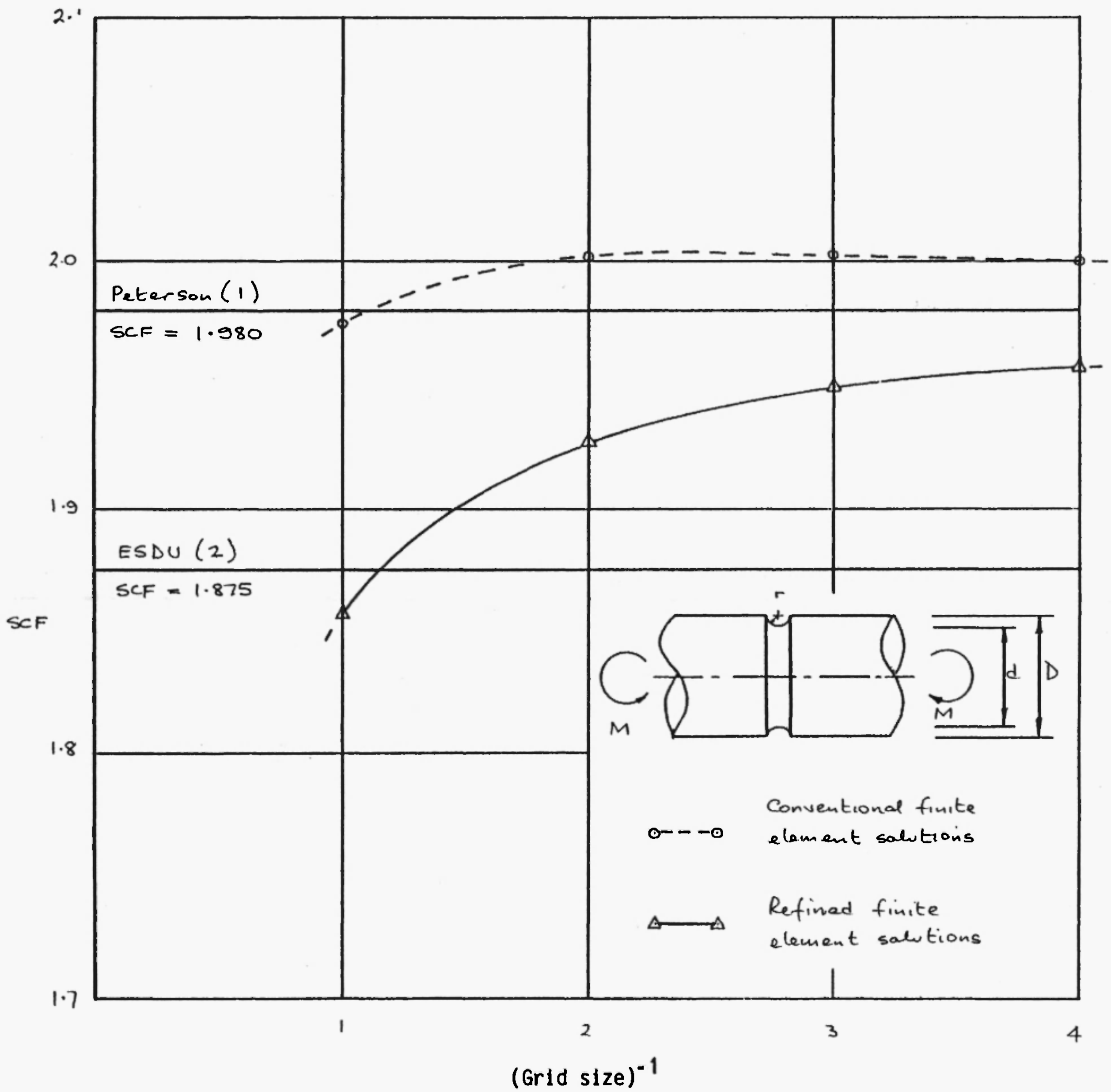

Fig. 8. Maximum stress concentration factor (SCF) for rod with a U-groove ($r/d = 0.1$ and $D/d = 1.2$) subjected to pure bending, obtained by conventional and refied finite element solutions.

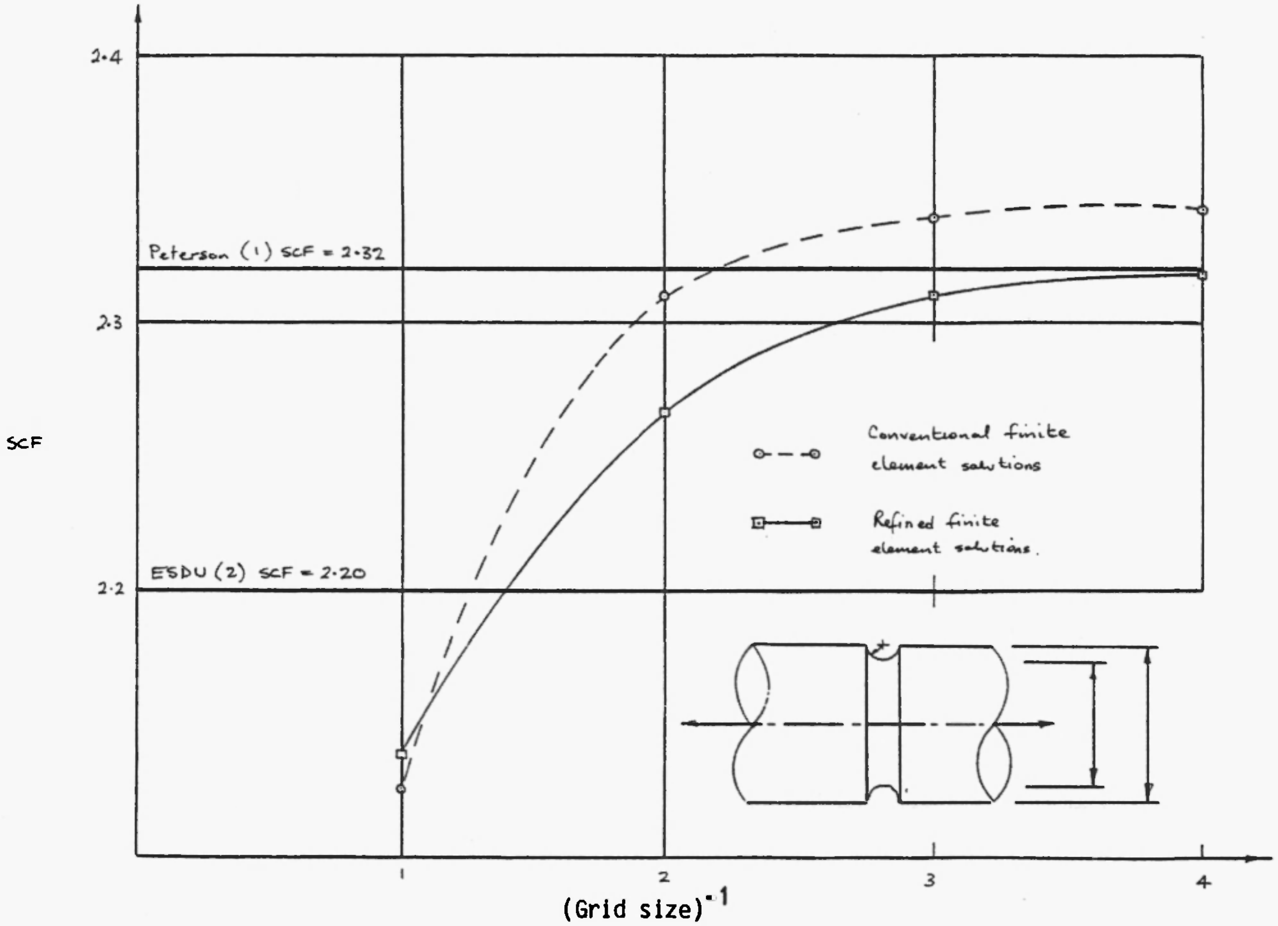

Fig. 9. Maximum stress concentration factor (SCF) for a tension rod with a U-groove. (r/d = 0.1 and D/d = 1.2) obtained from a solid model by conventional and refined finite element solutions.

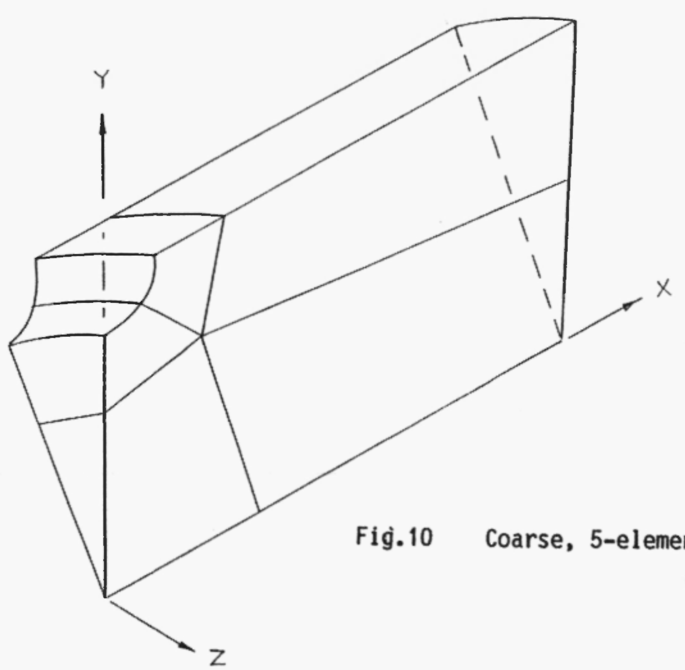

Fig.10 Coarse, 5-element, 165-dof. solid model (grid size = 1)

established finite element code as a self-contained set of routines called, as desired, by appropriate control statements. There is thus no question of modifying existing well proven code.

8. References

1. PETERSON, R.E., Stress Concentration design factors, 1953 (John Wiley, New York).

2. Engineering Science Data Unit, 'Elastic stress concentration factors. Geometric discontinuities in rods and tubes of isotropic materials', Structures Sub-Series, Item 69021, Vol. 7, 1970.

3. ROARK, R.J. and YOUNG, W.C., Formulas for Stress and Strain, 5th Edn. 1975 (McGraw Hill).

4. ROOKE, D.P. and CARTWRIGHT, D.J., Stress Intensity Factors, HMSO, 1976.

5. ZIENKIEWICZ, O.C., The finite element method, 3rd Edition, 1977 (McGraw-Hill, UK).

6. RICHARDS, T.H.E. and DANIELS, M.J., Enhancing Finite Element Boundary Stress Predictions for Plane and Axisymmetric Situations, J. Strain Analysis, Vol. 21, No. 1, 1986, 33-44.

7. RICHARDS, T.H.E. and DANIELS, M.J., Enhancing Finite Element Surface Stress Predictions: A Semi-Analytic Technique for Axisymmetric Solids, J. Strain Analysis, Vol, 22, No. 2, 1987, 75-86.

8. MARTIN, H.C. and CARNEY, G.F., Introduction to Finite Element Analysis, 1973 (McGraw Hill).

9. IRONS, B. and AHMAD, S., Techniques of Finite Elements, 1980 (Ellis Horwood Limited).

10. RICHARDS, T.H.E., Energy Methods in Stress Analysis, 1977 (Ellis Horwood Limited).

11. IRONS, B., 'Least squares surface fitting by finite elemernts, and applications to stress smoothing', Rolls-Royce Ltd., Aero Stress Memorandum 1524, 1967, 1-5.

12. HOLLAWAY, L.C., Photoelastic and finite element studies of the elastic stresses in two-dimensional composites, PhD thesis, University of London, 1970.

13. ALLISON, I.M. and SOH, A.K., 'On the determination of boundary stresses by the finite element method', Strain, 1981, 55-59.

14. BARLOW, J., 'Optimal stress locations in finite element models', Int. J. numer. Meth. Engng, 1976, **10**, 243-251.

15. HINTON, E. and CAMPBELL, J.S., 'Local and global smoothing of discontinuous finite element functions using a least squares method', Int. J. numer. Meth. Engng, 1974, **8**, 461-480.

16. HINTON, E., SCOTT, F.C., and RICKETTS, R.E., 'Local least squares smoothing for parabolic isoparametric elements', Int. J. numer. Meth. Engng, 1975, **9**, 235-238.

17. FRIED, I., 'Accuracy of complex finite elements', AIAA J., 1972, **10**, 347-349.

18. HENSHELL, R.D., WALTERS, D. and WARBURTON, G.B., 'On possible loss of accuracy in curved finite elements', J. Sound Vibration, 1972, **23**, 510-513.

19. THOMAS, K., 'Effects of geometric distortion on the accuracy of plane quadratic isoparametric finite elements', Guidelines for finite element idealisation, meeting preprint 2504, ASCE, 1975, 161-204.

20. HENSHELL, R.D. and SHAW, K.G., 'Crack tip elements are unnecessary', Int. J. numer. Meth. Engng, 1975, **9**, 495-509.

21. KNOTT, J.F., Fundamentals of Fracture Merchanics, Butterworths, 1973.

22. WILSON, E.L. 'Structural analysis of axisymmetric solids', J. Amer. Inst. Aero. Astr., 1965, **3**, 2269-2274.