Appendix A

A.1 Equivalent expression for the GlueVaR distortion function

Details on the definition of the GlueVaR distortion function $\kappa_{\beta,\alpha}^{h_1,h_2}(u)$ as a linear combination of the distortion functions of TVaR at confidence levels β and α, and VaR at confidence level α are provided, i.e. an explanation of how to obtain expression (3.3) can be found here. Expression (3.1) of the distortion function $\kappa_{\beta,\alpha}^{h_1,h_2}(u)$ can be rewritten as,

$$\begin{aligned}\kappa_{\beta,\alpha}^{h_1,h_2}(u) = h_1 \cdot \gamma_\beta(u) \cdot \mathbb{1}\left[0 \leqslant u < 1-\beta\right] \\ + \left(h_1 + \frac{h_2-h_1}{\beta-\alpha} \cdot (1-\alpha) \cdot \gamma_\alpha(u) - \frac{h_2-h_1}{\beta-\alpha} \cdot (1-\beta)\right) \cdot \\ \mathbb{1}\left[1-\beta \leqslant u < 1-\alpha\right] + \psi_\alpha(u),\end{aligned} \tag{A.1}$$

where $\mathbb{1}[x_1 \leqslant u < x_2]$ is an indicator function, so it takes a value of 1 if $u \in [x_1, x_2)$ and 0 otherwise.
Note that

$$\gamma_\beta(u) \cdot \mathbb{1}[0 \leqslant u < 1-\beta] = \gamma_\beta(u) - \psi_\beta(u), \tag{A.2}$$

$$\mathbb{1}[1-\beta \leqslant u < 1-\alpha] = \psi_\beta(u) - \psi_\alpha(u), \tag{A.3}$$

$$\begin{aligned}\gamma_\alpha(u) \cdot \mathbb{1}[1-\beta \leqslant u < 1-\alpha] = \\ \gamma_\alpha(u) - \psi_\alpha(u) - \left(\frac{1-\beta}{1-\alpha}\right) \cdot \left[\gamma_\beta(u) - \psi_\beta(u)\right].\end{aligned} \tag{A.4}$$

Taking into account expressions (A.2), (A.3) and (A.4), expression (A.1) may be rewritten as,

$$\kappa_{\beta,\alpha}^{h_1,h_2}(u) = \left[h_1 - \frac{(h_2-h_1)\cdot(1-\beta)}{\beta-\alpha}\right] \cdot \gamma_\beta(u) +$$

$$\begin{aligned}\left[-h_1+h_1-\frac{(h_2-h_1)\cdot(1-\beta)}{\beta-\alpha}+\frac{(h_2-h_1)\cdot(1-\beta)}{\beta-\alpha}\right]\cdot\psi_\beta(u)+\\ \frac{h_2-h_1}{\beta-\alpha}\cdot(1-\alpha)\cdot\gamma_\alpha(u)+\\ \left[1-h_1+\frac{(h_2-h_1)\cdot(1-\beta)}{\beta-\alpha}-\frac{h_2-h_1}{\beta-\alpha}\cdot(1-\alpha)\right]\cdot\psi_\alpha(u).\end{aligned} \tag{A.5}$$

Given that $\omega_1 = h_1 - \frac{(h_2-h_1)\cdot(1-\beta)}{\beta-\alpha}$, $\omega_2 = \frac{h_2-h_1}{\beta-\alpha}\cdot(1-\alpha)$ and $\omega_3 = 1-h_2$, expression (3.3) follows directly from (A.5). □

A.2 Bijective relationship between heights and weights as parameters for GlueVaR risk measures

Pairs of GlueVaR heights (h_1, h_2) and weights (ω_1, ω_2) are linearly related to each other. The parameter relationships are $(h_1, h_2)' = H\cdot(\omega_1, \omega_2)'$ and, inversely, $(\omega_1, \omega_2)' = H^{-1}\cdot(h_1, h_2)'$, where H and H^{-1} matrices are $H = \begin{pmatrix} 1 & \frac{1-\beta}{1-\alpha} \\ 1 & 1 \end{pmatrix}$ and $H^{-1} = \begin{pmatrix} \frac{1-\alpha}{\beta-\alpha} & \frac{\beta-1}{\beta-\alpha} \\ \frac{\alpha-1}{\beta-\alpha} & \frac{1-\alpha}{\beta-\alpha} \end{pmatrix}$, respectively.

A.3 Relationship between GlueVaR and Tail Distortion risk measures

This section of the appendix is intended to present the proof of Proposition 4.1. Following the notation introduced along this work, as for any random variable X it holds that $\mathrm{GlueVaR}_{\beta,\alpha}^{\omega_1,\omega_2}(X) = \int X\,d\mu$ with $\mu = \kappa_{\beta,\alpha}^{\omega_1,\omega_2}\circ P$ and $\mathrm{T}_{g,\alpha}(X) = \int X\,d\eta$ with $\eta = g_\alpha \circ P$, proving Proposition 4.1 is equivalent to proving that $\kappa_{\beta,\alpha}^{\omega_1,\omega_2} = g_\alpha$ under the proper conditions on ω_1, ω_2 and g.
On one hand, suppose that $\omega_2 = 1-\omega_1$ and that g is given by expression (4.2). First of all, let us rewrite g as

$$\begin{aligned}g(t) = \left(\frac{\omega_1\cdot(1-\alpha)}{1-\beta}+\omega_2\right)\cdot t\cdot\mathbb{1}\left[0\leqslant t<(1-\alpha)^{-1}\cdot(1-\beta)\right]+\\ (\omega_1+\omega_2\cdot t)\cdot\mathbb{1}\left[(1-\alpha)^{-1}\cdot(1-\beta)\leqslant t\leqslant 1\right]\end{aligned}$$

Recall that g_α is built as $g\left(\frac{u}{1-\alpha}\right)\cdot\mathbb{1}[0\leqslant u<1-\alpha]+\mathbb{1}[1-\alpha\leqslant u\leqslant 1]$. If u is less than $1-\beta$ therefore $t=\frac{u}{1-\alpha}$ is less than $(1-\alpha)^{-1}\cdot(1-\beta)$; if u is comprised between $1-\beta$ and $1-\alpha$, then $t = \frac{u}{1-\alpha}$ satisfies that $(1-\alpha)^{-1} \cdot (1-\beta) \leqslant t \leqslant 1$. Summarizing,

$$g_\alpha(u) = \begin{cases} \left[\dfrac{\omega_1}{1-\beta} + \dfrac{\omega_2}{1-\alpha}\right] \cdot u & \text{if } 0 \leqslant u < 1-\beta \\ \omega_1 + \dfrac{\omega_2}{1-\alpha} \cdot u & \text{if } 1-\beta \leqslant u < 1-\alpha \\ 1 & \text{if } 1-\alpha \leqslant u \leqslant 1 \end{cases} \tag{A.6}$$

which is the definition of distortion function $\kappa_{\beta,\alpha}^{\omega_1,\omega_2}$ as shown in (3.5).
On the other hand, consider as starting point the aforementioned expression (3.5) of $\kappa_{\beta,\alpha}^{\omega_1,\omega_2}$. As pointed out, g_α is always continuous in $1-\alpha$. Consequently parameters of $\kappa_{\beta,\alpha}^{\omega_1,\omega_2}$ must be such that guaranty continuity of the equivalent g_α in $1-\alpha$. In other words, $\lim_{u\uparrow(1-\alpha)} \kappa_{\beta,\alpha}^{\omega_1,\omega_2}(u) = \omega_1 + \omega_2 = 1 = \lim_{u\downarrow(1-\alpha)} \kappa_{\beta,\alpha}^{\omega_1,\omega_2}(u)$. This is exactly condition $\omega_2 = 1-\omega_1$. Now, forcing $g_\alpha = \kappa_{\beta,\alpha}^{\omega_1,\omega_2}$, it is straightforward to go backwards from expression (A.6) to expression (4.2) to complete the proof. $\square$