7 Capital allocation based on GlueVaR

In Section 6.1 of the previous chapter a set of elements to fully describe a capital allocation problem were identified. Nonetheless, two of those elements are of main importance: the assignment criterion and the functions used to simplify the information provided by each random loss. So, one could think that guidelines about how capital should be shared among firm's units are basically defined in terms of two components: (1) a capital allocation criterion and (2) a risk measure. The choice of the specific form for each component is essential as different capital allocation solutions result from the specific selected combinations.

In this chapter we consider the framework suggested by Dhaene *et al.* [2012b]. Under this framework, capital allocation principles are interpreted as solutions to optimization problems. This approach has been followed in the recent literature [see, for instance You and Li, 2014; Zaks and Tsanakas, 2014].

7.1 A capital allocation framework

Most of the proportional allocation principles can be described in the framework suggested by Dhaene *et al.* [2012b]. Under this unifying framework a capital allocation problem is represented by means of three elements: a non-negative function (which is usually linked to a norm), a set of weights, and a set of auxiliary random variables. However, the Haircut allocation principle could not be fitted into this framework despite its simplicity: the Haircut allocation principle combines a stand-alone proportional capital allocation criterion with the classical Value at Risk.

Here, the extension of the framework due to Dhaene *et al.* [2012b] is described. This was suggested in Belles-Sampera *et al.* [2014b]. A slight modification of the original framework was proposed, consisting in relaxing some

of the conditions in order to allow the inclusion of the Haircut capital allocation principle.

Assume that a capital K > 0 has to be allocated across n business units denoted by i = 1, ..., n. Any capital allocation problem can be described as the optimization problem given by

$$\min_{K_1, K_2, \dots, K_n} \sum_{j=1}^n \nu_j \cdot \mathbb{E}\left[\zeta_j \cdot D\left(\frac{X_j - K_j}{\nu_j}\right)\right] \quad \text{s.t.} \quad \sum_{j=1}^n K_j = K, \tag{7.1}$$

with the following characterizing elements:

- (a) a function $D: \mathbb{R} \to \mathbb{R}^+$;
- (b) a set of positive weights v_i , $i=1,\ldots,n$, such that $\sum_{i=1}^n v_i=1$; and
- (c) a set of random variables ζ_i , i = 1, ..., n, with $\mathbb{E}[\zeta_i] < +\infty$.

Unlike the original framework provided by Dhaene *et al.* [2012b], a distinction is made in (c) so that each ζ_i is now no longer forced to be positive with each $\mathbb{E}[\zeta_i]$ equal to 1.

To conclude, there exist a relationship between this capital allocation framework and aggregation functions. Aggregation functions may be defined as solutions to optimization problems, as proposed in De Baets [2013]. Capital allocation problems are disaggregation problems and therefore, to some extent, the goal of capital allocation principles is the opposite of the goal of aggregation functions, which is a summarizing purpose. Nonetheless, the optimization perspective taken into account in expression (7.1) involves aggregation operators in the objective function. For instance, one can think of the function $\mathbb{E}\left[\sum_{j=1}^{n} v_j \cdot \zeta_j \cdot D\left(\frac{X_j - K_j}{v_j}\right)\right]$ to be minimized in (7.1) as the composition of two main aggregation operators: one aggregation operator is given by expression $\sum_{j=1}^{n} v_j \cdot \zeta_j \cdot D\left(\frac{X_j - K_j}{v_j}\right)$ and the other one is the mathematical expectation E. It has to be noted that a similar perspective is proposed in Xu and Hu [2012], where the first aggregation function may be represented as $\Psi(L(\vec{K})) = \Psi(\sum_{j=1}^{n} \psi(X_j - K_j))$, where ψ is a function usually linked to a distance and Ψ an increasing function (which could be the identity function, for instance).

7.2 The Haircut capital allocation principle

Following the modification that we proposed in Belles-Sampera $et\,al.$ [2014b], the Haircut capital allocation solution can be obtained from the minimization problem (7.1). If a capital K>0 has to be allocated across n business units, the Haircut allocation principle states that the capital K_i to be assigned to each business unit must be

$$K_i = K \cdot \frac{F_{X_i}^{-1}(\alpha)}{\sum\limits_{j=1}^{n} F_{X_j}^{-1}(\alpha)}, \quad \forall i = 1, ..., n,$$
 (7.2)

where X_i is the random loss linked to the ith business unit, $F_{X_i}^{-1}$ is the inverse of the cumulative distribution function of X_i and $\alpha \in (0,1)$ is a given confidence level.

Let us consider $d_i = \min \left\{ d \ge 1 \mid 0 < |M^d[X_i]| < +\infty \right\}$ for all $i=1,\ldots,n$, where $M^d[X_i] = \mathbb{E} \left[X_i^d \right]$ is the moment of order d>0 of random variable X_i . Note that $d_i \ge 1$ for each i to face a feasible capital allocation problem. In other words, if a business unit presents a random loss with no finite moments, then the risk taken by that business unit is not insurable/hedgeable. The approach for fitting the Haircut allocation principle in the framework linked to the optimization problem (7.1) can be summarized as follows: if a constant r_i must be expressed as $r_i = \mathbb{E}[\zeta_i \cdot X_i]$, then using $\zeta_i = \frac{X_i^{d_i-1}}{M^{d_i}[X_i]} \cdot r_i$, a solution is found because $\mathbb{E}[\zeta_i \cdot X_i] = \mathbb{E}\left[\left(X_i^{d_i}/M^{d_i}[X_i]\right)\right] \cdot r_i = r_i$. Although this is an elegant approach, the interpretation of the transformation made by ζ_i on X_i is intricate.

Proposition 7.1. Let us consider a confidence level $\alpha \in (0,1)$. Then, the three characterizing elements required to represent the Haircut allocation principle in the general framework defined by (7.1) are:

(a)
$$D(x) = x^2$$
,

(b)
$$v_i = \frac{\mathbb{E}[\zeta_i \cdot X_i]}{\sum\limits_{j=1}^{n} \mathbb{E}[\zeta_j \cdot X_j]}, i = 1, \dots, n; and$$

(c)
$$\zeta_i = \frac{X_i^{d_i-1}}{M^{d_i}[X_i]} \cdot F_{X_i}^{-1}(\alpha), i = 1, ..., n.$$

Proof. In this setting, it is straightforward to show that the solution $\vec{K} = (K_1, K_2, ..., K_n)$ to the minimization problem (7.1) is the Haircut allocation solution expressed by (7.2). Dhaene *et al.* [2012b] show that, if function D is the squared Euclidean norm ($D(x) = x^2$), then any solution to (7.1) can be written as

$$K_i = \mathbb{E}[\zeta_i \cdot X_i] + \nu_i \cdot \left(K - \sum_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j]\right), \text{ for all } i = 1, \dots, n.$$
 (7.3)

In this setting, $v_i = \mathbb{E}[\zeta_i \cdot X_i] / \sum_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j]$ for each i, so

$$K_i = \mathbb{E}[\zeta_i \cdot X_i] + K \cdot \frac{\mathbb{E}[\zeta_i \cdot X_i]}{\sum\limits_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j]} - \mathbb{E}[\zeta_i \cdot X_i] = K \cdot \frac{\mathbb{E}[\zeta_i \cdot X_i]}{\sum\limits_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j]}.$$

And, finally, for all i it holds that $\mathbb{E}[\zeta_i \cdot X_i] = F_{X_i}^{-1}(\alpha)$ because of (c). Therefore, each K_i in the solution \vec{K} is given by

$$K_i = K \cdot \frac{F_{X_i}^{-1}(\alpha)}{\sum\limits_{j=1}^{n} F_{X_j}^{-1}(\alpha)}.$$

Some comments on v_i weights and ζ_i auxiliary random variables follow. Capital allocation principles driven by (7.3) can be thought of as two step allocation procedures: in a first step, a particular quantity $(C_i = \mathbb{E}[\zeta_i \cdot X_i])$ is allocated to each business unit. As the sum of all these quantities does not necessarily equal K (i.e., $\sum_{j=1}^n C_j \neq K$), in the second step the difference $\left(K - \sum_{j=1}^n C_j\right)$ is allocated to the business units considering weights v_i . From this perspective, C_i capitals are expected values of X_i losses restricted to particular events of interest and, therefore, ζ_i auxiliary random variables are used to select those events of interest for each business unit. On the other hand, v_i weights are related to the second step of the procedure, indicating how the difference between K and $\sum_{j=1}^n C_j$ must be distributed among business units. For a deeper interpretation of v_i weights and ζ_i auxiliary random variables in more general cases, the interested reader is referred to Dhaene *et al.* [2012b].

A remark on the gradient allocation principle. This principle can be fitted into the framework introduced by Dhaene *et al.* [2012b] following a similar

strategy than the one in Proposition 7.1, but changing $F_{X_i}^{-1}(\alpha)$ by $\frac{\partial \rho}{\partial u_i}(S)$ for all i = 1, ..., n. Or, in other words,

$$D(x) = x^{2}, \quad v_{i} = \frac{\mathbb{E}\left[\zeta_{i} \cdot X_{i}\right]}{\sum_{j=1}^{n} \mathbb{E}\left[\zeta_{j} \cdot X_{j}\right]} \quad \text{and} \quad \zeta_{i} = \left(\frac{X_{i}^{d_{i}-1}}{M^{d_{i}}[X_{i}]}\right) \cdot \frac{\partial \rho}{\partial u_{i}}(S)$$

for all i = 1, ..., n. Therefore, we find again that the gradient allocation principle is a proportional principle based on partial contributions, although we have now used a side track to arrive to this conclusion.

7.3 Proportional risk capital allocation principles using GlueVaR

The three characteristic elements of the framework suggested by Dhaene et al. [2012b] are function D, weights v_i and a set of appropriate ζ_i , for all $i=1,\ldots,n$. According to the notation used by Dhaene et al. [2012b], we deal with business unit driven proportional allocation principles when ζ_i depends on X_i . If ζ_i depends on $S=\sum_{i=1}^n X_i$ then we have aggregate portfolio driven proportional allocation principles. In the former case, the marginal risk contributions of business units to the overall risk of the portfolio are not taken into account; in the latter, they are. Adopting the notation introduced in the previous chapter, principles belonging to the first category are here denoted as stand-alone proportional allocation principles while principles in the second category are denoted as proportional allocation principles based on partial contributions.

In this chapter, two GlueVaR based proportional capital allocation principles that we suggested in Belles-Sampera *et al.* [2014b] are presented. Both principles share the expressions for two of the three characterizing elements:

$$D(x) = x^2$$
 and $v_i = \frac{\mathbb{E}[\zeta_i \cdot X_i]}{\sum\limits_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j]}$, for all $i = 1, ..., n$.

They differ in the set of random variables ζ_i , i = 1,...,n, which are presented below for the case of continuous random variables X_i .

7.3.1 Stand-alone proportional allocation principles using GlueVaR

Given two confidence levels α and β in (0, 1), $\alpha < \beta$, and two distorted survival probabilities h_1 and h_2 , if ζ_i is fixed as

$$\zeta_{i} = \omega_{1} \cdot \frac{\mathbb{1}\left[X_{i} \geqslant F_{X_{i}}^{-1}(\beta)\right]}{1 - \beta} + \omega_{2} \cdot \frac{\mathbb{1}\left[X_{i} \geqslant F_{X_{i}}^{-1}(\alpha)\right]}{1 - \alpha} + \omega_{3} \cdot \frac{X_{i}^{d_{i}-1}}{M^{d_{i}}[X_{i}]} \cdot F_{X_{i}}^{-1}(\alpha), \text{ for all } i = 1, ..., n,$$
(7.4)

then the stand-alone proportional allocation principle using as risk measure the GlueVaR $_{\beta,\alpha}^{h_1,h_2}$ can be represented in the modified capital allocation framework explained in Section 7.1. Components of the solution (K_1,K_2,\ldots,K_n) are expressed as

$$K_{i} = K \cdot \frac{\text{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(X_{i})}{\sum_{j=1}^{n} \text{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(X_{j})}, \text{ for all } i = 1,\dots,n.$$
 (7.5)

7.3.2 Proportional allocation principles based on partial contributions using GlueVaR

Similarly, if there exists a confidence level $\alpha^* \in (0,1)$ such that $F_S^{-1}(\alpha) = \sum_{j=1}^n F_{X_j}^{-1}(\alpha^*)$, the proportional allocation principle based on partial contributions using GlueVaR $_{\beta,\alpha}^{h_1,h_2}$ can be fitted to the modified capital allocation framework detailed in Section 7.1. In this case, ζ_i has to be equal to

$$\zeta_{i} = \omega_{1} \cdot \frac{\mathbb{1}\left[S \geqslant F_{S}^{-1}(\beta)\right]}{1 - \beta} + \omega_{2} \cdot \frac{\mathbb{1}\left[S \geqslant F_{S}^{-1}(\alpha)\right]}{1 - \alpha} + \omega_{3} \cdot \frac{X_{i}^{d_{i}-1}}{M^{d_{i}}[X_{i}]} \cdot F_{X_{i}}^{-1}(\alpha^{*}), \text{ for all } i = 1, ..., n.$$
(7.6)

Each component of the solution $(K_1, K_2, ..., K_n)$ is then obtained as

$$K_{i} = K \cdot \left[\omega_{1} \cdot \frac{\mathbb{E}\left[X_{i} \mid S \geq F_{S}^{-1}(\beta)\right]}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} + \omega_{2} \cdot \frac{\mathbb{E}\left[X_{i} \mid S \geq F_{S}^{-1}(\alpha)\right]}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} + \omega_{3} \cdot \frac{F_{X_{i}}^{-1}(\alpha^{*})}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} \right].$$
(7.7)

Alternatively, another approach can be considered. There exists a set of confidence levels $\alpha_j \in (0,1)$, for all $j=1,\ldots,n$, such that $F_S^{-1}(\alpha) = \sum_{j=1}^n F_{X_j}^{-1}(\alpha_j)$. Therefore, the proportional allocation principle based on partial contributions using GlueVaR $_{\beta,\alpha}^{h_1,h_2}$ can also be fitted to the modified capital allocation framework. In this case, ζ_i have to be equal to

$$\zeta_{i} = \omega_{1} \cdot \frac{\mathbb{1}\left[S \geqslant F_{S}^{-1}(\beta)\right]}{1 - \beta} + \omega_{2} \cdot \frac{\mathbb{1}\left[S \geqslant F_{S}^{-1}(\alpha)\right]}{1 - \alpha} + \omega_{3} \cdot \frac{X_{i}^{d_{i}-1}}{M^{d_{i}}[X_{i}]} \cdot F_{X_{i}}^{-1}(\alpha_{i}), \text{ for all } i = 1, ..., n.$$
(7.8)

Each component of the solution $(K_1, K_2, ..., K_n)$ is then obtained as

$$K_{i} = K \cdot \left[\omega_{1} \cdot \frac{\mathbb{E}\left[X_{i} \mid S \geq F_{S}^{-1}(\beta)\right]}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} + \omega_{2} \cdot \frac{\mathbb{E}\left[X_{i} \mid S \geq F_{S}^{-1}(\alpha)\right]}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} + \omega_{3} \cdot \frac{F_{X_{i}}^{-1}(\alpha_{i})}{\operatorname{GlueVaR}_{\beta,\alpha}^{h_{1},h_{2}}(S)} \right].$$
(7.9)

A final comment related to non-proportional capital allocation principles using GlueVaR. It has to be mentioned that it is possible and straightforward to obtain non-proportional principles using any of the auxiliary random variables ζ_i described in expressions (7.4), (7.6) or (7.8). If function $D(x) = x^2$, then the only thing that must be taken into account is that at least one of the weights v_i , i = 1, ..., n, must be different from $\mathbb{E}[\zeta_i \cdot X_i]/(\sum_{j=1}^n \mathbb{E}[\zeta_j \cdot X_j])$. Under these restrictions, whatever set of auxiliary random variables $\vec{\zeta}$ is chosen among expressions (7.4), (7.6) or (7.8), non-proportional capital allocation principles \vec{K} using GlueVaR are obtained through expression (7.3).

7.4 An example of insurance risk capital allocation using GlueVaR on claim costs

Data of previous chapters are used to illustrate the application of capital allocation principles based on GlueVaR risk measures. Table 7.1 shows risk values for this example. The last column presents diversification benefit, which is the difference between the sum of the risks of X_1 , X_2 and X_3 and the risk of $X_1 + X_2 + X_3$. In this example, VaR_{95%} and one of the GlueVaR risk measures are not subadditive in the whole domain.

	X_1	X ₂	X ₃	$X_1 + X_2 + X_3$	Difference ^(*)
	(a)	(b)	(c)	(d)	(a)+(b)+(d)-(c)
VaR _{95%}	2.5	0.6	1.1	5.9	-1.7
TVaR _{95%}	12.5	8.0	1.3	19.7	2.1
TVaR _{99.5%}	40.8	42.1	1.8	81.1	3.6
GlueVaR _{99.5%,95%}	18.6	16.9	1.4	35.6	1.3
GlueVaR _{99.5%,95%}	9.4	4.2	1.2	12.9	1.9
GlueVaR _{99.5%,95%}	4.9	2.9	1.1	10.2	-1.3

Table 7.1 Risk assessment of claim costs using GlueVaR risk measures

Next, a capital allocation application is illustrated where total capital has to be allocated between the three units of risk, X_1 , X_2 and X_3 . Table 7.2 shows particular allocation solutions for two proportional risk capital allocation principles using GlueVaR.

A different pattern is observed for the three GlueVaR risk measures when the stand-alone criterion or the partial contribution criterion is considered. In the case of the stand-alone criterion, the capital is allocated primarily to risk X_1 , followed by X_2 and X_3 , respectively. Let us focus on capital allocation solutions involving the partial contribution criterion in which confidence levels α_j , j=1,2,3, are not forced to be equal across the risk units. A notable increase in the risk allocated to X_2 is observed if a partial contribution criterion with no constant level α_j and GlueVaR $_{99.5\%,95\%}^{1/20,2/8}$ is chosen.

This result is obtained because the impact on the quantile of X_2 is the opposite of that on X_1 and X_3 when α_j , j=1,2,3, are estimated as $F_S^{-1}(95\%) = F_{X_1}^{-1}(\alpha_1) + F_{X_2}^{-1}(\alpha_2) + F_{X_3}^{-1}(\alpha_3)$. These confidence levels are equal to $\alpha_1 = 26\%$, $\alpha_2 = 98\%$ and $\alpha_3 = 43\%$. So, the associated quantiles for individual variables are $\text{VaR}_{26\%}(X_1)$, $\text{VaR}_{98\%}(X_2)$ and $\text{VaR}_{43\%}(X_3)$. The risk contribution of X_1 and X_3 are underweighted compared to the risk contribution of X_2 . If we interpret the GlueVaR risk measure as a linear combina-

^(*) Benefit of diversification.

The partial contribution criterion with constant level is not calculated in this example. However, there is a $\alpha^* = 95,42\%$ such that $\mathrm{VaR}_{95\%}(Z) \simeq \sum_{j=1}^3 \mathrm{VaR}_{95,42\%}(X_j)$.

Table 7.2 Proportional capital allocation solutions using GlueVaR for the claim costs data

	Proportion allocated to X ₁	Proportion allocated to X2	Proportion allocated to X ₃				
Stand-alone criterion							
GlueVaR _{99.5%,95%} ^{11/30,2/3}	50.41%	45.80%	3.79%				
GlueVaR _{99.5%,95%}	63.51%	28.38%	8.11%				
$\rm GlueVaR_{99.5\%,95\%}^{1/20,1/8}$	54.44%	32.22%	12.22%				
Partial contribution criterion with non constant $^{(a)}$ $lpha_j$							
GlueVaR _{99.5%,95%} (b)	46.42%	51.74%	1.84%				
GlueVaR _{99.5%,95%} (b)	68.19%	26.86%	4.95%				
GlueVaR _{99.5%,95%} (b)	25.11%	73.11%	2.78%				

⁽a) Confidence levels $\alpha_j \in (0,1)$ are selected to satisfy $F_S^{-1}(95\%) = F_{X_1}^{-1}(\alpha_1) + F_{X_2}^{-1}(\alpha_2) + F_{X_3}^{-1}(\alpha_3)$. In this case $\alpha_1 = 26\%$, $\alpha_2 = 98\%$ and $\alpha_3 = 43\%$.

tion of $\omega_1 \cdot TVaR_{99.5\%} + \omega_2 \cdot TVaR_{95\%} + \omega_3 \cdot VaR_{95\%}$, the associated weights of the GlueVaR $_{99.5\%,95\%}^{1/20,2/8}$ are $\omega_1 = 1/24$, $\omega_2 = 1/12$ and $\omega_3 = 21/24$. So, the GlueVaR $_{99.5\%,95\%}^{1/20,1/8}$ reflects a risk measurement attitude just a bit more conservative than VaR $_{95\%}$, giving the largest weight to this risk value. Bearing in mind the quantitative tools that we have proposed in Chapter 5 to assess aggregate risk attitudes, the latter statement is reinforced by the following fact: the area under the distortion function of GlueVaR $_{99.5\%,95\%}^{1/20,2/8}$ is

$$\frac{1}{24} \cdot \frac{1+99.5\%}{2} + \frac{1}{12} \cdot \frac{1+95\%}{2} + \frac{21}{24} \cdot 95\% = 0.042 + 0.081 + 0.831$$
$$= 95.4\%$$

which is, effectively, slightly higher than 95%, the size of the area associated to $VaR_{95\%}$.

7.5 Exercices

- 1. Consider two risks that are Normally distributed with means, μ_1 and μ_2 , both non-negative and covariance matrix S. Write a program to implement the GlueVaR stand-alone allocation principle assuming that the inputs are the distribution parameters and, in addition, α , β , ω_1 , ω_2 and ω_3 .
- **2.** Assume n random variables, each one is uniformly distributed in the interval [0, 100]. Consider the GlueVaR stand-alone allocation principle, where $\omega_1 = \omega_2 = \omega_3$, express the result of the allocation to each component in terms of n, α and β .
- 3. Assume n random variables, each one is uniformly distributed in the interval [0,100]. Consider the GlueVaR stand-alone allocation principle, where $\alpha=0.95$ and $\beta=0.99$, express the result of the allocation to each component in terms of n, ω_1 , ω_2 and ω_3 and discuss the particular cases when $\omega_1=1$ or $\omega_2=1$.
- **4.** Consider the example described in Section 7.4 and consider a change of monetary units, which means that each variable is multiplied by a constant E > 0, where E is the exchange rate. How would that modification affect the capital allocation results if nothing else changes?