

# Contents

Preface — VII

Acknowledgment — XIII

|          |                                                                                                |
|----------|------------------------------------------------------------------------------------------------|
| <b>1</b> | <b>Retrospective problem of mean field games — 1</b>                                           |
| 1.1      | Introduction — 1                                                                               |
| 1.2      | A brief outline of the derivation of the mean field games system — 2                           |
| 1.2.1    | Derivation — 2                                                                                 |
| 1.2.2    | Summary of notation — 6                                                                        |
| 1.2.3    | The mean field games system and the statement of the retrospective problem — 7                 |
| 1.2.4    | Four main challenges of working with MFGS (1.21) — 8                                           |
| 1.3      | Two integral identities — 9                                                                    |
| 1.4      | Two Carleman estimates — 12                                                                    |
| 1.4.1    | The first Carleman estimate — 12                                                               |
| 1.4.2    | The second Carleman estimate — 15                                                              |
| 1.5      | Lipschitz stability and uniqueness — 17                                                        |
| <b>2</b> | <b>Stability and uniqueness for semiforecasting and forecasting problems — 23</b>              |
| 2.1      | Introduction — 23                                                                              |
| 2.2      | Statements of semiforecasting and forecasting problems — 24                                    |
| 2.3      | Two Carleman estimates — 25                                                                    |
| 2.3.1    | The Carleman estimate for the operator $\partial_t + \beta\Delta$ — 25                         |
| 2.3.2    | The quasi-Carleman estimate — 28                                                               |
| 2.4      | Lipschitz stability and uniqueness for the semiforecasting problem — 30                        |
| 2.5      | Hölder stability estimate for the forecasting problem — 35                                     |
| <b>3</b> | <b>Hölder stability and uniqueness for the case of lateral Cauchy data — 38</b>                |
| 3.1      | Introduction — 38                                                                              |
| 3.2      | Problem statement — 38                                                                         |
| 3.3      | Two forms of the kernel $K(x, y)$ — 40                                                         |
| 3.3.1    | The first form — 40                                                                            |
| 3.3.2    | The second form — 40                                                                           |
| 3.3.3    | The Carleman weight function and estimates of the integral operators in (3.17) and (3.20) — 41 |
| 3.4      | Carleman estimates for operators $\partial_t \pm \beta\Delta$ — 43                             |
| 3.4.1    | Step 1. Evaluate terms with $v_t$ in (3.31) — 44                                               |
| 3.4.2    | Step 2. Evaluate terms with $v_{x_i}$ in (3.31) — 45                                           |
| 3.4.3    | Step 3. Estimate $(u_t - \beta\Delta u) u\varphi_\lambda$ — 46                                 |

- 3.4.4 Step 4. Multiply (3.36) by  $5\lambda\beta$ , sum up with (3.35) and integrate the resulting inequality over  $Q_T$  — **46**
- 3.4.5 Step 5. Incorporating  $u_{x_i x_j}^2$  and  $u_t^2$  in the right-hand side of (3.38) — **47**
- 3.5 Hölder stability and uniqueness — **49**

## **4 Hölder stability for a coefficient inverse problem for the mean field games system — 58**

- 4.1 Introduction — **58**
- 4.2 Statement of coefficient inverse problem — **59**
- 4.3 Two forms of the kernel  $K(x, y)$  of the integral operator in (4.3) — **60**
  - 4.3.1 The first form — **60**
  - 4.3.2 The second form — **61**
  - 4.3.3 Estimates for two integral operators — **61**
- 4.4 Formulations of theorems — **62**
  - 4.4.1 Two Carleman estimates — **62**
  - 4.4.2 Hölder stability and uniqueness — **63**
- 4.5 Proof of Theorem 4.2 — **65**
  - 4.5.1 Estimating functions  $v$  and  $q$  — **70**
  - 4.5.2 Estimating functions  $w$  and  $r$  — **71**
  - 4.5.3 Estimate all four functions  $v, q, w, r$  simultaneously — **74**
  - 4.5.4 The target Hölder stability estimate — **74**
- 4.6 Proof of Theorem 4.3 — **77**

## **5 Convexification numerical method for the retrospective problem for mean field games — 78**

- 5.1 Introduction — **78**
- 5.2 Statement of the problem — **78**
- 5.3 Carleman estimates — **79**
- 5.4 Convexification — **81**
  - 5.4.1 The minimization problem — **81**
  - 5.4.2 The global strong convexity of the functional  $J_{\lambda, a}(u, p)$  on  $B(R)$  — **81**
  - 5.4.3 The accuracy of the minimizer — **87**
  - 5.4.4 The global convergence of the gradient descent method — **91**
- 5.5 Numerical studies — **93**
  - 5.5.1 Numerical data generation — **93**
  - 5.5.2 Numerical testing for the minimization problem — **94**

## **6 Convexification for a coefficient inverse problem for mean field games — 100**

- 6.1 Introduction — **100**
- 6.2 Problem statement — **100**
- 6.3 Convexification — **102**

|       |                                                                                       |            |
|-------|---------------------------------------------------------------------------------------|------------|
| 6.4   | Convergence analysis —                                                                | <b>106</b> |
| 6.4.1 | Carleman estimates —                                                                  | <b>106</b> |
| 6.4.2 | Global strong convexity and uniqueness —                                              | <b>107</b> |
| 6.4.3 | The accuracy of the minimizer and uniqueness —                                        | <b>108</b> |
| 6.4.4 | The gradient descent method —                                                         | <b>111</b> |
| 6.5   | Proof of Theorem 6.1 —                                                                | <b>112</b> |
| 6.6   | Proof of Theorem 6.2 —                                                                | <b>115</b> |
| 6.7   | Proof of Theorem 6.3 —                                                                | <b>118</b> |
| 6.7.1 | Analysis of $J_{1,\lambda}$ —                                                         | <b>119</b> |
| 6.7.2 | Analysis of $\lambda^{3/2}J_{1,\lambda} + J_{2,\lambda}$ and of $J_{\lambda,\beta}$ — | <b>122</b> |
| 6.8   | Proof of Theorem 6.4 —                                                                | <b>123</b> |
| 6.9   | Numerical studies —                                                                   | <b>127</b> |
| 6.9.1 | Numerical data generation —                                                           | <b>127</b> |
| 6.9.2 | Numerical arrangements —                                                              | <b>128</b> |
| 6.9.3 | Numerical tests —                                                                     | <b>131</b> |

## **Bibliography — 135**

## **Index — 139**