

Contents

Preface — V

Introduction — VII

Notation — XXIX

1 Preliminaries — 1

1.1 Basic concepts and research tools — 1

1.1.1 Lebesgue spaces with variable exponents $L^{p(x)}$ — 9

Part I: Multi-dimensional ρ -almost periodicity

2 Multi-dimensional ρ -almost periodic type functions and their Stepanov generalizations — 17

2.1 Multi-dimensional ρ -almost periodic type functions — 19

2.1.1 T -almost periodic type functions in finite-dimensional spaces — 30

2.1.2 $\mathbb{D}$ -asymptotically Bohr $(\mathcal{B}, I', \rho)$ -almost periodic type functions — 32

2.1.3 (ω, ρ) -periodic functions and $(\omega_j, \rho_j)_{j \in \mathbb{N}_n}$ -periodic functions — 37

2.1.4 Liouville type results for anti-periodic linear operators — 40

2.1.5 Applications to the heat equation in $\mathbb{R}^n$, the fractional diffusion-wave equation in $\mathbb{R}^n$ and the iterated polyharmonic operators in $\mathbb{R}^n$ — 42

2.1.6 New applications in one-dimensional setting — 44

2.1.7 Heteroclinic period blow-up — 45

2.2 (ω, ρ) -periodic solutions of abstract integro-differential impulsive equations on Banach space — 47

2.2.1 Nonhomogeneous linear impulsive problem — 50

2.2.2 (ω, ρ) -periodic solutions of integro-differential impulsive problem — 64

2.2.3 (ω, ρ) -BVP for impulsive integro-differential equations of fractional order on Banach space — 69

2.3 Stepanov multi-dimensional ρ -almost periodic functions in Lebesgue spaces with variable exponents — 75

2.3.1 Stepanov multi-dimensional Bochner transform, Stepanov distance and Stepanov norm — 75

2.3.2 Stepanov multi-dimensional ρ -almost periodicity — 76

2.3.3 $\mathbb{D}$ -asymptotically Stepanov ρ -almost periodic type functions — 85

2.4 Some new classes of ρ -almost periodic type functions in $\mathbb{R}^n$ — 88

2.4.1 $(S, \mathbb{D}, \mathcal{B})$ -asymptotically (ω, ρ) -periodic type functions and $(S, \mathcal{B})$ -asymptotically $(\omega_j, \rho_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$ -periodic type functions — 88

2.4.2	Multi-dimensional quasi-asymptotically ρ -almost periodic type functions — 92
2.4.3	Remotely ρ -almost periodic type functions — 93
2.4.4	ρ -Slowly oscillating type functions in $\mathbb{R}^n$ — 95
2.4.5	Applications to the abstract Volterra integro-differential equations — 97
2.4.6	Semi- $(\rho_j, \mathcal{B})_{j \in \mathbb{N}_n}$ -periodic functions — 102
3	Multi-dimensional ρ-almost periodic type functions: Weyl, Besicovitch and Doss generalizations — 104
3.1	Multi-dimensional Weyl ρ -almost periodic type functions — 104
3.1.1	Applications to the abstract Volterra integro-differential equations — 108
3.2	Multi-dimensional Doss ρ -almost periodic type functions — 110
3.2.1	Relationship between Weyl almost periodicity and Doss almost periodicity — 126
3.2.2	Invariance of Doss ρ -almost periodicity under the actions of convolution products — 129
3.2.3	Applications to the abstract Volterra integro-differential equations and partial differential equations — 131
3.3	Multi-dimensional Besicovitch almost periodic type functions and applications — 135
3.3.1	Multi-dimensional Besicovitch almost periodic type functions — 139
3.3.2	Multi-dimensional Besicovitch normal type functions — 147
3.3.3	Besicovitch–Doss almost periodicity — 150
3.3.4	On condition (B) — 155
3.3.5	Applications to the abstract Volterra integro-differential equations — 160
3.4	Besicovitch multi-dimensional almost automorphic type functions and applications — 171
3.4.1	Multi-dimensional Besicovitch almost automorphy in Lebesgue spaces with variable exponent — 173
3.4.2	Applications to the abstract Volterra integro-differential equations — 182
3.5	Multi-dimensional ρ -almost periodic type distributions — 187
3.5.1	ρ -almost periodic type distributions in $\mathbb{R}^n$ — 190
3.5.2	$\mathbb{D}$ -asymptotically ρ -almost periodic type distributions in $\mathbb{R}^n$ — 194
3.5.3	Some applications — 199
3.6	Notes and appendices — 201

Part II: Metrical ρ -almost periodicity

4	Metrically ρ-almost periodic type functions and applications — 235
4.1	$(R_X, \mathcal{B}, \mathcal{P}, L)$ -multi-almost periodic type functions — 235
4.1.1	Generalization of multi-dimensional (Stepanov) almost automorphy — 246

4.1.2	Bohr $(\mathcal{B}, I', \rho, \mathcal{P})$ -multi-almost periodic type functions —	250
4.1.3	Generalization of multi-dimensional (Stepanov) ρ -almost periodicity —	257
4.1.4	Applications to the abstract Volterra integro-differential equations —	259
4.2	Stepanov ρ -almost periodic functions in general metric —	261
4.2.1	Invariance of metrical Stepanov c -almost periodicity under the actions of infinite convolution products —	264
4.2.2	An application to the abstract semilinear fractional differential equations —	266
4.3	Weyl ρ -almost periodic functions in general metric —	269
4.3.1	Metrical Weyl distance —	270
4.3.2	Metrical Weyl ρ -almost periodic type functions —	273
4.3.3	Invariance of metrical Weyl ρ -almost periodicity under actions of infinite convolution products —	282
4.3.4	Applications to the abstract Volterra integro-differential equations —	284
5	Asymptotical ρ-almost periodicity in general metric —	288
5.1	Asymptotically ρ -almost periodic type functions in general metric —	288
5.1.1	Metrical $(S, \mathbb{D}, \mathcal{B})$ -asymptotical (ω, ρ) -periodicity and metrical $(S, \mathcal{B})$ -asymptotical $(\omega_j, \rho_j, \mathbb{D}_j)_{j \in \mathbb{N}_n}$ -periodicity —	288
5.1.2	Metrically ρ -slowly oscillating type functions in $\mathbb{R}^n$ —	291
5.1.3	Metrically quasi-asymptotically ρ -almost periodic type functions —	294
5.1.4	Applications to the abstract Volterra integro-differential equations —	298
5.2	Multi-dimensional weighted ergodic components in general metric —	300
5.2.1	Stepanov weighted ergodic components in general metric —	302
5.2.2	Weyl weighted ergodic components in general metric —	307
5.2.3	Weighted pseudo-ergodic components in general metric —	312
6	Metrical approximations of functions —	318
6.1	Metrical approximations: the main concept —	319
6.1.1	Metrical normality and metrical Bohr type definitions —	325
6.1.2	Metrically semi- $(c_j, \mathcal{B})_{j \in \mathbb{N}_n}$ -periodic functions —	330
6.2	Metrical approximations: Stepanov, Weyl, Besicovitch and Doss concepts —	332
6.2.1	Stepanov and Weyl metrical approximations —	332
6.2.2	Besicovitch and Doss metrical approximations —	340
6.3	Further results and applications —	345
6.3.1	Invariance under the actions of infinite convolution products —	346
6.4	Metrical almost periodicity: Poisson, Levitan and Bebutov concepts —	351
6.4.1	Levitan (N, c) -almost periodic functions and uniformly Poisson c -stable functions —	361
6.4.2	Multi-dimensional Levitan N -almost periodic functions —	363
6.4.3	Applications to the abstract Volterra integro-differential equations —	365

- 7 Special classes of metrically almost periodic functions — 371**
 - 7.1 Hölder ρ -almost periodic type functions in $\mathbb{R}^n$ — **371**
 - 7.1.1 Multi-dimensional Hölder ρ -almost periodic type functions — **372**
 - 7.1.2 Hölder- α -($\mathbb{F}$, $\mathcal{B}$)-boundedness and Hölder- α -($\mathbb{F}$, $\mathcal{B}$, Λ' , ρ)-continuity — **385**
 - 7.1.3 Extensions of Hölder ρ -almost periodic type functions — **391**
 - 7.1.4 Invariance of Hölder ρ -almost periodicity under the actions of convolution products — **393**
 - 7.1.5 Further applications and examples — **395**
 - 7.2 Generalized almost periodic functions with values in ordered Banach spaces — **399**
 - 7.2.1 Generalized Bohr $(\mathcal{B}, I'_1, I'_2, \rho_1, \rho_2)$ -almost periodic type functions — **400**
 - 7.2.2 Some subclasses of generalized Bohr $(\mathcal{B}, I'_1, I'_2, \rho_1, \rho_2, \mathcal{P})$ -almost periodic type functions — **404**
 - 7.2.3 Applications to the abstract Volterra integro-differential equations — **408**
 - 7.3 Notes and appendices to Part II — **417**

Part III: Almost periodic type solutions to integro-differential inclusions

- 8 Abstract fractional equations with proportional Caputo fractional derivatives — 427**
 - 8.1 Proportional fractional integrals and proportional Caputo fractional derivatives — **428**
 - 8.2 Abstract proportional Caputo fractional differential inclusions — **433**
 - 8.2.1 Solution operator families for $(\text{DFP})_{\mathbb{R}}^{\zeta}$ and $(\text{DFP})_{\mathbb{L}}^{\zeta}$ — **434**
 - 8.2.2 Some applications to the abstract Volterra integro-differential inclusions — **440**
 - 8.3 Almost periodic type solutions to semilinear proportional Caputo fractional differential equations — **443**
 - 8.4 Nonexistence of (ω, c) -periodic solutions of (286) and nonexistence of Poisson stable like solutions of (286) — **447**
 - 8.4.1 On quasi-periodic properties of proportional fractional integrals — **447**
- 9 Almost periodic type solutions to abstract impulsive Volterra integro-differential inclusions — 452**
 - 9.1 Abstract impulsive Volterra integro-differential inclusions — **452**
 - 9.1.1 Abstract impulsive differential inclusions of integer order — **454**
 - 9.1.2 Abstract impulsive higher-order Cauchy problems — **461**
 - 9.1.3 On abstract impulsive fractional differential inclusions — **463**
 - 9.1.4 Abstract Volterra integro-differential inclusions with impulsive effects — **466**

9.2	The existence and uniqueness of almost periodic type solutions —	471
9.2.1	Piecewise continuous almost periodic functions —	472
9.2.2	$(\mathcal{B}, \rho)$ -piecewise continuous almost periodic type functions —	474
9.2.3	Relations with Stepanov almost periodic type functions —	479
9.2.4	Composition principles for $(\mathcal{B}, (t_k))$ -piecewise continuous almost periodic type functions —	484
9.2.5	Asymptotically almost periodic type solutions of $(ACP)_{1;1}$ —	487
9.2.6	Asymptotically Weyl almost periodic type solutions of $(ACP)_{1;1}$ —	488
9.2.7	Besicovitch–Doss almost periodic type solutions of $(ACP)_{1;1}$ —	492
9.2.8	Almost periodic type solutions of the abstract higher-order impulsive Cauchy problems —	495
9.2.9	Almost periodic type solutions of the abstract Volterra integro-differential inclusions with impulsive effects —	497

Bibliography — 503

Index — 537

