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Spectral Flow

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A Functional Analytic and Index-Theoretic Approach

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Preface

Spectral flow as a mathematical concept and term was introduced by Atiyah and Lusztig in an unpublished work on paths of self-adjoint Fredholm operators on a Hilbert space, given explicitly as elliptic differential operators on a compact manifold. The basic idea was then picked up and further developed in the work of Atiyah, Patodi, and Singer on the index theory of pseudo-differential operators on a compact manifold with boundary [13, 14]. The definition given in [14] is of topological nature, applying only to closed paths (that is, loops). However, the work also contains an intuitive description of what spectral flow is: let $t \in [0,1] \mapsto H_t$ be a continuous path of self-adjoint Fredholm operators so that their spectrum is real and discrete close to 0, or otherwise stated, the spectrum consists of bands depending on t and bounded away from 0, complemented by low-lying eigenvalues, see Figure 1 for a sketch. Then the spectral flow of the path is simply the sum of all eigenvalues crossing through 0 weighted by the orientation of the passage.

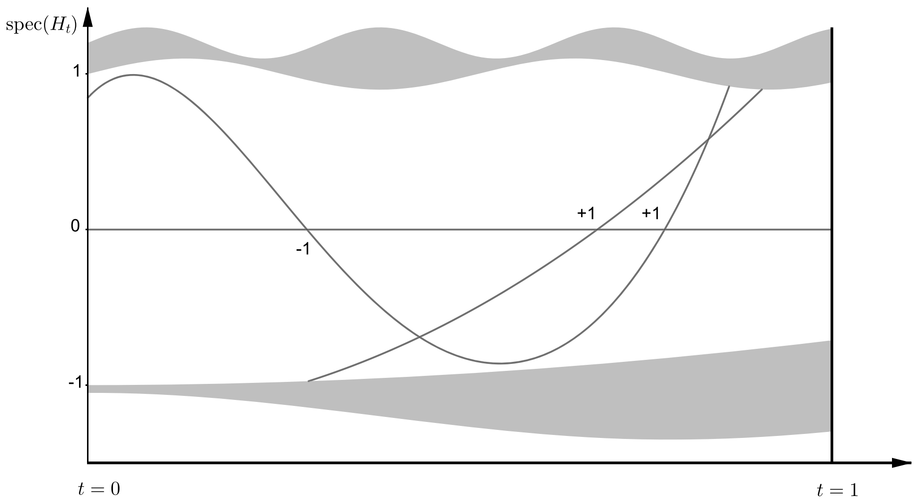

Figure 1: Schematic representation of the spectrum of a path of bounded self-adjoint Fredholm operators. The eigenvalues crossing through 0 give contributions to the spectral flow which are either −1 or +1, depending on the orientation of the crossing. The gray region shows the essential (continuous) spectrum. The spectral flow of $t \in [0,1] \mapsto H_t$ with spectrum as given in the picture is $-1 + 1 + 1 = 1$.

This book is about

(i) how to give a precise mathematical description of spectral flow;
(ii) what its fundamental properties are;
(iii) how to derive formulas for spectral flow;
(iv) how to connect spectral flow to index theory;

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(v) how to use spectral flow to extract topological information;
(vi) how to study the spectral flow of a continuous spectrum;
(vii) how spectral flow can be put to work in applications.

If the path $t \in [0,1] \mapsto H_t$ is real analytic, also the eigenvalues are real analytic (even at eigenvalue crossings by Rellich's theorem) and it is fairly immediate to properly define the spectral flow and hence settle item (i). But within the continuous category, it is not as obvious. For instance, defining the winding number of a continuous invertible function typically may take a full two-hour lecture in a topology class. In a more conceptual approach developed in the field of differential topology [105], one argues that the function can be assumed to be in a so-called generic transversal position. This approach has also been applied to other intersection numbers in finite-dimensional contexts [105] and the intersection theory of Lagrangian subspaces developed by Bott [35], Maslov [133], and Arnold [9]. In some wide sense, these contributions can be considered as finite-dimensional predecessors of the spectral flow as discussed here.

It turns out that there is a relatively elementary analytical approach to the definition of a spectral flow of arbitrary continuous paths. It is sketched in the work of Floer [87] (see p. 230 therein) and with full details in the independent work of Phillips [147]. It provides a solid analytical framework for the study of the spectral flow. Once it is achieved, one readily deduces the main structural properties of the spectral flow, most notably its homotopy invariance, a concatenation property, and monotonicity. Furthermore, one can also prove various formulas for the spectral flow, e. g., integral expressions [55, 56] and sums of intersection numbers, using crossing forms and the reduction to relative Morse indices [84].

In the last decades numerous variations and generalizations of the initial concept of a spectral flow have been developed. First of all, the spectral flow has also been studied for paths of unbounded self-adjoint Fredholm operators [31, 126, 196]. Secondly, the $\mathbb{R}$-valued spectral flow of paths of self-adjoint Breuer–Fredholm operators in semifinite von Neumann algebras has been introduced [144, 148, 26, 197]. In real Hilbert spaces or for paths having real symmetries, various $\mathbb{Z}_2$-valued variations of the spectral flow have been studied [57, 70, 76, 75]. Another variant considered a Clifford-algebra-valued spectral flow [37]. Spectral flows taking values in K-theory groups have been introduced as higher spectral flows [68, 195], see also [124, 68, 109, 193]. Pairing such K-theoretic spectral flows with cyclic cocycles one obtains a multiparameter spectral flow which is $\mathbb{Z}$-valued if one works with standard Fredholm operators and $\mathbb{R}$-valued in a semifinite setting [171]. Other notable additions to the theory of spectral flow concern the connections to η-invariants [14, 208, 109, 56] and the spectral shift function [155, 94].

As mentioned above, already in the initial work of Atiyah, Patodi, and Singer [14], a tight connection between spectral flow and index theory became apparent. More precisely, let $t \in \mathbb{R} \mapsto H_t$ be a path of self-adjoint Fredholm operators with well-defined asymptotics at $t = \pm\infty$ (as well as some additional properties), then $D_H = \partial_t - H_t$ is a Fredholm operator with index equal to minus the spectral flow of the family $t \in \mathbb{R} \mapsto H_t$.

This theorem about the equality "*spectral flow* = *Fredholm index*" has been very influential, with numerous generalizations in various directions [27, 176, 160, 1, 94]. Robbin and Salamon allowed for unbounded H_t with compact resolvent converging at $t = \pm\infty$ [160]. Moreover, Callias' index theorem [46] can be considered as a higher-dimensional generalization (i. e., t consists of more parameters) of [14] with finite-dimensional fibers. It has also been generalized to multiparameter families of self-adjoint Fredholm operators [43, 109, 193], and finally, a semifinite and noncommutative version was proved recently [171]. Another aspect of index theory are Kato's Fredholm pairs of projections and their index [112] (also called essential codimension in [42]). Furthermore, this index can be computed as a spectral flow, as first shown by Wojciechowski [207] and in more generality by Phillips [148], see also [70].

While initially the spectral flow was introduced as a tool for the index theory of differential operators on manifolds with boundary, most of the results described above were driven by the desire to get a better mathematical understanding of the spectral flow and its basic properties. There are by now also quite diverse applications of spectral flow. Many still concern the index theory on classical manifolds, but let us also mention a few applications to other fields:

– Spectral flow has served as a starting point for semifinite index theory in noncommutative geometry [51, 53, 54, 171]. Spectral flow is by now an established tool for the proof of numerous index-theoretic statements [130, 170, 75, 171].
– Spectral flow has lead to new perspectives and new results on the Bott–Maslov and Conley–Zehnder indices [30, 90, 113, 168, 203].
– In topological insulators (such as quantum Hall systems) and Wilson–Dirac lattice gauge theories, the so-called Laughlin argument is a result about a certain spectral flow [82, 153, 71]. Higher-dimensional versions of the Laughlin argument connect nonabelian monopoles to spectral flow [58].
– Spectral flow is linked to the vortex dynamics in Fermi superfluids [186, 116] (no rigorous results on this seem to be available).
– Spectral flow (of the unitary angles of Lagrangian subspaces) is used for the oscillation theory of Hamiltonian systems [166, 167, 173] and the spectral theory of surface states in topological insulators [174]. This allowed extending the theory to semifinite oscillation theory [101].
– Spectral flow leads to novel criteria in bifurcation theory [84, 85, 146, 151].

Many, but not all, of these aspects are addressed in this book.

The theory of the spectral flow is by now in a fairly mature state. This is documented by the numerous contributions listed in the references, and the bibliography is certainly not even exhaustive. There is a growing number of applications, see the list above. Some parts of the theory have been reviewed [26, 201], others are covered by sections or appendices in books of a broader scope [136] or more specialized nature [52]. Nevertheless, to date there does not seem to exist a book fully dedicated to the subject.

This book offers an in-depth and yet elementary treatise of spectral flow. We tried to carry out mathematical arguments with full details. Admittedly, this makes the text at some points pedantically precise. We hope that this makes the book accessible to master students and newcomers, both with a background in mathematics and physics. The experienced reader can certainly skim many pages and locate the points of interest. The presentation is of functional-analytic nature and on purpose restricted to a Hilbert space framework. We avoid (or circumvent) the use of K-theory, let alone KK-theory, even though several results in the book can naturally be formulated in that language. Hence all the reader should master is what is typically taught in a two-semester course in functional analysis, apart from some familiarity with notions of topology. More specifically, we suppose that Riesz' theory of compact operators, as well as the spectral calculus of bounded and unbounded self-adjoint operators, is known. On the other hand, a detailed treatment of the theory of Fredholm operators on Hilbert spaces is included, even though it may already be part of many introductory lectures on functional analysis.

Let us conclude this introduction with an overview of the contents of the book, chapter by chapter. This also allows stressing some novelties of the results, proofs, and presentation.

Chapter 1. The first introductory chapter is meant to give an intuitive understanding of what a spectral flow is in a restricted finite-dimensional context and to present and discuss many of the results to come later on. Hence the knowledge of linear algebra is a sufficient basis for understanding this chapter. Nevertheless, Phillips' construction of the spectral flow is already explained here and it is shown how it can be used to construct the winding number via the spectral flow through $-1 \in \mathbb{S}^1$. The chapter also contains a section on $\mathbb{Z}_2$-valued spectral flows in a finite-dimensional context and another section on multiparameter spectral flow. These two topics are not further developed later in the book, and the reader is referred to [57, 76, 75, 77] and [171], respectively, for a functional-analytic treatment for Fredholm operators.

Chapter 2. The second chapter uses the finite-dimensional spectral flow for the study of what is in general called the Maslov index, but is here referred to as Bott–Maslov index due to the much earlier contribution of Bott [35]. To consistently use the identification of the Lagrangian subspaces with their unitary phase via the stereographic projection makes the presentation considerably more transparent than in parts of the literature. Moreover, this serves as a preparation for the treatment of the infinite-dimensional Bott–Maslov index later on in Chapter 9. The same can be said about the finite-dimensional theory of the Conley–Zehnder index which is also spelled out in Chapter 2. Furthermore, the chapter contains a description of oscillation theory of block Jacobi matrices and scattering systems as an application of the Bott–Maslov and Conley–Zehnder indices. This is in spirit close to the original work of Bott.

Chapter 3. This chapter provides a detailed description of Fredholm operators in a two-Hilbert space setting. It provides a characterization by the essential spectrum and specializes the results to self-adjoint and unitary Fredholm operators. All of this can be found in many lecture notes and textbooks, so the chapter is included for the convenience of the less experienced readers. It allows stating well-known results that will be used later on, and it introduces some notations. Somewhat less well-known may be the content of Section 3.5 which proves an index theorem for the finite-dimensional spectral flow as introduced in Chapter 1. This result will be considerably generalized later on in the book.

Chapter 4. This chapter is central to the book. The reader already well-acquainted with Fredholm theory can jump directly to Chapter 4. Here the formal definition of the spectral flow of a continuous path of bounded self-adjoint Fredholm operators is given. Then the main properties are derived. Most of this follows closely the presentation of Phillips [147]. Moreover, the spectral flow is extended to the wider class of paths of essentially hyperbolic operators and then transferred to paths of essentially gapped unitary operators. Finally, Section 4.6 shows that the spectral flow of paths of self-adjoint Fredholm operators can be deduced from the spectral flow of a path of essentially gapped unitaries, and vice versa. This procedure is not obvious and apparently not worked out in detail elsewhere.

Chapter 5. This chapter begins with a self-contained modern presentation of Kato's Fredholm pairs of projections. Many elements are taken from the influential work of Avron, Seiler, and Simon [18], others have their roots in the literature on K-theory of operator algebras. Then the connection between the indices of Fredholm pairs and the spectral flow along the linear path connecting them is discussed in detail, together with various variations on the theme. This allows deriving various further formulas for the spectral flow of paths given by compact perturbations of the initial point. Furthermore, one can then rewrite the spectral flow as a sum of indices of Fredholm pairs, a formulation which also goes back to Phillips [148] and generalizes to the semifinite index (Chapter 11). Moreover, another formulation of the spectral flow in terms of relative Morse indices is presented, as put forward by Fitzpatrick, Pejsachowicz, and Recht [84].

Chapter 6. The chapter discusses various topologies on the set of unbounded Fredholm operators. In particular, the gap topology induced by the norm topology on the graph projections is analyzed. It is argued that on the set of self-adjoint Fredholm operators this is natural from numerous perspectives, more specifically by studying the topologies induced by the Cayley transform, as well as the bounded transform. Moreover, the Riesz topology is studied as it is often more readily accessible. The chapter also contains several results on the homotopy theory of the set of self-adjoint Fredholm operators equipped with the gap topology, for example, that it is connected and can be retracted to the subset of self-adjoint Fredholm operators with compact resolvent, see Section 6.4.

Chapter 7. This chapter introduces the spectral flow for paths of possibly unbounded self-adjoint Fredholm operators, essentially following the work of Booß–Bavnbek, Lesch, and Phillips [31]. Under certain summability assumptions, the spectral flow of paths of self-adjoint Fredholm operators with compact resolvent can be related to the η-invariants of its endpoints. The idea for this connection goes back to the work of Atiyah, Patodi, and Singer [14]. Section 7.2 provides a proof of this connection by elaborating on the work of Getzler [96] and Carey and Phillips [55, 56]. For particular paths of such self-adjoint Fredholm operators with compact resolvent stemming from families of Hamiltonian systems, the spectral flow is then connected to the Conley–Zehner index of the system. This is due to Robbin and Salamon [160] and is explained in Section 7.3. Finally, Section 7.4 returns to the topic "*spectral flow = Fredholm index*" and proves the most general result in this direction given in this book. We believe that the technique of proof is novel, apart from the parallel joint work with Stoiber on the more general Callias-type operators [172]. The technique gives a new perspective on the result itself. In fact, it is similar to Witten's semiclassical proof of the Morse inequalities [206, 67] and locally connects the eigenvalue crossings of $t \in \mathbb{R} \mapsto H_t$ to the low-lying spectrum of $D_{H,\kappa} = \kappa\partial_t - H_t$ for a small semiclassical parameter κ, and then simply turns up κ to 1.

Chapter 8. In this chapter it is shown that the spectral flow restricted to closed loops actually establishes a bijection with the fundamental group of the self-adjoint Fredholm operators. It is also shown that the spectral flow can be uniquely characterized by a few of its structural properties derived in Chapter 4. Several of the results of this chapter are considerably more general than needed for these applications to the spectral flow. In particular, all homotopy groups for the sets of Fredholm and self-adjoint Fredholm operators are obtained. For the unbounded self-adjoint Fredholm operators equipped with the gap topology, this is a somewhat surprising and deep result due to Joachim [108]. A recent preprint by Prokhorova [154] allowed to considerably simplify the argument presented in Section 8.6. Let us also note that Section 8.5 computes the homotopy groups of Fredholm pairs.

Chapter 9. This chapter can be seen as an application of Chapter 4, in the same way as Chapter 2 is an application of the spectral flow in finite dimension described in Chapter 1. It develops the theory of the Bott–Maslov and Conley–Zehnder index in an infinite-dimensional Krein space framework, condensing the by now numerous contributions to the topic. Somewhat novel is the characterization of (not necessarily Lagrangian) maximally isotropic subspaces in terms of the Krein signature, see Section 9.2. The chapter also includes an application of the Bott–Maslov index in infinite dimension to the computation of bound states of scattering systems by means of oscillation theory.

Chapter 10. This chapter is included to show the strength of spectral flow as a tool for proving statements in index theory. In a series of recent works [128, 129, 130, 131, 170], it was shown that index pairings resulting from pairing an even or odd unbounded Fredholm module with a differentiable projection or unitary, respectively, can be computed as the half-signature of a finite-dimensional matrix called the spectral localizer. The proof of this fact as presented here is based on a series of deformations of paths of self-adjoint Fredholm operators so that in the end remains a path of finite-dimensional self-adjoint matrices, for which by Chapter 1 the spectral flow is equal to the half-signature of the spectral localizer. The chapter contains also a discussion of the η-invariant in this context.

Chapter 11. This chapter begins by generalizing the theory of Fredholm operators described in Chapter 3 to semifinite von Neumann algebras and skew corners thereof. While this Breuer–Fredholm theory is widely used, detailed proofs are not available in the literature. Hence the chapter can be seen as an addendum to Takesaki's monumental work [189]. Based on this theory of semifinite Fredholm operators, the semifinite spectral flow is introduced and again its basic properties are described. Then formulas connecting the semifinite spectral flow to the semifinite index are presented. As an application, generalizations of the results of Chapter 10 are given.

Chapter 12. The final chapter is dedicated to yet another application of spectral flow, notably to variational bifurcation theory. There is relatively vast literature on this topic and various applications to differential equations have appeared. In the present introductory presentation, particular focus is on the bifurcation of branches of periodic orbits of Hamiltonian systems.

The book also contains a few technical appendices, a list of acronyms and notations, as well as an extended bibliography. Let us point out that the list of references is definitely not exhaustive. If there are notable contributions that do not appear here, it is due to the ignorance of the authors rather than any form of bad intentions.

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Halle, February 2023 — Nils Waterstraat

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Contents

1 Spectral flow in finite dimension

This introductory chapter presents the elementary theory of spectral flow through $0 \in \mathbb{R}$ for continuous paths of self-adjoint matrices. In this finite-dimensional setting, the spectral flow only depends on the signature of the matrices at the endpoints. Nevertheless, many of the structural properties, as well as useful formulas, for the spectral flow can already be understood in this elementary framework. Furthermore, one can rewrite the spectral flow in a form that is susceptible to be generalized to paths of unitary matrices where the spectral flow is then considered through the point $-1 \in \mathbb{S}^1$. This is carried out in detail and provides an alternative way to construct the winding number. Let us also note that this approach will later on (in Chapter 4) be extended to Fredholm operators on an infinite-dimensional Hilbert space. The chapter contains two further sections, one on a $\mathbb{Z}_2$-valued analogue of spectral flow and one on a multiparameter spectral flow, which are included as an outlook and guide to the literature. In the rest of the book, we will not elaborate on how these two sections extend to a Hilbert space framework.

1.1 From intuition to definition

Let us begin by recalling some basic facts from linear algebra. For an $N \times N$ matrix $H = (H_{n,m})_{n,m=1,\ldots,N} \in \mathbb{C}^{N\times N}$ with complex entries, the spectrum $\mathrm{spec}(H)$ consists of all complex numbers λ such that $\lambda \mathbf{1}_N - H$ is not an invertible matrix, namely such that $\det(\lambda\mathbf{1}_N - H) = 0$. The matrix is self-adjoint if it is equal to its adjoint $H^* = (\overline{H_{m,n}})_{n,m=1,\ldots,N}$. Now let $t \in [0,1] \mapsto H_t \in \mathbb{C}^{N\times N}$ be a path of self-adjoint matrices which for the moment is assumed to be real analytic. Then the eigenvalues $\lambda_1(t), \ldots, \lambda_N(t)$ are also real analytic paths provided that one chooses the correct branches at level crossings [112]. Intuitively, the spectral flow along this path counts the number of eigenvalues crossing 0 from left to right, minus the number of those eigenvalues crossing 0 from right to left. This makes sense as long as the endpoints H_0 and H_1 are invertible matrices, so that no ambiguity remains. As the analyticity allows clearly distinguishing the eigenvalue curves, one can also just look at the crossings of a single eigenvalue along the full path, and the sum over all its crossings (at 0) is equal to half the difference of the signs at the endpoints of the path. Hence the spectral flow of a real analytic path $t \in [0,1] \mapsto H_t = H_t^* \in \mathbb{C}^{N\times N}$ of self-adjoint matrices with invertible endpoints H_0 and H_1 is defined by

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \sum_{n=1}^{N} \frac{1}{2}\big(\mathrm{sgn}(\lambda_n(1)) - \mathrm{sgn}(\lambda_n(0))\big), \tag{1.1}$$

where $\mathrm{sgn}(\lambda) \in \{-1, +1\}$ is the sign of a real nonvanishing number λ. Thus the spectral flow effectively counts the spectrum flowing from the negative to the positive spectral semiaxis, minus the spectrum flowing from the positive to the negative semiaxis. Using the fundamental theorem of calculus and a smooth increasing function $g : \mathbb{R} \to [-1, 1]$

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which is equal to -1 on $(-\infty, -\epsilon]$ and to 1 on $[\epsilon, \infty)$, one can rewrite the spectral flow using the derivative $\partial_s g(\lambda_n(s))$ as

$$\begin{aligned}\mathrm{Sf}(t \in [0,1] \mapsto H_t) &= \sum_{n=1}^{N} \frac{1}{2} \int_0^1 ds\, \partial_s g(\lambda_n(s)) \\ &= \frac{1}{2} \int_0^1 ds\, \partial_s \operatorname{Tr}(g(H_s)) \\ &= \frac{1}{2} \int_0^1 ds\, \operatorname{Tr}(g'(H_s)\partial_s H_s), \end{aligned} \tag{1.2}$$

where $\operatorname{Tr}(A)$ denotes the trace of $A \in \mathbb{C}^{N\times N}$, the spectral theorem was used and ϵ was supposed to be sufficiently small so that neither H_1 nor H_0 has spectrum in $[-\epsilon, \epsilon]$.

Next let us express the spectral flow in terms of the signature. Recall that for any invertible self-adjoint matrix, the signature is defined as the difference between the number of positive and negative eigenvalues,

$$\mathrm{Sig}(H) = \#\{\lambda \in \mathrm{spec}(H) : \lambda > 0\} - \#\{\lambda \in \mathrm{spec}(H) : \lambda < 0\},$$

where each eigenvalue is counted with its multiplicity. If now $P^{>} = \chi(H > 0)$ as well as $P^{<} = \chi(H < 0)$ are the spectral projections of the positive/negative spectrum of H, then

$$\mathrm{Sig}(H) = \operatorname{Tr}(P^{>}) - \operatorname{Tr}(P^{<}).$$

Note that $\mathrm{Sig}(H) \in \{-N, -N+2, -N+4, \dots, N-2, N\}$. One other basic fact about the signature is Sylvester's theorem stating that for any invertible matrix $A \in \mathbb{C}^{N\times N}$,

$$\mathrm{Sig}(A^* H A) = \mathrm{Sig}(H).$$

Furthermore, by (1.1) the following result holds.

Proposition 1.1.1. *Given a real-analytic path $t \in [0,1] \mapsto H_t$ of self-adjoint matrices with invertible endpoints, its spectral flow satisfies*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}(\mathrm{Sig}(H_1) - \mathrm{Sig}(H_0)). \tag{1.3}$$

Let us note that, as both endpoints are invertible, the difference $\mathrm{Sig}(H_1) - \mathrm{Sig}(H_0)$ is indeed even so that the right-hand side of (1.3) is an integer. Proposition 1.1.1 has an immediate, important corollary.

Corollary 1.1.2. *The spectral flow of a curve of self-adjoint matrices with invertible endpoints only depends on the endpoints of the path. In particular, one can homotopically deform the path with fixed endpoints, and this does not change the spectral flow.*

Up to now we only considered real-analytic paths and actually it seems hopeless to count the eigenvalue crossings of a merely continuous path. On the other hand, as the spectral flow only depends on the endpoints due to Proposition 1.1.1, it is reasonable to simply define it by the right-hand side of (1.3). Furthermore, it will be helpful and convenient to drop the assumption that the endpoints are invertible. Then one has to make a choice on how to count zero eigenvalues of the endpoints. Other than in many standard works [147, 31], our choice is symmetric around 0.

Definition 1.1.3. Let $I = [t_0, t_1]$ be a bounded interval. Given $t \in I \mapsto H_t \in \mathbb{C}^{N\times N}$, a continuous path of self-adjoint matrices, the spectral flow is defined by

$$\mathrm{Sf}(t \in I \mapsto H_t) = \frac{1}{2}(\mathrm{Sig}(H_{t_1}) - \mathrm{Sig}(H_{t_0})), \tag{1.4}$$

where the signature of a self-adjoint matrix is given by the difference of the number of its positive and its negative eigenvalues.

Let us stress that $\mathrm{Sf}(t \in I \mapsto H_t)$ may take half-integer values. Figure 1.1 illustrates in which situations such half-integer values appear. A sufficient condition for having an integer-valued spectral flow is that the endpoints are invertible. Often we will choose the bounded interval I to be $[0, 1]$, a case to which one can restrict after applying an affine transformation. Let us also stress again that the definition (1.4) of the spectral flow of finite-dimensional matrices only depends on its endpoints and not on the path in between them so that one could also simply use the more compact notation $\mathrm{Sf}(H_0, H_1)$. Let us, however, already point out at this point that in infinite dimensions the spectral flow *does* depend on the path, and not only the endpoints. Therefore $\mathrm{Sf}(H_0, H_1)$ will stand for the spectral flow along straight-line paths, both in finite and infinite dimension,

$$\mathrm{Sf}(H_0, H_1) = \mathrm{Sf}(t \in [0, 1] \mapsto (1 - t)H_0 + tH_1).$$

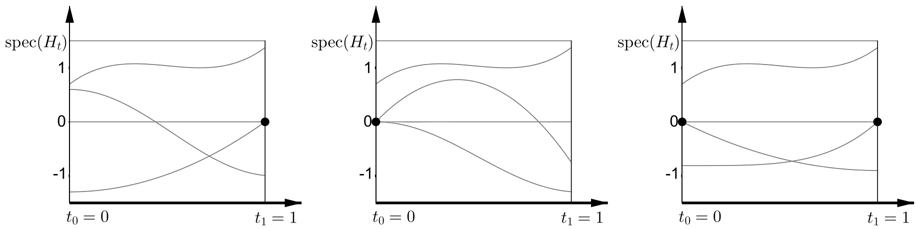

Figure 1.1: Schematic representation of how the kernels at the endpoints are accounted for. The spectral flow in the three figures is $-\frac{1}{2}$, -1, and 0, respectively.

Often it will also be of relevance to have unbounded intervals $I \subset \mathbb{R}$. Then an additional assumption is necessary to define the spectral flow:

Definition 1.1.4. Let $I \subset \mathbb{R}$ be a possibly unbounded interval and $t \in I \mapsto H_t \in \mathbb{C}^{N\times N}$ a continuous path of self-adjoint matrices such that $\sup_{t\in I\cap(t_+,\infty)} \|(H_t)^{-1}\| < \infty$ and $\sup_{t\in I\cap(-\infty,t_-)} \|(H_t)^{-1}\| < \infty$ for suitable $t_- < t_+$. Then the spectral flow of the path is defined as

$$\mathrm{Sf}(t \in I \mapsto H_t) = \frac{1}{2}(\mathrm{Sig}(H_{t_+}) - \mathrm{Sig}(H_{t_-})). \tag{1.5}$$

1.2 Structural properties of the spectral flow

Here we collect a number of basic properties of the spectral flow. Neither of them is difficult to prove in finite dimensions. All of these properties have generalizations to infinite dimensions. The proof of the first two statements follow immediately from the definition.

Proposition 1.2.1 (Path reversal). *Let $t \in [0,1] \mapsto H_t$ be a path of self-adjoint matrices. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_{1-t}) = -\mathrm{Sf}(t \in [0,1] \mapsto H_t).$$

Proposition 1.2.2 (Path reflection). *Let $t \in [0,1] \mapsto H_t$ be a path of self-adjoint matrices. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto -H_t) = -\mathrm{Sf}(t \in [0,1] \mapsto H_t).$$

Proposition 1.2.3 (Concatenation). *Let $t \in [0,2] \mapsto H_t$ be a path of self-adjoint matrices. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [1,2] \mapsto H_t) = \mathrm{Sf}(t \in [0,2] \mapsto H_t).$$

Proof. By Definition 1.1.3,

$$\begin{aligned}
&\mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [1,2] \mapsto H_t)\\
&\quad = \frac{1}{2}(\mathrm{Sig}(H_1) - \mathrm{Sig}(H_0)) + \frac{1}{2}(\mathrm{Sig}(H_2) - \mathrm{Sig}(H_1))\\
&\quad = \frac{1}{2}(\mathrm{Sig}(H_2) - \mathrm{Sig}(H_0))\\
&\quad = \mathrm{Sf}(t \in [0,2] \mapsto H_t)
\end{aligned}$$

showing the claim. □

Proposition 1.2.4 (Homotopy invariance). *For $s \in [0,1]$ let $t \in [0,1] \mapsto H_t(s)$ be a continuous path of self-adjoint matrices such that $s \in [0,1] \mapsto H_0(s)$ and $s \in [0,1] \mapsto H_1(s)$ are paths in the invertible matrices. Then*

$$s \in [0,1] \mapsto \mathrm{Sf}(t \in [0,1] \mapsto H_t(s))$$

is constant.

Proof. As $s \in [0,1] \mapsto \mathrm{Sig}(H_0(s))$ and $s \in [0,1] \mapsto \mathrm{Sig}(H_1(s))$ are constant, the claim follows from Definition 1.1.3. □

Proposition 1.2.5 (Invariance under conjugation). *Let $t \in [0,1] \mapsto H_t$ be a path of self-adjoint matrices and A an invertible matrix. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto A^* H_t A) = \mathrm{Sf}(t \in [0,1] \mapsto H_t).$$

Proof. By Sylvester's theorem, $\mathrm{Sig}(A^* H_0 A) = \mathrm{Sig}(H_0)$ and $\mathrm{Sig}(A^* H_1 A) = \mathrm{Sig}(H_1)$. Definition 1.1.3 allows us to conclude the argument. □

Proposition 1.2.6 (Additivity). *Let $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t$ be paths of self-adjoint matrices. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t \oplus H'_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [0,1] \mapsto H'_t).$$

Proof. As $\mathrm{Sig}(H_0 \oplus H'_0) = \mathrm{Sig}(H_0) + \mathrm{Sig}(H'_0)$ and $\mathrm{Sig}(H_1 \oplus H'_1) = \mathrm{Sig}(H_1) + \mathrm{Sig}(H'_1)$, the claim holds by Definition 1.1.3. □

Proposition 1.2.7 (Monotonicity). *Let $t \in [0,1] \mapsto H_t$ be an increasing path of self-adjoint matrices. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) \geq 0.$$

Proof. As the path is increasing, $H_1 \geq H_0$. Therefore $\mathrm{Sig}(H_1) \geq \mathrm{Sig}(H_0)$, and Definition 1.1.3 allows us to conclude. □

1.3 Alternative expressions for the spectral flow

The concatenation procedure stated in Proposition 1.2.3 can be iterated, namely if $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ is a finite partition of $[0,1]$, then

$$\begin{aligned}\mathrm{Sf}(t \in [0,1] \mapsto H_t) &= \sum_{m=1}^{M} \mathrm{Sf}(H_{t_m}, H_{t_{m-1}}) \\ &= \frac{1}{2} \sum_{m=1}^{M} (\mathrm{Sig}(H_{t_m}) - \mathrm{Sig}(H_{t_{m-1}})).\end{aligned}$$

This can be further modified to an expression that will be the starting point for the definition of the spectral flow of paths of unitaries (in Section 1.5), as well as of self-adjoint Fredholm operators on an infinite-dimensional Hilbert space, as given by Phillips [147] (see Chapter 4). For each interval $[t_{m-1}, t_m]$, let us next choose an $a_m \geq 0$ such that $a_m \notin \operatorname{spec}(H_t)$ and $-a_m \notin \operatorname{spec}(H_t)$ for all $t \in [t_{m-1}, t_m]$. Then the spectral projections

$$P_{a_m,t} = \chi_{[-a_m,a_m]}(H_t) \tag{1.6}$$

have constant rank for all $t \in [t_{m-1}, t_m]$. Note that these spectral projections are orthogonal, namely self-adjoint. Furthermore, the operators

$$P_{a_m,t}H_tP_{a_m,t} = P_{a_m,t}H_t = H_tP_{a_m,t}$$

are self-adjoint for all $t \in [t_{m-1}, t_m]$ and, because no eigenvalues leave or enter the spectral interval $[-a_m, a_m]$, one has

$$\operatorname{Sig}(H_{t_m}) - \operatorname{Sig}(H_{t_{m-1}}) = \operatorname{Sig}(H_{t_m}P_{a_m,t_m}) - \operatorname{Sig}(H_{t_{m-1}}P_{a_m,t_{m-1}}).$$

Therefore

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\sum_{m=1}^{M}\left(\operatorname{Sig}(H_{t_m}P_{a_m,t_m}) - \operatorname{Sig}(H_{t_{m-1}}P_{a_m,t_{m-1}})\right). \tag{1.7}$$

One can further manipulate this expression by setting

$$P^{>}_{a_m,t} = \chi_{(0,a_m]}(H_t), \quad P^{<}_{a_m,t} = \chi_{[-a_m,0)}(H_t),$$

as well as

$$P^{\geq}_{a_m,t} = \chi_{[0,a_m]}(H_t), \quad P_{0,t} = \chi_{\{0\}}(H_t).$$

Then, if $a \notin \operatorname{spec}(H_t)$,

$$\operatorname{Sig}(H_tP_{a,t}) = \operatorname{Tr}(P^{>}_{a,t} - P^{<}_{a,t}).$$

Thus with the a_m as above,

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\sum_{m=1}^{M}\operatorname{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}). \tag{1.8}$$

Now

$$\operatorname{Tr}(P^{<}_{a_m,t}) = \operatorname{Tr}(P_{a_m,t} - P^{\geq}_{a_m,t}) = \operatorname{Tr}(P_{a_m,t} - P^{>}_{a_m,t} - P_{0,t}),$$

so that $\operatorname{Tr}(P_{a_m,t_m}) = \operatorname{Tr}(P_{a_m,t_{m-1}})$ substituted into (1.8) implies

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\operatorname{Tr}(P_{0,0}) + \sum_{m=1}^{M}\operatorname{Tr}(P^{\geq}_{a_m,t_m} - P^{\geq}_{a_m,t_{m-1}}) - \frac{1}{2}\operatorname{Tr}(P_{0,1}). \tag{1.9}$$

As already pointed out, all three expressions (1.7), (1.8), (1.9) coincide, extend to self-adjoint Fredholm operators, and essentially directly lead to Phillips' definition of the spectral flow [147], the difference being the boundary terms $\frac{1}{2}\,\mathrm{Tr}(P_{0,0} - P_{0,1})$ which can be half-integer. The present symmetric treatment of the kernels at the endpoints assures the path reflection property. On the other hand, the map $s \mapsto \mathrm{Sf}(t \in [0,s] \mapsto H_t)$ is, due to these terms, in general neither right nor left continuous, and furthermore the map $s \mapsto \mathrm{Sf}(t \in [0,1] \mapsto H_t + s\mathbf{1})$ is not right continuous.

Another approach to generalize the notion of spectral flow to infinite dimensions, actually also going back to the work [14], is to still use (1.4), but with a modified definition of the signature. On an infinite-dimensional Hilbert space, one often has the situation that there are an infinite number of positive eigenvalues, as well as an infinite number of negative eigenvalues. To take the difference with these two infinities, one can attempt to work with a ζ-function regularization and this leads to the so-called η-invariant of an invertible self-adjoint matrix H,

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\big(H|H|^{-s-1}\big).$$

Now for an invertible matrix, one readily finds that

$$\eta(H) = \mathrm{Sig}(H). \tag{1.10}$$

Moreover, due to the integral identity

$$|\lambda|^{-s-1} = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt\, t^{\frac{s-1}{2}} e^{-t\lambda^2},$$

one can rewrite the above definition of the η-invariant as follows:

$$\begin{aligned} \eta(H) &= \lim_{s\to 0} \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt\, t^{\frac{s-1}{2}}\, \mathrm{Tr}\big(He^{-tH^2}\big) \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty dt\, t^{-\frac{1}{2}}\, \mathrm{Tr}\big(He^{-tH^2}\big), \end{aligned} \tag{1.11}$$

because $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In this form, the identity $\eta(H) = \mathrm{Sig}(H)$ results from

$$\frac{1}{\sqrt{\pi}} \int_0^\infty dt\, t^{-\frac{1}{2}} \lambda e^{-t\lambda^2} = \mathrm{sgn}(\lambda), \quad \lambda \in \mathbb{R}, \tag{1.12}$$

which can readily be checked by a change of variables. If now H is a self-adjoint operator on an infinite-dimensional Hilbert space having compact resolvent, then often e^{-tH^2} is of trace class and, under suitable further properties, it may be possible to show that the integral in (1.11) is convergent. If this is the case, one says that the η-invariant is well-defined. Then the spectral flow is given by $\frac{1}{2}(\eta(H_1) - \eta(H_0))$, similar as in (1.4). In this book, the spectral flow will be defined using Phillips approach and then the connection to the η-invariant is a consequence under particular assumptions.

1.4 Spectral flow as a sum of eigenvalue crossings

The following approximation result will allow computing the spectral flow for generic paths by counting eigenvalue crossings, see Propositions 1.4.3 and 1.4.5 below.

Proposition 1.4.1. *Let $t \in [0,1] \mapsto H_t = H_t^* \in \mathbb{C}^{N\times N}$ be a continuous path of self-adjoint matrices. For any $\epsilon > 0$, there exists a real-analytic path $t \in [0,1] \mapsto \hat{H}_t = \hat{H}_t^* \in \mathbb{C}^{N\times N}$ with $\|\hat{H}_t - H_t\| < \epsilon$ uniformly in t such that all eigenvalue crossings are simple and transversal, namely $\dim(\mathrm{Ker}(\hat{H}_t)) \le 1$ and $\partial_t \hat{H}_t|_{\mathrm{Ker}(\hat{H}_t)} \ne 0$, and, moreover, with invertible endpoints $\hat{H}_0$ and $\hat{H}_1$.*

Proof. The Weierstrass approximation theorem implies that there exists a real-analytic path $t \in [0,1] \mapsto \tilde{H}_t = (\tilde{H}_t)^* \in \mathbb{C}^{N\times N}$ with $\|\tilde{H}_t - H_t\| < \frac{\epsilon}{2}$ uniformly in t. By Theorem II.1.10 in [112], there is a real-analytic path of unitaries $t \in [0,1] \mapsto U_t$ such that one has $U_t^* \tilde{H}_t U_t = \mathrm{diag}(\lambda_1(t), \dots, \lambda_N(t))$, where $t \mapsto \lambda_k(t)$ are real-analytic functions representing the eigenvalues of $\tilde{H}_t$. By Sard's theorem, the complement of the set of regular values (points with nonvanishing derivative) of the eigenvalues λ_k, with $k = 1, \dots, N$, in $(-\frac{\epsilon}{2}, +\frac{\epsilon}{2})$ has measure zero. Thus there are $\delta_1, \dots, \delta_N \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ such that 0 is a common regular value of the functions $t \mapsto \lambda_k(t) + \delta_k$ for $k = 1, \dots, N$ and such that $\dim(\mathrm{Ker}(\mathrm{diag}(\lambda_1(t) + \delta_1, \dots, \lambda_N(t) + \delta_N))) \le 1$ for all $t \in [0,1]$, as well as $\dim(\mathrm{Ker}(\mathrm{diag}(\lambda_1(t) + \delta_1, \dots, \lambda_N(t) + \delta_N))) = 0$ for $t = 0, 1$. Then the continuous path $t \in [0,1] \mapsto \hat{H}_t = U_t \,\mathrm{diag}(\lambda_1(t) + \delta_1, \dots, \lambda_N(t) + \delta_N) U_t^*$ has the desired properties. In particular, the crossing eigenvalues $\hat{\lambda}_k(t)$ of $\hat{H}_t$ have derivatives $\hat{\lambda}_k'(t) = \partial_t \hat{\lambda}_k(t)$ given by

$$\hat{\lambda}_k'(t) = \langle \phi_k(t) \mid (\partial_t \hat{H}_t) \phi_k(t) \rangle,$$

where $\phi_k(t)$ is the (only) unit eigenvector of $\hat{\lambda}_k(t) = 0$, namely $\hat{H}_t \phi_k(t) = 0$ and $\|\phi_k(t)\| = 1$. □

Remark 1.4.2. Let us note the path $\hat{H}_t$ is *not* constructed by first diagonalizing H_t and then approximating the eigenvalues and eigenfunctions by smoothened versions. Indeed, this procedure is impossible because Example II.5.3 in [112] shows that the eigenvectors of H_t may even not be continuous. Instead, the above proof first approximates the matrix elements by analytic objects. ◇

By homotopy invariance, see Proposition 1.2.4, of the spectral flow as defined in Definition 1.1.3, one can now compute the spectral flow by the real-analytic path,

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto \hat{H}_t),$$

provided that there is a path of invertible self-adjoints connecting $\hat{H}_0$ to H_0 and a path of invertible self-adjoints connecting $\hat{H}_1$ to H_1 (which, of course, requires that H_0 and H_1 are invertible themselves). In particular, this is the case for $\epsilon < \min\{\|H_0^{-1}\|^{-1}, \|H_1^{-1}\|^{-1}\}$. By Proposition 1.1.1, the spectral flow of the path $t \in [0,1] \mapsto \hat{H}_t$ can also be computed by the alternative formulas (1.1) and (1.2). These formulas lead to yet another expression for the spectral flow.

Proposition 1.4.3. *If $t \in [0,1] \mapsto H_t$ is a continuously differentiable path with simple and transversal eigenvalue crossings and invertible endpoints as given in Proposition* 1.4.1, *then*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \sum_{\lambda_j(t)=0} \mathrm{sgn}(\lambda_j'(t)), \tag{1.13}$$

where $t \in [0,1] \mapsto \lambda_j(t)$ denote the eigenvalue curves of H_t that are continuously differentiable at any eigenvalue crossing and the sum runs over pairs (j,t) such that $\lambda_j(t) = 0$.

As a preamble to the proof, let us recall the Kato continuity and selection theorem. Thus consider a continuously differentiable path $t \in [0,1] \mapsto H_t \in \mathbb{C}^{N\times N}$ with invertible endpoints. At the endpoints 0 and 1, the derivatives are defined as the left, resp. right, limits of the derivatives, or alternatively using only left, resp. right, difference quotients. Then by Theorem II 6.8 in [112], the eigenvalues $\lambda_1(t), \dots, \lambda_N(t)$ are continuously differentiable provided that one chooses the correct branches at level crossings. Moreover, for $t_0 \in [0,1]$ let $\lambda_{k_1}(t), \dots, \lambda_{k_n}(t)$ be the eigenvalues of H_t such that $\lambda_{k_1}(t_0) = \dots = \lambda_{k_n}(t_0) = \delta \in \mathbb{R}$. Then (e. g., by Theorem II 5.4 in [112]) the derivatives of these eigenvalues coincide with the spectrum of $\chi_{\{\delta\}}(H_{t_0})(\partial_t H)_{t_0}\chi_{\{\delta\}}(H_{t_0})$ seen as an operator on $\mathrm{Ran}(\chi_{\{\delta\}}(H_{t_0}))$. More precisely,

$$\{\lambda_{k_1}'(t_0), \dots, \lambda_{k_n}'(t_0)\} = \mathrm{spec}(\chi_{\{\delta\}}(H_{t_0})(\partial_t H)_{t_0}\chi_{\{\delta\}}(H_{t_0})).$$

Proof. Let us first note that the sum on the right-hand side of (1.13) is finite by the genericity assumption, which also implies that the signs $\mathrm{sgn}(\lambda_j'(t))$ at these points are well defined. Consider $t_0 \in (0,1)$ such that $\mathrm{Ker}(H_{t_0}) \neq \{0\}$ and choose $a > 0$ such that $\mathrm{spec}(H_{t_0}) \cap [-a,a] = \{0\}$. Then there is $\epsilon > 0$ such that $\pm a \notin \mathrm{spec}(H_t)$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Let $\lambda : (t_0 - \epsilon, t_0 + \epsilon) \to (-a,a)$ be the continuously differentiable function representing the eigenvalue of H_t in $[-a,a]$. Because $\lambda'(t_0) \neq 0$ there is $0 < \eta < \frac{\epsilon}{2}$ such that $\lambda(t) \neq 0$ for $t \in (t_0 - 2\eta, t_0 + 2\eta) \setminus \{t_0\}$. This implies

$$\mathrm{sgn}(\lambda(t_0 + \eta)) = -\mathrm{sgn}(\lambda(t_0 - \eta)) = \mathrm{sgn}(\lambda'(t_0)),$$

and therefore

$$\operatorname{sgn}(\lambda'(t_0)) = \operatorname{Tr}(\chi_{[0,a]}(H_{t_0+\eta})) - \operatorname{Tr}(\chi_{[0,a]}(H_{t_0-\eta})).$$

As $t \in (t_0 - 2\eta, t_0 + 2\eta) \mapsto \chi_{(a,\infty)}(H_t)$ is continuous and thus the integer-valued map $t \in (t_0 - 2\eta, t_0 + 2\eta) \mapsto \operatorname{Tr}(\chi_{(a,\infty)}(H_t))$ is constant, one has

$$\begin{aligned}
\operatorname{sgn}(\lambda'(t_0)) &= \operatorname{Tr}(\chi_{[0,\infty)}(H_{t_0+\eta})) - \operatorname{Tr}(\chi_{[0,\infty)}(H_{t_0-\eta})) \\
&= \frac{1}{2}(\operatorname{Sig}(H_{t_0+\eta}) - \operatorname{Sig}(H_{t_0-\eta})) \\
&= \operatorname{Sf}(t \in [t_0 - \eta, t_0 + \eta] \mapsto H_t).
\end{aligned}$$

The concatenation property of the spectral flow, see Proposition 1.2.3, implies the claim. □

In some situations, one is confronted with paths which are not generic in the above sense, and one would not like to deform them into a generic one as in Proposition 1.4.1. A typical example is a path with certain symmetry properties. Under a weaker genericity assumption (so-called regular crossings), it is nevertheless possible to find a generalization of (1.13) which uses the notion of crossing form [160, 84, 200].

Definition 1.4.4. Let $t \in [0,1] \mapsto H_t = H_t^*$ be a continuously differentiable path of self-adjoint matrices. An instant $t \in [0,1]$ is called a crossing if $\operatorname{Ker}(H_t) \neq \{0\}$. Then the crossing form at t is the quadratic form

$$\Gamma_t : \operatorname{Ker}(H_t) \to \mathbb{R}, \quad \Gamma_t(\phi) = \langle \phi | (\partial_t H)_t \phi \rangle.$$

A crossing is called regular if Γ_t is nondegenerate.

By the above, a crossing t_0 is regular if and only if all derivatives $\lambda_k'(t_0)$ of eigenvalues contributing to the kernel of H_{t_0} do not vanish at the point $t = t_0$. In particular, regular crossings are isolated. Moreover, if H_0, H_1 are invertible, then there is $\epsilon > 0$ such that $H_0 + \delta\mathbf{1}$ and $H_1 + \delta\mathbf{1}$ are invertible for all $\delta \in (-\epsilon, \epsilon)$. Then, by Proposition 1.2.4, $\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \operatorname{Sf}(t \in [0,1] \mapsto H_t + \delta\mathbf{1})$. Again appealing to Sard's theorem, the complement of the set of common regular values of the eigenvalues λ_k, $k = 1, \dots, N$ in $(-\epsilon, \epsilon)$ has measure zero. Therefore, for the computation of the spectral flow it is sufficient to consider paths $t \in [0,1] \mapsto H_t$ that have only regular crossings. The next result provides the formula expressing the spectral flow in terms of all crossing forms, in particular the endpoints are not necessarily invertible. The reader is invited to check that the boundary terms are correct by inspecting once again Figure 1.1.

Proposition 1.4.5. *Let $t \in [0,1] \mapsto H_t \in \mathbb{C}^{N\times N}$ be a continuously differentiable path of self-adjoint matrices with only regular crossings. Then the spectral flow of this path is*

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\operatorname{Sig}(\Gamma_0) + \sum_{t\in(0,1)} \operatorname{Sig}(\Gamma_t) + \frac{1}{2}\operatorname{Sig}(\Gamma_1), \tag{1.14}$$

where $\operatorname{Sig}(\Gamma_t)$ *denotes the signature of the quadratic form* Γ_t.

Proof. Let us suppose for simplicity that H_0 and H_1 are invertible. As already stated, a look at Figure 1.1 shows how to deal with nontrivial kernels at these points (details are also given in the proof of Proposition 4.3.6 below). First note that by the above regular crossings are isolated, thus the sum on the right-hand side of (1.14) is finite. Moreover, if the path $t \in [0,1] \mapsto H_t$ consists of invertibles, both sides of (1.14) vanish. Consider one regular crossing $t_0 \in (0,1)$. Choose $a > 0$ such that $\mathrm{spec}(H_{t_0}) \cap [-a,a] = \{0\}$. Then there is $\epsilon > 0$ such that $\pm a \notin \mathrm{spec}(H_t)$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. For $m = \dim(\mathrm{Ker}(H_{t_0}))$, let

$$\lambda_1, \dots, \lambda_m : (t_0 - \epsilon, t_0 + \epsilon) \to (-a, a)$$

be the continuously differentiable eigenvalues of H_t in $[-a,a]$. Because t_0 is a regular crossing, $\lambda_k'(t_0) \neq 0$ for $k = 1, \dots, m$. Therefore, there is $0 < \eta < \frac{\epsilon}{2}$ such that $\lambda_k(t) \neq 0$ for $k = 1, \dots, m$ and $t \in (t_0 - 2\eta, t_0 + 2\eta) \setminus \{t_0\}$. This implies

$$\mathrm{sgn}(\lambda_k(t_0 + \eta)) = -\,\mathrm{sgn}(\lambda_k(t_0 - \eta)) = \mathrm{sgn}(\lambda_k'(t_0)), \quad k = 1, \dots, m.$$

Summing over all eigenvalues λ_k shows

$$\mathrm{Sig}(\Gamma_t) = \mathrm{Tr}((\chi_{[0,a]}(H_{t_0+\eta})) - \mathrm{Tr}(\chi_{[0,a]}(H_{t_0-\eta})).$$

As $t \in (t_0 - 2\eta, t_0 + 2\eta) \mapsto \chi_{(a,\infty)}(H_t)$ is continuous and therefore the integer-valued continuous map $t \in (t_0 - 2\eta, t_0 + 2\eta) \mapsto \mathrm{Tr}(\chi_{(a,\infty)}(H_t))$ is constant, one has

$$\begin{aligned}
\mathrm{Sig}(\Gamma_t) &= \mathrm{Tr}(\chi_{[0,\infty)}(H_{t_0+\eta})) - \mathrm{Tr}(\chi_{[0,\infty)}(H_{t_0-\eta})) \\
&= \frac{1}{2}(\mathrm{Sig}(H_{t_0+\eta}) - \mathrm{Sig}(H_{t_0-\eta})) \\
&= \mathrm{Sf}(t \in [t_0 - \eta, t_0 + \eta] \mapsto H_t).
\end{aligned}$$

The concatenation property of the spectral flow, see Proposition 1.2.3, implies the claim. □

1.5 The spectral flow for paths of unitaries

Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a (not necessarily closed) continuous path of unitary $N \times N$ matrices. For each t, the spectrum of U_t lies on the unit circle $\mathbb{S}^1$. The aim of this section is to define a spectral flow of this path through $-1 \in \mathbb{S}^1$ in the positive sense. For the particular case of a closed path, this spectral flow is nothing but the standard winding number. There are several ways to approach the definition of the spectral flow of a path of unitaries. One way is to first show that the path can be approximated by a smooth one (the so-called generic position in differential topology [105]) and then resile to a formula similar to (1.2). Another approach introduces self-adjoint matrices $H_t = -\imath \log(U_t)$ by choosing a suitable branch of the logarithm and then applies Definition 1.1.3 to it.

However, suppose one chooses the branch cut on the positive real axis, then 1 is a special point which, after all, should intuitively be irrelevant for the spectral flow through −1 that only depends on eigenvalues close to −1 (this is avoided by the so-called homotopy lifting lemma). This expectation turns out to be true, see Section 4.5 which shows that one may even allow for essential spectrum away from −1, provided the suitable definition of spectral flow is used which is presented in this section. The basic idea of this construction goes back to Phillips [147] and will be used later on. Hence this section serves as an intuitive preparation for the following. Before starting out, let us point the interested reader to a leisurely introduction of the winding number and its applications by Roe [161].

Let us begin with a technical preparation. For $a \in [0,\pi)$, the spectral projections are denoted by

$$P_{a,t} = \chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(U_t), \tag{1.15}$$

where χ_S denotes the characteristic function onto the set $S \subset \mathbb{C}$. These are the natural counterparts of the spectral projections (1.6) in the case of paths of self-adjoint matrices.

Lemma 1.5.1. *For a unitary matrix $U \in \mathrm{U}(N)$, there are a number $a \in [0,\pi)$ and a neighborhood $\mathcal{N}$ of U in $\mathrm{U}(N)$ such that $V \mapsto \chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(V)$ is a norm-continuous, projection-valued map on $\mathcal{N}$.*

Proof. There is an $a \in [0,\pi)$ such that $e^{\imath(\pi\pm a)}$ are not in the spectrum of U. Then there exists $\pi - a > \epsilon > 0$ such that the set

$$\mathcal{N}_{a,\epsilon} = \{e^{\imath(\pi+b)} : b \in [-a-\epsilon,-a] \cup [a, a+\epsilon]\}$$

is disjoint from $\mathrm{spec}(U)$. The set

$$\mathcal{N} = \{V \in \mathrm{U}(N) : \mathcal{N}_{a,\epsilon} \cap \mathrm{spec}(V) = \emptyset\}$$

is open and on this set the function $V \mapsto \chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(V)$ is norm-continuous as $\chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}$ agrees with the continuous function $f : \mathbb{S}^1 \to \mathbb{C}$ defined by

$$\begin{aligned} e^{\imath\varphi} \mapsto \chi_{[\pi-a,\pi+a]}(\varphi) &- (\varphi - (\pi + a + \epsilon))\frac{1}{\epsilon}\chi_{[\pi+a,\pi+a+\epsilon]}(\varphi) \\ &+ (\varphi - (\pi - a - \epsilon))\frac{1}{\epsilon}\chi_{[\pi-a-\epsilon,\pi-a]}(\varphi). \end{aligned}$$

This concludes the argument. □

By compactness and the previous lemma, it is possible to choose a finite partition

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1,$$

of $[0,1]$ and $a_m \in [0,\pi)$, $m = 1,\dots,M$, such that

$$t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$$

is continuous with constant rank. Each projection $P_{a,t}$ can be decomposed into

$$P^{>}_{a,t} = \chi_{\{e^{\imath b}:b\in(\pi,\pi+a]\}}(U_t), \quad P^{<}_{a,t} = \chi_{\{e^{\imath b}:b\in[\pi-a,\pi)\}}(U_t).$$

This is illustrated in Figure 1.2.

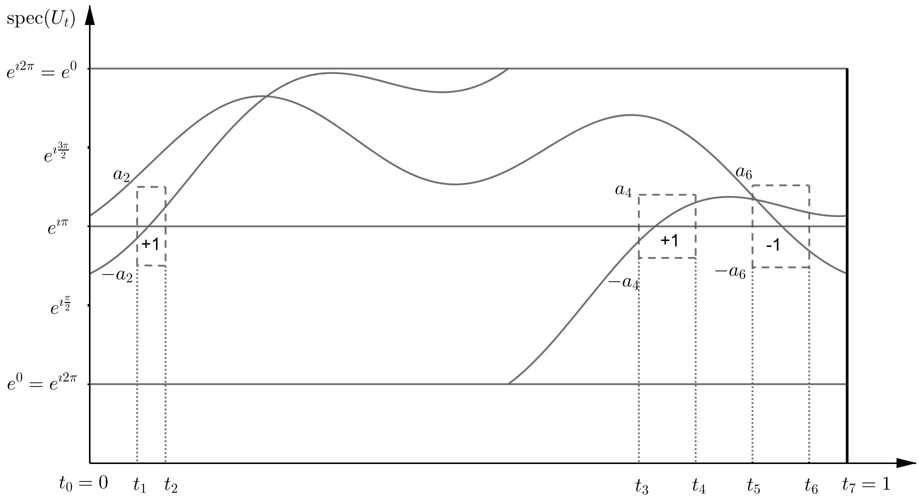

Figure 1.2: Schematic representation of the spectrum of a closed path of unitary matrices and the objects used in Definition 1.5.2.

Definition 1.5.2. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and $a_m \in [0, \pi)$, $m = 1, \ldots, M$ as above, the spectral flow of the path $t \in [0, 1] \mapsto U_t \in \mathrm{U}(N)$ is defined as

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}).$$

Let us first explain how this definition fits together with Definition 1.1.3. For that purpose, let us first of all note that

$$\mathrm{Tr}(P^{>}_{a,t} - P^{<}_{a,t}) = \mathrm{Sig}(\mathfrak{Im}(U^*_t)P_{a,t}),$$

where $\mathfrak{Im}(A) = \frac{1}{2\imath}(A - A^*)$ is the imaginary part of a matrix A and $P_{a,t}$ is the spectral projection defined in (1.15) (note that this is also the spectral projection of the self-adjoint operator $\mathfrak{Re}(U_t)$ onto the interval $[-1, -\cos(a)]$). As it is a self-adjoint matrix, $\mathfrak{Im}(U^*_t)P_{a,t} = P_{a,t}\mathfrak{Im}(U^*_t)P_{a,t}$, on the right-hand side, one hence takes the signature of a self-adjoint matrix given by the difference of its positive and negative eigenvalues. The projection $P_{a,t}$ restricts to the spectrum around -1 in a symmetric way, see once again Figure 1.2. Note that the self-adjoint matrix $P_{a,t}\Im m(U_t^*)P_{a,t} = \Im m(U_t^*)P_{a,t}$ therefore has potentially a large kernel, but this kernel does not influence its signature (which only counts positive and negative eigenvalues). Now replacing into Definition 1.5.2 gives

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\sum_{m=1}^{M}\left(\mathrm{Sig}(\Im m(U_{t_m}^*)P_{a_m,t_m}) - \mathrm{Sig}(\Im m(U_{t_{m-1}}^*)P_{a_m,t_{m-1}})\right).$$

Comparing with (1.7), this clearly shows the similarities with the spectral flow of self-adjoint matrices. The basic result about the spectral flow is that it is well defined by the above procedure and it is homotopy invariant.

Theorem 1.5.3. *The definition of* $\mathrm{Sf}(t \in [0,1] \mapsto U_t)$ *is independent of the choice of the partition* $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ *of* $[0,1]$ *and values* $a_m \in [0,\pi)$ *such that* $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ *is continuous.*

Proof. For each point $t_* \in [t_{m-1}, t_m]$ for $m \in \{1, 2, \dots, M\}$ added to the partition, the number $\mathrm{Tr}(P^>_{a_m,t_*} - P^<_{a_m,t_*})$ is both added and subtracted, thus $\mathrm{Sf}(t \in [0,1] \mapsto U_t)$ does not change. Therefore the definition of the spectral flow is independent of the choice of the partition.

For $m \in \{1, 2, \dots, M\}$, let us compare a_m to a'_m where $t \in [t_{m-1}, t_m] \mapsto P_{a'_m,t}$ is continuous with constant rank. Without loss of generality, one may assume $a'_m > a_m$. As $e^{\imath(\pi \pm a_m)}$ and $e^{\imath(\pi \pm a'_m)}$ are not in the spectrum of U_t for any $t \in [t_{m-1}, t_m]$, it follows that both $t \in [t_{m-1}, t_m] \mapsto P^>_{a'_m,t} - P^>_{a_m,t}$ and $t \in [t_{m-1}, t_m] \mapsto P^<_{a'_m,t} - P^<_{a_m,t}$ are continuous projection-valued functions and hence of constant rank, say $k^>$ and $k^<$. Thus

$$\begin{aligned}
&\mathrm{Tr}(P^>_{a'_m,t_m} - P^<_{a'_m,t_m} - P^>_{a'_m,t_{m-1}} + P^<_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^>_{a'_m,t_m}) - \mathrm{Tr}(P^<_{a'_m,t_m}) - \mathrm{Tr}(P^>_{a'_m,t_{m-1}}) + \mathrm{Tr}(P^<_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^>_{a_m,t_m}) + k^> - \mathrm{Tr}(P^<_{a_m,t_m}) - k^< - \mathrm{Tr}(P^>_{a_m,t_{m-1}}) - k^> + \mathrm{Tr}(P^<_{a_m,t_{m-1}}) + k^< \\
&\quad = \mathrm{Tr}(P^>_{a_m,t_m} - P^<_{a_m,t_m} - P^>_{a_m,t_{m-1}} + P^<_{a_m,t_{m-1}}).
\end{aligned}$$

Therefore the definition of the spectral flow is independent of the choice of the values $a_m \in [0,\pi)$ such that $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ is continuous. □

Remark 1.5.4. Note that the spectral flow of paths of unitaries does not only depend on the endpoints of the path. In particular there are closed paths of unitaries with non-vanishing spectral flow. For example consider the path $t \in [0,1] \mapsto U_t = e^{2\pi\imath t}$ of complex numbers on the unit circle identified with unitary matrices acting on $\mathbb{C}$. This path is closed with endpoints $U_0 = U_1 = \mathbf{1}$ but its spectral flow $\mathrm{Sf}(t \in [0,1] \mapsto U_t) = 1$ does not vanish. ◇

Some elementary properties of the spectral flow are collected in the following result.

Theorem 1.5.5. *Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a continuous path.*
(i) *If $-1 \notin \mathrm{spec}(U_t)$ for all $t \in [0,1]$, then $\mathrm{Sf}(t \in [0,1] \mapsto U_t) = 0$.*
(ii) *The spectral flow has a concatenation property, namely if $t \in [1,2] \mapsto U_t \in \mathrm{U}(N)$ is a second continuous path, composable to the first one in the sense that the endpoint of the first path is the initial point of the second path, then*

$$\mathrm{Sf}(t \in [0,2] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t) + \mathrm{Sf}(t \in [1,2] \mapsto U_t).$$

(iii) *Changing the orientation of the path leads to a change of the sign of the spectral flow*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto U_{1-t}).$$

(iv) *The spectral flow has the reflection property*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto U_t^*).$$

(v) *The spectral flow has an additivity property under direct sums, namely if one has a second continuous path $t \in [0,1] \mapsto V_t \in \mathrm{U}(N')$, then*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t \oplus V_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t) + \mathrm{Sf}(t \in [0,1] \mapsto V_t).$$

(vi) *The spectral flow is invariant under conjugation of the given path by another path $t \in [0,1] \mapsto W_t \in \mathrm{U}(N)$ of unitaries*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto W_t U_t W_t^*).$$

Proof. All items follow directly form the definition of the spectral flow. □

Theorem 1.5.6. *Let $t \in [0,1] \mapsto U_t$ and $t \in [0,1] \mapsto U'_t$ be two continuous paths in $\mathrm{U}(N)$ such that $U_0 = U'_0$ and $U_1 = U'_1$ and such that there exists a continuous homotopy between the two paths leaving the endpoints fixed. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto U'_t).$$

Proof. Let us first note that for $U_0, U_1 \in \mathrm{U}(N)$, both in the same neighborhood $\mathcal{N}$ of the type given in Lemma 1.5.1, and any path $t \in [0,1] \mapsto U_t$ of unitaries from U_0 to U_0 lying entirely in $\mathcal{N}$, the spectral flow is

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\,\mathrm{Tr}(P^{>}_{a,1} - P^{<}_{a,1} - P^{>}_{a,0} + P^{<}_{a,0}),$$

where $a = a_1$ is chosen as in Lemma 1.5.1 and the partition is trivial, namely $t_0 = 0$ and $t_1 = 1$. Therefore the spectral flow is independent of the path in $\mathcal{N}$ connecting U_0 to U_1.

Let us denote the homotopy between the two paths by $h : [0,1] \times [0,1] \to \mathrm{U}(N)$, more precisely, h is continuous, $h(t,0) = U_t$ and $h(t,1) = U'_t$ for all $t \in [0,1]$, and one has $h(0,s) = U_0 = U'_0$ and $h(1,s) = U_1 = U'_1$ for all $s \in [0,1]$. By compactness, one can cover the image of h by a finite set $\{\mathcal{N}_1, \dots, \mathcal{N}_k\}$ of neighborhoods as in Lemma 1.5.1. The preimages $\{h^{-1}(\mathcal{N}_1), \dots, h^{-1}(\mathcal{N}_k)\}$ of these neighborhoods form a finite cover of the set $[0,1] \times [0,1]$. For the Lebesgue number $\epsilon_0 > 0$ of this cover, any subset of $[0,1] \times [0,1]$ of diameter less than ϵ_0 is contained in some element of this finite cover of $[0,1] \times [0,1]$. Thus, if we partition $[0,1] \times [0,1]$ into a grid of squares of diameter less than ϵ_0, then the image of each square will lie entirely within some $\mathcal{N}_l$ for $l \in \{1, \dots, k\}$. By compactness, it is sufficient to show that

$$\mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) = \mathrm{Sf}(t \in [0,1] \mapsto h(t,s''))$$

for $s', s'' \in [0,1]$ with $|s' - s''| < \frac{\epsilon_0}{\sqrt{2}}$. Without loss of generality, one may assume $s' < s''$. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ such that $|t_m - t_{m-1}| < \frac{\epsilon_0}{\sqrt{2}}$ for all $m \in \{1, \dots, M\}$, the image $h([t_{m-1}, t_m] \times [s', s''])$ is contained in one of the neighborhoods $\mathcal{N}_l$ for $l \in \{1, \dots, k\}$. Therefore, by the first paragraph of this proof, one has

$$\begin{aligned} &\mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s')) + \mathrm{Sf}(s \in [s', s''] \mapsto h(t_m, s)) \\ &\quad = \mathrm{Sf}(s \in [s', s''] \mapsto h(t_{m-1}, s')) + \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t, s'')) \end{aligned}$$

for all $m \in \{1, \dots, M\}$. In conclusion,

$$\begin{aligned} \mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) &= \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s')) \\ &= \sum_{m=1}^{M} \mathrm{Sf}(s \in [s', s''] \mapsto h(t_{m-1}, s)) \\ &\quad + \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s'')) - \mathrm{Sf}(s \in [s', s''] \mapsto h(t_m, s)) \\ &= \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s'')) \\ &= \mathrm{Sf}(t \in [0,1] \mapsto h(t,s'')), \end{aligned}$$

where the third step follows from $\mathrm{Sf}(s \in [s', s''] \mapsto h(0,s)) = \mathrm{Sf}(s \in [s', s''] \mapsto h(1,s)) = 0$ as the considered paths are constant. □

Remark 1.5.7. For a path $t \in [0,1] \mapsto A_t \in \mathbb{C}^{N \times N}$ of invertible matrices, one can define the spectral flow as the spectral flow of the unitary phase $U_t = A_t |A_t|^{-1}$,

$$\mathrm{Sf}(t \in [0,1] \mapsto A_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t).$$

If A_t is normal for all $t \in [0,1]$, it is possible to label the spectral curves $\lambda_j(t)$ such that each varies continuously in t. When t increases, the spectral curves $t \mapsto \lambda_j(t)$ can cross the segment $[0,1]\times(-\infty,0)$. One has a spectral crossing of positive signature if there is a passage through the negative real axis from the upper half-plane to the lower half-plane or a spectral crossing of negative signature if the passage is from the lower half-plane to the upper half-plane. If there is a finite number of crossings and no crossings at the boundaries $t = 0$ and $t = 1$, the sum of these signatures over all crossings is equal to $\mathrm{Sf}(t \in [0,1] \mapsto A_t)$. ◇

Up to now, all paths of unitaries were merely assumed to be continuous. In complete analogy with Proposition 1.4.1, one can deform the path into a real-analytic one with simple regular crossings. A detailed proof is omitted.

Proposition 1.5.8. *Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a continuous path of unitary matrices. For any $\epsilon > 0$, there exists a real-analytic path $t \in [0,1] \mapsto V_t \in \mathrm{U}(N)$ of unitary matrices with $\|V_t - U_t\| < \epsilon$ uniformly in t such that all eigenvalue crossings are simple and transversal, namely $\dim(\mathrm{Ker}(V_t)+\mathbf{1}_N) \leq 1$ and $-\imath V_t^*\partial_t V_t|_{\mathrm{Ker}(V_t+\mathbf{1})} \neq 0$, and, moreover, with endpoints V_0 and V_1 having no spectrum at -1.*

Of course, much more can be said if the path is differentiable. First of all, there is an integral formula related to (1.2) for the spectral flow of self-adjoint matrices.

Proposition 1.5.9. *Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a continuously differentiable path of unitaries such that -1 is no eigenvalue of its endpoints. Let $(\lambda_j(t))_{j=1,\dots,N}$ be the eigenvalues of U_t with a enumeration such that each eigenvalue is differentiable in t. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \sum_{j=1}^{N} \frac{1}{2\imath} \int_0^1 dt\, g'(-\imath(\log_-(\lambda_j(t)) - \imath\pi))\partial_t \log_-(\lambda_j(t)),$$

where $\log_- : \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$ is a branch of the logarithm such that $\log_-(-1) = \imath\pi$, and $g : \mathbb{R} \to [-1,1]$ is as above a smooth function which is equal to -1 on $(-\infty,-\epsilon]$ and equal to 1 on $[\epsilon,\infty)$ where $\epsilon > 0$ is such that the endpoints U_0 and U_1 have no spectrum in $\{e^{\imath b} : b \in [\pi-\epsilon,\pi+\epsilon]\}$. Note that in the integrand on the right-hand side one has $g'(-\imath(\log_-(\lambda_j(t)) - \imath\pi)) = 0$ unless $|\lambda_j(t)-1| \leq |e^{\imath\epsilon}-1|$ and for those t and j also $\log_-(\lambda_j(t))$ is well-defined and differentiable.

Just as (1.2), this follows directly from the fundamental theorem. It is also possible to write out formulas for the spectral flow using crossing forms. Again the proofs are not spelled out.

Definition 1.5.10. Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a differentiable path of unitary matrices. An instant $t \in [0,1]$ is called a crossing if $\mathrm{Ker}(U_t + \mathbf{1}_N) \neq \{0\}$. Then the crossing form at t is the quadratic form

$$\Gamma_t : \mathrm{Ker}(U_t + \mathbf{1}_N) \to \mathbb{R}, \quad \Gamma_t(\phi) = -\imath\langle\phi|U_t^*\partial_t U_t\phi\rangle.$$

A crossing is called regular, if Γ_t is nondegenerate.

Proposition 1.5.11. *Let $t \in [0,1] \mapsto U_t \in \mathrm{U}(N)$ be a continuously differentiable path of unitary matrices that has only regular crossings. Then the spectral flow of this path is*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\,\mathrm{Sig}(\Gamma_0) + \sum_{t\in(0,1)} \mathrm{Sig}(\Gamma_t) + \frac{1}{2}\,\mathrm{Sig}(\Gamma_1), \tag{1.16}$$

where $\mathrm{Sig}(\Gamma_t)$ denotes the signature of the quadratic form Γ_t.

For a closed path, the spectral flow of a path of unitaries reduces to the winding number which for differentiable paths has further well-known expressions.

Proposition 1.5.12. *Let $t \in [0,1] \mapsto U_t$ be a closed path in $\mathrm{U}(N)$ which is piecewise continuously differentiable. Then*

$$\begin{aligned} \mathrm{Sf}(t \in [0,1] \mapsto U_t) &= \frac{1}{2\pi\imath} \int_0^1 dt\, \mathrm{Tr}(U_t^* \partial_t U_t) \\ &= \frac{1}{2\pi\imath} \int_0^1 dt\, \partial_t \log(\det(U_t)). \end{aligned} \tag{1.17}$$

Proof. The spectral flow on the left-hand side of (1.17) is a homotopy invariant by Theorem 1.5.6. Moreover, it is also well known that the winding number integral on the right-hand side of (1.17) is a homotopy invariant on the set of differentiable closed paths. Indeed, consider a path $s \in [0,1] \mapsto U_{t,s}$ of piecewise continuously differentiable closed loops (in t) of unitaries that is continuously differentiable in s for any t. Then

$$\begin{aligned} \partial_s \int_0^1 dt\, \mathrm{Tr}(U_{t,s}^* \partial_t U_{t,s}) &= \int_0^1 dt\, \mathrm{Tr}(\partial_s U_{t,s}^* \partial_t U_{t,s} + U_{t,s}^* \partial_t \partial_s U_{t,s}) \\ &= \int_0^1 dt\, \mathrm{Tr}(\partial_s U_{t,s}^* \partial_t U_{t,s} - \partial_t U_{t,s}^* \partial_s U_{t,s}) \\ &= \int_0^1 dt\, \mathrm{Tr}(-U_{t,s}^* \partial_s U_{t,s} U_{t,s}^* \partial_t U_{t,s} + U_{t,s}^* \partial_t U_{t,s} U_{t,s}^* \partial_s U_{t,s}) \\ &= 0, \end{aligned}$$

where in the first step the derivatives ∂_s and ∂_t were exchanged, the second step used integration by parts, and the final step uses the cyclicity of the trace. Now one can deform $t \in [0,1] \mapsto U_t$ to a path $t \in [0,1] \mapsto e^{2\pi\imath nt} P + \mathbf{1} - P$ where n is the winding number and $P \in \mathbb{C}^{N\times N}$ is a one-dimensional projection. For the latter path, the equality can be readily checked and as both sides are constant along the homotopy, the claim follows. □

1.6 The spectral flow through the imaginary axis

Up to now, the spectral flow of paths of self-adjoint and unitary matrices was considered. This section briefly discusses what can be done for more general paths $t \in [0,1] \mapsto A_t$ of matrices. The most fruitful generalization is to look at the spectral flow through the imaginary axis where the passage of each eigenvalue is weighted by the orientation and the algebraic multiplicity of the eigenvalue. The algebraic multiplicity is given by the dimension of the Riesz projection of the eigenvalue (see Appendix A.1). Therefore let us introduce several Riesz projections of a given matrix A, namely $P^{>}(A)$, $P^{\geq}(A)$, $P^{=}(A)$, $P^{<}(A)$, and $P^{\leq}(A)$ on all eigenvalues having positive, nonnegative, vanishing, negative, and nonpositive real part. Then the signature of a matrix A will be defined as the number of eigenvalues with positive real part minus the number of eigenvalues with negative real part, both counted with their algebraic multiplicities,

$$\mathrm{Sig}(A) = \mathrm{Tr}\big(P^{>}(A)\big) - \mathrm{Tr}\big(P^{<}(A)\big).$$

Based on this, the spectral flow can be defined as in Definition 1.1.3 by

$$\mathrm{Sf}\big(t \in [0,1] \mapsto A_t\big) = \frac{1}{2}\big(\mathrm{Sig}(A_1) - \mathrm{Sig}(A_0)\big). \tag{1.18}$$

It is possible to rewrite this as in (1.9), namely

$$\mathrm{Sf}\big(t \in [0,1] \mapsto A_t\big) = \frac{1}{2}\,\mathrm{Tr}\big(P^{=}(A_1)\big) + \mathrm{Tr}\big(P^{>}(A_1)\big) - \mathrm{Tr}\big(P^{>}(A_0)\big) - \frac{1}{2}\,\mathrm{Tr}\big(P^{=}(A_0)\big).$$

With these definitions, one can again verify the basic properties from Section 1.2, except for the invariance under conjugation (Proposition 1.2.5) and the comparison (Proposition 1.2.7). On the other hand, one has the following invariance under a continuous path $t \in [0,1] \mapsto B_t$ of invertible basis changes:

$$\mathrm{Sf}\big(t \in [0,1] \mapsto B_t A_t B_t^{-1}\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto A_t\big).$$

Clearly, the definition (1.18) reduces to Definition 1.1.3 if all A_t are self-adjoint. Moreover, if all A_t are normal, then by the spectral theorem the real part of an eigenvalue is given by the eigenvalue of the self-adjoint matrix $\Re e(A) = \frac{1}{2}(A + A^*)$, also called the real part of A. Therefore, for a path $t \in [0,1] \mapsto A_t$ of normal matrices A_t, one has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto A_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto \Re e(A_t)\big),$$

where the right-hand side is a spectral flow in the sense of Definition 1.1.3. In general, however, such a connection does not hold as is shown by the next example.

Example 1.6.1. Consider

$$t \in [-a, a] \mapsto A_t = \begin{pmatrix} at & 1 \\ 0 & at \end{pmatrix}, \quad a \in \mathbb{R},$$

which has a spectral flow ±2 depending on the sign of a for all $a > 0$. But

$$\Re e(A_t) = at\mathbf{1} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has spectrum $\{at + \frac{1}{2}, at - \frac{1}{2}\}$. Therefore, for a sufficiently small, the spectral flow of $t \in [-a, a] \mapsto \Re e(A_t)$ vanishes. ◇

1.7 $\mathbb{Z}_2$-valued orientation flow of skew-adjoint real matrices

This section considers real matrices $A \in \mathbb{R}^{N\times N}$. They can either be considered as $\mathbb{R}$-linear operators on a finite-dimensional real Hilbert space $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^N$, or as $\mathbb{C}$-linear operators on $\mathbb{C}^N$ that commute with the componentwise complex conjugation, namely satisfying $\overline{A} = A$. If considered as $\mathbb{C}$-linear, then one can dispose the spectral theory of A and the spectrum is invariant under complex conjugation. Furthermore, let us now suppose that the matrix $A \in \mathbb{R}^{N\times N}$ is skew-adjoint, namely that it satisfies $A^* = -A$. As $H = \imath A$ is self-adjoint and has real spectrum, the spectrum of A lies on the imaginary axis. Together with the invariance of the spectrum under complex conjugation, one deduces the reflection symmetry

$$\mathrm{spec}(A) = -\,\mathrm{spec}(A) \subset \imath\mathbb{R}.$$

This implies that a skew-adjoint real matrix can only be invertible if N is even. For odd N, the kernel is necessarily of dimension greater than or equal to 1. Moreover, given two invertible real skew-adjoint matrices A_0 and A_1, the spectral flow between the associated self-adjoints $H_0 = \imath A_0$ and $H_1 = \imath A_1$ vanishes simply because both H_0 and H_1 have a vanishing signature due to the spectral symmetry. Nevertheless, one can extract a $\mathbb{Z}_2$-valued flow as in [57, 75]. It was called $\mathbb{Z}_2$-valued spectral flow in [57], but here we rather follow the terminology of [75] because there is no spectral flow involved (see the example below).

Definition 1.7.1. Suppose given two invertible skew-adjoint matrices $A_0, A_1 \in \mathbb{R}^{N\times N}$. The $\mathbb{Z}_2$-valued orientation flow (along the straight line path) from A_0 to A_1 is then defined by

$$\mathrm{Of}(A_0, A_1) = \mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \in \mathbb{Z}_2.$$

Here $\mathbb{Z}_2$ is viewed as a multiplicative group $\{-1, 1\}$ and $\mathrm{pf}(A_i)$ denotes the Pfaffian of A_i for $i = 0, 1$.

Example 1.7.2. Let $\mathcal{H}_\mathbb{R} = \mathbb{R}^2$ and consider two paths, one linear and a second nonanalytic path $t \in [0,1] \mapsto \tilde{A}_t$ of skew-adjoint matrices:

$$A_t = (2t-1)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{A}_t = |2t-1|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.19}$$

The spectra of A_t and $\tilde{A}_t$ are

$$\mathrm{spec}(A_t) = \mathrm{spec}(\tilde{A}_t) = \{(1-2t)\imath, (2t-1)\imath\}.$$

Thus both eigenvalues form a crossing with a double degenerate kernel at $t = \frac{1}{2}$, and the associated spectral flow vanishes. Nevertheless, there is a difference between the two paths. In fact, for $\tilde{A}_t$, one can consider the homotopy $s \in [0,1] \mapsto \tilde{A}_t(s)$ of paths of skew-adjoints given by

$$\tilde{A}_t(s) = |2ts-1|\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

that leaves the endpoints fixed. Then $\tilde{A}_t(1) = \tilde{A}_t$, while $\tilde{A}_t(0)$ is a constant path with spectrum $\mathrm{spec}(\tilde{A}_t(0)) = \{-\imath, \imath\}$ which is actually the straight-line path between $\tilde{A}_0$ and $\tilde{A}_1$. Consequently the spectral crossing of the path $\tilde{A}_t$ can be homotopically lifted. On the other hand, it is impossible to lift the kernel of A_t. This defect is encoded in the eigenfunctions as follows. Viewing A_0 and A_1 as nondegenerate skew-symmetric bilinear forms, results from linear algebra imply that there exists a real invertible matrix B such that

$$A_1 = B^* A_0 B.$$

Actually, here $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which exchanges the eigenvectors of the upper and lower branch of A_t at $t = \frac{1}{2}$. This is reflected by the sign of $\det(B)$ and this sign is the $\mathbb{Z}_2$-valued orientation flow $\mathrm{Of}(A_0, A_1)$ between the points A_0 and A_1 along the straight line path as

$$\mathrm{pf}(B^* A_0 B) = \det(B)\,\mathrm{pf}(A_0).$$

Let us stress again that due to the above, this $\mathbb{Z}_2$-valued orientation flow is not only determined by the spectrum of the path, but rather depends on the eigenfunctions as well. However, we will show further below that a path having vanishing kernel throughout necessarily has a trivial $\mathbb{Z}_2$-valued orientation flow. Another important difference between the two cases in (1.19) is that A_t is analytic in t, while $\tilde{A}_t$ is not. Let us also note that in the complex matrices one can deform A_t into $\tilde{A}_t$ by the homotopy

$$s \in [0,1] \mapsto A_t(s) = e^{\imath\pi(1-s)\chi(t<\frac{1}{2})}|2t-1| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence the reality restriction is necessary for the definition of the $\mathbb{Z}_2$-valued orientation flow. ◇

Next let us analyze a few properties of the orientation flow.

Lemma 1.7.3. *For invertible skew-adjoint matrices $A_0, A_1 \in \mathbb{R}^{N\times N}$ and any invertible matrices $B, C \in \mathbb{R}^{N\times N}$ with $\det(C) > 0$, one has*

$$\mathrm{Of}(A_0, A_1) = \mathrm{Of}(A_1, A_0) = \mathrm{Of}(BA_0B^*, BA_1B^*) = \mathrm{Of}(CA_0C^*, A_1).$$

Moreover, if $A_0', A_1' \in \mathbb{R}^{M\times M}$ are skew-adjoint invertibles,

$$\mathrm{Of}(A_0 \oplus A_0', A_1 \oplus A_1') = \mathrm{Of}(A_0, A_1)\,\mathrm{Of}(A_0', A_1'),$$

with multiplication in $(\mathbb{Z}_2, \cdot)$.

Proof. By definition, $\mathrm{Of}(A_0, A_1) = \mathrm{Of}(A_1, A_0)$. Moreover,

$$\begin{aligned}
\mathrm{Of}(BA_0B^*, BA_1B^*) &= \mathrm{sgn}(\mathrm{pf}(BA_0B^*))\,\mathrm{sgn}(\mathrm{pf}(BA_1B^*)) \\
&= \mathrm{sgn}(\det(B))\,\mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\det(B))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \\
&= \mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \\
&= \mathrm{Of}(A_0, A_1),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Of}(CA_0C^*, A_1) &= \mathrm{sgn}(\mathrm{pf}(CA_0C^*))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \\
&= \mathrm{sgn}(\det(C))\,\mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \\
&= \mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_1)) \\
&= \mathrm{Of}(A_0, A_1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Of}(A_0 \oplus A_0', A_1 \oplus A_1') &= \mathrm{sgn}(\mathrm{pf}(A_0 \oplus A_0'))\,\mathrm{sgn}(\mathrm{pf}(A_1 \oplus A_1')) \\
&= \mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_0'))\,\mathrm{sgn}(\mathrm{pf}(A_1))\,\mathrm{sgn}(\mathrm{pf}(A_1')) \\
&= \mathrm{Of}(A_0, A_1)\,\mathrm{Of}(A_0', A_1'),
\end{aligned}$$

proving the last claim. □

The following proposition indicates that the $\mathbb{Z}_2$-valued orientation flow can be used as an obstruction.

Proposition 1.7.4. *Let $A_0, A_1 \in \mathbb{R}^{N\times N}$ be skew-adjoint invertible matrices. Let there exist a path $t \in [0,1] \mapsto A_t$ of skew-adjoint invertibles from A_0 to A_1. Then*

$$\mathrm{Of}(A_0, A_1) = 1.$$

Proof. As $t \in [0,1] \mapsto \mathrm{pf}(A_t)$ is continuous and $\mathrm{pf}(A_t) \neq 0$ for all $t \in [0,1]$, one has $\mathrm{sgn}(\mathrm{pf}(A_0)) = \mathrm{sgn}(\mathrm{pf}(A_1))$. Therefore $\mathrm{Of}(A_0, A_1) = \mathrm{sgn}(\mathrm{pf}(A_0))\,\mathrm{sgn}(\mathrm{pf}(A_1)) = 1$. □

Next let us discuss the concatenation property of the $\mathbb{Z}_2$-valued orientation flow.

Proposition 1.7.5. *For skew-adjoint invertibles $A_0, A_1, A_2 \in \mathbb{R}^{N\times N}$,*

$$\mathrm{Of}(A_0, A_2) = \mathrm{Of}(A_0, A_1)\,\mathrm{Of}(A_1, A_2). \tag{1.20}$$

Proof. The claim directly follows from Definition 1.7.1. □

It requires supplementary thought to introduce the $\mathbb{Z}_2$-valued orientation flow if the endpoints of the considered path are not invertible [77]. On the other hand, this issue is not of importance for the concatenation of a subdivision of a path $t \in [0,2] \mapsto A_t$ where A_0 and A_2 are invertibles. Then if A_1 is not invertible, one can add a skew-adjoint perturbation W_1 on the kernel of W_1 such that $A_1 + W_1$ is invertible, and then (1.20) holds if A_1 is replaced by $A_1 + W_1$. This is independent of the choice of W_1 because the two modifications cancel out. This fact is important for the definition of the $\mathbb{Z}_2$-valued orientation flow for arbitrary paths in infinite dimension.

However, in this book the definition of the $\mathbb{Z}_2$-valued orientation flow for skew-adjoint real Fredholm operators is not carried out. The reader is referred to the references [57, 75, 77]. Let us note though that it is straightforward to combine the above description of the finite dimensional case with the Phillips approach to spectral flow described in Chapter 4. Furthermore, let us note that it is possible to define various other $\mathbb{Z}_2$-valued flows for paths of matrices (or Fredholm operators) having other symmetry properties. The earliest is the parity introduced in [86] and further studied in the spirit above in [76], another one with a symmetry in the time parameter t was introduced in [71] and finally an exhaustive and unifying treatment of $\mathbb{Z}_2$-valued flows was given recently in [75]. An alternative approach to the same problem is based on a Clifford algebra valued flow [37].

1.8 Multiparameter spectral flow

Proposition 1.5.12 states that the spectral flow of a closed differentiable path in the unitary group $\mathrm{U}(N)$ is equal to its winding number. On the other hand, it is well known that the fundamental group of $\mathrm{U}(N)$ is equal to $\mathbb{Z}$ for all N and that each connected component of the loops in $\mathrm{U}(N)$ is precisely labeled by the winding number (e. g., [161]). Hence the spectral flow establishes a concrete form of the homomorphism

$$\mathrm{Sf} : \pi_1(\mathrm{U}(N)) \to \mathbb{Z}, \tag{1.21}$$

where the independence of the choice of representative is guaranteed by Theorem 1.5.6. On the other hand, recall that the higher homotopy groups $\pi_k(\mathrm{U}(N))$ for $k \in \mathbb{N}$ are the pointed homotopy equivalence classes of continuous maps $z \in \mathbb{S}^k \mapsto U_z \in \mathrm{U}(N)$ from the k-sphere $\mathbb{S}^k$ to the unitary matrices, with a group structure given by glueing [103]. A famous result of Bott [35] states that for N sufficiently large (in the so-called stable range)

$$\pi_k(\mathrm{U}(N)) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Just as the spectral flow establishes the group homomorphism (1.21), one would now like an explicit map providing a group homomorphism

$$\mathrm{Sf}_k : \pi_k(\mathrm{U}(N)) \to \mathbb{Z}, \quad k \text{ odd}.$$

Similar as the winding number cocycle, this map applied to the dense set of differentiable functions $z \in \mathbb{S}^k \mapsto U_z \in \mathrm{U}(N)$ will naturally involve integrals and derivatives of all k variables on $\mathbb{S}^k$. It will therefore be called the k-multiparameter spectral flow and is given by

$$\mathrm{Sf}_k(z \in \mathbb{S}^k \mapsto U_z) = \left(\frac{\imath}{\pi}\right)^{\frac{k+1}{2}} \frac{1}{2^k k!!} \int_{\mathbb{S}^k} \mathrm{Tr}\big((U^* \mathrm{d}U)^{\wedge k}\big), \quad k \text{ odd},$$

where $k!! = k(k-2)\cdots 3 \cdot 1$. Another term often used in the physics literature for this object is higher winding number or an odd Chern number. Of course, the choice of the normalization factor is crucial to guarantee that the integral is an integer, and the above is the standard choice [63, 152, 58]. In fact, there is an index theorem showing that $\mathrm{Sf}_k(z \in \mathbb{S}^k \mapsto U_z)$ is an integer. Moreover, one can compute the integral for the generator of $\pi_k(\mathrm{U}(N))$ which is given by

$$U_z = \imath z_0 \mathbf{1} + \sum_{j=1}^{k} z_j \gamma_j.$$

Here $z = (z_0, \dots, z_k) \in \mathbb{S}^k \subset \mathbb{R}^{k+1}$ and $\gamma_1, \dots, \gamma_k$ is an irreducible self-adjoint representation of the Clifford algebra with k generators, namely $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{i,j}$ and each γ_j is a matrix of size $2^{\frac{k-1}{2}}$. A lengthy, but explicit computation then shows $\mathrm{Sf}_k(z \in \mathbb{S}^k \mapsto U_z) = 1$ (e. g., [58]). A similar construction can also be done in even dimensions [58], see also [119]. Furthermore, higher-dimensional generalizations of the index theorem of [14] are so-called Callias-type index theorems, see [171, 172]. Building on all the above, it is possible to construct a k-multiparameter spectral flow for paths of self-adjoint Fredholm operators, anticommuting with a symmetry (see [171, 172]).

2 Applications of finite-dimensional spectral flow

This chapter is about the Bott–Maslov and Conley–Zehnder indices in a finite-dimensional setting. Both of them can be defined as a spectral flow of unitary matrices, but what makes these indices particularly interesting is an interpretation as an intersection number of Lagrangian subspaces. Especially simple is the situation of closed paths for which the spectral flow reduces to the winding number, see Section 1.5. Hence closed paths are often considered as a special example, but no further applications of the winding number are given (see the book of [161]). On the other hand, the application of the Bott–Maslov and Conley–Zehnder indices that we focus on considers open paths. It is based on the intersection theory interpretation and, in particular, addresses transversality and monotonicity aspects: the Sturm–Liouville oscillation theory for the spectral theory of matrix-valued Jacobi matrices. Such Jacobi matrices are technically less involved than their continuous analogues, namely Sturm–Liouville operators and Hamiltonian systems, but nevertheless allow us to illustrate all essential features.

This chapter is based on Chapter 1, but otherwise essentially self-contained. Later on, it is only relevant for Chapter 9, so the reader mainly interested in spectral flow in an infinite-dimensional setting may directly jump to Chapter 3 or even Chapter 4.

2.1 Bott–Maslov index in finite dimension

The Bott–Maslov index was introduced independently by Bott [35] and Maslov [133], and consecutively studied by numerous authors [9, 11, 47, 61, 97, 127, 159, 164, 209, 166, 167, 106]. It is an intersection number associated to paths of Lagrangian planes in a finite-dimensional Krein space. In many applications (such as classical mechanics), the Krein space is real and thus reduces to a symplectic vector space. The classical theory of the Maslov index is developed in this context [133, 9, 11, 159], but no real structure will be used here. It turns out that the intersection number defining the Bott–Maslov index is tightly linked to the spectral flow of unitary matrices as discussed in Section 1.5, via the so-called stereographic projection. Bott's initial motivation was to study the spectral properties of matrix-valued Sturm–Liouville operators [35] and the discrete analogue of this, oscillation theory for block Jacobi matrices, will be discussed as an application in the following Sections 2.4 and 2.5. Maslov's motivation rather came from semiclassical analysis where the Bott–Maslov index is relevant for a correct choice of phase factors [133]. Very influential was also the paper by Arnold analyzing the Bott–Maslov index from the point of view of differential topology [9]. Here the theory is developed for finite-dimensional complex Krein spaces, following [166, 167]. In Chapter 9 it is then extended to infinite-dimensional Krein spaces.

Let us consider the even-dimensional complex vector space $\mathbb{C}^{2N}$. The euclidean scalar product of $\phi, \psi \in \mathbb{C}^{2N}$ is denoted by $\langle \phi | \psi \rangle = \phi^* \psi$ where the second notation

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alludes to ϕ and ψ as frames for one-dimensional subspaces spanned by these vectors (more general frames will be introduced on the next page). On $\mathbb{C}^{2N}$ act the matrices

$$J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{2.1}$$

where all block entries are of size $N \times N$. Both define a sesquilinear form on $\mathbb{C}^{2N}$ by $\langle\phi|\psi\rangle_J = \phi^* J\psi$ and $\langle\phi|\psi\rangle_I = \phi^* I\psi$ which equip $\mathbb{C}^{2N}$ with a so-called Krein space structure [29, 20, 100]. When I is restricted to $\mathbb{R}^{2N} \subset \mathbb{C}^{2N}$, one also speaks of a real symplectic vector space [115]. Let us also note that J and I induce quadratic forms $\phi \in \mathbb{C}^{2N} \mapsto \phi^* J\phi$ and $\phi \in \mathbb{C}^{2N} \mapsto \phi^* I\phi$ on $\mathbb{C}^{2N}$. Of course, these forms and the two matrices J and I are related, namely

$$\imath I = \mathcal{C}^* J \mathcal{C}, \tag{2.2}$$

where $\mathcal{C}$ is the Cayley transform given by

$$\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\imath\mathbf{1} \\ \mathbf{1} & \imath\mathbf{1} \end{pmatrix}. \tag{2.3}$$

This allows working in either representation and we choose to focus on J. All of the concepts and results below directly transfer to results formulated with I.

The central object of this section and actually most of the chapter are J-Lagrangian projections. First recall that a square matrix P is called an orthogonal projection provided that $P = P^* = P^2$. Further recall that there is a tight connection between projections and subspaces, namely associated to each projection is the subspace $\mathrm{Ran}(P)$ given by its range, and inversely associated to every subspace there is an orthogonal projection. Moreover, the dimension of a projection is the dimension $\dim(\mathrm{Ran}(P))$.

Definition 2.1.1. An orthogonal projection $P \in \mathbb{C}^{2N\times 2N}$ is called J-isotropic if and only if $PJP = 0$. Further, N-dimensional J-isotropic projections are called J-Lagrangian. The set of all J-Lagrangian projections is denoted by $\mathbb{P}(\mathbb{C}^{2N}, J)$ and called the J-Lagrangian Grassmannian.

Note that the dimension of any J-isotropic projection is less than or equal to N. Moreover, J-Lagrangian projections are maximally J-isotropic projections in the sense that for any J-isotropic projection P there is a projection P' such that $P + P'$ is a J-Lagrangian projection.

Based on the equivalence between subspaces and projections, one calls a subspace $\mathcal{E} \subset \mathbb{C}^{2N}$ J-Lagrangian if the restriction $J|_{\mathcal{E}}$ of J to $\mathcal{E}$ vanishes, namely if $\phi^* J\psi = 0$ for all $\phi, \psi \in \mathcal{E}$. For the description of $P \in \mathbb{P}(\mathbb{C}^{2N}, J)$, the concept of a J-Lagrangian frame is useful. By definition, this is an $2N \times N$ matrix Φ of full rank N such that $\Phi^* J\Phi = 0$. The frame is called normalized if, moreover, $\Phi^*\Phi = \mathbf{1}$. Associated to a frame is then a J-Lagrangian projection $P = \Phi(\Phi^*\Phi)^{-1}\Phi^*$ and a J-Lagrangian subspace $\mathcal{E} = \mathrm{Ran}(\Phi)$. Note that, if Φ is a J-Lagrangian frame for P, then so is ΦA where $A \in \mathrm{GL}(N,\mathbb{C})$ is an invertible matrix.

Lemma 2.1.2. *An orthogonal projection $P \in \mathbb{P}(\mathbb{C}^{2N})$ is J-Lagrangian if and only if $JPJ = \mathbf{1} - P$.*

Proof. Let $(\phi_n)_{n=1,\dots,N}$ be an orthonormal basis of $\mathrm{Ran}(P)$. Then $\Phi = (\phi_1,\dots,\phi_N) \in \mathbb{C}^{2N\times N}$ is a normalized frame for P, and $(J\phi_n)_{n=1,\dots,N}$ is an orthonormal set of vectors which are all orthogonal to $\mathrm{Ran}(P)$. Hence $(\phi_n, J\phi_n)_{n=1,\dots,N}$ are $2N$ orthonormal vectors which hence form a basis of $\mathbb{C}^{2N}$. Moreover, $(\Phi, J\Phi)$ is unitary and $J\Phi$ is a frame for $\mathbf{1} - P$, that is, $\mathbf{1} - P = J\Phi(J\Phi)^* = J\Phi\Phi^* J = JPJ$. Conversely, multiplying $JPJ = \mathbf{1} - P$ by P from the left and J from the right shows $PJP = 0$. □

Let us note that $JPJ = \mathbf{1} - P$ can be rewritten as $J(\mathbf{1} - P)J = P$ so that J-Lagrangian projections always come in pairs which span orthogonal subspaces. Another comment concerns the symmetry $Q = \mathbf{1} - 2P$ associated to the projection P. If P is J-Lagrangian, it satisfies $JQJ = -Q$. Hence Q is odd with respect to J, but often the relation is also called a chiral symmetry of Q.

For the following, one needs to specify a reference J-Lagrangian projection P_{ref} with associated normalized reference frame Φ_{ref} which we choose to be

$$P_{\mathrm{ref}} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \quad \Phi_{\mathrm{ref}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}. \tag{2.4}$$

Then also set $\mathcal{E}_{\mathrm{ref}} = \mathrm{Ran}(P_{\mathrm{ref}}) = \mathrm{Ran}(\Phi_{\mathrm{ref}})$, and furthermore $P^{\perp}_{\mathrm{ref}} = JP_{\mathrm{ref}}J$, $\Phi^{\perp}_{\mathrm{ref}} = J\Phi_{\mathrm{ref}}$, as well as $\mathcal{E}^{\perp}_{\mathrm{ref}} = J\mathcal{E}_{\mathrm{ref}}$.

Definition 2.1.3. The singular cycle $\mathbb{P}_{\mathrm{sing}}(\mathbb{C}^{2N}, J)$ in $\mathbb{P}(\mathbb{C}^{2N}, J)$ consists of those J-Lagrangian projections having a nontrivial intersection with $\mathcal{E}^{\perp}_{\mathrm{ref}}$,

$$\mathbb{P}_{\mathrm{sing}}(\mathbb{C}^{2N}, J) = \{P \in \mathbb{P}(\mathbb{C}^{2N}, J) : \mathcal{E} = \mathrm{Ran}(P) \ \text{satisfies} \ \dim(\mathcal{E} \cap \mathcal{E}^{\perp}_{\mathrm{ref}}) \geq 1\}.$$

The singular cycle has a natural stratification by $\dim(\mathcal{E} \cap \mathcal{E}^{\perp}_{\mathrm{ref}})$. As will be shown below, it is moreover two-sided in the sense that a path $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N}, J)$ can pass through $\mathbb{P}_{\mathrm{sing}}(\mathbb{C}^{2N}, J)$ from left to right or from right to left. Following [9], the Bott–Maslov index is then defined as the weighted sum over all intersections along the path. Here we give an equivalent definition which is based on the following fact:

Proposition 2.1.4. *The J-Lagrangian Grassmannian is bijectively mapped onto the unitary matrices $\mathbb{U}(N)$ by the stereographic projection $\Pi : \mathbb{P}(\mathbb{C}^{2N}, J) \to \mathbb{U}(N)$ defined by*

$$\Pi(P) = U, \quad P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix}, \tag{2.5}$$

or alternatively by

$$\Pi^{-1}(U) = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix}. \tag{2.6}$$

Moreover,

$$\Pi(\mathbb{P}_{\text{sing}}(\mathbb{C}^{2N}, J)) = \{U \in \mathbb{U}(N) : \dim(\text{Ker}(U + \mathbf{1})) \geq 1\}.$$

More precisely, if $\mathcal{E} = \text{Ran}(P)$ *and* $U = \Pi(P)$,

$$\dim(\mathcal{E} \cap \mathcal{E}_{\text{ref}}^{\perp}) = \dim(\text{Ker}(U + \mathbf{1})). \tag{2.7}$$

Proof. Let us consider the symmetry $Q = \mathbf{1} - 2P$. The relation $JQJ = -Q$ then implies that $Q = \begin{pmatrix} 0 & -U \\ -U^* & 0 \end{pmatrix}$ is off-diagonal in the grading of J, and $Q^2 = \mathbf{1}$ then shows that the off-diagonal entry $U \in \mathbb{C}^{N\times N}$ is indeed unitary, as the notation suggests. Thus $P = \frac{1}{2}(\mathbf{1} - Q)$ is of the form given in (2.5). Note that a normalized form for P is then given by

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}. \tag{2.8}$$

Let us now verify (2.7). First of all, let $\mathcal{E} = \text{Ran}(\Phi)$ and $\mathcal{E}_{\text{ref}}^{\perp} = \text{Ran}(\Phi_{\text{ref}}^{\perp})$. One readily checks $U + \mathbf{1} = 2(\Phi_{\text{ref}}^{\perp})^* J\Phi$. Hence the claim is

$$\dim(\text{Ran}(\Phi) \cap \text{Ran}(\Phi_{\text{ref}}^{\perp})) = \dim(\text{Ker}((\Phi_{\text{ref}}^{\perp})^* J\Phi)),$$

as the kernel on the right-hand side is clearly independent of the choice of the frame Φ representing $\mathcal{E}$. Let us begin with the inequality $\leq$. Suppose there are $N \times k$ matrices c, d of rank k such that $\Phi c = \Phi_{\text{ref}}^{\perp} d$. Then $(\Phi_{\text{ref}}^{\perp})^* J\Phi c = (\Phi_{\text{ref}}^{\perp})^* J\Phi_{\text{ref}}^{\perp} d = 0$, showing that indeed $(\Phi_{\text{ref}}^{\perp})^* J\Phi$ has at least a kernel of dimension k. Conversely, if c is an $N \times k$ matrix of rank k such that $0 = (\Phi_{\text{ref}}^{\perp})^* J\Phi c = (\Phi_{\text{ref}})^*(\Phi c)$, then a k-dimensional subspace of $\text{Ran}(\Phi)$ is orthogonal to $\text{Ran}(\Phi_{\text{ref}})$ (with respect to the euclidean scalar product) and thus lies in the orthogonal complement $\text{Ran}(\Phi_{\text{ref}}^{\perp})$. □

For a J-Lagrangian plane $\mathcal{E}$ given by the range of a J-Lagrangian projection $P_{\mathcal{E}}$, its stereographic projection is defined by

$$\Pi(\mathcal{E}) = \Pi(P_{\mathcal{E}}). \tag{2.9}$$

Remark 2.1.5. Let us spell out the algebraic relations that Proposition 2.1.4 provides for a J-Lagrangian frame Φ for $P \in \mathbb{P}(\mathbb{C}^{2N}, J)$, namely $P = \Phi(\Phi^*\Phi)^{-1}\Phi^*$. Let $a, b \in \mathbb{C}^{N\times N}$ denote the matrix entries of $\Phi = \binom{a}{b}$. Then J-unitary is equivalent to $a^*a - b^*b = 0$. As Φ is of rank N, one can hence conclude that $0 < \Phi^*\Phi = a^*a + b^*b = 2a^*a = 2b^*b$, which in turn implies that a and b are both invertible. Moreover, one has $U = \Pi(P) = ab^{-1}$. ◇

Based on Proposition 2.1.4, Arnold's two-sidedness of $\mathbb{P}_{\text{sing}}(\mathbb{C}^{2N}, J)$ is easily explained: All elements of a small neighborhood of P_{ref} are such that $\Pi(P)$ has an eigenvalue close to -1 and it is to the left if its imaginary part is positive and to its right if its imaginary part is negative (of course, the choice of left and right is arbitrary here). A path through P_{ref} now either passes from left to right, or vice versa, depending on the direction in which the eigenvalue passes through −1. All this supposes that the intersection is simple and transversal, namely in a so-called generic position (which can be assured by differential topological arguments). On the other hand, these issues become irrelevant if the Bott–Maslov index is directly defined as a spectral flow:

Definition 2.1.6. Let $t \in [0,1] \mapsto P_t$ be a path in the J-Lagrangian Grassmannian $\mathbb{P}(\mathbb{C}^{2N}, J)$. Its Bott–Maslov index is defined by

$$\mathrm{BM}\big(t \in [0,1] \mapsto P_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto \Pi(P_t)\big).$$

Let us stress that Definition 2.1.6 does not require the path to be closed, which differs from part of the literature and is needed for several applications in which one naturally has to deal with open paths. Of course, closed paths lead to stronger stability results which will hence be stated separately. From its definition, the Bott–Maslov index nevertheless inherits all the properties of the spectral flow stated in Section 1.5: path reversal, concatenation, homotopy invariance, additivity, etc. No further detail will be given here. Also the Bott–Maslov index of paths of I-Lagrangian planes and its properties are not spelled out explicitly.

Another important point is that the Bott–Maslov index depends on the choice of the reference J-Lagrangian subspace $\mathcal{E}_{\text{ref}}$. The choice (2.4) leads to $\Pi(P_{\text{ref}}) = \mathbf{1}$. If one is interested in a general situation of intersections through an arbitrary given J-Lagrangian subspace $\mathcal{F}$, then the following statement is of interest. Its proof is essentially identical to that of Proposition 2.1.4.

Proposition 2.1.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be J-Lagrangian subspaces with associated J-Lagrangian projections $P_{\mathcal{E}}, P_{\mathcal{F}} \in \mathbb{P}(\mathbb{C}^{2N}, J)$. Then*

$$\dim(\mathcal{E} \cap \mathcal{F}^{\perp}) = \dim\big(\mathrm{Ker}\big(\Pi(P_{\mathcal{F}})^* \Pi(P_{\mathcal{E}}) + \mathbf{1}\big)\big). \tag{2.10}$$

Therefore the following is a straightforward generalization of Definition 2.1.6.

Definition 2.1.8. Let $t \in [0,1] \mapsto P_t$ be a path in the J-Lagrangian Grassmannian $\mathbb{P}(\mathbb{C}^{2N}, J)$ and let $\mathcal{F}$ be a J-Lagrangian subspace with associated J-Lagrangian projection $P_{\mathcal{F}} \in \mathbb{P}(\mathbb{C}^{2N}, J)$. Its Bott–Maslov index through $\mathcal{F}$ is defined by

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, P_t)\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto \Pi(P_{\mathcal{F}})^* \Pi(P_t)\big).$$

We will later on see that the transitive Möbius action of J-unitaries always allows choosing $\mathcal{F} = \mathcal{E}_{\text{ref}}$ so that then $P_{\mathcal{F}} = P_{\text{ref}}$, see Corollary 2.2.11.

Let $t \in [0,1] \mapsto \mathcal{E}_t$ be a path of J-Lagrangian subspaces and let $P_{\mathcal{E}_t}$ denote the J-Lagrangian projection onto $\mathcal{E}_t$. If the path $t \in [0,1] \mapsto P_{\mathcal{E}_t}$ is continuous the Bott–Maslov index of the path $t \in [0,1] \mapsto \mathcal{E}_t$ through a J-Lagrangian subspace $\mathcal{F}$ is given by

$$\mathrm{BM}\big(t \in [0,1] \mapsto (\mathcal{F}, \mathcal{E}_t)\big) = \mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, P_{\mathcal{E}_t})\big). \tag{2.11}$$

From this formula it is clear that another straightforward generalization is to consider the Bott–Maslov index also for paths $t \in [0,1] \mapsto (\mathcal{F}_t, \mathcal{E}_t)$ of pairs of Lagrangian subspace. Also this case can be reduced to Definition 2.1.6, see also Chapter 9.

Let us now consider the special case of a differentiable path of J-Lagrangian subspaces. Then one can use the following formulas. The first is useful to analyze the transversality of the path, the second for the computation of the winding number integral.

Lemma 2.1.9. *Let $t \in [0,1] \mapsto \Phi_t = \binom{a_t}{b_t}$ be a differentiable path of (not necessarily normalized) J-Lagrangian frames with associated projections $P_t = \Phi_t(\Phi_t^*\Phi_t)^{-1}\Phi_t^*$. Then $U_t = \Pi(P_t)$ satisfies*

$$U_t^*\partial_t U_t = (b_t^{-1})^*(\Phi_t^* J \partial_t \Phi_t)(b_t^{-1})$$

and

$$\mathrm{Tr}\big(U_t^*\partial_t U_t\big) = \mathrm{Tr}\big((a_t)^{-1}\partial_t a_t - (b_t)^{-1}\partial_t b_t\big).$$

Proof. As $U_t = a_t b_t^{-1}$ by Remark 2.1.5, one has

$$U_t^*\partial_t U_t = (b_t^{-1})^* a_t^*\big((\partial_t a_t) - a_t b_t^{-1}(\partial_t b_t)\big)b_t^{-1}.$$

As

$$a_t^* a_t b_t^{-1} = a_t^* U_t = a_t^*(U_t^*)^{-1} = a_t^*(a_t^*)^{-1} b_t^* = b_t^*,$$

this concludes the proof of the first identity. As to the second one,

$$\begin{aligned}\mathrm{Tr}\big(U_t^*\partial_t U_t\big) &= \mathrm{Tr}\big(b_t(a_t)^{-1}\big((\partial_t a_t)(b_t)^{-1} - a_t(b_t)^{-1}(\partial_t b_t)(b_t)^{-1}\big)\big)\\ &= \mathrm{Tr}\big((a_t)^{-1}\partial_t a_t - (\partial_t b_t)(b_t)^{-1}\big).\end{aligned}$$

Alternatively, one can take the trace of the first formula and use the cyclicity together with the identity $b_t^{-1}(b_t^{-1})^* a_t^* = a_t^{-1}$. □

Combined with Proposition 1.5.12, one deduces the following:

Corollary 2.1.10. *Let $t \in [0,1] \mapsto \Phi_t = \binom{a_t}{b_t}$ be a closed path of J-Lagrangian frames which is piecewise continuously differentiable. Then $P_t = \Phi_t(\Phi_t^*\Phi_t)^{-1}\Phi_t^*$ satisfies*

$$\mathrm{BM}\big(t \in [0,1] \mapsto P_t\big) = \frac{1}{2\pi\imath}\int_0^1 dt\, \mathrm{Tr}\big((a_t)^{-1}\partial_t a_t - (b_t)^{-1}\partial_t b_t\big).$$

2.2 *J*-unitary matrices

For the application of the Bott–Maslov index in the next section, it will be necessary to introduce invertible linear maps that preserve the Krein space structures. These are the so-called J-unitary and I-unitary $2N \times 2N$ matrices T and M which satisfy $T^*JT = J$ and $M^*IM = I$, respectively. Note that these relations imply that T and M are invertible. The set of all matrices satisfying these relations form two subgroups of the general linear group $\mathrm{GL}(\mathbb{C}, 2N)$:

$$\mathbb{U}(\mathbb{C}^{2N}, J) = \{T \in \mathrm{GL}(2N, \mathbb{C}) : T^*JT = J\},$$
$$\mathbb{U}(\mathbb{C}^{2N}, I) = \{M \in \mathrm{GL}(2N, \mathbb{C}) : M^*IM = I\}.$$

The group $\mathbb{U}(\mathbb{C}^{2N}, J)$ is also called the generalized Lorentz group of signature (N, N) and often denoted by $\mathrm{U}(N, N)$. It follows from (2.2) that the Cayley transform connects these two groups via

$$\mathcal{C}^* \mathbb{U}(\mathbb{C}^{2N}, J)\mathcal{C} = \mathbb{U}(\mathbb{C}^{2N}, I). \tag{2.12}$$

The group $\mathbb{U}(\mathbb{C}^{2N}, I)$ contains the symplectic group as the following real subgroup:

$$\mathrm{SP}(2N, \mathbb{R}) = \mathbb{U}(\mathbb{C}^{2N}, I) \cap \mathrm{GL}(2N, \mathbb{R}).$$

This section is only about the complex theory so that this reality constraint will not play any role. Furthermore, everything will be spelled out for the Lorentz group $\mathbb{U}(\mathbb{C}^{2N}, J)$. Based on (2.12), it can readily translated into claims on $\mathbb{U}(\mathbb{C}^{2N}, I)$.

Let us note that when $T \in \mathbb{U}(\mathbb{C}^{2N}, J)$, then also the inverse T^{-1} is in $\mathbb{U}(\mathbb{C}^{2N}, J)$. Taking the inverse of the relation $T^*JT = J$ also shows that T^* is in $\mathbb{U}(\mathbb{C}^{2N}, J)$ so that this group is $*$-invariant. The group can be written out more explicitly using $N \times N$ matrices A, B, C, D. More precisely,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{U}(\mathbb{C}^{2N}, J) \tag{2.13}$$

if and only if

$$A^*A - C^*C = \mathbf{1}, \quad D^*D - B^*B = \mathbf{1}, \quad A^*B = C^*D. \tag{2.14}$$

One reads off that $D^*D = \mathbf{1} + B^*B \geq \mathbf{1}$ and thus D is invertible, Moreover,

$$\|D^{-1}\| \leq 1, \quad \|BD^{-1}\| < 1.$$

Due to $*$-invariance, one further has $DD^* = \mathbf{1} + CC^*$ so that $D^{-1}C(D^{-1}C)^* = \mathbf{1} - D^{-1}(D^{-1})^* < \mathbf{1}$. In particular, $\|D^{-1}C\| < 1$. Similarly, A is invertible. Let us also note that $\mathbb{U}(\mathbb{C}^{2N}, J)$ has a subgroup

$$\mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(2N) = \mathbb{U}(N) \oplus \mathbb{U}(N), \tag{2.15}$$

explicitly given by the diagonal J-unitaries $\mathrm{diag}(U_0, U_1)$ where $U_0, U_1 \in \mathbb{U}(N)$.

Remark 2.2.1. In the above and also in the following, the focus is on the Lie group of J-unitary matrices and their spectral and geometric properties. The group $\mathbb{U}(\mathbb{C}^{2N}, J)$ has the Lie algebra $\mathbb{B}_{\mathrm{sa}}(\mathbb{C}^{2N}, J)$ formed by the J-self-adjoint $2N \times 2N$ matrices H satisfying $JH^*J = H$. Note that $\mathbb{B}_{\mathrm{sa}}(\mathbb{C}^{2N}, J)$ is a real vector space and for all $H \in \mathbb{B}_{\mathrm{sa}}(\mathbb{C}^{2N}, J)$ with $\|H\| < 1$ one has

$$e^{\imath H} \in \mathbb{U}(\mathbb{C}^{2N}, J)$$

and

$$\mathfrak{C}(H) = (H - \imath\mathbf{1})(H + \imath\mathbf{1})^{-1} \in \mathbb{U}(\mathbb{C}^{2N}, J),$$

namely exponential and Cayley transform of J-self-adjoints are J-unitary. ◇

In the following, some elements of the spectral theory of J-unitaries will be used. Even though this can be found in the monographs [29, 20, 100], we include the basic properties needed here. The spectrum $\mathrm{spec}(T)$ of a J-unitary T has the reflection property

$$\mathrm{spec}(T) = \left(\overline{\mathrm{spec}(T)}\right)^{-1}, \tag{2.16}$$

which follows directly from the identity

$$T - z\mathbf{1} = J((T^*)^{-1} - z\mathbf{1})J = -zJ(T^*)^{-1}(T - \overline{z}^{-1}\mathbf{1})^*J.$$

Also some facts about spectral projections of a J-unitary T will be relevant. Let hence $\Delta \subset \mathrm{spec}(T)$ be a (separated) spectral subset and recall that the Riesz projection of T on Δ is

$$R_\Delta = \oint_\Gamma \frac{dz}{2\pi\imath}(z\mathbf{1} - T)^{-1}, \tag{2.17}$$

where Γ is a curve in $\mathbb{C} \setminus \mathrm{spec}(T)$ with winding number 1 around each point of Δ and 0 around all points of $\mathrm{spec}(T) \setminus \Delta$. Let us stress that R_Δ is (in general) not self-adjoint. Standard facts about Riesz projections are recalled in Appendix A.1.

Proposition 2.2.2. *Let T be a J-unitary, $\Delta \subset \mathbb{C}$ a spectral subset and its $\mathbb{S}^1$-reflected set given by $(\overline{\Delta})^{-1} = \{z \in \mathbb{C} : \overline{z}^{-1} \in \Delta\}$. Then*

$$(R_\Delta)^* = JR_{(\overline{\Delta})^{-1}}J$$

and

$$\operatorname{Ker}(R_\Delta)^\perp = J\,\operatorname{Ran}(R_{(\overline{\Delta})^{-1}}),$$

where the orthogonal complement is with respect to the euclidean scalar product. In particular,

$$\dim\big(\operatorname{Ran}(R_\Delta)\big) = \dim\big(\operatorname{Ran}(R_{(\overline{\Delta})^{-1}})\big). \tag{2.18}$$

Proof. First of all, let us note that indeed $(\overline{\Delta})^{-1}$ is in the spectrum of T, and thus by the spectral mapping theorem one also knows that $\overline{\Delta}$ is in the spectrum of T^{-1}. Let us take the adjoint of formula (2.17),

$$(R_\Delta)^* = \oint_{\overline{\Gamma}} \frac{dz}{2\pi\imath}(z\mathbf{1} - T^*)^{-1},$$

where $\overline{\Gamma}$ is the complex conjugate of Γ, hence encircling $\overline{\Delta}$ instead of Δ. It is also positively oriented even though the complex conjugated of the path Γ would have inverse orientation, but the imaginary factor compensates this. Thus $R_\Delta(T)^* = R_{\overline{\Delta}}(T^*)$ if one adds the initial operator as an argument to the Riesz projection. As $T^* = JT^{-1}J$,

$$(R_\Delta)^* = J\oint_{\overline{\Gamma}} \frac{dz}{2\pi\imath}(z\mathbf{1} - T^{-1})^{-1}J,$$

concluding the proof of the first identity. As to the second,

$$\operatorname{Ker}(R_\Delta)^\perp = \operatorname{Ran}(R_\Delta^*) = \operatorname{Ran}(JR_{(\overline{\Delta})^{-1}}J) = J\,\operatorname{Ran}(R_{(\overline{\Delta})^{-1}}),$$

so that the proof is complete. □

The following result now shows how one can construct two J-Lagrangian subspaces from a hyperbolic J-unitary (one which has no eigenvalues of unit modulus).

Proposition 2.2.3. *Let T be a J-unitary with* $\operatorname{spec}(T) \cap \mathbb{S}^1 = \emptyset$. *For* $\Delta = \operatorname{spec}(T) \cap B_1(0)$, *let* $\mathcal{E}^< = \operatorname{Ran}(R_\Delta)$ *and* $\mathcal{E}^> = \operatorname{Ran}(R_{\mathbb{C}\setminus\Delta})$ *be the subspaces of contracting and expanding directions for T. Then $\mathcal{E}^<$ and $\mathcal{E}^>$ are J-Lagrangian subspaces.*

Proof. By hypothesis and (2.18), both $\mathcal{E}^<$ and $\mathcal{E}^>$ are half-dimensional. Moreover, Proposition 2.2.2 shows that they are J-isotropic. □

Let us note that the orthogonal projections on $\mathcal{E}^<$ and $\mathcal{E}^>$ can be constructed from the Riesz projections. More generally, from a Riesz projection R_Δ, one can now construct a (self-adjoint) projection P_Δ onto $\operatorname{Ran}(R_\Delta)$ by setting

$$P_\Delta = R_\Delta\big(R_\Delta^* R_\Delta\big)^{-1} R_\Delta^*.$$

Remark 2.2.4. All the spectral properties of J-unitaries have a counterpart for the J-self-adjoint operators, the essential difference being that the reflection on the unit circle becomes reflection on the real axis. ◇

Next let us turn to the polar decomposition in $\mathbb{U}(\mathbb{C}^{2N},J)$ and an important corollary of it.

Proposition 2.2.5. *Let $T \in \mathbb{U}(\mathbb{C}^{2N},J)$ have the polar decomposition $T = W|T|$ where $|T| = (T^*T)^{\frac{1}{2}}$ and W is unitary. Then $|T| \in \mathbb{U}(\mathbb{C}^{2N},J)$ and $W \in \mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(2N)$.*

Proof. As $\mathbb{U}(\mathbb{C}^{2N},J)$ is a $*$-invariant group, also $T^* \in \mathbb{U}(\mathbb{C}^{2N},J)$ and therefore one has $T^*T \in \mathbb{U}(\mathbb{C}^{2N},J)$. By Lemma 2.2.6 below, one also has $|T| \in \mathbb{U}(\mathbb{C}^{2N},J)$. Finally, it follows that $W = T|T|^{-1} \in \mathbb{U}(\mathbb{C}^{2N},J)$ as a product of two J-unitaries. □

Lemma 2.2.6. *Let $T = T^* \in \mathbb{U}(\mathbb{C}^{2N},J)$ be a self-adjoint J-unitary and $f : \mathbb{R} \to \mathbb{C}$ a function that is continuous on a neighborhood of* $\mathrm{spec}(T) \cup \mathrm{spec}(T^{-1})$. *Then one has $Jf(T)^*J = \overline{f}(T^{-1})$. In particular, if $T > 0$ and $s \in \mathbb{R}$, then T^s is also a self-adjoint J-unitary.*

Proof. Let $g : \mathbb{R} \to \mathbb{C}$, $g(x) = \sum_{m=0}^{M} a_m x^m$ be a polynomial that agrees with f on the set $\mathrm{spec}(T) \cup \mathrm{spec}(T^{-1})$, namely such that $f|_{\mathrm{spec}(T)\cup\mathrm{spec}(T^{-1})} = g|_{\mathrm{spec}(T)\cup\mathrm{spec}(T^{-1})}$. Then, as T is J-unitary, one obtains

$$Jf(T)^*J = Jg(T)^*J = \sum_{m=0}^{M} \overline{a_m}(JT^*J)^m = \sum_{m=0}^{M} \overline{a_m}(T^{-1})^m = \overline{g}(T^{-1}) = \overline{f}(T^{-1}).$$

This is then applied to $f(z) = z^s$ as function on the half-space $\mathbb{R}e(z) > 0$ which is real on the real axis, so that $J(T^s)^*J = T^{-s} = (T^s)^{-1}$. □

Corollary 2.2.7. *The group $\mathbb{U}(\mathbb{C}^{2N},J)$ is path connected.*

Proof. The path $s \in [0,1] \mapsto W^s|T|^s$ connects a J-unitary $T = W|T|$ to the identity. Here W^s is defined using any branch cut, e. g., on the negative real axis. Due to Proposition 2.2.5 and Lemma 2.2.6, one has $|T|^s \in \mathbb{U}(\mathbb{C}^{2N},J)$. As to W^s, one can argue similarly, or use that $W \in \mathbb{U}(\mathbb{C}^{2N},J)$ is equivalent to $W = JWJ = \mathrm{diag}(W_+, W_-)$ with $W_\pm \in \mathrm{U}(N)$ so that also $W^s = \mathrm{diag}(W_+^s, W_-^s) \in \mathbb{U}(\mathbb{C}^{2N},J)$. In conclusion, the path lies in $\mathbb{U}(\mathbb{C}^{2N},J)$. □

The group $\mathbb{U}(\mathbb{C}^{2N},J)$ of J-unitaries naturally acts on the J-Lagrangian Grassmannian $\mathbb{P}(\mathbb{C}^{2N},J)$. On J-Lagrangian subspaces $\mathcal{E}$, the action is easy to write out:

$$(T,\mathcal{E}) \mapsto T\mathcal{E}.$$

If Φ is a normalized frame for $\mathcal{E}$, then the action becomes

$$(T,\Phi) \mapsto T\Phi|T\Phi|^{-1},$$

where the positive factor $|T\Phi|^{-1} \in \mathbb{C}^{N\times N}$ assures that the right-hand side is again a normalized frame. On projections, the formula looks a little more involved, which is why the notation $T \cdot P$ is introduced by

$$(T,P) \in \mathbb{U}(\mathbb{C}^{2N},J) \times \mathbb{P}(\mathbb{C}^{2N},J) \mapsto T \cdot P = TPT^*|TPT^*|^{-2}TPT^* \in \mathbb{P}(\mathbb{C}^{2N},J).$$

Note here that $|TPT^*|$ is not an invertible matrix, but it is bijective on the range of TPT^*. Let us also comment that one can check that this is indeed a group action, namely $S \cdot (T \cdot P) = (ST) \cdot P$ for $S, T \in \mathbb{U}(\mathbb{C}^{2N},J)$. Under the stereographic projection Π, this action becomes the action of the group $\mathbb{U}(\mathbb{C}^{2N},J)$ via operator Möbius transformation (also called canonical transformation or fractional transformation) on the unitary group which is defined by the following equation and also denoted by a dot:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot U = (AU + B)(CU + D)^{-1}.$$

Note that indeed $CU + D = D(D^{-1}CU + \mathbf{1})$ is invertible because $\|D^{-1}C\| < 1$ and the right-hand side $(AU + B)(CU + D)^{-1}$ is unitary.

Proposition 2.2.8. *The Möbius action implements the group action of* $\mathbb{U}(\mathbb{C}^{2N},J)$ *on* $\mathbb{P}(\mathbb{C}^{2N},J)$ *under the stereographic projection*

$$\Pi(T \cdot P) = T \cdot \Pi(P). \tag{2.19}$$

Proof. One way to check this is to realize that the action on frames in the form (2.8) can be read off the identity

$$T\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} AU + B \\ CU + D \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} T \cdot U \\ \mathbf{1} \end{pmatrix}(CU + D).$$

This directly implies the claim. □

Proposition 2.2.9. *The action of* $\mathbb{U}(\mathbb{C}^{2N},J)$ *on* $\mathbb{P}(\mathbb{C}^{2N},J)$ *is transitive. More precisely, for each pair P_0 and P_1 of J-Lagrangian projections there is a* $T \in \mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(2N)$ *such that $T \cdot P_0 = P_1$. The J-Lagrangian Grassmannian* $\mathbb{P}(\mathbb{C}^{2N},J)$ *is path connected.*

Proof. Let $U_0 = \Pi(P_0)$ and $U_1 = \Pi(P_1)$. Then $T = \operatorname{diag}(U_1, U_0)$ is J-unitary and satisfies $T \cdot U_0 = U_1$. The second claim now follows from Corollary 2.2.7. □

Now let us come to a first application of the above action. Suppose given a path $t \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N},J)$ and a J-unitary $T \in \mathbb{U}(\mathbb{C}^{2N},J)$, one naturally obtains another path $t \mapsto T \cdot P_t \in \mathbb{P}(\mathbb{C}^{2N},J)$. Its Bott–Maslov index can be computed in terms of that of $t \mapsto P_t$, albeit with respect to a different reference Lagrangian plane.

Proposition 2.2.10. *Let* $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N},J)$, $\mathcal{F}$ *a J-Lagrangian reference plane and* $T \in \mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(2N)$. *Then*

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, T \cdot P_t)\big) = \mathrm{BM}\big(t \in [0,1] \mapsto (P_{T^*\mathcal{F}}, P_t)\big). \tag{2.20}$$

Proof. Set $V = \Pi(P_{\mathcal{F}})$. Furthermore, let $\mathcal{E}_t = \mathrm{Ran}(P_t)$ and $U_t = \Pi(P_t)$. According to (2.10), $\mathrm{BM}(t \in [0,1] \mapsto (P_{\mathcal{F}}, T \cdot P_t))$ is given by counting the intersections $T\mathcal{E}_t \cap \mathcal{F}^\perp$. Now

$$\begin{aligned}(T^*\mathcal{F})^\perp &= \{\phi \in \mathbb{C}^{2N} : \phi^* T^* \psi = 0 \ \ \forall \psi \in \mathcal{F}\}\\ &= \{\phi \in \mathbb{C}^{2N} : (T\phi)^* \psi = 0 \ \ \forall \psi \in \mathcal{F}\}\\ &= T^{-1}\mathcal{F}^\perp,\end{aligned}$$

so that $\mathcal{F}^\perp = T(T^*\mathcal{F})^\perp$ and

$$T\mathcal{E}_t \cap \mathcal{F}^\perp = T\big(\mathcal{E}_t \cap (T^*\mathcal{F})^\perp\big).$$

As T is bijective,

$$\dim\big(T\mathcal{E}_t \cap \mathcal{F}^\perp\big) = \dim\big(\mathcal{E}_t \cap (T^*\mathcal{F})^\perp\big),$$

showing that the intersections of the two sides of (2.20) are of the same dimension. It remains to show that they also have the same orientation. For this purpose, let us first note that Proposition 1.4.1 allows approximating the path by a differentiable one (even only with simple transversal eigenvalue crossings; strictly speaking, Proposition 1.4.1 addresses paths of self-adjoints, but it readily transposes to paths of unitaries). For the computation of the two spectral flows as given by Definition 2.1.8, one can then invoke Lemma 2.1.9 to analyze the orientation of the eigenvalue crossings of the two paths in (2.20). For that purpose, let $T = \mathrm{diag}(W_+, W_-)$ with $W_\pm \in \mathbb{U}(N)$. For the path on the left-hand side, the crossing form (without factor $\imath$) is

$$\begin{aligned}&(V^* T \cdot U_t)^* \partial_t (V^* T \cdot U_t)|_{\mathrm{Ker}(V^* T \cdot U_t + \mathbf{1})}\\ &\qquad = (V^* W_+ U_t W_-^*)^* \partial_t (V^* W_+ U_t W_-^*)|_{\mathrm{Ker}(V^* W_+ U_t W_-^* + \mathbf{1})}\\ &\qquad = W_-(U_t^* \partial_t U_t) W_-^*|_{W_- \,\mathrm{Ker}(V^* W_+ U_t + W_-)},\end{aligned}$$

while for the right-hand side it is

$$\begin{aligned}((T^* \cdot V)^* U_t)^* \partial_t ((T^* \cdot V)^* U_t)|_{\mathrm{Ker}((T^*\cdot V)^* U_t + \mathbf{1})} &= U_t^* \partial_t U_t|_{\mathrm{Ker}(W_-^* V^* W_+ U_t + \mathbf{1})}\\ &= U_t^* \partial_t U_t|_{\mathrm{Ker}(V^* W_+ U_t + W_-)}.\end{aligned}$$

Hence the eigenvalue crossings are also in the same direction. □

Corollary 2.2.11. *Let $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N}, J)$ and $\mathcal{F}$ be a reference J-Lagrangian subspace. Then there exists a $T \in \mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(2N)$ such that*

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, P_t)\big) = \mathrm{BM}\big(t \in [0,1] \mapsto T \cdot P_t\big).$$

Proof. Due to Proposition 2.2.9, $(T^{-1})^*\mathcal{F} = \mathcal{E}_{\mathrm{ref}}$ for a suitable $T \in \mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(2N)$. Thus applying Proposition 2.2.10 concludes the proof. □

For closed paths, the following can now be said.

Proposition 2.2.12. *Let $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N},J)$ be a closed path. Then for any J-Lagrangian reference subspace $\mathcal{F}$ and any $T \in \mathbb{U}(\mathbb{C}^{2N},J)$, one has*

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, T \cdot P_t)\big) = \mathrm{BM}\big(t \in [0,1] \mapsto P_t\big).$$

Proof. By Corollary 2.2.11, one can choose $T' \in \mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(2N)$ such that

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_{\mathcal{F}}, T \cdot P_t)\big) = \mathrm{BM}\big(t \in [0,1] \mapsto (T'T) \cdot P_t\big).$$

By Definition 2.1.6 and the hypothesis, the Bott–Maslov index is given by the winding number of a closed path. As $\mathbb{U}(\mathbb{C}^{2N},J) \cap \mathbb{U}(\mathbb{C}^{2N})$ is path connected by Corollary 2.2.7, $T'T$ can be homotopically deformed into the identity. Hence the homotopy invariance of the spectral flow implies the claim. □

Another scenario to obtain a path of J-Lagrangians (and thus an associated Bott–Maslov index) is to have a path of J-unitaries

$$t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}, \tag{2.21}$$

and then, given a fixed J-Lagrangian projection $P \in \mathbb{P}(\mathbb{C}^{2N},J)$, to consider $t \mapsto T_t \cdot P$. After applying the stereographic projection, this provides a path $t \mapsto U_t = \Pi(T_t \cdot P) = T_t \cdot \Pi(P)$ of unitaries. For the analysis of transversality of this path and an explicit computation of its winding number (and thus the associated Bott–Maslov index as in Corollary 2.1.10), the following result is then useful.

Lemma 2.2.13. *Let $t \mapsto T_t$ be a differentiable path in $\mathbb{U}(\mathbb{C}^{2N},J)$ and $P \in \mathbb{P}(\mathbb{C}^{2N},J)$. Then $U_t = \Pi(T_t \cdot P) \in \mathbb{U}(N)$ satisfies*

$$U_t^* \partial_t U_t = \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}^* (\partial_t T_t J T_t^*) \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix},$$

and

$$\mathrm{Tr}\big(U_t^* \partial_t U_t\big) = 2\,\mathrm{Tr}\big((\mathbf{1} - P_t)(\partial_t T_t J T_t^*)\big),$$

where $P_t = T_t \cdot P$.

Proof. Let $\Phi = \binom{a}{b}$ be a frame for P so that $V = \Pi(P) = ab^{-1}$. Lemma 2.1.9 applied to the (not normalized) frame $\Phi_t = T_t \Phi$ implies

$$U_t^* \partial_t U_t = \big((C_t a + D_t b)^{-1}\big)^* \Phi^* (T_t^* J \partial_t T_t) \Phi (C_t a + D_t b)^{-1}. \tag{2.22}$$

Using the identity

$$\Phi(C_t a + D_t b)^{-1} = \begin{pmatrix} V \\ \mathbf{1} \end{pmatrix} (C_t V + D_t)^{-1} = T_t^{-1} \begin{pmatrix} U_t \\ \mathbf{1} \end{pmatrix},$$

together with

$$(T_t^{-1})^* (T_t^* J \partial_t T_t) T_t^{-1} = J(\partial_t T_t J T_t^*) J,$$

implies the first claim. This immediately implies the second claim. An alternative proof is given in Proposition 9.5.11 below. □

For closed paths of the type $t \mapsto T_t \cdot P$, it is again particularly simple to compute the Bott–Maslov index. It will be shown in Section 2.3 that the outcome is in fact the Conley–Zehnder index.

Proposition 2.2.14. *Let $t \in [0,1] \mapsto T_t \in \mathbb{U}(\mathbb{C}^{2N}, J)$ be a closed differentiable path. Then*

$$\begin{aligned} &\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P) \\ &\quad = \frac{1}{2\pi\imath} \int_0^1 dt \,\mathrm{Tr}\big((A_t + B_t)^{-1} \partial_t (A_t + B_t) - (C_t + D_t)^{-1} \partial_t (C_t + D_t)\big), \end{aligned}$$

where the matrix entries of T_t are denoted as in (2.21). *In particular, the Bott–Maslov index* $\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P)$ *is independent of $P \in \mathbb{P}(\mathbb{C}^{2N}, J)$, a fact that also holds without the differentiability assumption.*

Proof. Because the path is closed, the Bott–Maslov index is given by a winding number and thus is homotopy invariant. Therefore, one may deform P to P_{ref} for which $U_t = \Pi(T_t \cdot P_{\mathrm{ref}}) = (A_t + B_t)(C_t + D_t)^{-1}$ as $T_t \Phi_{\mathrm{ref}} = 2^{-\frac{1}{2}} \binom{A_t + B_t}{C_t + D_t}$. Replacing in the expression for the winding number leads to the claimed formula (alternatively, one can use Corollary 2.1.10). As to the very last claim on the independence of P for only continuous paths, one can use the homotopy invariance of the winding number under a homotopy $s \in [0,1] = (T)^s \cdot P_{\mathrm{ref}}$ from P to a fixed reference Lagrangian projection P_{ref}, where $T \in \mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(2N)$ is such that $P = T \cdot P_{\mathrm{ref}}$. □

One can further combine Lemma 2.2.13 with Lemma 2.1.9 and consider paths in $\mathbb{P}(\mathbb{C}^{2N}, J)$ of the type $t \mapsto T_t \cdot P_t$. This leads to a formula that allows to analyze their transversality. Closed paths of this type are considered below.

Lemma 2.2.15. *Let $t \in [0,1] \mapsto \Phi_t$ be a differentiable path of (not necessarily normalized) J-Lagrangian frames with associated $P_t = \Phi_t (\Phi_t^* \Phi_t)^{-1} \Phi_t^*$. Further let $t \mapsto T_t \in \mathbb{U}(\mathbb{C}^{2N}, J)$ be differentiable and consider the path $t \mapsto T_t \cdot P_t \in \mathbb{P}(\mathbb{C}^{2N}, J)$. Then $U_t = \Pi(T_t \cdot P_t)$ satisfies*

$$U_t^* \partial_t U_t = \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}^* (\partial_t T_t J T_t^*) \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix} + (b_t^{-1})^* (\Phi_t^* J \partial_t \Phi_t) b_t^{-1},$$

where now b_t is the entry of $T_t \Phi_t = \binom{a_t}{b_t}$.

Proof. Again using the first identity of Lemma 2.1.9, one finds

$$U_t^* \partial_t U_t = (b_t^{-1})^* ((T_t \Phi_t)^* J \partial_t (T_t \Phi_t)) b_t^{-1}.$$

Using the Leibniz rule and the arguments of the proofs of Lemmas 2.1.9 and 2.2.13 allows to conclude the computation. □

Proposition 2.2.16. *Let $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N}, J)$ and $t \in [0,1] \mapsto T_t \in \mathbb{U}(\mathbb{C}^{2N}, J)$ be closed paths. Then for any fixed $P \in \mathbb{P}(\mathbb{C}^{2N}, J)$, one has*

$$\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P_t) = \mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P) + \mathrm{BM}(t \in [0,1] \mapsto P_t).$$

Proof. Because the path is closed and the Bott–Maslov index is given by a winding number, one can deform the path homotopically to

$$\widehat{P}_t = \begin{cases} T_0 \cdot P_{2t}, & t \in [0, \frac{1}{2}], \\ T_{2t-1} \cdot P_1, & t \in [\frac{1}{2}, 1], \end{cases}$$

without changing the winding number. The concatenation property of the winding number combined with Propositions 2.2.12 and 2.2.14 then allows to conclude the proof. □

2.3 Conley–Zehnder index in finite dimension

It is a well-known fact that the graph of a symplectic matrix is a Lagrangian subspace with respect to a suitable quadratic form. This algebraic fact transposes to Krein spaces $(\mathbb{C}^{2N}, J)$. The graph of a J-unitary is then a subspace of $\mathbb{C}^{4N}$ on which one has to choose a suitable sesquilinear form $\widehat{J}$ such that the graph is a $\widehat{J}$-Lagrangian subspace. Then the theory of Section 2.1 readily transfers. In particular (and similar as in Proposition 2.1.4), there is a stereographic projection of the graph providing a unitary matrix on $\mathbb{C}^{2N}$ which allows studying the intersection with a suitable reference $\widehat{J}$-Lagrangian subspace $\widehat{\Phi}_{\mathrm{ref}}$ in a convenient manner. Given a path of J-unitaries, its Conley–Zehnder index is then nothing but a Bott–Maslov index of the path of the $\widehat{J}$-Lagrangian subspaces given by graphs of J-unitaries.

Let T be a J-unitary matrix. Then its graph

$$\mathcal{G}_T = \mathrm{Ran}\left((\mathbf{1} \oplus T) \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \right) = \mathrm{Ran}\left(\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix} \right) \subset \mathbb{C}^{4N}$$

is Lagrangian with respect to the Krein form $(-J) \oplus J$ on $\mathbb{C}^{4N}$. Moreover, the diagonal $\mathrm{Ran}(\binom{\mathbf{1}}{\mathbf{1}}) \subset \mathbb{C}^{4N}$ is another $(-J) \oplus J$-Lagrangian subspace and therefore

$$\mathrm{Ran}\left(\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix}\right) \cap \mathrm{Ran}\left(\begin{pmatrix} \mathbf{1} \\ \pm\mathbf{1} \end{pmatrix}\right) = \left\{\begin{pmatrix} \phi \\ T\phi \end{pmatrix} : \phi \in \mathrm{Ker}(T \mp \mathbf{1})\right\} \tag{2.23}$$

and, in particular,

$$\dim(\mathrm{Ker}(T \mp \mathbf{1})) = \dim\left(\mathrm{Ran}\left(\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix}\right) \cap \mathrm{Ran}\left(\begin{pmatrix} \mathbf{1} \\ \pm\mathbf{1} \end{pmatrix}\right)\right)$$

can be computed as the dimension of the intersection of two $(-J) \oplus J$-Lagrangian subspaces in $\mathbb{C}^{4N}$. In order to apply the intersection theory developed in Section 2.1, let us use the basis change to the canonical Krein form $\widehat{J} = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$ on $\mathbb{C}^{4N}$:

$$\widehat{F}((-J) \oplus J)\widehat{F} = \widehat{J}, \quad \widehat{F} = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}. \tag{2.24}$$

Then it is natural to associate to $T \in \mathbb{U}(\mathbb{C}^{2N}, J)$ the operator

$$\widehat{T} = \widehat{F}(\mathbf{1} \oplus T)\widehat{F} \in \mathbb{U}(\mathbb{C}^{4N}, \widehat{J}).$$

Its range $\mathrm{Ran}(\widehat{T})$ is then equal to the transformed graph $\widehat{\mathcal{G}}_T = \widehat{F}\mathcal{G}_T$.

Let $\widehat{\Pi}$ denote the associated stereographic projection in the Krein space $(\mathbb{C}^{4N}, \widehat{J})$. As a reference projection and reference frame, we will then use the $\widehat{F}$-transformed diagonal $\widehat{F}\binom{\mathbf{1}}{\mathbf{1}} = \binom{\mathbf{1}}{\mathbf{1}}$:

$$\widehat{P}_{\mathrm{ref}} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \quad \widehat{\Phi}_{\mathrm{ref}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}. \tag{2.25}$$

Note that while this looks the same as in (2.4), the entries are of double size here. One has $\widehat{\Pi}(\widehat{P}_{\mathrm{ref}}) = \mathbf{1}$ and $\widehat{F}\widehat{P}_{\mathrm{ref}}\widehat{F} = \widehat{P}_{\mathrm{ref}}$, as well as $\widehat{\mathcal{G}}_T = \mathrm{Ran}(\widehat{T}\widehat{\Phi}_{\mathrm{ref}})$, so that $\widehat{T} \cdot \widehat{P}_{\mathrm{ref}} \in \mathbb{P}(\mathbb{C}^{4N}, \widehat{J})$ is the orthogonal projection on $\widehat{\mathcal{G}}_T$. Moreover, the reference plane $\widehat{\mathcal{E}}_{\mathrm{ref}} = \mathrm{Ran}(\widehat{P}_{\mathrm{ref}})$ satisfies

$$\dim(\mathrm{Ker}(T - \mathbf{1})) = \dim(\widehat{\mathcal{G}}_T \cap \widehat{\mathcal{E}}_{\mathrm{ref}}), \quad \dim(\mathrm{Ker}(T + \mathbf{1})) = \dim(\widehat{\mathcal{G}}_T \cap \widehat{J}\widehat{\mathcal{E}}_{\mathrm{ref}}). \tag{2.26}$$

Now it is natural to look at the stereographic projection of the graph which is denoted $S(T)$ because of its connections to scattering theory explained further down.

Theorem 2.3.1. *To a given $T \in \mathbb{U}(\mathbb{C}^{2N}, J)$ let us associate a unitary $S(T)$ by*

$$S(T) = \widehat{\Pi}(\widehat{\mathcal{G}}_T) = \widehat{\Pi}(\widehat{P}_{\mathrm{ref}})^* \widehat{\Pi}(\widehat{T} \cdot \widehat{P}_{\mathrm{ref}}) \in \mathbb{U}(2N). \tag{2.27}$$

If $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *then*

$$S(T) = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix} = \begin{pmatrix} (A^*)^{-1} & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix}.$$

The nonlinear map $T \in \mathbb{U}(\mathbb{C}^{2N}, J) \mapsto S(T) \in \mathbb{U}(2N)$ *is a continuous dense embedding with image*

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}(2N) : \alpha, \delta \in \mathbb{C}^{N \times N} \text{invertible} \right\}. \tag{2.28}$$

Proof. As noted after (2.13), one has $D^*D \geq \mathbf{1}$ and $DD^* \geq \mathbf{1}$ so that D is invertible (similarly A is invertible). Now as

$$\widehat{F}\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & \mathbf{1} \\ \mathbf{1} & 0 \\ C & D \end{pmatrix}, \tag{2.29}$$

the lower 2×2 block is invertible. Hence one can normalize $\widehat{F}\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix}$ to a frame by multiplying by its inverse. According to Remark 2.1.5, the upper 2×2 entry is the associated stereographic projection, namely

$$\begin{aligned} S(T) &= \widehat{\Pi}\left(\operatorname{Ran}\left(\widehat{F}\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} A & B \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ C & D \end{pmatrix}^{-1} \\ &= \begin{pmatrix} A & B \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix}. \end{aligned}$$

From this the first formula for $S(T)$ follows, and the second results from the relations in $\mathbb{U}(\mathbb{C}^{2N}, J)$. Clearly, its upper left and lower right entries, the matrices denoted by α and δ in (2.28), are invertible.

Finally let us show that the map $T \in \mathbb{U}(\mathbb{C}^{2N}, J) \mapsto S(T) \in \mathbb{U}(2N)$ is surjective onto the set (2.28). Indeed, given an element of this set, it is natural to set $D = \delta^{-1}$, $B = \beta\delta^{-1}$, $C = -\delta^{-1}\gamma$, and $A = \alpha - \beta\delta^{-1}\gamma$. With some care one then checks that the defining equations stated after (2.13) indeed hold.

To show that the set given in (2.28) is dense in $\mathbb{U}(2N)$ let us consider a unitary matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}(2N),$$

where α is not invertible. Because U is unitary, $\alpha^*\alpha + \gamma^*\gamma = \mathbf{1}$ and therefore γ maps $\operatorname{Ker}(\alpha)$ bijectively onto $\gamma \operatorname{Ker}(\alpha)$. Thus there is $\mu : \mathbb{C}^N \to \mathbb{C}^N$ mapping $\gamma \operatorname{Ker}(\alpha)$ onto

$\mathrm{Ker}(\alpha^*)$ and such that $\mu\gamma : \mathbb{C}^N \to \mathbb{C}^N$ is a partial isometry with $\mathrm{Ker}(\mu\gamma) = \mathrm{Ker}(\alpha)^\perp$ and $\mathrm{Ran}(\mu\gamma) = \mathrm{Ker}(\alpha^*)$. Clearly,

$$V_t = \exp\left(\imath t \begin{pmatrix} 0 & \mu \\ \mu^* & 0 \end{pmatrix}\right)$$

is unitary for all $t \in \mathbb{R}$ and

$$V_t U = \begin{pmatrix} \alpha + \imath t\mu\gamma & \beta + \imath t\mu\delta \\ \gamma + \imath t\mu^*\alpha & \delta + \imath t\mu^*\beta \end{pmatrix} + o(t).$$

As α maps $\mathrm{Ker}(\alpha)^\perp$ bijectively onto $\mathrm{Ran}(\alpha) = \mathrm{Ker}(\alpha^*)^\perp$ and $\mu\delta$ maps $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\mu\delta)^\perp$ isometrically onto $\mathrm{Ran}(\mu\delta) = \mathrm{Ker}(\alpha^*)$, the map $\alpha + \imath t\mu\gamma : \mathbb{C}^N \to \mathbb{C}^N$ is invertible for $t \neq 0$ and its inverse is bounded by $\|(\alpha + \imath t\mu\gamma)^{-1}\| \leq C|t|^{-1}$ for some constant $C > 0$ and $|t| \leq 1$. Therefore the upper left entry of $V_t U$ is invertible for $t \neq 0$ sufficiently small. A similar argument shows that for $t \neq 0$ sufficiently small and $\epsilon > 0$ there is a unitary W such that $\|\mathbf{1} - W\| < \epsilon$ and such that the diagonal entries of WV_tU are invertible. This implies the last claim. □

Proposition 2.3.2. *Given $T \in \mathbb{U}(\mathbb{C}^{2N}, J)$, one has*

$$S(-T) = -JS(T)J$$

and

$$S(T)^* = S(T)^{-1} = S(T^{-1}) = JS(T^*)J.$$

Proof. The first claim follows directly from the definition. The first equality of the second set of identities holds as $S(T)$ is unitary. The second can directly be checked using the defining equations of $\mathbb{U}(\mathbb{C}^{2N}, J)$. Using $T^{-1} = JT^*J$, one next finds

$$S(T^*) = \begin{pmatrix} A^{-1} & C^*(D^*)^{-1} \\ -(D^*)^{-1}B^* & (D^*)^{-1} \end{pmatrix} = \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & (D^*)^{-1} \end{pmatrix}$$

and

$$S(T^{-1}) = \begin{pmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & (D^*)^{-1} \end{pmatrix} = \begin{pmatrix} A^{-1} & -C^*(D^*)^{-1} \\ (D^*)^{-1}B^* & (D^*)^{-1} \end{pmatrix}. \tag{2.30}$$

This shows the last claim. □

The following result states that there is a tight connection between the eigenvalues 1 and −1 of T and $S(T)$.

Theorem 2.3.3. *Let T and $S(T)$ be as in Theorem* 2.3.1. *Then*

$$\mathrm{Ker}(T - \mathbf{1}) = \mathrm{Ker}\big(S(T) - \mathbf{1}\big), \quad \mathrm{Ker}(T + \mathbf{1}) = J\,\mathrm{Ker}\big(S(T) + \mathbf{1}\big).$$

Proof. While this follows from (2.23) and the results of Section 2.1, let us provide a direct proof. One has for all vectors $\phi, \phi', \psi, \psi' \in \mathcal{H}$,

$$S(T)\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}\phi'\\ \phi\end{pmatrix} \quad \Longleftrightarrow \quad T\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \begin{pmatrix}\phi'\\ \psi'\end{pmatrix}, \tag{2.31}$$

as can readily be seen by writing everything out:

$$\begin{pmatrix}A - BD^{-1}C & BD^{-1}\\ -D^{-1}C & D^{-1}\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}\phi'\\ \phi\end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \begin{pmatrix}\phi'\\ \psi'\end{pmatrix}.$$

In particular, studying an eigenvalue λ of T, one has

$$S(T)\begin{pmatrix}\psi\\ \lambda\phi\end{pmatrix} = \begin{pmatrix}\lambda\psi\\ \phi\end{pmatrix} \quad \Longleftrightarrow \quad T\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \lambda\begin{pmatrix}\psi\\ \phi\end{pmatrix}, \tag{2.32}$$

or similarly for eigenvalues λ of $S(T)$:

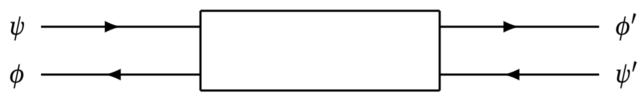
$$S(T)\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \lambda\begin{pmatrix}\psi\\ \psi'\end{pmatrix} \quad \Longleftrightarrow \quad T\begin{pmatrix}\psi\\ \lambda\psi'\end{pmatrix} = \begin{pmatrix}\lambda\psi\\ \psi'\end{pmatrix}. \tag{2.33}$$

Both equations are particularly interesting in the case $\lambda = \pm 1$:

$$S(T)\begin{pmatrix}\psi\\ \pm\phi\end{pmatrix} = \pm\begin{pmatrix}\psi\\ \pm\phi\end{pmatrix} \quad \Longleftrightarrow \quad T\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \pm\begin{pmatrix}\psi\\ \phi\end{pmatrix}. \tag{2.34}$$

This equivalence proves the theorem. □

Remark 2.3.4. The transformation (2.31) from T to $S(T)$ can be visualized as follows:



In a quantum-mechanical setup, the box in the middle is referred to as the sample. The (transfer) matrix T transfers left states to right states, while the (scattering) matrix $S(T)$ maps incoming states to outgoing states. Having this picture in mind, the eigenvalue 1 of T appearing in Theorem 2.3.3 allows constructing periodic solutions of a periodized system in which the same sample is repeated periodically. Similarly, the eigenvalue −1 of T corresponds to antiperiodic solutions (having a double period). Let us note that the conventions are different than in [21] where a formulation closer to scattering theory was chosen. This implies that the off-diagonal entries in $S(T)$ each have an extra sign. ◇

Remark 2.3.5. If $\binom{\psi}{\phi}$ is an eigenvector of $T \in \mathbb{U}(\mathbb{C}^{2N}, J)$ with eigenvalue λ off the unit circle, then $\|\psi\| = \|\phi\|$. Indeed, by (2.32) and the fact that $S(T)$ is unitary and therefore isometric, it follows that

$$\|\psi\|^2 + |\lambda|^2\|\phi\|^2 = |\lambda|^2\|\psi\|^2 + \|\phi\|^2.$$

As $|\lambda| \neq 1$, the claim follows. ◇

Theorem 2.3.3, as well as the connection between eigenvectors, can easily be adapted to study other eigenvalues on the unit circle. Indeed, if $T\phi = z\phi$ for $z \in \mathbb{S}^1$, then also $(\overline{z}T)\phi = \phi$. But the operator $\overline{z}T$ is also J-unitary so that one can apply the above again to construct an associated unitary. This shows the following:

Proposition 2.3.6. *Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a J-unitary so that, for $z \in \mathbb{S}^1$,*

$$S(\overline{z}T) = \begin{pmatrix} \overline{z}(A^*)^{-1} & BD^{-1} \\ -D^{-1}C & zD^{-1} \end{pmatrix}. \tag{2.35}$$

Then

$$\mathrm{Ker}(T - z\mathbf{1}) = \mathrm{Ker}\big(S(\overline{z}T) - \mathbf{1}\big).$$

Therefore, the unitaries $S(\overline{z}T)$ are a tool to study the eigenvalues of T which lie on the unit circle. Let us focus again on $z = \pm 1$. Theorem 2.3.3 concerns the kernel of $S(T) \mp \mathbf{1}$. It is natural to analyze how much more spectrum $S(T)$ has close to ± 1, or, what is equivalent, how much spectrum the self-adjoint operator

$$\mathbb{R}e\big(S(T)\big) = \frac{1}{2}\big(S(T) + S(T)^*\big)$$

has close to ± 1. For this purpose it is useful to have an explicit expression for $\mathbb{R}e(S(T))$.

Proposition 2.3.7. *Let T be a J-unitary and $S(T)$ as above. Then*

$$\mathbb{R}e\big(S(T)\big) = (\mathbf{1} + T)\big(\mathbf{1} + T^*T\big)^{-1}(\mathbf{1} + T)^* - \mathbf{1}.$$

Proof. Let us begin by calculating

$$\mathbb{R}e\big(S(T)\big) = \frac{1}{2}\big(S(T) + S(T)^*\big) = \frac{1}{2}\big(S(T) + \mathbf{1}\big)\big(S(T) + \mathbf{1}\big)^* - \mathbf{1}.$$

Next let us rewrite (2.29) as

$$\widehat{F}\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix} = \begin{pmatrix} S(T) \\ \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ C & D \end{pmatrix}.$$

Hence

$$S(T) + \mathbf{1} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}^* \widehat{F}\begin{pmatrix} \mathbf{1} \\ T \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ C & D \end{pmatrix}^{-1} = (\mathbf{1} + T)\begin{pmatrix} \mathbf{1} & 0 \\ C & D \end{pmatrix}^{-1},$$

so that

$$\mathbb{R}e(S(T)) = \frac{1}{2}(\mathbf{1}+T)\begin{pmatrix}\mathbf{1} & 0\\ C & D\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{1} & C^*\\ 0 & D^*\end{pmatrix}^{-1}(\mathbf{1}+T)^* - \mathbf{1}.$$

But using the identities (2.14), one finds

$$\begin{pmatrix}\mathbf{1} & 0\\ C & D\end{pmatrix}^*\begin{pmatrix}\mathbf{1} & 0\\ C & D\end{pmatrix} = \frac{1}{2}(\mathbf{1}+T^*T).$$

Replacing shows the claim. □

Now we can define the Conley–Zehnder index of a path of $t \mapsto T_t$ of J-unitaries [62] (which, strictly speaking, also goes back to the work of Bott [35]). There are two possible choices, namely one can either choose to focus on the eigenvalues +1 or −1 of T_t. Due to (2.26), this corresponds to the intersections of the graph $\widehat{\mathcal{G}}_{T_t}$ with $\widehat{\mathcal{E}}_{\mathrm{ref}}$ or with $\widehat{J}\widehat{\mathcal{E}}_{\mathrm{ref}}$, respectively, or yet otherwise stated, the spectral flow of $t \mapsto S(T_t)$ through 1 or −1, respectively. Here we will choose the second possibility, which then fits with the spectral flow of unitaries as defined in Section 1.5. Let us stress that this is merely a choice in the present finite-dimensional setup, but in the infinite-dimensional setting of Chapter 9 the Fredholm condition is chosen such that one has to consider the spectral flow through −1 (as is done here).

Definition 2.3.8. Given a path $t \in [0,1] \mapsto T_t \in \mathbb{U}(\mathbb{C}^{2N}, J)$, the Conley–Zehnder index is defined as

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \mathrm{BM}(t \in [0,1] \mapsto (\widehat{J}\widehat{\mathcal{E}}_{\mathrm{ref}}, \widehat{\mathcal{G}}_{T_t})). \tag{2.36}$$

The Bott–Maslov index on the right-hand side of (2.36) is taken in the Krein space $(\mathbb{C}^{4N}, \widehat{J})$. Of course, one can also come back to the Krein space $(\mathbb{C}^{4N}, (-J) \oplus J)$ by multiplying by $\widehat{F}$ and then

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \mathrm{BM}(t \in [0,1] \mapsto (\mathcal{F}_-, \mathcal{G}_{T_t})),$$

where $\mathcal{F}_\pm = \mathrm{Ran}(\binom{\mathbf{1}}{\pm\mathbf{1}})$, in which the minus sign results from the choice of counting eigenvalue passages of T_t through −1. Alternatively, if one is interested in counting the eigenvalue passages through 1 (often corresponding to periodic solutions), then

$$\mathrm{CZ}(t \in [0,1] \mapsto -T_t) = \mathrm{BM}(t \in [0,1] \mapsto (\mathcal{F}_+, \mathcal{G}_{T_t})).$$

Furthermore, applying the stereographic projection $\widehat{\Pi}$, one then immediately deduces that

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \mathrm{Sf}(t \in [0,1] \mapsto S(T_t)),$$

where the spectral flow of unitaries on the right-hand side is the spectral flow through -1, see Section 1.5. The following formula allows to analyze transversality issues of a given differentiable path of J-unitaries.

Lemma 2.3.9. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a differentiable path in $\mathbb{U}(\mathbb{C}^{2N}, J)$. Then*

$$S(T_t)^* \partial_t S(T_t) = \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix}^* (T_t^* J \partial_t T_t) \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix}.$$

For a vector $\phi_t \in \mathbb{C}^{2N}$ satisfying $T_t\phi_t = \phi_t$, one has $S(T_t)\phi_t = \phi_t$ by (2.34) *and*

$$\phi_t^* S(T_t)^* \partial_t S(T_t)\phi_t = \phi_t^* T_t^* J \partial_t T_t \phi_t.$$

For a vector $\phi_t \in \mathbb{C}^{2N}$ satisfying $T_t\phi_t = -\phi_t$, one has $S(T_t)J\phi_t = -J\phi_t$ by (2.34) *and*

$$\phi_t^* J S(T_t)^* \partial_t S(T_t) J\phi_t = \phi_t^* T_t^* J \partial_t T_t \phi_t.$$

Proof. First of all,

$$S(T_t)^* \partial_t S(T_t) = \widehat{\Pi}(\widehat{T}_t \cdot \widehat{P}_{\text{ref}})^* \partial_t \widehat{\Pi}(\widehat{T}_t \cdot \widehat{P}_{\text{ref}}).$$

Now one can apply (2.22) to get

$$S(T_t)^* \partial_t S(T_t) = (b_t^{-1})^* \widehat{\Phi}_{\text{ref}}^* (\widehat{T}_t^* \widehat{J} \partial_t \widehat{T}_t) \widehat{\Phi}_{\text{ref}} b_t^{-1},$$

where b_t is the lower component of $\widehat{T}_t \Phi_{\text{ref}}$, namely

$$b_t^{-1} = \sqrt{2} \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix}.$$

Next

$$\widehat{T}_t^* \widehat{J} \partial_t \widehat{T}_t = F(\mathbf{1} \oplus T_t^*)FF(J \oplus J)FF(0 \oplus \partial_t T_t)F = F(0 \oplus T_t^* J \partial_t T_t)F,$$

and the formula for $S(T_t)^* \partial_t S(T_t)$ now follows from the identity

$$\widehat{\Phi}_{\text{ref}}^* F(0 \oplus A) F \widehat{\Phi}_{\text{ref}} = \frac{1}{2} A,$$

substituted into the above. Finally, if $T_t\phi_t = \phi_t$, then

$$\begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix} \phi_t = \phi_t,$$

as one checks directly by decomposing ϕ_t in upper and lower components. The last claim is verified in a similar manner. □

For the computation of the Conley–Zehnder index of a closed path it is again useful to have an explicit formula for the winding number cocycle at one's disposal, see Proposition 2.3.11 below.

Lemma 2.3.10. *Let $t \mapsto T_t$ be a differentiable path of J-unitaries with diagonal entries A_t and D_t. Then*

$$\mathrm{Tr}\big(S(T_t)^*\partial_t S(T_t)\big) = \mathrm{Tr}\big((A_t)^{-1}\partial_t A_t - (D_t)^{-1}\partial_t D_t\big).$$

Proof. It is possible to derive this result from Lemma 2.3.9, but we provide a direct proof (as in [21]) because it is not any longer. Let us drop the index t and also simply write $\partial = \partial_t$. The J-unitarity of T and T^* is equivalent to the following identities:

$$\begin{aligned} A^*A &= \mathbf{1} + C^*C, \quad & D^*D &= \mathbf{1} + B^*B, \quad & A^*B &= C^*D,\\ AA^* &= \mathbf{1} + BB^*, \quad & DD^* &= \mathbf{1} + CC^*, \quad & AC^* &= BD^*. \end{aligned}$$

As already noted, A and D are thus invertible. Now

$$\begin{aligned} \mathrm{Tr}\big(S(T)^*\partial S(T)\big) &= \mathrm{Tr}\left(\begin{pmatrix} A^{-1} & -A^{-1}B \\ (D^*)^{-1}B^* & (D^*)^{-1}\end{pmatrix}\partial\begin{pmatrix} (A^*)^{-1} & BD^{-1} \\ -B^*(A^*)^{-1} & D^{-1}\end{pmatrix}\right)\\ &= \mathrm{Tr}\big(A^{-1}\partial(A^*)^{-1} + A^{-1}B\partial B^*(A^*)^{-1} + A^{-1}BB^*\partial(A^*)^{-1}\\ &\quad + (D^*)^{-1}B^*\partial BD^{-1} + (D^*)^{-1}B^*B\partial D^{-1} + (D^*)^{-1}\partial D^{-1}\big). \end{aligned}$$

Now let us replace BB^* and B^*B by the above expressions in the third and fifth summands:

$$\begin{aligned} &\mathrm{Tr}\big(S(T)^*\partial S(T)\big)\\ &\quad = \mathrm{Tr}\big(A^*\partial(A^*)^{-1} + A^{-1}B\partial B^*(A^*)^{-1} + (D^*)^{-1}B^*\partial BD^{-1} + D\partial D^{-1}\big)\\ &\quad = \mathrm{Tr}\big(A^*\partial(A^*)^{-1} + (A^*)^{-1}A^{-1}B\partial B^* + D^{-1}(D^*)^{-1}B^*\partial B + D\partial D^{-1}\big)\\ &\quad = \mathrm{Tr}\big(A^*\partial(A^*)^{-1} + (AA^*)^{-1}B\partial B^* + (D^*D)^{-1}B^*\partial B + D\partial D^{-1}\big). \end{aligned}$$

Now replace AA^* and D^*D in terms of B and use $(\mathbf{1} + B^*B)^{-1}B^* = B^*(\mathbf{1} + BB^*)^{-1}$. Again using the cyclicity, one finds

$$\begin{aligned} \mathrm{Tr}\big(S(T)^*\partial S(T)\big) &= \mathrm{Tr}\big(A^*\partial(A^*)^{-1} + (\mathbf{1} + BB^*)^{-1}\partial(BB^*) + D\partial D^{-1}\big)\\ &= \mathrm{Tr}\big(A^*\partial(A^*)^{-1} + (AA^*)^{-1}\partial(AA^*) + D\partial D^{-1}\big)\\ &= \mathrm{Tr}\big(-(A^*)^{-1}\partial A^* + (AA^*)^{-1}(\partial AA^* + A\partial A^*) - D^{-1}\partial D\big), \end{aligned}$$

which implies the result. □

Proposition 2.3.11. *Let $t \in [0,1] \mapsto T_t$ be a closed path of J-unitaries with diagonal entries A_t and D_t which are piecewise continuously differentiable. Then*

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \frac{1}{2\pi\imath}\int_0^1 dt\,\mathrm{Tr}\big((A_t)^{-1}\partial_t A_t - (D_t)^{-1}\partial_t D_t\big). \tag{2.37}$$

Proof. Proposition 1.5.12 combined with Lemma 2.3.10 implies the claim. □

Corollary 2.3.12. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a closed path in $\mathbb{U}(\mathbb{C}^{2N}, J)$. Then for $P \in \mathbb{P}(\mathcal{K}, J)$,*

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P).$$

Proof. By an approximation argument, one can assume the path to be continuously differentiable. Let us deform the path $t \mapsto T_t$ via the homotopy $h_s(T_t) = T_t|T_t|^{-s}$ for $s \in [0,1]$. This homotopy is indeed inside the J-unitary matrices (see Proposition 2.2.5). Then $h_0(T_t) = T_t$ and $h_1(T_t) = T_t|T_t|^{-1} \in \mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(\mathbb{C}^{2N})$. Matrices in $\mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(\mathbb{C}^{2N})$ are diagonal, so that one has a path of diagonal matrices. For such diagonal matrices in $\mathbb{U}(\mathbb{C}^{2N}, J) \cap \mathbb{U}(\mathbb{C}^{2N})$, the formula in Proposition 2.2.14 coincides with (2.37). Therefore, using the homotopy invariance of the winding numbers defining the Conley–Zehnder and Bott–Maslov indices, one deduces

$$\begin{aligned}\mathrm{CZ}(t \in [0,1] \mapsto T_t) &= \mathrm{CZ}(t \in [0,1] \mapsto h_1(T_t)) \\ &= \mathrm{BM}(t \in [0,1] \mapsto h_1(T_t) \cdot P_{\mathrm{ref}}) \\ &= \mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P_{\mathrm{ref}}) \\ &= \mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P),\end{aligned}$$

concluding the proof. □

Finally, let us combine Corollary 2.3.12 with Proposition 2.2.16.

Corollary 2.3.13. *Let $t \in [0,1] \mapsto P_t \in \mathbb{P}(\mathbb{C}^{2N}, J)$ and $t \in [0,1] \mapsto T_t \in \mathbb{U}(\mathbb{C}^{2N}, J)$ be closed paths. Then*

$$\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P_t) = \mathrm{CZ}(t \in [0,1] \mapsto T_t) + \mathrm{BM}(t \in [0,1] \mapsto P_t).$$

2.4 Oscillation theory for finite Jacobi matrices

Classical Sturm–Liouville oscillation theory [16, 8] shows that the number of zeros of a formal solution to a Sturm–Liouville equation at a given energy (a second-order ordinary differential equation of a particular type) is equal to the number of bound states below that energy. This number of zeros can also be understood as the spectral flow of the Prüfer phase associated to the solution which in turn is the Bott–Maslov index of the solution if the equation is understood as a first order Hamiltonian system. This point also allows us to deal with matrix-valued Sturm–Liouville equations [35, 10] which is of importance for many applications. For example, linearizing the geodesic equation leads to the Jacobi equation which is a matrix-valued Sturm–Liouville equation [35]. Instead of analyzing the oscillations of one solution in the space variable, it is also of great interest to study the oscillation of the solution on the energy variable. This provides an effective approach to the spectral theory of the Sturm–Liouville operator. Both types of oscillation are linked and provide complementary insight [173].

It is well-known that tridiagonal Jacobi operators are the discrete analogues of Sturm–Liouville operators. In particular, their spectral theory can be understood via oscillation theory in the energy variable [166, 167, 78]. As an application of the Bott–Maslov index and hence spectral flow, this will be explained in detail in this section. A matrix Jacobi operator of finite length $L \geq 3$ is a matrix of the form

$$H_L = \begin{pmatrix} V_1 & A_2 & & & & \\ A_2^* & V_2 & A_3 & & & \\ & A_3^* & V_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & V_{L-1} & A_L \\ & & & & A_L^* & V_L \end{pmatrix}, \tag{2.38}$$

where $(V_n)_{n=1,\ldots,L}$ are self-adjoint complex $N \times N$ matrices and $(A_n)_{n=2,\ldots,L}$ are invertible complex $N \times N$ matrices. The scalar case corresponds to $N = 1$. The aim in the following is to compute the spectrum of H_L, namely to find those $E \in \mathbb{R}$ for which there exists a nonvanishing state $\psi^E \in \mathbb{C}^{LN}$ such that the Schrödinger equation holds:

$$H_L \psi^E = E\psi^E. \tag{2.39}$$

Remark 2.4.1. It is always possible to consider the particular setup where the A_n are positive. Indeed, one can attain this by a gauge transformation, namely a strictly local unitary $G = \mathrm{diag}(G_1, \ldots, G_L)$ with $N \times N$ unitary matrices G_n, $n = 1, \ldots, L$. Then

$$GH_LG^* = \begin{pmatrix} G_1V_1G_1^* & G_1A_2G_2^* & & & & \\ (G_1A_2G_2^*)^* & G_2V_2G_2^* & G_2A_3G_3^* & & & \\ & (G_2A_3G_3^*)^* & G_3V_3G_3^* & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & G_{L-1}V_{L-1}G_{L-1}^* & G_{L-1}A_LG_L^* \\ & & & & (G_{L-1}A_LG_L^*)^* & G_LV_LG_L^* \end{pmatrix}.$$

Now one can iteratively choose the G_n. Start out with $G_1 = \mathbf{1}$. Then choose G_2 to be the (unitary) phase in the polar decomposition of $A_2 = G_2|A_2|$, next let G_3 be the phase of

$G_2A_3 = G_3|G_2A_3|$, and so on. One concludes that GH_LG^* is again of the form of H_L given in (2.38), but with positive off-diagonal terms. From now on, one may thus suppose that $A_n > 0$ for all $n = 2, \dots, L$. For periodic Jacobi matrices as considered in Section 2.5, it is not possible to construct such a gauge transformation G. In fact, a periodic Jacobi matrix models a ring through which there can be a magnetic flux. In order to directly use the formalism also in Section 2.5, we will therefore keep the A_n as general invertible self-adjoint matrices. ◇

Next let us introduce the $2N \times 2N$ transfer matrices M_n^E by

$$M_n^E = \begin{pmatrix} (E\mathbf{1} - V_n)A_n^{-1} & -A_n^* \\ A_n^{-1} & 0 \end{pmatrix}, \quad n = 1, \dots, L, \tag{2.40}$$

with $A_1 = \mathbf{1}$. Of crucial importance is that, for a real energy $E \in \mathbb{R}$, the transfer matrices are in the group $\mathbb{U}(\mathbb{C}^{2N}, I)$,

$$(M_n^E)^* I M_n^E = I,$$

with I as in (2.1). Also their products

$$M^E(n, m) = M_n^E \cdots M_{m+1}^E, \quad n > m,$$

are I-unitary. It is also useful to set $M^E(n, n) = \mathbf{1}$ and $M^E(n, m) = M^E(m, m)^{-1}$ for $n < m$. The transfer matrix $M^E(n, 0)$ is the equivalent of the fundamental solution of a Sturm–Liouville operator (rewritten as a first order system).

The eigenvalue problem (2.39) at energy $E \in \mathbb{R}$ will now be considered as an equation for vectors $\psi^E = (\psi_n^E)_{n=1,\dots,L} \in \mathbb{C}^{NL}$ composed of vectors $\psi_n^E \in \mathbb{C}^N$. The tridiagonal form of H_L then leads to

$$A_{n+1}\psi_{n+1}^E + V_n\psi_n^E + A_n^*\psi_{n-1}^E = E\psi_n^E, \tag{2.41}$$

for $n = 2, \dots, L - 1$, together with the (Dirichlet) boundary conditions

$$A_2\psi_2^E + V_1\psi_1^E = E\psi_1^E, \quad V_L\psi_L^E + A_L^*\psi_{L-1}^E = E\psi_L^E. \tag{2.42}$$

Equation (2.41) is also called the three-term recurrence relation because ψ_{n+1}^E can be computed from ψ_n^E and ψ_{n-1}^E. In particular, if two neighboring values are known, then all others can be computed. This produces a vector which, however, typically does not satisfy both boundary conditions (it can only do so if E happens to be an eigenvalue of H_L). Regrouping two neighboring vectors into

$$\Psi_n^E = \begin{pmatrix} A_{n+1}\psi_{n+1}^E \\ \psi_n^E \end{pmatrix},$$

one can then rewrite (2.41) using the above I-unitary transfer matrices M_n^E as

$$\Psi_n^E = M_n^E \Psi_{n-1}^E. \tag{2.43}$$

This equation will also be used for $2N\times N$ matrix-valued Ψ_n^E, which are then I-Lagrangian frames (if the initial condition Ψ_0^E is I-Lagrangian). Furthermore, (2.43) can be iterated

$$\Psi_n^E = M_n^E \Psi_{n-1}^E, \quad n = 1, \dots, L. \tag{2.44}$$

This forces one to fix the initial condition Ψ_0^E. In order to satisfy the first boundary condition in (2.42) automatically, let us therefore choose

$$\Psi_0^E = \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix} \in \mathbb{C}^{2N\times N}, \tag{2.45}$$

which is a left Dirichlet boundary condition. Clearly, the rank of Ψ_0^E is N and it satisfies $(\Psi_0^E)^* I \Psi_0^E = 0$, so that it is an I-Lagrangian frame. Hence its range spans an I-Lagrangian subspace of $\mathbb{C}^{2N}$. As the transfer matrices are I-unitary, it follows that also the range of Ψ_n^E spans an I-Lagrangian subspace. In particular, Ψ_L^E is an I-Lagrangian frame. Now the dimension of the intersection of the associated I-Lagrangian subspace $\mathrm{Ran}(\Psi_L^E)$ with the right boundary condition

$$\Psi_{\mathrm{bd}} = \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix},$$

which is also an I-Lagrangian subspace, is equal to the multiplicity m^E of E as the eigenvalue of H_L,

$$m^E = \dim(\mathrm{Ran}(\Psi_L^E) \cap \mathrm{Ran}(\Psi_{\mathrm{bd}})). \tag{2.46}$$

Indeed, any vector in the intersection yields a solution of the Schrödinger equation (2.39) also satisfying the right boundary condition in (2.42), and vice versa. This establishes the connection between the eigenvalue problem of matrix Jacobi operators and the intersection theory of Lagrangian subspaces.

In order to apply the theory of the Bott–Maslov index developed in Section 2.1 directly, let us apply the Cayley transform $\mathcal{C}$ to pass to J-unitary transfer matrices lying in the generalized Lorentz group $\mathbb{U}(\mathbb{C}^{2N}, J)$

$$T_n^E = \mathcal{C} M_n^E \mathcal{C}^*, \quad T^E(n,m) = \mathcal{C} M^E(n,m) \mathcal{C}^*,$$

as well as to J-unitary frames and projections

$$\Phi_n^E = \mathcal{C}\Psi_n^E, \quad P_n^E = \Phi_n^E((\Phi_n^E)^* \Phi_n^E)^{-1} (\Phi_n^E)^*.$$

Explicitly, one finds

$$T_n^E = \frac{1}{2} \begin{pmatrix} (E - V_n)A_n^{-1} - \imath(A_n^* + A_n^{-1}) & (E - V_n)A_n^{-1} + \imath(A_n^* - A_n^{-1}) \\ (E - V_n)A_n^{-1} - \imath(A_n^* - A_n^{-1}) & (E - V_n)A_n^{-1} + \imath(A_n^* + A_n^{-1}) \end{pmatrix}.$$

The right boundary condition then becomes

$$\Phi_{\mathrm{bd}} = \mathcal{C}\Psi_{\mathrm{bd}},$$

namely $\Phi_{\mathrm{bd}} = \imath\binom{-\mathbf{1}}{\mathbf{1}} = -\imath\Phi_{\mathrm{ref}}^{\perp}$, so that comparing with (2.46) gives

$$\begin{aligned} m^E &= \dim(\mathrm{Ran}(\Phi_L^E) \cap \mathrm{Ran}(\Phi_{\mathrm{ref}}^{\perp})) \\ &= \dim(\mathrm{Ker}(\Pi(P_L^E)) + \mathbf{1}), \end{aligned}$$

where the second equality follows from (2.7). It is hence natural to set

$$U_n^E = \Pi(P_n^E) \in \mathbb{U}(N).$$

This unitary is called the matrix Prüfer phase. By the above, the matrix U_L^E has an eigenvalue -1 if and only if the matrix H_L has an eigenvalue E. To count all eigenvalues below some $E \in \mathbb{R}$, one hence has to count the number of intersections of the path of J-Lagrangian planes $e \in (-\infty, E] \mapsto P_L^e$ with $P_{\mathrm{ref}}^{\perp}$, or equivalently the number of passages of eigenvalues of $e \in (-\infty, E] \mapsto U_L^e$ by -1. This is not automatically the Bott–Maslov index though which takes into account the orientations of the passages. However, the following result, a core fact of oscillation theory, is that all these passages are in the same direction and that they are transversal.

Theorem 2.4.2. *The multiplicity of E as eigenvalues H_L is equal to the multiplicity of -1 as eigenvalue of U_L^E. Moreover,*

$$\frac{1}{\imath}(U_L^E)^* \partial_E U_L^E > 0.$$

As a function of the energy E, the eigenvalues of U_L^E rotate around the unit circle in the positive sense and with nonvanishing speed. Furthermore, for $E \in \mathbb{R} \setminus \mathrm{spec}(H_L)$

$$\begin{aligned} \#\{\text{eigenvalues of } H_L \leq E\} &= \mathrm{BM}(e \in (-\infty, E] \mapsto P_L^e) \\ &= \mathrm{Sf}(e \in (-\infty, E] \mapsto U_L^e \text{ through } -1). \end{aligned}$$

Proof. The first claim was already proved above. For the proof of the positivity, let us introduce $N \times N$ matrices a^E and b^E by

$$\Phi_L^E = \begin{pmatrix} a^E \\ b^E \end{pmatrix}.$$

They are invertible and $U_L^E = a^E(b^E)^{-1} = ((a^E)^{-1})^*(b^E)^*$. Now

$$(U_L^E)^* \partial_E U_L^E = ((b^E)^{-1})^* [(a^E)^* \partial_E a^E - (b^E)^* \partial_E b^E](b^E)^{-1}.$$

Thus it is sufficient to verify positive definiteness of

$$\frac{1}{\imath}[(a^E)^*\partial_E a^E - (b^E)^*\partial_E b^E] = \frac{1}{\imath}(\Phi_L^E)^* J \partial_E \Phi_L^E = (\Psi_L^E)^* I \partial_E \Psi_L^E,$$

where (2.2) was used. From the product rule, it follows that

$$\partial_E \Psi_L^E = \sum_{n=1}^{L} M_L^E \cdots M_{n+1}^E (\partial_E M_n^E) M_{n-1}^E \cdots M_1^E \Psi_0^E.$$

This implies

$$(\Psi_L^E)^* I \partial_E \Psi_L^E = \sum_{n=1}^{L} (\Psi_0^E)^* (M_{n-1}^E \cdots M_1^E)^* (M_n^E)^* I (\partial_E M_n^E)(M_{n-1}^E \cdots M_1^E)\Psi_0^E.$$

One checks that

$$(M_n^E)^* I (\partial_E M_n^E) = \begin{pmatrix} (A_n A_n^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

and thus

$$(\Psi_L^E)^* I \partial_E \Psi_L^E = \sum_{n=1}^{L} (\Psi_0^E)^* (M_{n-1}^E \cdots M_1^E)^* \begin{pmatrix} (A_n A_n^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} (M_{n-1}^E \cdots M_1^E)\Psi_0^E.$$

Clearly, each of the summands is positive semidefinite. In order to prove a strict lower bound, it is sufficient that the first two terms $n = 1, 2$ give a strictly positive contribution. Hence let us verify that

$$\begin{pmatrix} (A_1 A_1^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + (M_1^E)^* \begin{pmatrix} (A_2 A_2^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} M_1^E > 0.$$

As A_1 and A_2 are invertible, this positivity is equivalent to

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} + (M_1^E)^* \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} M_1^E > 0.$$

Using the notation $B^E = (E - V_1)A_1^{-1}$, one thus just has to note the invertibility

$$\begin{pmatrix} \mathbf{1} + (B^E)^* B^E & -(A_1 B^E)^* \\ -A_1 B^E & A_1 A_1^* \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -(B^E)^* \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -B^E & A_1^* \end{pmatrix}.$$

This proves the claimed positivity. All other claims now follow from the discussion above. □

In view of Theorem 2.4.2, it is of interest to study the whole path $E \in \mathbb{R} \mapsto U_L^E$. As H_L has NL eigenvalues, the spectral flow of this path has to be equal to NL. The following proposition shows that this path is actually closed so that the spectral flow of the path $E \in \mathbb{R} \mapsto U_L^E$ reduces to a winding number.

Proposition 2.4.3. *The matrix Prüfer phases satisfy, for $n \geq 1$,*

$$\lim_{E\to\pm\infty} U_n^E = \mathbf{1}.$$

Hence the paths $E \in \mathbb{R} \mapsto U_n^E$ and $E \in \mathbb{R} \mapsto P_n^E$ are closed and the Bott–Maslov index of the latter is

$$\mathrm{BM}(E \in \mathbb{R} \mapsto P_n^E) = nN.$$

Proof. We will freely use the notations of the proof of Theorem 2.4.2 with $n = L$. The crucial observation is that all objects Ψ_n^E, Φ_n^E, a^E, and b^E are polynomials in E. Hence U_n^E is a rational function in E. As U_n^E is unitary for all $E \in \mathbb{R}$, it is clear that the limits $\lim_{E\to\pm\infty} U_n^E$ exist. Now

$$\begin{aligned}\Phi_n^E &= \mathcal{C}M^E(n,0)\Psi_0^E \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{1} & \imath\mathbf{1}\\ \mathbf{1} & -\imath\mathbf{1}\end{pmatrix}\begin{pmatrix}E^n A_n^{-1}\cdots A_1^{-1} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ 0\end{pmatrix} + \mathcal{O}(E^{n-1}) \\ &= \frac{1}{\sqrt{2}}E^n\begin{pmatrix}(A_1\cdots A_n)^{-1}\\ (A_1\cdots A_n)^{-1}\end{pmatrix} + \mathcal{O}(E^{n-1}).\end{aligned}$$

Thus $a^E = \frac{1}{\sqrt{2}}E^n(A_1\cdots A_n)^{-1} + \mathcal{O}(E^{n-1})$ and $b^E = \frac{1}{\sqrt{2}}E^n(A_1\cdots A_n)^{-1} + \mathcal{O}(E^{n-1})$ so that $U_n^E = a^E(b^E)^{-1} = \mathbf{1} + \mathcal{O}(E^{-1})$. As already indicated above, the last claim follows from Theorem 2.4.2, but it is also possible to carry out an explicit computation. □

Let us stress once again that this section only considered the oscillation theory of Jacobi operators in the energy variable. For (renormalized) space oscillation theory, the reader is referred to [8, 78, 190, 173].

2.5 Oscillation theory for periodic Jacobi matrices

In this section, the spectral theory of a periodic Jacobi matrix of the form

$$H_L^{\mathrm{per}} = \begin{pmatrix} V_1 & A_2 & & & & A_1^* \\ A_2^* & V_2 & A_3 & & & \\ & A_3^* & V_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & V_{L-1} & A_L \\ A_1 & & & & A_L^* & V_L \end{pmatrix} \tag{2.47}$$

will be analyzed. Just as in (2.38), $(V_n)_{n=1,\dots,L}$ are self-adjoint complex $N\times N$ matrices and $(A_n)_{1=2,\dots,L}$ are invertible complex $N \times N$ matrices, the only difference is the additional entries in the upper right and lower left corner. They periodize the system which can thus be thought of as a ring. Other than for H_L given by (2.38), the gauge transformation as in Remark 2.4.1 only allows rendering A_n, $n = 2, \dots, L$, positive. Hence all phases (corresponding to magnetic fields) are concentrated in A_1. The periodic Jacobi matrix is the discrete analogue of a periodic Sturm–Liouville operator. Bott's work [35] considered precisely such operators because they appear naturally as the Jacobi equation for a closed geodesic. For the study of their spectral theory, Bott developed the intersection theory as described in Sections 2.1 and 2.3.

Again the aim here will be to find eigenvalues $E \in \mathbb{R}$ of H_L^{per} for which there exists a nonvanishing state $\psi^E \in \mathbb{C}^{LN}$ such that the following Schrödinger equation holds:

$$H_L^{\text{per}} \psi^E = E\psi^E. \tag{2.48}$$

Again these eigenfunctions can be constructed using the transfer matrices M_n^E, with $n = 1, \dots, L$, defined in (2.40), but now A_1 being as in (2.47). The Schrödinger equation is equivalent to

$$\Psi_0^E = (M_L^E \cdots M_1^E)\Psi_0^E,$$

for a nonvanishing vector $\Psi_0^E = \binom{A_1\psi_1^E}{\psi_0^E}$. Therefore one is lead to study the eigenvalue 1 of the full transfer matrix

$$M^E = M_L^E \cdots M_1^E.$$

Then set

$$T^E = \mathcal{C} M^E \mathcal{C}^*.$$

According to Theorem 2.3.3, this can be achieved by studying the eigenvalue 1 of the unitary $S(T^E) \in \mathbb{U}(2N)$, namely

$$\dim(\text{Ker}(H_L^{\text{per}} - E\mathbf{1})) = \dim(\text{Ker}(S(T^E) - \mathbf{1})).$$

This implies the first statement of the following result:

Theorem 2.5.1. *The multiplicity of E as eigenvalues H_L^{per} is equal to the multiplicity of* 1 *as eigenvalue of $S(T^E)$. Moreover,*

$$\frac{1}{\imath} S(T^E)^* \partial_E S(T^E) > 0.$$

As a function of the energy E, the eigenvalues of $S(T^E)$ rotate around the unit circle in the positive sense and with nonvanishing speed. Then for $E \in \mathbb{R} \setminus \text{spec}(H_L^{\text{per}})$,

$$\#\{\text{eigenvalues of } H_L^{\text{per}} \leq E\} = \text{Sf}(e \in (-\infty, E] \mapsto S(T^e) \text{ through } 1).$$

Proof. Based on Lemma 2.3.9, the claimed positivity follows from the positivity of $\frac{1}{\imath}(T^E)^* J \partial_E T^E$. This was already checked in the proof of Theorem 2.4.2. All other claims follow immediately from the setup. □

2.6 Bound states for scattering systems

There are numerous extensions of the basic energy oscillation theory presented in Section 2.4 to block Jacobi operators on infinite-dimensional Hilbert spaces. Here block Jacobi operators H on the infinite discrete line are considered which are perturbations of a given periodic block Jacobi operator H_{per}. The perturbation $H - H_{\text{per}}$ is supposed to be of finite rank. This is the most elementary setup of quantum scattering theory, already considered in [21]. It is known that the perturbation leads to bound states and it is the object of this section to access these bound states by oscillation theory which again uses the theory of the Bott–Maslov index and the spectral flow. The result is also in the spirit of relative oscillation theory [7, 81] where one compares two Jacobi operators, here given by H and H_{per}. As part of the preparations for the main result (Theorem 2.6.5), also bound states of half-space restrictions of Jacobi operators will be considered. This section also serves as a preparation for Section 9.7 where bound states of higher-dimensional scattering systems are analyzed which then requires the Bott–Maslov index in an infinite-dimensional setting.

Let us begin by describing the matrix Jacobi operator H. Formally, it is a (two-sided) infinite matrix of the form (2.38) associated to two sequences $(A_n)_{n\in\mathbb{Z}}$ and $(V_n)_{n\in\mathbb{Z}}$ of respectively positive and self-adjoint $N \times N$ matrices. It will be considered as a self-adjoint operator on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^N)$. The Schrödinger equation $H\psi^E = E\psi^E$ will be considered for all sequences $\psi^E = (\psi_n^E)_{n\in\mathbb{Z}}$ of vectors $\psi_n^E \in \mathbb{C}^N$, and not only square-integrable states from $\ell^2(\mathbb{Z}, \mathbb{C}^N)$. Explicitly written out, it becomes

$$A_{n+1}\psi_{n+1}^E + V_n\psi_n^E + A_n\psi_{n-1}^E = E\psi_n^E. \tag{2.49}$$

This is also called the three-term recurrence relation because ψ_{n+1}^E can be computed from ψ_n^E and ψ_{n-1}^E, so that the solution ψ^E is fixed by two neighboring values (which are often chosen to be ψ_0^E and ψ_1^E). Regrouping two neighboring vectors into

$$\Psi_n^E = \begin{pmatrix} A_{n+1}\psi_{n+1}^E \\ \psi_n^E \end{pmatrix},$$

one can then rewrite (2.49) as in (2.44) using the I-unitary transfer matrices M_n^E:

$$\Psi_n^E = M_n^E \Psi_{n-1}^E, \quad M_n^E = \begin{pmatrix} (E\mathbf{1} - V_n)A_n^{-1} & -A_n \\ A_n^{-1} & 0 \end{pmatrix}. \tag{2.50}$$

This equation will also be used for $2N \times N$ matrix-valued Ψ_n^E, which just as in Section 2.4 are then I-Lagrangian frames (if some initial condition is I-Lagrangian).

Let us now state the hypothesis specifying the scattering situation: there are $N \times N$ matrices $A > 0$ and $V = V^*$ such that for all $n \notin \{1, \dots, L\}$,

$$A_n = A, \quad V_n = V.$$

Thus H only has varying matrix elements on L sites. If also the matrix entries on those N sites are equal to A and V, one obtains a periodic Jacobi operator H_{per}. It can be analyzed using the transfer matrix

$$M^E = \begin{pmatrix} (E\mathbf{1} - V)A^{-1} & -A \\ A^{-1} & 0 \end{pmatrix}.$$

It is well known that the spectrum $\text{spec}(H_{\text{per}})$ of H_{per} is purely absolutely continuous spectrum and consists of at most N intervals, e. g., Appendix A in [17]. Furthermore, H has the same absolutely continuous spectrum, but may, moreover, have a finite number of further eigenvalues not lying in $\text{spec}(H_{\text{per}})$. Each such eigenvalue, also called bound state, corresponds to a square-integrable solution of (2.49) and it is the aim of this section to show how these eigenvalues can be accessed by oscillation theory.

The operator H_{per} is specified only by A and V and hence it is not surprising that its spectrum can be read off the transfer matrix.

Proposition 2.6.1. $E \in \text{spec}(H_{\text{per}}) \Longleftrightarrow \text{spec}(M^E) \cap \mathbb{S}^1 \neq \emptyset$.

Proof. We only prove the implication "$\Longleftarrow$" because the other is essentially obtained by the reverse procedure combined with Bloch–Floquet theory. Let $e^{\imath\theta}$ belong to the spectrum $\text{spec}(M^E)$. Let $w = (w_0, w_1) \in \mathbb{C}^N \oplus \mathbb{C}^N$ be the corresponding eigenvector, that is,

$$M^E w = e^{\imath\theta} w.$$

The second line of this equation is $A^{-1}w_0 = e^{\imath\theta} w_1$. As A^{-1} has trivial kernel, this shows that neither w_0 nor w_1 is vanishing. The first line then becomes

$$Ew_1 = e^{\imath\theta} Aw_1 + Vw_1 + e^{-\imath\theta} Aw_1.$$

Therefore $\psi \in \ell^\infty(\mathbb{Z}) \otimes \mathbb{C}^N$ defined by $\psi(n_1) = e^{\imath\theta n} w_1$ satisfies $H_{\text{per}}\psi = E\psi$. From this one can now readily construct a Weyl sequence for H_{per} at energy E. Let $\chi_L \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ be the indicator function to $[-L, L]$. Then $\|\chi_L \psi\| = \mathcal{O}(L^{\frac{1}{2}})$ and set $\psi_L = \chi_L \psi / \|\chi_L \psi\|$. It follows that $\|(H_{\text{per}} - E)\psi_L\| = \mathcal{O}(L^{-\frac{1}{2}})$, and we conclude that $E \in \text{spec}(H_{\text{per}})$. Let us note that by translating the ψ_L one can also obtain an orthonormal Weyl sequence so that E is actually in the essential spectrum of H_{per}. □

Proposition 2.6.2. *For $E \notin \text{spec}(H_{\text{per}})$, the subspaces $\mathcal{E}^{E,<}$ and $\mathcal{E}^{E,>}$ spanned by all eigenspaces of M^E with eigenvalues of modulus less than 1 or larger than 1 respectively are I-Lagrangian.*

Proof. Let $R^{E,<}$ and $R^{E,>}$ be the corresponding Riesz projections. If

$$\Delta = \{z \in \text{spec}(M^E) : |z| < 1\},$$

then in the notations of Proposition 2.2.2 one has $R^{E,<} = R_\Delta$ and $R^{E,>} = R_{\overline{\Delta}^{-1}}$. Proposition 2.2.2 holds for an I-unitary just as for a J-unitary, so that one concludes $(R^{E,<})^* = I^* R^{E,>} I$. Thus

$$(R^{E,<})^* I R^{E,<} = (R^{E,<})^* (R^{E,>})^* I = (R^{E,>} R^{E,<})^* I = 0,$$

implying the claim. □

Let us now introduce the two unitaries

$$W^{E,<} = \Pi(\mathcal{C}\mathcal{E}^{E,<}), \quad W^{E,>} = \Pi(\mathcal{C}\mathcal{E}^{E,>}),$$

where here the subspaces are identified with their orthogonal range projections (in order to avoid yet another notation). For their analysis, it is helpful to provide another expression and this also leads to another useful interpretation of these unitaries. Let H^+_{per} and H^-_{per} be the (Dirichlet) restrictions of H_{per} to $\ell^2(\mathbb{N}, \mathbb{C}^N)$ and $\ell^2(\mathbb{N}^-, \mathbb{C}^N)$, respectively, where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^- = \{\dots, -1, 0\}$. Clearly, one has $\text{spec}(H^+_{\text{per}}) \supset \text{spec}(H_{\text{per}})$ and $\text{spec}(H^-_{\text{per}}) \supset \text{spec}(H_{\text{per}})$, by a standard Weyl sequence argument. All new points of the spectrum are called bound states of H^+_{per} and H^-_{per}, respectively. Such a bound state with energy $E \in \text{spec}(H^+_{\text{per}}) \setminus \text{spec}(H_{\text{per}})$ is always given when the Dirichlet boundary condition leads to a square integrable solution (which in this case will be exponentially decaying at $+\infty$). The Dirichlet boundary condition at sites 0 and 1 is given by the I-Lagrangian frame $\binom{\mathbf{1}}{0}$. Therefore the

$$m^{+,E}_{\text{per}} = \text{multiplicity of } E \text{ as eigenvalue of } H^+_{\text{per}}$$

is given by

$$m^{+,E}_{\text{per}} = \dim\left(\mathcal{E}^{E,<} \cap \text{Ran}\begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix}\right).$$

This is an intersection of two I-Lagrangian subspaces. Applying the Cayley transform $\mathcal{C}$ and using $\mathcal{C}\binom{\mathbf{1}}{0} = \Phi_{\text{ref}}$ with Φ_{ref} as in (2.4), this can be rewritten as

$$m^{+,E}_{\text{per}} = \dim(\mathcal{C}\mathcal{E}^{E,<} \cap \text{Ran}(\Phi_{\text{ref}})) = \dim(J\mathcal{C}\mathcal{E}^{E,<} \cap \mathcal{E}^{\perp}_{\text{ref}}),$$

and hence, due to Proposition 2.1.4 and $\Pi(J\mathcal{C}\mathcal{E}^{E,<}) = -\Pi(\mathcal{C}\mathcal{E}^{E,<}) = -W^{E,<}$, one concludes

$$m^{+,E}_{\text{per}} = \dim(\text{Ker}(W^{E,<} - \mathbf{1})).$$

Similarly, the multiplicity $m_{\text{per}}^{-,E}$ of E as eigenvalue of H_{per}^{-} is given by the intersection of the Dirichlet boundary condition $\binom{0}{\mathbf{1}}$ with $\mathcal{E}^{E,>}$ so that

$$m_{\text{per}}^{-,E} = \dim(\mathcal{C}\mathcal{E}^{E,>} \cap \mathcal{E}_{\text{ref}}^{\perp}) = \dim(\operatorname{Ker}(W^{E,>} + \mathbf{1})).$$

Further let $\pi_n : \mathbb{C}^N \to \ell^2(\mathbb{Z}, \mathbb{C}^N)$ be the partial isometry onto the nth site, and similarly for $\ell^2(\mathbb{N}, \mathbb{C}^N)$ and $\ell^2(\mathbb{N}^-, \mathbb{C}^N)$ (in later chapters, π_n also denotes homotopy groups, but we believe that no confusion can arise). For $E \in \mathbb{C}$ not in the spectrum of H_{per}^{+} and H_{per}^{-}, respectively, the $N \times N$ Green matrices of H_{per}^{+} and H_{per}^{-} are defined by

$$G^{+,E} = \pi_1^*(H_{\text{per}}^{+} - E)^{-1}\pi_1, \quad G^{-,E} = \pi_0^*(H_{\text{per}}^{-} - E)^{-1}\pi_0.$$

Note that for real E, these matrices are self-adjoint. On the other hand, for $\mathfrak{Im}(E) > 0$, one has $\mathfrak{Im}(G^{\pm,E}) = \frac{1}{2\imath}(G^{\pm,E} - (G^{\pm,E})^*) > 0$, while for $\mathfrak{Im}(E) < 0$, one has $\mathfrak{Im}(G^{\pm,E}) < 0$.

Proposition 2.6.3. *For real $E \notin \operatorname{spec}(H_{\text{per}}^{+})$ and $E \notin \operatorname{spec}(H_{\text{per}}^{-})$, the I-Lagrangian frames*

$$\Psi^{E,<} = \begin{pmatrix} AG^{+,E} \\ -A^{-1} \end{pmatrix}, \quad \Psi^{E,>} = \begin{pmatrix} -\mathbf{1} \\ G^{-,E} \end{pmatrix},$$

span the I-Lagrangian subspaces $\mathcal{E}^{E,<}$ and $\mathcal{E}^{E,>}$, respectively. For all real $E \notin \operatorname{spec}(H_{\text{per}})$, one has

$$W^{E,<} = (AG^{+,E}A + \imath\mathbf{1})(AG^{+,E}A - \imath\mathbf{1})^{-1},$$
$$W^{E,>} = -(G^{-,E} - \imath\mathbf{1})(G^{-,E} + \imath\mathbf{1})^{-1},$$

where the right-hand sides of these equations are understood as analytic extensions into $E \in \operatorname{spec}(H_{\text{per}}^{+}) \setminus \operatorname{spec}(H_{\text{per}})$ and $E \in \operatorname{spec}(H_{\text{per}}^{-}) \setminus \operatorname{spec}(H_{\text{per}})$, respectively. One has for $E \in \mathbb{R} \setminus \operatorname{spec}(H_{\text{per}})$,

$$\frac{1}{\imath}(W^{E,<})^*\partial_E W^{E,<} < 0, \quad \frac{1}{\imath}(W^{E,>})^*\partial_E W^{E,>} > 0. \tag{2.51}$$

Proof. Let us consider

$$\psi_n^E = \pi_n^*(H_{\text{per}}^{+} - E)^{-1}\pi_1 \in \mathbb{C}^{N\times N}.$$

As $E \notin \operatorname{spec}(H_{\text{per}}^{+})$, the sequences $\psi^E = (\psi_n^E)_{n\geq 1}$ are square-integrable. More precisely, for any $w \in \mathbb{C}^N$ one has $\psi^E w \in \ell^2(\mathbb{N}, \mathbb{C}^N)$. Now for $n \geq 2$,

$$\pi_n^*(H_{\text{per}}^{+} - E)\psi^E = \pi_n^*(H_{\text{per}}^{+} - E)(H_{\text{per}}^{+} - E)^{-1}\pi_1 = \pi_n^*\pi_1 = 0.$$

As $\pi_n^* H_{\text{per}}\pi_m \neq 0$ only for $|m - n| \leq 1$, one gets

$$A\psi^E_{n-1} + (V - E)\psi^E_n + A\psi^E_{n+1} = 0,$$

which is equivalent to

$$\begin{pmatrix} A\psi^E_{n+1} \\ \psi^E_n \end{pmatrix} = \begin{pmatrix} (E - V)A^{-1} & -A \\ A^{-1} & 0 \end{pmatrix} \begin{pmatrix} A\psi^E_n \\ \psi^E_{n-1} \end{pmatrix} = M^E \begin{pmatrix} A\psi^E_n \\ \psi^E_{n-1} \end{pmatrix}.$$

Furthermore,

$$\pi_1^*(H^+_{\mathrm{per}} - E)\psi^E = \pi_1^*(H^+_{\mathrm{per}} - E)(H^+_{\mathrm{per}} - E)^{-1}\pi_1 = \pi_1^*\pi_1 = \mathbf{1},$$

where the $\mathbf{1}$ on the right-hand side is the $N \times N$ identity matrix. Writing out the left-hand side with the three-term recurrence relation, one gets

$$(V - E)\psi^E_1 + A\psi^E_2 = \mathbf{1}.$$

This can be rewritten as

$$\begin{pmatrix} A\psi^E_2 \\ \psi^E_1 \end{pmatrix} = \begin{pmatrix} (E - V)A^{-1} & -A \\ A^{-1} & 0 \end{pmatrix} \begin{pmatrix} A\psi^E_1 \\ -A^{-1} \end{pmatrix} = M^E \begin{pmatrix} A\psi^E_1 \\ -A^{-1} \end{pmatrix}.$$

Successively applying M^E leads to decaying solutions so that the frame on the right-hand side has to span the contracting directions $\mathcal{E}^{E,<}$ of M^E. As $\psi^E_1 = G^{+,E}$, the first claim follows for all E not being a bound state of H^+_{per}. For the second, let $E \notin \mathrm{spec}(H^-_{\mathrm{per}})$ and set

$$\phi^E_n = \pi_n^*(H^-_{\mathrm{per}} - E)^{-1}\pi_0, \quad n \le 0.$$

Then one finds as above $(V - E)\phi^E_0 + A\phi^E_{-1} = \mathbf{1}$ so that

$$\begin{pmatrix} -\mathbf{1} \\ \phi^E_0 \end{pmatrix} = M^E \begin{pmatrix} A\phi^E_0 \\ \phi^E_{-1} \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} A\phi^E_0 \\ \phi^E_{-1} \end{pmatrix} = (M^E)^{-1} \begin{pmatrix} -\mathbf{1} \\ \phi^E_0 \end{pmatrix}.$$

Now $\phi^E = (\phi^E_n)_{n \le 0}$ is square integrable (at $-\infty$). As the expanding subspace $\mathcal{E}^{E,>}$ of M^E is the contracting subspace of $(M^E)^{-1}$, this implies the claim for $\mathcal{E}^{>,E}$.

Next the expressions for the unitaries $W^{E,<} = \Pi(\mathcal{C}\mathcal{E}^{E,<})$ and $W^{E,>} = \Pi(\mathcal{C}\mathcal{E}^{E,>})$ can readily read off for all real energies not being bound states. Let E_* be a bound state of H^+_{per}. Then $E \mapsto W^{E,<}$ is analytic on a pointed neighborhood of $B_\epsilon(E_*) \setminus \{E_*\} \subset \mathbb{C}$. To prove that one can apply the Riemann theorem on removable singularities (following the argument in [169]), one needs to prove a uniform bound on the $W^{E,<}$. For real E, this follows from the unitarity; for $\mathfrak{Im}(E) > 0$, one readily checks that $\mathfrak{Im}(G^{+,E}) > 0$ that $(W^{E,<})^* W^{E,<} < \mathbf{1}$ (notably the Cayley transform maps the matrix upper half-plane of matrices with $\mathfrak{Im}(G) > 0$ bijectively onto the Siegel disc of matrices with $W^* W < \mathbf{1}$).

For $\Im m(E) < 0$, let us use the identity $W^{\overline{E},<} = ((W^{E,<})^{-1})^*$. Now the inverse can be computed by the quotient of subdeterminants of $W^{E,<}$ by $\det(W^{E,<})$. This latter determinant $\det(W^{E,<})$ takes values in $\mathbb{S}^1$ for real E and is continuous, therefore bounded away from 0 on the pointed neighborhood. Hence also $W^{E,<}$ is bounded for $\Im m(E) < 0$. This shows that the singularity is indeed removable. For $W^{E,>}$, one proceeds in the same manner. Let us also note that the singularities of $G^{+,E}$ and $G^{-,E}$ lead to an eigenvalue 1 in $W^{E,<}$ and -1 in $W^{E,>}$ of the same multiplicity, respectively, which agrees with the above formulas for $m^{+,E}$ and $m^{-,E}$.

Proceeding as in the proof of Theorem 2.4.2 (with Ψ_L^E replaced by $\Psi^{E,<}$), one finds

$$\frac{1}{\imath}(W^{E,<})^*\partial_E W^{E,<} = (AG^{+,E}A + \imath\mathbf{1})^{-1}(\Psi^{E,<})^* I\partial_E\Psi^{E,<}(AG^{+,E}A - \imath\mathbf{1})^{-1}. \tag{2.52}$$

It is thus sufficient to show the negativity of

$$\begin{aligned}(\Psi^{E,<})^* I\partial_E\Psi^{E,<} &= \begin{pmatrix} AG^{+,E} \\ -A^{-1}\end{pmatrix}^* \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0\end{pmatrix}\begin{pmatrix} A\partial_E G^{+,E} \\ 0\end{pmatrix} \\ &= -\partial_E G^{+,E} \\ &= -\pi_1^*(H_{\text{per}}^+ - E)^{-2}\pi_1,\end{aligned}$$

which indeed holds for $E \notin \text{spec}(H_{\text{per}}^+)$. Substituting into (2.52), this proves the first bound (2.51) for all but a finite set of energies in $\text{spec}(H_{\text{per}}^+) \setminus \text{spec}(H_{\text{per}})$. By continuity in E, one obtains that the first bound (2.51) holds with an $\leq$ instead of $<$. To obtain a strict bound at a bound state energy $E_* \in \text{spec}(H_{\text{per}}^+) \setminus \text{spec}(H_{\text{per}})$, one needs again a supplementary argument. Let $P_* = \chi_{\{E_*\}}(H_{\text{per}}^+)$ be the spectral projection on the bound state and set $P_*^c = \mathbf{1} - P_*$. Then $\pi_1^* P_* \pi_1$ is nonnegative, but typically with nontrivial kernel. Then set $\epsilon = E - E_*$ and $B = A^{\frac{1}{2}}$, as well as

$$\begin{aligned}C_\epsilon &= B\frac{1}{\epsilon}(AG^{+,E}A - \imath\mathbf{1})^{-1}B \\ &= (-B\pi_1^* P_*\pi_1 B + \epsilon B\pi_1^* P_*^c(H_{\text{per}}^+ - E)^{-1}P_*^c\pi_1 B - \imath\epsilon A^{-1})^{-1},\end{aligned}$$

where the second equality follows from $H_{\text{per}}^+ P_* = E_* P_*$ so that $(H_{\text{per}}^+ - E)^{-1}P_* = -\frac{1}{\epsilon}P_*$. By (2.52), one then has

$$\begin{aligned}&B\frac{1}{2\imath}(W^{E,<})^*\partial_E W^{E,<}B \\ &\quad = -C_\epsilon^*[B^{-1}\pi_1^* P_*\pi_1 B^{-1} + \epsilon^2 B^{-1}\pi_1^* P_*^c(H_{\text{per}}^+ - E)^{-2}P_*^c\pi_1 B^{-1}]C_\epsilon.\end{aligned}$$

Now on the range of $B^{-1}\pi_1^* P_*\pi_1 B^{-1}$ and ϵ sufficiently small, the right-hand side is given by $(B^{-1}\pi_1^* P_*\pi_1 B^{-1})^{-1} + \mathcal{O}(\epsilon)$ and thus clearly negative. On the orthogonal complement, namely the kernel of $B^{-1}\pi_1^* P_*\pi_1 B^{-1}$, one can again redistribute the ϵ's as before to check that also on this subspace the right-hand side is negative. For the second inequality in (2.51), one proceeds similarly using

$$\begin{aligned}(\Psi^{E,>})^* I \partial_E \Psi^{E,>} &= \begin{pmatrix} -\mathbf{1} \\ G^{-,E} \end{pmatrix}^* \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \partial_E G^{-,E} \end{pmatrix} \\ &= \partial_E G^{-,E} \\ &= \pi_0^* (H_{\text{per}}^- - E)^{-2} \pi_0,\end{aligned}$$

which is positive. □

Similar as in Theorem 2.4.2 and based on the above formulas for $m^{\pm,E}$, one now concludes the following:

Corollary 2.6.4. *As a function of the energy E, the eigenvalues of $W^{E,<}$ and $W^{E,>}$ rotate around the unit circle in the negative and positive sense, respectively. For an interval $[E_0, E_1] \subset \mathbb{R} \backslash \text{spec}(H_{\text{per}})$ such that E_0, E_1 are not bound states of H_{per}^+ and H_{per}^-, respectively, one has*

$$\#\{\text{eigenvalues of } H_{\text{per}}^+ \in [E_0, E_1]\} = -\,\text{Sf}(E \in [E_0, E_1] \mapsto W^{E,<} \text{ through } 1)$$

and

$$\#\{\text{eigenvalues of } H_{\text{per}}^- \in [E_0, E_1]\} = \text{Sf}(E \in [E_0, E_1] \mapsto W^{E,>} \text{ through } -1).$$

Now let us come back to the scattering situation described at the beginning of the section and set

$$m^E = \text{multiplicity of } E \text{ as eigenvalue of } H.$$

Each eigenstate $\psi^E \in \ell^2(\mathbb{Z}, \mathbb{C}^N)$ decays both at $-\infty$ and $+\infty$. Outside of the interval $[1, L] \cap \mathbb{Z}$, the decaying solution satisfies (2.50) with $M_n^E = M^E$. Hence neighboring sites must produce vectors lying in $\mathcal{E}^{E,>}$ on $(-\infty, 0] \cap \mathbb{Z}$ and lying in $\mathcal{E}^{E,<}$ on $[L+1, \infty) \cap \mathbb{Z}$. In-between the solutions must match. Therefore

$$m^E = \dim(M^E(L,1)\mathcal{E}^{E,>} \cap \mathcal{E}^{E,<}).$$

Note that this is again the intersection of two I-Lagrangian subspaces because of Proposition 2.6.2 and because $M^E(L,1)$ is I-unitary. One can therefore directly apply Proposition 2.1.7 (after transforming I-Lagrangian subspaces into J-Lagrangian subspaces with the Cayley transform) to deduce

$$m^E = \dim(\text{Ker}(\Pi(\mathcal{C}\mathcal{E}^{E,<})^* \Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>}) - \mathbf{1})). \tag{2.53}$$

Therefore let us set

$$U^E = -\Pi(\mathcal{C}\mathcal{E}^{E,<})^*\,\Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>}). \tag{2.54}$$

The special case $A = \mathbf{1}$ and $V = 0$ of the following result is contained in Section 7 of [21].

Theorem 2.6.5. *One has*

$$\frac{1}{\imath}(U^E)^*\partial_E U^E > 0. \tag{2.55}$$

Suppose that $[E_0, E_1] \cap \mathrm{spec}(H_{\mathrm{per}}) = \emptyset$ *and that* E_0 *and* E_1 *are not eigenvalues of* H*. Then the number of bound states of* H *in* $[E_0, E_1]$ *is given by*

$$\#\{\text{eigenvalues of } H \text{ in } [E_0,E_1]\} = \mathrm{Sf}(E \in [E_0,E_1] \mapsto U^E \text{ through } -1).$$

Proof. Once (2.55) is verified, the second claim follows from (2.53) (just as in the proof of Theorem 2.4.2). Let us first note that, due to Proposition 2.6.3, the definition (2.54) can be rewritten as

$$U^E = -(W^{E,<})^*\,\Pi(\mathcal{C}M^E(L,1)\Psi^{E,>}).$$

When deriving the definition (2.54) of U^E, one readily realizes that it is sufficient to show the positivity of

$$\frac{1}{\imath}(W^{E,<})\partial_E(W^{E,<})^* = (W^{E,<})\Big(-\frac{1}{\imath}(W^{E,<})^*\partial_E W^{E,<}\Big)(W^{E,<})^* \tag{2.56}$$

and

$$\frac{1}{\imath}\Pi(\mathcal{C}M^E(L,1)\Psi^{E,>})^*\partial_E\Pi(\mathcal{C}M^E(L,1)\Psi^{E,>}). \tag{2.57}$$

The first expression is indeed positive by Proposition 2.6.3. For the expression (2.57), one can proceed as in the proof of Theorem 2.4.2 (with Ψ^E_L replaced by $M^E(L,1)\Psi^{E,>}$), and conclude that it is sufficient to show the positivity of

$$\begin{aligned}&(M^E(L,1)\Psi^{E,>})^* I\partial_E(M^E(L,1)\Psi^{E,>})\\ &\quad = (\Psi^{E,>})^*(M^E(L,1)^* I\partial_E M^E(L,1))\Psi^{E,>} + (\Psi^{E,>})^* I\partial_E\Psi^{E,>}.\end{aligned}$$

The positivity of $M^E(L,1)^* I\partial_E M^E(L,1)$ was already checked in the proof of Theorem 2.4.2, and the positivity of $(\Psi^{E,>})^* I\partial_E\Psi^{E,>}$ in Proposition 2.6.3. □

3 Bounded Fredholm operators

Spectral flow on infinite-dimensional Hilbert spaces cannot be understood without basic knowledge of compact and Fredholm operators. This chapter covers these essentials which are typically taught in a class on linear functional analysis. There is, of course, an exhaustive literature on the subject. Let us mention the excellent standard books [162, 157, 123] which cover most of the material of this chapter. A more detailed account of Fredholm operators is contained in [80]. We decided to include this chapter for several reasons. First of all, a detailed account of notations is needed anyway. Secondly, it is convenient to have clear statements of what is needed later in the book readily available. And last but not least, we hope the chapter helps newcomers to rapidly enter the heart of the matter. On the other hand, we did not include detailed proofs of standard facts which can be found in the above mentioned text books. Merely Section 3.5, which shows an index theorem for the finite-dimensional spectral flow, is not standard textbook material.

3.1 Compact operators and their spectral theory

Let us begin by fixing some notations. Let $\mathcal{H}$ and $\mathcal{H}'$ be separable complex Hilbert spaces of infinite dimension. The scalar product of two vectors $\phi, \psi \in \mathcal{H}$ or $\mathcal{H}'$ is denoted by $\langle\phi|\psi\rangle_{\mathcal{H}} \in \mathbb{C}$ or $\langle\phi|\psi\rangle_{\mathcal{H}'} \in \mathbb{C}$, respectively, and is chosen to be linear in the second argument, and antilinear in the first. The associated norm is $\|\psi\|_{\mathcal{H}} = \langle\psi|\psi\rangle_{\mathcal{H}}^{\frac{1}{2}}$ or respectively $\|\psi\|_{\mathcal{H}'} = \langle\psi|\psi\rangle_{\mathcal{H}'}^{\frac{1}{2}}$. Given a linear operator $T : \mathcal{H} \to \mathcal{H}'$, its operator norm is defined by

$$\|T\| = \sup_{\psi \neq 0} \frac{\|T\psi\|_{\mathcal{H}'}}{\|\psi\|_{\mathcal{H}}} = \sup_{\|\psi\|_{\mathcal{H}}=1} \|T\psi\|_{\mathcal{H}'}.$$

The operator T is called bounded if $\|T\| < \infty$ and the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted by $\mathbb{B}(\mathcal{H}, \mathcal{H}')$. For $\mathcal{H}' = \mathcal{H}$, the set

$$\mathbb{B}(\mathcal{H}) = \mathbb{B}(\mathcal{H}, \mathcal{H})$$

of bounded operators is a Banach $*$-algebra with involution given by the adjoint operator. In particular, it is complete and the operator norm satisfies $\|TS\| \leq \|T\|\|S\|$ for $T, S \in \mathbb{B}(\mathcal{H})$. It is also a C^*-algebra because the C^*-equation $\|T\|^2 = \|T^*T\|$ holds. Next let us introduce the closed unit ball in $\mathcal{H}$ by

$$B_{\mathcal{H}} = \{\psi \in \mathcal{H} : \|\psi\| \leq 1\},$$

where here and below the subscript $\mathcal{H}$ on the norm is dropped. Note that in the following also the subscript on the scalar product is dropped. It is well known that $B_{\mathcal{H}}$ is a compact

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set if and only if $\mathcal{H}$ is finite dimensional. The compact operators can now be introduced as the set of those bounded linear operators which map $B_{\mathcal{H}}$ into a precompact set.

Definition 3.1.1. An operator $K \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ is called compact if and only if its image of the unit ball $K(B_{\mathcal{H}})$ has a compact closure. The set of all compact operators from $\mathcal{H}$ to $\mathcal{H}'$ is denoted by $\mathbb{K}(\mathcal{H}, \mathcal{H}')$. For $\mathcal{H}' = \mathcal{H}$, we set $\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H}, \mathcal{H})$.

A property equivalent to compactness of K is the following: for every bounded sequence $(\psi_n)_{n\geq 1}$ in $\mathcal{H}$, the sequence $(K\psi_n)_{n\geq 1}$ in $\mathcal{H}'$ has a convergent subsequence. The following results are basic.

Theorem 3.1.2. *For $K \in \mathbb{K}(\mathcal{H}, \mathcal{H}')$, $A \in \mathbb{B}(\mathcal{H}', \mathcal{H}'')$ and $B \in \mathbb{B}(\mathcal{H}'', \mathcal{H})$, where $\mathcal{H}''$ is another separable Hilbert space, $AK \in \mathbb{B}(\mathcal{H}, \mathcal{H}'')$ and $KB \in \mathbb{B}(\mathcal{H}'', \mathcal{H}')$ are compact. Moreover, the adjoint operator $K^* \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ is compact.*

Theorem 3.1.3. *The set $\mathbb{K}(\mathcal{H})$ is a closed two-sided $*$-ideal in $\mathbb{B}(\mathcal{H})$. Let $i : \mathbb{K}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ denote the embedding. Then the quotient $\mathbb{Q}(\mathcal{H}) = \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ is a C^*-algebra called the Calkin algebra. Together with $\mathbb{K}(\mathcal{H})$ and $\mathbb{B}(\mathcal{H})$, it forms a short exact sequence of C^*-algebras*

$$0 \longrightarrow \mathbb{K}(\mathcal{H}) \xrightarrow{\;i\;} \mathbb{B}(\mathcal{H}) \xrightarrow{\;\pi\;} \mathbb{Q}(\mathcal{H}) \longrightarrow 0,$$

which is called the Calkin exact sequence. The projection π onto the quotient $\mathbb{Q}(\mathcal{H})$ is called the Calkin projection.

Proofs of Theorems 3.1.2, 3.1.3 and the following results can be found in the above mentioned textbooks. The next theorem shows that eigenspaces and Jordan blocks of compact operators are always finite dimensional.

Theorem 3.1.4. *For $K \in \mathbb{K}(\mathcal{H})$, let us set $T = \mathbf{1} - K$. The following statements hold:*
(i) *There exists $n \in \mathbb{N}$ such that* $\operatorname{Ker}(T^k) = \operatorname{Ker}(T^n)$ *for all $k \geq n$.*
(ii) $\operatorname{Ran}(T) = T(\mathcal{H})$ *is a closed subspace.*
(iii) $\dim(\operatorname{Ker}(T)) = \dim(\operatorname{Ker}(T^*)) < \infty$.

Definition 3.1.5. The spectrum $\operatorname{spec}(T)$ of a bounded operator consists of all points $\lambda \in \mathbb{C}$ for which $\lambda\mathbf{1} - T$ is not invertible. The point spectrum $\operatorname{spec}_p(T)$ of T consists of all eigenvalues of T, namely all $\lambda \in \mathbb{C}$ for which $\operatorname{Ker}(\lambda\mathbf{1} - T)$ is nontrivial.

Theorem 3.1.6 (Riesz' spectral theory of compact operators). *The spectrum* $\operatorname{spec}(K)$ *of every compact operator $K \in \mathbb{K}(\mathcal{H})$ is a countable set $\{\lambda_j : j \geq 1\} \cup \{0\}$ where all $\lambda_j \neq 0$ are eigenvalues of finite multiplicity which can only accumulate at 0. Moreover, 0 can be an eigenvalue of either infinite or finite multiplicity, and in the latter case 0 is an accumulation point of the sequence $(\lambda_j)_{j\geq 1}$.*

3.2 Basic properties of bounded Fredholm operators

Definition 3.2.1. An operator $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ is Fredholm if and only if
(i) $\dim(\operatorname{Ker}(T)) < \infty$,
(ii) $\dim(\operatorname{Ker}(T^*)) < \infty$,
(iii) $\operatorname{Ran}(T)$ is closed in $\mathcal{H}'$.

The set of Fredholm operators is denoted by $\mathbb{FB}(\mathcal{H}, \mathcal{H}')$.

Theorem 3.2.2. *For $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, the following are equivalent:*
(i) *T is a Fredholm operator.*
(ii) $\dim(\operatorname{Ker}(T)) < \infty$ *and* $\dim(\mathcal{H}'/\operatorname{Ran}(T)) < \infty$.
(iii) *There exists a unique $S_0 \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ with*

$$\operatorname{Ker}(S_0) = \operatorname{Ker}(T^*), \quad \operatorname{Ker}(S_0^*) = \operatorname{Ker}(T),$$

such that $S_0 T$ and TS_0 are orthogonal projections onto $\operatorname{Ker}(T)^\perp$ *and* $\operatorname{Ker}(T^*)^\perp$ *and*

$$\dim\big(\operatorname{Ran}(\mathbf{1} - S_0 T)\big) < \infty, \quad \dim\big(\operatorname{Ran}(\mathbf{1} - TS_0)\big) < \infty.$$

(iv) *There exists a pseudoinverse for T, namely $S \in \mathbb{B}(\mathcal{H}', \mathcal{H})$, such that $TS - \mathbf{1} \in \mathbb{K}(\mathcal{H}')$ and $ST - \mathbf{1} \in \mathbb{K}(\mathcal{H})$.*

Proof. (i) $\Longrightarrow$ (ii). This is obvious as $\operatorname{Ker}(T^*) = \operatorname{Ran}(T)^\perp$ is finite dimensional and $\operatorname{Ran}(T)$ is closed.

(ii) $\Longrightarrow$ (i). As $\operatorname{Ker}(T^*) = \operatorname{Ran}(T)^\perp$, it remains to show that $\dim(\mathcal{H}'/\operatorname{Ran}(T)) < \infty$ already implies that $\operatorname{Ran}(T)$ is closed. For that purpose, let us consider the restriction $\widetilde{T} = T|_{\operatorname{Ker}(T)^\perp} : \operatorname{Ker}(T)^\perp \to \mathcal{H}'$. It is continuous, injective, and has the same image $\operatorname{Ran}(\widetilde{T}) = \operatorname{Ran}(T)$. Hence it is sufficient to prove the claim for an injective map with a finite-dimensional cokernel, and we denote this map again by T. Let now $\{\phi_1, \ldots, \phi_N\}$ be a basis of $\mathcal{H}'/\operatorname{Ran}(T)$. Then we define a linear map $\widehat{T} : \mathbb{C}^N \oplus \mathcal{H} \to \mathcal{H}'$ by

$$\widehat{T}(\lambda_1, \ldots, \lambda_N, \psi) = \sum_{n=1}^{N} \lambda_n \phi_n + T\psi.$$

This map $\widehat{T}$ is bijective and continuous. Thus the inverse mapping theorem implies that also $\widehat{T}^{-1}$ is continuous. Hence $\operatorname{Ran}(T) = \widehat{T}((0, \mathcal{H})) = (\widehat{T}^{-1})^{-1}((0, \mathcal{H}))$ is closed.

(i) $\Longrightarrow$ (iii). As $T|_{\operatorname{Ker}(T)^\perp} : \operatorname{Ker}(T)^\perp \to \operatorname{Ran}(T)$ is by assumption a bijective continuous linear map between Hilbert spaces, the inverse mapping theorem implies the existence of a continuous inverse $S_0 : \operatorname{Ran}(T) \to \operatorname{Ker}(T)^\perp$. It can be extended to all of $\mathcal{H}'$ by $S_0\psi = 0$ for $\psi \in \operatorname{Ran}(T)^\perp$. Then

$$TS_0 = \text{orthogonal projection in } \mathcal{H}' \text{ onto } \operatorname{Ran}(T) = \overline{\operatorname{Ran}(T)} = \operatorname{Ker}(T^*)^\perp,$$

$$S_0 T = \text{orthogonal projection in } \mathcal{H} \text{ onto } \operatorname{Ker}(T)^\perp = \overline{\operatorname{Ran}(T^*)}.$$

This implies all the stated properties. Uniqueness is obvious.

(iii) $\Longrightarrow$ (iv). This is obvious as every bounded operator with a finite-dimensional range is compact.

(iv) $\Longrightarrow$ (i). Suppose that $(\psi_n)_{n\geq 1}$ is an infinite orthonormal basis of $\mathrm{Ker}(T)$. As these vectors are all eigenvectors of the compact operator $K = ST - \mathbf{1}$ to the eigenvalue -1, this is a contradiction to Theorem 3.1.6. For $\mathrm{Ker}(T^*)$, one can argue in the same manner by using the compact operator $\tilde{K} = (TS - \mathbf{1})^*$. It remains to show that $\mathrm{Ran}(T)$ is closed. Let $K = ST - \mathbf{1}$ as above. Choose $L \in \mathbb{K}(\mathcal{H})$ with a finite-dimensional range and such that

$$\|K - L\| \leq \frac{1}{2}.$$

Then for all $\phi \in \mathrm{Ker}(L)$,

$$\begin{aligned}\|S\|\|T\phi\| \geq \|ST\phi\| &= \|(\mathbf{1} + K)\phi\| \\ &\geq \|\phi\| - \|K\phi\| \geq \|\phi\| - \|(K - L)\phi\| - \|L\phi\| \\ &\geq \frac{1}{2}\|\phi\|.\end{aligned}$$

Thus $\|\phi\| \leq 2\|S\|\|T\phi\|$ for all $\phi \in \mathrm{Ker}(L)$. This, first of all, implies that $T(\mathrm{Ker}(L))$ is closed. Indeed, if $(T\phi_n)_{n\geq 1}$ is a sequence with $\phi_n \in \mathrm{Ker}(L)$ and $\psi = \lim_n T\phi_n$, then

$$\|\phi_n - \phi_m\| \leq 2\|S\|\|T\phi_n - T\phi_m\|.$$

Thus $(\phi_n)_{n\geq 1}$ is a Cauchy sequence and hence has a limit point $\phi = \lim \phi_n \in \mathrm{Ker}(L)$, where it is used that $\overline{\mathrm{Ker}(L)} = \mathrm{Ker}(L)$. As T is continuous, it follows that $\psi = T\phi \in T(\mathrm{Ker}(L))$. On the other hand,

$$T(\mathrm{Ker}(L)^{\perp}) = T(\mathrm{Ran}(L^*)),$$

where it is used that $\mathrm{Ran}(L^*)$ is of finite dimension and hence closed. Consequently, $T(\mathrm{Ker}(L)^{\perp})$ is finite dimensional. Hence $\mathrm{Ran}(T) = T(\mathrm{Ker}(L)) + T(\mathrm{Ker}(L)^{\perp})$ is closed as the sum of a closed and a finite-dimensional subspace is closed. □

Corollary 3.2.3. *If $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ is a Fredholm operator and $K \in \mathbb{K}(\mathcal{H}, \mathcal{H}')$ is compact, then $T + K$ is a Fredholm operator.*

Proof. Indeed, any pseudoinverse of T is also a pseudo-inverse of $T + K$. □

Theorem 3.2.4. *An operator $T \in \mathbb{B}(\mathcal{H})$ is Fredholm if and only if the image $\pi(T)$ of T in the Calkin algebra is invertible.*

Proof. Let T be a Fredholm operator. By item (iv) in Theorem 3.2.2, there is an operator $S \in \mathbb{B}(\mathcal{H})$ such that $TS - \mathbf{1}, ST - \mathbf{1} \in \mathbb{K}(\mathcal{H})$. As π is an algebra homomorphism and $\pi(K) = 0$ for all $K \in \mathbb{K}(\mathcal{H})$,

$$0 = \pi(T)\pi(S) - \pi(\mathbf{1}) = \pi(T)\pi(S) - \mathbf{1},$$
$$0 = \pi(S)\pi(T) - \pi(\mathbf{1}) = \pi(S)\pi(T) - \mathbf{1}.$$

Hence $\pi(T)$ is invertible with inverse $\pi(S)$.

Conversely, let $\widehat{T} = \pi(T) \in \mathbb{Q}(\mathcal{H})$ be invertible with inverse $\widehat{S}$, namely

$$\widehat{T}\widehat{S} - \mathbf{1} = 0 = \widehat{S}\widehat{T} - \mathbf{1}.$$

As π is surjective, there exists $S \in \mathbb{B}(\mathcal{H})$ such that $\pi(S) = \widehat{S}$. Since π is a homomorphism, it follows that

$$\pi(TS - \mathbf{1}) = 0 = \pi(ST - \mathbf{1}).$$

Consequently, $TS - \mathbf{1},\ ST - \mathbf{1} \in \mathbb{K}(\mathcal{H})$ and thus T is a Fredholm operator by item (iv) of Theorem 3.2.2. □

Remark 3.2.5. In Definition 3.2.1, it is not possible to drop the condition of closedness on the range, as item (ii) in Theorem 3.2.2 might erroneously suggest. Indeed, consider the example of the self-adjoint operator

$$T = \sum_{n\geq 1} \frac{1}{n} |n\rangle\langle n|,$$

on $\ell^2(\mathbb{N})$, where $|n\rangle$ is the state localized at $n \geq 1$ and the Dirac ket-bra notation is used. The kernel of T and (the equal) kernel of T^* are finite dimensional, but T is compact and hence not Fredholm. ◇

There are two widely used criteria for a bounded operator to be a Fredholm operator. One will be given in Theorem 3.4.1 further down, the other is stated in the following proposition.

Proposition 3.2.6. *Let $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ be a bounded linear operator. If there are a compact linear operator $K \in \mathbb{K}(\mathcal{H}, \mathcal{H}'')$, where $\mathcal{H}''$ is another separable Hilbert space, and a constant $c > 0$ such that*

$$\|\phi\| \leq c\big(\|T\phi\| + \|K\phi\|\big)$$

for all $\phi \in \mathcal{H}$, then T has a closed range and a finite-dimensional kernel.

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ such that $T\phi_n$ is convergent, namely there is a $\psi \in \mathcal{H}'$ such that $\lim_{n\to\infty} T\phi_n = \psi$. As K is compact, there is a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ such that $K\phi_{n_k}$ is convergent. Then $(K\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and as $\lim_{k\to\infty} T\phi_{n_k} = \psi$, also $(T\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. Therefore for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\max\{\|T\phi_{n_k} - T\phi_{n_m}\|, \|K\phi_{n_k} - K\phi_{n_m}\|\} < \frac{\epsilon}{2c}$ for all $k, m > N$. Therefore

$$\|\phi_{n_k} - \phi_{n_m}\| \le c\big(\|T\phi_{n_k} - T\phi_{n_m}\| + \|K\phi_{n_k} - K\phi_{n_m}\|\big) < \epsilon$$

or all $k, m > N$. Thus $(\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent.

Suppose that the kernel of T is infinite dimensional and that $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis of it. Then $(\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{H}$ such that $T\phi_n$ is constant (equal to 0) and therefore convergent. As there is no convergent subsequence of $(\phi_n)_{n\in\mathbb{N}}$ this is a contradiction to the above. Thus $\mathrm{Ker}(T)$ is finite dimensional. Moreover, there is a constant $c_1 > 0$ such that $\|\psi\| \le c_1\|T\psi\|$ for all $\psi \in \mathrm{Ker}(T)^\perp$, because otherwise there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathrm{Ker}(T)^\perp$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\|T\psi_n\| \le \frac{1}{n}$ for all $n \in \mathbb{N}$. As $(T\psi_n)_{n\in\mathbb{N}}$ is convergent, by the above there is a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ converging to some vector $\psi \in \mathrm{Ker}(T)^\perp$ with $\|\psi\| = 1$. This is a contradiction to $T\psi = \lim_{k\to\infty} T\psi_{n_k} = 0$. Finally, let $(\theta_n)_{n\ge1}$ be a sequence in $\mathrm{Ran}(T)$ converging to some θ. Then there are $\phi_n \in \mathrm{Ker}(T)^\perp$ with $T\phi_n = \theta_n$. By the previous argument, one has $\|\phi_n - \phi_m\| \le c_1\|\theta_n - \theta_m\|$ so that $(\phi_n)_{n\ge1}$ is Cauchy and thus converges to some ϕ. Consequently, $T\phi = \theta$ so that $\theta \in \mathrm{Ran}(T)$ and hence $\mathrm{Ran}(T)$ is closed. □

3.3 The index of a Fredholm operator

Definition 3.3.1. The index of a Fredholm operator $T \in \mathbb{FB}(\mathcal{H}, \mathcal{H}')$ is

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*)).$$

As $\mathrm{Ker}(T^*) = \mathrm{Ran}(T)^\perp$ and $\mathrm{Ran}(T)$ is closed for a Fredholm operator, one can rewrite the index as

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathcal{H}'/\,\mathrm{Ran}(T)).$$

Let us add a word of justification on the terminology. Most textbooks speak of the Fredholm index, and not an index, a notable exception being the book of Lax [123]. Indeed, Fredholm believed that the index always vanishes as it does for square matrices. Fritz Noether showed in a 1921 paper [140] that this is not true. In the same work he also proved the first index theorem connecting the winding number of an invertible complex function to an index. In the Russian literature, these contributions are honored by speaking of Noether operators. To us it seems more adequate to attribute the index to Noether, and thus speak of a Noether index, but we refrain from doing so here.

The following elementary properties of Fredholm operators and the index are immediate consequences of Theorem 3.2.2 and Definition 3.3.1.

Corollary 3.3.2. (i) *For* $T \in \mathbb{FB}(\mathcal{H}, \mathcal{H}')$, $T' \in \mathbb{FB}(\mathcal{H}'', \mathcal{H})$, *also* $TT' \in \mathbb{FB}(\mathcal{H}'', \mathcal{H}')$.
(ii) *If* $T \in \mathbb{FB}(\mathcal{H}, \mathcal{H}')$, *then* $T^* \in \mathbb{FB}(\mathcal{H}', \mathcal{H})$ *and* $\mathrm{Ind}(T^*) = -\,\mathrm{Ind}(T)$.
(iii) *If* $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ *is invertible, then* $A \in \mathbb{FB}(\mathcal{H}, \mathcal{H}')$ *and* $\mathrm{Ind}(A) = 0$.

(iv) *For* $T \in \mathbb{FB}(\mathcal{H},\mathcal{H}')$ *and invertible operators* $A \in \mathbb{B}(\mathcal{H}',\mathcal{H}'')$ *and* $B \in \mathbb{B}(\mathcal{H}'',\mathcal{H})$*, one has* $\mathrm{Ind}(AT) = \mathrm{Ind}(TB) = \mathrm{Ind}(T)$.

(v) *For* $T \in \mathbb{FB}(\mathcal{H},\mathcal{H}')$,

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T^*T)) - \dim(\mathrm{Ker}(TT^*)).$$

(vi) *For* $T \in \mathbb{FB}(\mathcal{H},\mathcal{H}')$ *and* $T' \in \mathbb{FB}(\mathcal{H}'',\mathcal{H}''')$*, one has* $T \oplus T' \in \mathbb{FB}(\mathcal{H}\oplus\mathcal{H}'',\mathcal{H}'\oplus\mathcal{H}''')$ *and*

$$\mathrm{Ind}(T \oplus T') = \mathrm{Ind}(T) + \mathrm{Ind}(T').$$

Example 3.3.3. The standard example of a Fredholm operator with nonvanishing index is the unilateral shift S on $\ell^2(\mathbb{N})$ defined by $S|k\rangle = \delta_{k\geq 2}|k-1\rangle$ which has a one-dimensional kernel spanned by $|1\rangle$ and trivial cokernel. Hence $\mathrm{Ind}(S) = 1$. Similarly, it follows that $\mathrm{Ind}(S^*) = -1$ and $\mathrm{Ind}(S^n) = n$ for $n \geq 1$. ◇

The following theorem proves the key property of the index, namely its homotopy invariance. As a prelude, let us show that this already is a nontrivial fact in finite dimension. Hence let $T \in \mathbb{C}^{N\times M}$ be a matrix which is not necessarily square. By the rank theorem,

$$\begin{aligned} M &= \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Ran}(T)) \\ &= \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Ker}(T^*)^{\perp}) \\ &= \dim(\mathrm{Ker}(T)) + (N - \dim(\mathrm{Ker}(T^*))). \end{aligned}$$

Thus

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*)) = M - N,$$

which, in particular, shows the homotopy invariance of $\mathrm{Ind}(T)$. This will now be generalized to infinite-dimensional Hilbert spaces.

Theorem 3.3.4. *Let* $T \in \mathbb{FB}(\mathcal{H},\mathcal{H}')$, $T' \in \mathbb{FB}(\mathcal{H}'',\mathcal{H})$ *be Fredholm,* $K \in \mathbb{K}(\mathcal{H},\mathcal{H}')$ *compact, and* $t \mapsto T_t \in \mathbb{FB}(\mathcal{H},\mathcal{H}')$ *a norm-continuous path of Fredholm operators. Then:*
(i) $\mathrm{Ind}(T+K) = \mathrm{Ind}(T)$*, namely* Ind *is compactly stable.*
(ii) $t \mapsto \mathrm{Ind}(T_t)$ *is constant, namely* Ind *is homotopy invariant.*
(iii) $\mathrm{Ind}(TT') = \mathrm{Ind}(T) + \mathrm{Ind}(T')$.
(iv) $\mathrm{Ind} : (\mathbb{FB}(\mathcal{H}),\circ) \to (\mathbb{Z},+)$ *is a homomorphism between semigroups.*

For the proof, but also later use, let us introduce the notation

$$\mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}') = \{T \in \mathbb{FB}(\mathcal{H},\mathcal{H}') : \mathrm{Ind}(T) = n\}, \quad n \in \mathbb{Z},$$

and $\mathbb{F}_n\mathbb{B}(\mathcal{H}) = \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}) = \mathrm{Ind}^{-1}(\{n\})$.

Proof.

Claim 1. For $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}')$, there exists a partial isometry $V \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ of finite rank such that $T+V$ is invertible.

Indeed, as $\dim(\mathrm{Ker}(T)) = \dim(\mathrm{Ker}(T^*)) = N < \infty$, there are two orthonormal bases $(\phi_n)_{n=1,\dots,N}$ and $(\psi_n)_{n=1,\dots,N}$ of $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$, respectively. Setting

$$V = \sum_{n=1}^{N} |\psi_n\rangle\langle\phi_n|$$

gives $V(\mathrm{Ker}(T)) = \mathrm{Ker}(T^*)$. Furthermore, V^*V is the orthogonal projection onto $\mathrm{Ker}(T)$ and VV^* is the orthogonal projection onto $\mathrm{Ker}(T^*)$. Now $T+V$ is injective as $(T+V)\psi = 0$ implies $T\psi = -V\psi \in \mathrm{Ran}(T) \cap \mathrm{Ran}(V)$ and

$$\mathrm{Ran}(T) \cap \mathrm{Ran}(V) = \mathrm{Ran}(T) \cap \mathrm{Ker}\big(T^*\big) = \mathrm{Ran}(T) \cap \mathrm{Ran}(T)^\perp = \{0\},$$

so that $T\psi = 0$ and $V^*V\psi = 0$. That implies

$$\psi \in \mathrm{Ker}(T) \cap \mathrm{Ker}\big(V^*V\big) = \mathrm{Ker}(T) \cap \mathrm{Ker}(T)^\perp = \{0\}.$$

Furthermore $T+V$ is surjective, as

$$(T+V)(\mathcal{H}) = (T+V)\big(\mathrm{Ker}(T)^\perp \oplus \mathrm{Ker}(T)\big) = \mathrm{Ran}(T) \oplus \mathrm{Ker}\big(T^*\big) = \mathcal{H}'.$$

Hence $T+V$ is bijective and thus invertible.

Claim 2. For $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}')$ and $K \in \mathbb{K}(\mathcal{H},\mathcal{H}')$, one has $T+K \in \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}')$.

Indeed, with V as in Claim 1, it follows that

$$\begin{aligned}\mathrm{Ind}(T+K) &= \mathrm{Ind}\big((T+V)(\mathbf{1} + (T+V)^{-1}(K-V))\big)\\ &= \mathrm{Ind}\big(\mathbf{1} + (T+V)^{-1}(K-V)\big)\\ &= 0,\end{aligned}$$

where in the second equality the invertibility of $T+V$ was exploited, and in the last the compactness of $(T+V)^{-1}(K-V)$ combined with Theorem 3.1.4.

Now let us consider $T \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H},\mathcal{H}')$ for $n > 0$. Then by Corollary 3.3.2(vi) one has $T \oplus S^n \in \mathbb{F}_0\mathbb{B}(\mathcal{H} \oplus \ell^2(\mathbb{N}), \mathcal{H}' \oplus \ell^2(\mathbb{N}))$, where S is the unilateral shift on $\ell^2(\mathbb{N})$ as introduced in Example 3.3.3. Hence due to Claim 2,

$$(T+K) \oplus S^n = T \oplus S^n + K \oplus 0 \in \mathbb{F}_0\mathbb{B}\big(\mathcal{H} \oplus \ell^2(\mathbb{N}), \mathcal{H}' \oplus \ell^2(\mathbb{N})\big),$$

as $K \oplus 0$ is compact. Thus $\mathrm{Ind}(T+K) + n = 0$ again by Corollary 3.3.2(vi) and therefore $T+K \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H}',\mathcal{H})$. Finally, for $T \in \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ one has $T^* \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H}',\mathcal{H})$ by Corollary 3.3.2(ii). Thus, by the above, $T^* + K^* \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H}', \mathcal{H})$. Again by Corollary 3.3.2(ii), $T + K \in \mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ follows.

For the proof of (ii), let us first show that $\mathbb{F}_0\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open with respect to the operator norm. Let $A \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ and V as in Claim 1. Then

$$A + V = T + V + A - T = (T + V)\big(\mathbf{1} + (T + V)^{-1}(A - T)\big),$$

is invertible for $\|T - A\|$ sufficiently small by an obvious Neumann series argument. Thus, by Claim 2, $A \in \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}')$ and $\mathrm{Ind}(A) = 0$, namely $A \in \mathbb{F}_0\mathbb{B}(\mathcal{H}, \mathcal{H}')$. To show that $\mathbb{F}_{-n}\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open for $n > 0$, one can repeat the argument for $T \oplus S^n$ and $A \oplus S^n$. Then taking adjoints and exchanging $\mathcal{H}$ and $\mathcal{H}'$ shows that $\mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open. Thus $t \mapsto \mathrm{Ind}(T_t)$ is continuous and therefore constant.

Next let us address (iii), first for $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H})$ so that V as in Claim 1 exists. Due to Corollary 3.3.2(iv),

$$\mathrm{Ind}(T') = \mathrm{Ind}((T + V)T') = \mathrm{Ind}(TT' + VT') = \mathrm{Ind}(TT'),$$

where the last equality follows from item (i) as VT' is compact. For $\mathrm{Ind}(T) = -n < 0$, one has again $\mathrm{Ind}(T \oplus S^n) = 0$ and thus from the above

$$\mathrm{Ind}(TT' \oplus S^n) = \mathrm{Ind}((T \oplus S^n)(T' \oplus \mathbf{1})) = \mathrm{Ind}(T' \oplus \mathbf{1}) = \mathrm{Ind}(T').$$

On the other hand,

$$\mathrm{Ind}(TT' \oplus S^n) = \mathrm{Ind}(TT') + \mathrm{Ind}(S^n) = \mathrm{Ind}(TT') + n = \mathrm{Ind}(TT') - \mathrm{Ind}(T),$$

what concludes the proof if $\mathrm{Ind}(T) < 0$. For $\mathrm{Ind}(T) = n > 0$, one has $\mathrm{Ind}(T \oplus (S^*)^n) = 0$ by Corollary 3.3.2(ii). Thus, we can argue as above where S is replaced by S^*. Finally, it is clear that (iii) implies (iv). □

Theorem 3.3.5. *The set $\mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open and connected with respect to the operator norm. Moreover, the space $\mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is homotopy equivalent to $\mathbb{F}_0\mathbb{B}(\mathcal{H}, \mathcal{H}')$.*

Proof. That $\mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open with respect to the operator norm was already shown in the proof of item (ii) of Theorem 3.3.4.

To show that $\mathbb{F}_n\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is connected, let us first consider the case $n = 0$ and $\mathcal{H} = \mathcal{H}'$. For $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H})$, let V be as in Claim 1 of the proof of Theorem 3.3.4. Then $t \in [0, 1] \mapsto T + tV$ is a continuous path in $\mathbb{F}_0\mathbb{B}(\mathcal{H})$ from T to some invertible operator T_1. Using its polar decomposition $T_1 = U|T_1|$ with a unitary operator U, the path can be continued by $t \in [1, 2] \mapsto T_t = U|T_1|^{2-t}$ to $T_2 = U$. Finally, choose some branch of the logarithm and set $H = -\imath \log(U)$ by spectral calculus. Then $t \in [2, 3] \mapsto T_t = e^{\imath(3-t)H}$ is a continuous path from U to the identity. In summary, any $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H})$ is homotopic to $\mathbf{1}$ within $\mathbb{F}_0\mathbb{B}(\mathcal{H})$, a fact that we henceforth denote by $T \sim \mathbf{1}$.

Now let n be arbitrary. For $T, T' \in \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ by Theorem 3.2.2, there is a pseudoinverse $S \in \mathbb{B}(\mathcal{H}',\mathcal{H})$ such that $ST' = \mathbf{1} + K$ for $K \in \mathbb{K}(\mathcal{H})$. Moreover, as $S \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H}',\mathcal{H})$ by construction, $TS \in \mathbb{F}_0\mathbb{B}(\mathcal{H}')$ by Theorem 3.3.4. As shown in the previous paragraph, there is a continuous path $t \in [0,1] \mapsto A_t \in \mathbb{F}_0\mathbb{B}(\mathcal{H}')$ connecting $A_0 = TS$ to $A_1 = \mathbf{1}$. The path

$$t \in [0,1] \mapsto B_t = A_t T' - (1-t)T(ST' - \mathbf{1})$$

is in $\mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ as $A_tT' \in \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ by Theorem 3.3.4 and $ST' - \mathbf{1}$ is compact. It connects $B_0 = T$ to $B_1 = T'$. Consequently, $T \sim T'$.

It remains to show that $\mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ is homotopy equivalent to $\mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}')$. Consider a fixed operator $T_n \in \mathbb{F}_n\mathbb{B}(\mathcal{H}')$ (the existence of such an operator is guaranteed by Example 3.3.3 above), and define the continuous map

$$f : \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}') \to \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}'), \quad T \mapsto T_nT.$$

Let $S_n \in \mathbb{F}_{-n}\mathbb{B}(\mathcal{H}')$ be a pseudoinverse of T_n. Then

$$g : \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}') \to \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}'), \quad S \mapsto S_nS$$

is continuous and

$$(g\circ f)(T) = S_nT_nT = (\mathbf{1}+K_1)T, \quad (f\circ g)(S) = T_nS_nS = (\mathbf{1}+K_2)S$$

for compact operators $K_1, K_2 \in \mathbb{K}(\mathcal{H}')$. The map

$$h_1 : \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}') \times [0,1] \to \mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}'), \quad h_1(T,t) = (\mathbf{1}+tK_1)T$$

is a homotopy connecting $g\circ f$ to the identity map on $\mathbb{F}_0\mathbb{B}(\mathcal{H},\mathcal{H}')$. Analogously,

$$h_2 : \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}') \times [0,1] \to \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}'), \quad h_2(S,t) = (\mathbf{1}+tK_2)S$$

defines a homotopy connecting $f\circ g$ to the identity map on $\mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$, which completes the proof. □

Corollary 3.3.6. *The index map* $\mathrm{Ind} : \mathbb{F}\mathbb{B}(\mathcal{H},\mathcal{H}') \to \mathbb{Z}$ *is a bijection between the path-connected components of* $\mathbb{F}\mathbb{B}(\mathcal{H},\mathcal{H}') = \bigcup_{n\in\mathbb{Z}} \mathbb{F}_n\mathbb{B}(\mathcal{H},\mathcal{H}')$ *and* $\mathbb{Z}$.

Theorem 3.2.2(iii) exhibited a special pseudoinverse S_0 for a given $T \in \mathbb{F}\mathbb{B}(\mathcal{H},\mathcal{H}')$. In terms of S_0, the index can readily be calculated by

$$\begin{aligned}\mathrm{Ind}(T) &= \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*)) \\ &= \dim(\mathrm{Ran}(\mathbf{1}-S_0T)) - \dim(\mathrm{Ran}(\mathbf{1}-TS_0)) \\ &= \mathrm{Tr}(\mathbf{1}-S_0T) - \mathrm{Tr}(\mathbf{1}-TS_0)\end{aligned}$$

$$= \mathrm{Tr}\big((\mathbf{1} - S_0 T)^p\big) - \mathrm{Tr}\big((\mathbf{1} - TS_0)^p\big),$$

for any $p > 0$ as $\mathbf{1} - S_0 T$ and $\mathbf{1} - TS_0$ are finite-dimensional orthogonal projections. Often the special inverse S_0 is not known, but one may find other pseudoinverses S for which $\mathbf{1} - ST$ and $\mathbf{1} - TS$ have trace class properties, namely are in one of the Schatten ideals $\mathcal{L}^p(\mathcal{H})$ of compact operators $K \in \mathbb{K}(\mathcal{H})$ such that $\mathrm{Tr}((K^*K)^{\frac{p}{2}}) < \infty$, or $\mathcal{L}^p(\mathcal{H}')$ of compact operators $K' \in \mathbb{K}(\mathcal{H}')$ such that $\mathrm{Tr}(((K')^*K')^{\frac{p}{2}}) < \infty$, respectively. Then one has the following formula in which neither of the summands on the right-hand side is necessarily integer-valued.

Theorem 3.3.7 (Calderon–Fedosov formula [46, 83]). *Let $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, $S \in \mathbb{B}(\mathcal{H}', \mathcal{H})$, and $n \in \mathbb{N}$ be such that*

$$\mathbf{1} - ST \in \mathcal{L}^n(\mathcal{H}), \quad \mathbf{1} - TS \in \mathcal{L}^n(\mathcal{H}').$$

Then T is a Fredholm operator and for all $m \geq n$,

$$\mathrm{Ind}(T) = \mathrm{Tr}\big((\mathbf{1} - ST)^m\big) - \mathrm{Tr}\big((\mathbf{1} - TS)^m\big).$$

Proof. The Fredholm property clearly follows from Theorem 3.2.2. Let us now first consider the case $m = n = 1$. Note that $T|_{\mathrm{Ker}(T)^\perp} : \mathrm{Ker}(T)^\perp \to \mathrm{Ran}(T)$ is bijective. Let P denote the orthogonal projection onto $\mathrm{Ker}(T)^\perp$. Then

$$TP(\mathbf{1} - ST) = T(\mathbf{1} - ST) = (\mathbf{1} - TS)T.$$

Therefore $(\mathbf{1} - TS)$ maps $\mathrm{Ran}(T)$ to $\mathrm{Ran}(T)$ and

$$\begin{aligned} \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)}\big) &= \mathrm{Tr}\big((T|_{\mathrm{Ker}(T)^\perp})^{-1}(\mathbf{1} - TS)T|_{\mathrm{Ker}(T)^\perp}\big) \\ &= \mathrm{Tr}\big(P(\mathbf{1} - ST)|_{\mathrm{Ker}(T)^\perp}\big). \end{aligned} \tag{3.1}$$

As $(\mathbf{1} - ST)|_{\mathrm{Ker}(T)} = \mathbf{1}_{\mathrm{Ker}(T)}$,

$$\begin{aligned} \mathrm{Tr}(\mathbf{1} - ST) &= \mathrm{Tr}\big((\mathbf{1} - ST)|_{\mathrm{Ker}(T)}\big) + \mathrm{Tr}\big(P(\mathbf{1} - ST)|_{\mathrm{Ker}(T)^\perp}\big) \\ &= \dim\big(\mathrm{Ker}(T)\big) + \mathrm{Tr}\big(P(\mathbf{1} - ST)|_{\mathrm{Ker}(T)^\perp}\big). \end{aligned}$$

Analogously, as $(\mathbf{1} - TS)|_{\mathrm{Ran}(T)^\perp} = \mathbf{1}_{\mathrm{Ran}(T)^\perp}$,

$$\begin{aligned} \mathrm{Tr}(\mathbf{1} - TS) &= \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)^\perp}\big) + \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)}\big) \\ &= \dim\big(\mathrm{Ran}(T)^\perp\big) + \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)}\big). \end{aligned}$$

By (3.1), this implies

$$\begin{aligned} \mathrm{Tr}(\mathbf{1} - ST) - \mathrm{Tr}(\mathbf{1} - TS) &= \dim\big(\mathrm{Ker}(T)\big) + \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)}\big) \\ &\quad - \dim\big(\mathrm{Ran}(T)^\perp\big) - \mathrm{Tr}\big((\mathbf{1} - TS)|_{\mathrm{Ran}(T)}\big) \\ &= \mathrm{Ind}(T). \end{aligned}$$

Hence, the formula is proved in the case $m = 1$. For $m > 1$, we set $K = \mathbf{1} - ST$, $L = \mathbf{1} - TS$ and replace S by $S_m = (\sum_{j=0}^{m-1} K^j)S$. Then

$$S_m T = \left(\sum_{j=0}^{m-1} K^j\right) ST = \left(\sum_{j=0}^{m-1} K^j\right)(\mathbf{1} - K) = \mathbf{1} - K^m.$$

Furthermore, $K^m = (\mathbf{1} - ST)^m \in \mathcal{L}^1(\mathcal{H})$ by hypothesis. Together with $TK = LT$, one also has

$$TS_m = T\left(\sum_{j=0}^{m-1} K^j\right)S = \left(\sum_{j=0}^{m-1} L^j\right)TS = \left(\sum_{j=1}^{m-1} L^j\right)(\mathbf{1} - L) = \mathbf{1} - L^m.$$

As $L^m \in \mathcal{L}^1(\mathcal{H}')$, it follows that

$$\mathrm{Ind}(T) = \mathrm{Tr}(K^m) - \mathrm{Tr}(L^m) = \mathrm{Tr}\big((1 - ST)^m\big) - \mathrm{Tr}\big((\mathbf{1} - TS)^m\big),$$

and this finishes the proof. □

3.4 The notion of essential spectrum

There is another characterization of Fredholm operators using the notion of essential spectrum of a normal operator. For a normal operator $T \in \mathbb{B}(\mathcal{H})$, the essential spectrum is by definition $\mathrm{spec}_{\mathrm{ess}}(T) = \mathrm{spec}(T) \setminus \mathrm{spec}_{\mathrm{dis}}(T)$, where the discrete spectrum $\mathrm{spec}_{\mathrm{dis}}(T)$ consists of all isolated eigenvalues of finite multiplicity. Further below in Corollary 3.4.5, it will be shown that this coincides with another standard definition of the essential spectrum.

Theorem 3.4.1. *An operator $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ is Fredholm if and only if $0 \notin \mathrm{spec}_{\mathrm{ess}}(T^*T)$ and $0 \notin \mathrm{spec}_{\mathrm{ess}}(TT^*)$.*

Let us note that Corollary 3.3.2(v) then gives the index in terms of the nullities of T^*T and TT^*. For the proof of Theorem 3.4.1, let us use the following lemma (as in [18]).

Lemma 3.4.2. *For $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$, the following are equivalent:*
(i) $\mathrm{Ran}(T)$ *is closed.*
(ii) *There is a constant $c > 0$ such that $\|T\phi\| \geq c\|\phi\|$ for all $\phi \in \mathrm{Ker}(T)^\perp$.*
(iii) 0 *is either not in* $\mathrm{spec}(T^*T)$ *or an isolated point of* $\mathrm{spec}(T^*T)$.

Proof. (i) $\Longrightarrow$ (ii). The map $T : \mathrm{Ker}(T)^\perp \to \mathrm{Ran}(T)$ is a bijection. If $\mathrm{Ran}(T)$ is closed, it is a bijection between two Hilbert spaces. By the inverse mapping theorem, the inverse is a bounded operator. This is restated in (ii).

(ii) $\Longrightarrow$ (i). Let $(\phi_n)_{n\geq 1}$ be a sequence in $\mathrm{Ran}(T)$ converging to $\phi \in \mathcal{H}'$. Then there are $\psi_n \in \mathrm{Ker}(T)^\perp$ with $T\psi_n = \phi_n$. By (ii), one has $\|\psi_n - \psi_m\| \leq \frac{1}{c}\|\phi_n - \phi_m\|$ so that $(\psi_n)_{n\geq 1}$ is

Cauchy and thus converges to some ψ. One has $T\psi = \phi$ so that $\phi \in \mathrm{Ran}(T)$ and $\mathrm{Ran}(T)$ is closed.

(ii) $\Longleftrightarrow$ (iii). Item (iii) is equivalent to $\langle \phi | T^* T \phi \rangle \geq c^2 \|\phi\|^2$ for some $c > 0$ and all $\phi \in \mathrm{Ker}(T)^\perp$, which is indeed equivalent to (ii). □

Proof of Theorem 3.4.1. By definition, if T is Fredholm, then $\mathrm{Ran}(T)$ is closed and $\mathrm{Ker}(T) = \mathrm{Ker}(T^*T)$ is finite dimensional and thus $0 \notin \mathrm{spec}_{\mathrm{ess}}(T^*T)$ by Lemma 3.4.2. As $\mathrm{spec}(TT^*) \setminus \{0\} = \mathrm{spec}(T^*T) \setminus \{0\}$, the point 0 is also isolated in $\mathrm{spec}(TT^*)$ or not in $\mathrm{spec}(TT^*)$. Since also $\mathrm{Ker}(T^*) = \mathrm{Ker}(TT^*)$ is finite dimensional, one concludes that $0 \notin \mathrm{spec}_{\mathrm{ess}}(TT^*)$. The inverse implication follows in the same manner from Lemma 3.4.2. □

Theorem 3.4.1 suggests to consider the self-adjoint operator

$$L = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

Then

$$L^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix}$$

shows that T is a Fredholm operator if and only if $\mathrm{spec}_{\mathrm{ess}}(L^2) \subset (0, \infty)$. Moreover,

$$\mathrm{Ker}(L) = \mathrm{Ker}(L^2) = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T^*).$$

Now L has a symmetry

$$JLJ = -L, \quad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

This shows that the spectrum of L is symmetric around 0. Of most interest is the kernel $\mathrm{Ker}(L)$ itself. It is invariant under J so that $J|_{\mathrm{Ker}(L)}$ is also a symmetry (self-adjoint unitary) squaring to $\mathbf{1}_{\mathrm{Ker}(L)}$. In particular, it has a well-defined signature which actually is equal to the index of T by the above expression of $\mathrm{Ker}(L)$. This leads to the so-called supersymmetric formulation of the index (this terminology is used, e. g., in [67]):

Corollary 3.4.3. *An operator $T \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ is Fredholm if and only if $0 \notin \mathrm{spec}_{\mathrm{ess}}(L)$. If this holds,*

$$\mathrm{Ind}(T) = \mathrm{Sig}(J|_{\mathrm{Ker}(L)}).$$

Now follow some further corollaries of Theorem 3.4.1.

Corollary 3.4.4. *A normal operator $T \in \mathbb{B}(\mathcal{H})$ is Fredholm if and only if $0 \notin \mathrm{spec}_{\mathrm{ess}}(T)$. The index of a normal Fredholm operator vanishes.*

Proof. Note that $\lambda \in \mathbb{C}$ is a point in the spectrum of T if and only if its modulus square $|\lambda|^2$ lies in $\operatorname{spec}(TT^*) = \operatorname{spec}(T^*T)$. Therefore $0 \in \operatorname{spec}_{\text{ess}}(T)$ is equivalent to $0 \in \operatorname{spec}_{\text{ess}}(TT^*)$. The first claim then follows from Theorem 3.4.1 as $T^*T = TT^*$. Obviously, the Fredholm index then vanishes. □

The following result shows that the above notion of essential spectrum (namely Weyl's notion of essential spectrum) coincides with another standard definition of the essential spectrum as the spectrum in the Calkin algebra. There are several other intermediate versions of essential spectra, see [80].

Corollary 3.4.5. *For a normal operator $T \in \mathbb{B}(\mathcal{H})$, one has $\operatorname{spec}_{\text{ess}}(T) = \operatorname{spec}(\pi(T))$ where the latter is the spectrum in the Calkin algebra.*

Proof. By adding a constant multiple of the identity, it is sufficient to analyze the essential spectrum at 0. But for 0, Corollary 3.4.4 implies the claim because by Theorem 3.2.4 the Fredholm property is equivalent to being invertible in the Calkin algebra. □

This immediately implies the compact stability of the essential spectrum. (Another proof of this stability can be given using the Weyl criterion below.)

Corollary 3.4.6. *For a normal operator $T \in \mathbb{B}(\mathcal{H})$ and a compact operator K such that $T + K$ is normal, one has $\operatorname{spec}_{\text{ess}}(T + K) = \operatorname{spec}_{\text{ess}}(T)$.*

Due to Theorem 3.2.2 and Corollary 3.4.4, the following criterion is often helpful.

Proposition 3.4.7 (Weyl criterion for essential spectrum). *A point $\lambda \in \mathbb{R}$ is in the essential spectrum of $H = H^* \in \mathbb{B}(\mathcal{H})$ if and only if there exists a singular Weyl sequence for λ, namely a sequence $(\phi_n)_{n\geq1}$ of unit vectors in $\mathcal{H}$ that converges weakly to 0 and such that $(H - \lambda\mathbf{1})\phi_n \to 0$.*

Proof. "$\Longrightarrow$" If λ is an eigenvalue of infinite multiplicity, then any orthonormal basis $(\phi_n)_{n\geq1}$ of the eigenspace is a Weyl sequence. If λ is an accumulation point of the spectrum, then there exists a sequence $(\lambda_n)_{n\geq1}$ of disjoint points in the spectrum $\operatorname{spec}(H)$ converging to λ. Now choose disjoint open intervals I_n centered at λ_n and of length $|I_n|$ converging to 0. The orthogonal projections $P_n = \chi_{I_n}(H)$ are nontrivial and pairwise orthogonal. Therefore there exist unit vectors $\phi_n \in \operatorname{Ran}(P_n)$ which are pairwise orthogonal and satisfy

$$\|(H - \lambda\mathbf{1})\phi_n\| \leq \|(H - \lambda_n\mathbf{1})\phi_n\| + |\lambda_n - \lambda| \leq |I_n| + |\lambda_n - \lambda|.$$

As this converges to 0, $(\phi_n)_{n\geq1}$ is a singular Weyl sequence.

"$\Longleftarrow$" For the converse, it is sufficient to show that for every $\epsilon > 0$ the spectral projection $P_\epsilon = \chi(|H-\lambda| < \epsilon)$ has an infinite-dimensional range. Let us suppose to the contrary. Then P_ϵ is compact for some $\epsilon > 0$ and hence $P_\epsilon\phi_n \to 0$ for any sequence $(\phi_n)_{n\geq1}$ weakly converging to 0. By spectral calculus, $\|(H - \lambda\mathbf{1})\phi_n\| \geq \epsilon(\|\phi_n\| - 2\|P_\epsilon\phi_n\|)$ and there could not exist a singular Weyl sequence. □

3.5 Spectral flow in finite dimension as index

This section provides a first example of a concrete Fredholm operator and shows that its index contains topological information given by a spectral flow. Hence it establishes a first connection between Fredholm operators and spectral flow, using only its finite-dimensional version described in the introductory Chapter 1, however. The result goes back at least to the work of Atiyah, Patodi, and Singer [14] who considered particular classes of paths of self-adjoint operators. The finite-dimensional case is dealt explicitly in the works of Ben-Artzi and Gohberg [27], Schwarz [176], as well as Robbin and Salamon [160]. It is also covered by the Callias index theorem [45] as the one-dimensional special case.

More concretely, let $t \in \mathbb{R} \mapsto H_t = H_t^* \in \mathbb{C}^{N\times N}$ be a continuous path of self-adjoint matrices with invertible limits

$$H_{\pm\infty} = \lim_{t\to\pm\infty} H_t. \tag{3.2}$$

Such a path has a well-defined spectral flow by Definition 1.1.4. In order to construct an associated Fredholm operator, let us consider the Sobolev space $W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ with norm $\|\phi\|_{W^{1,2}} = (\int_{\mathbb{R}} \|\phi(t)\|^2 dt)^{\frac{1}{2}} + (\int_{\mathbb{R}} \|\phi'(t)\|^2 dt)^{\frac{1}{2}} = \|\phi\|_{L^2} + \|\phi'\|_{L^2}$. The main object of study in this section is the operator

$$D_H : W^{1,2}(\mathbb{R}, \mathbb{C}^N) \to L^2(\mathbb{R}, \mathbb{C}^N)$$

given by

$$(D_H\phi)(t) = \phi'(t) - H_t\phi(t). \tag{3.3}$$

Theorem 3.5.1. *Let $t \in \mathbb{R} \mapsto H_t = H_t^* \in \mathbb{C}^{N\times N}$ be a continuous path such that the limits in* (3.2) *are invertible. Then D_H is a Fredholm operator of index*

$$\mathrm{Ind}(D_H) = -\,\mathrm{Sf}(t \in \mathbb{R} \mapsto H_t).$$

The proof below essentially follows the work of Robbin and Salamon [160]. Later on in Section 7.4, an alternative proof based on semiclassical ideas will be given. Before going into the proof of Theorem 3.5.1 let us note that if (3.3) is autonomous, namely H does not depend on t, then $\mathrm{Ind}(D_H) = 0$. As $\mathrm{Ker}(D_H)$ is then trivial by classical ODE theory, it follows that D_H is surjective in this case. Also let us comment that the modulus of $\mathrm{Ind}(D_H)$ is bounded above by N and any $k \in [-N, N] \cap \mathbb{Z}$ is the index of some D_H. Indeed, this follows from the additivity of the index and the spectral flow as well as the following examples for $N = 1$: the function $H_t = \mp \arctan(t)$ leads to $\mathrm{Ind}(D_H) = \pm 1$, while $H_t = 1$ gives $\mathrm{Ind}(D_H) = 0$.

Proof of Theorem 3.5.1. We first show that there are constants $a, c > 0$ such that

$$\|\phi\|_{W^{1,2}} \leq c\big(\|\phi\chi_{[-a,a]}\|_{L^2([-a,a])} + \|D_H\phi\|_{L^2}\big) \tag{3.4}$$

for all $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$. This estimate is shown in three steps. First note that

$$\begin{aligned}\|\phi\|_{W^{1,2}} &= \|\phi\|_{L^2} + \|\phi'\|_{L^2} \\ &= \|\phi\|_{L^2} + \big\|(D_H + H_t)\phi\big\|_{L^2} \\ &\leq c_1\big(\|\phi\|_{L^2} + \|D_H\phi\|_{L^2}\big),\end{aligned} \tag{3.5}$$

for some constant $c_1 > 0$. Second, assume that $H_t = H_0$ is constant, where $H_0 \in \mathbb{C}^{N\times N}$ is self-adjoint and invertible. Decomposing

$$\mathbb{C}^N = \operatorname{Ran}\big(\chi(H_0 > 0)\big) \oplus \operatorname{Ran}\big(\chi(H_0 < 0)\big),$$

one can assume without loss of generality that all eigenvalues of H_0 are of the same sign. If they are negative, then for $\eta \in L^2(\mathbb{R}, \mathbb{C}^N)$ the unique solution of $D_{H_0}\phi = \phi' - H_0\phi = \eta$ with $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ is

$$\phi(t) = \int_{-\infty}^{t} e^{H_0(t-s)}\eta(s)ds = (\psi * \eta)(t),$$

where $\psi(t) = e^{H_0 t}\chi(t \geq 0)$ is integrable (due to the negative spectrum of H_0). By Young's inequality,

$$\|\phi\|_{L^2} \leq \|\psi\|_{L^1}\|\eta\|_{L^2},$$

so that

$$\|\phi'\|_{L^2} = \|H_0\phi + \eta\|_{L^2} \leq \big(\|H_0\|\|\psi\|_{L^1} + 1\big)\|\eta\|_{L^2}.$$

As $\eta = D_{H_0}\phi$, this implies that there is a constant $c_2(H_0) > 0$ such that

$$\|\phi\|_{W^{1,2}} \leq c_2(H_0)\|D_{H_0}\phi\|_{L^2}. \tag{3.6}$$

If all eigenvalues of H_0 are positive, the argument is similar, namely the unique solution $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ of $D_{H_0}\phi = \phi' - H_0\phi = \eta$ for $\eta \in L^2(\mathbb{R}, \mathbb{C}^N)$ is

$$\phi(t) = \int_{t}^{\infty} e^{H_0(t-s)}\eta(s)ds = (\tilde{\psi} * \eta)(t)$$

where $\tilde{\psi}(t) = \psi(-t)$. It follows that (3.6) holds for all H_0 so that D_{H_0} is, in particular, injective. Let us note that by the above argument it is also surjective. Hence the operator D_{H_0} is bijective.

For a nonconstant path $t \in \mathbb{R} \mapsto H_t$, there is a constant $\tilde{a}$ such that

$$\|H_t - H_{\pm\infty}\| \leq \frac{1}{2c_2} \quad \text{for } \pm t \geq \tilde{a},$$

where $c_2 = \max\{c_2(H_{+\infty}), c_2(H_{-\infty})\}$. For $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ such that $\phi(t) = 0$ for all $t \in [-\tilde{a}, \tilde{a}]$, we define $\phi_+(t) = \phi(t)\chi(t > 0)$ and $\phi_-(t) = \phi(t)\chi(t < 0)$. Then by (3.6),

$$\begin{aligned}
\|\phi_+\|_{W^{1,2}} &\leq c_2\|D_{H_{+\infty}}\phi_+\|_{L^2} \\
&\leq c_2\big(\big\|(D_{H_{+\infty}} - D_H)\phi_+\big\|_{L^2} + \|D_H\phi_+\|_{L^2}\big) \\
&= c_2\big(\big\|(H_{+\infty} - H_t)\phi_+\big\|_{L^2} + \|D_H\phi_+\|_{L^2}\big) \\
&\leq \frac{1}{2}\|\phi_+\|_{L^2} + c_2\|D_H\phi_+\|_{L^2} \\
&\leq \frac{1}{2}\|\phi_+\|_{W^{1,2}} + c_2\|D_H\phi_+\|_{L^2}.
\end{aligned}$$

Therefore

$$\|\phi_+\|_{W^{1,2}} \leq 2c_2\|D_H\phi_+\|_{L^2}$$

and similarly

$$\|\phi_-\|_{W^{1,2}} \leq 2c_2\|D_H\phi_-\|_{L^2}.$$

In conclusion,

$$\|\phi\|_{W^{1,2}} \leq 4c_2\|D_H\phi\|_{L^2} \tag{3.7}$$

for all $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ such that $\phi(t) = 0$ for $t \in [-\tilde{a}, \tilde{a}]$. Now choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta(t) = 0$ for $|t| > \tilde{a} + 1$ and $\beta(t) = 1$ for $t \in [-\tilde{a}, \tilde{a}]$. For $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$, using (3.5) for $\beta\phi$ and (3.7) for $(1 - \beta)\phi$, one obtains

$$\begin{aligned}
\|\phi\|_{W^{1,2}} &\leq \|\beta\phi\|_{W^{1,2}} + \big\|(1-\beta)\phi\big\|_{W^{1,2}} \\
&\leq c_1\big(\|\beta\phi\|_{L^2} + \big\|D_H(\beta\phi)\big\|_{L^2}\big) + 4c_2\big\|D_H\big((1-\beta)\phi\big)\big\|_{L^2} \\
&\leq c_1\big(\|\chi_{[-a,a]}\phi\|_{L^2([-a,a])} + \big\|D_H(\beta\phi)\big\|_{L^2}\big) + 4c_2\big\|D_H\big((1-\beta)\phi\big)\big\|_{L^2} \\
&\leq c_3\big(\|\chi_{[-a,a]}\phi\|_{L^2([-a,a])} + \|D_H\phi\|_{L^2}\big),
\end{aligned}$$

where $a = \tilde{a} + 1$ and in the last step the inequality

$$\max\big\{\big\|D_H(\beta\phi)\big\|_{L^2}, \big\|D_H\big((1-\beta)\phi\big)\big\|_{L^2}\big\} \leq c_4\|\chi_{[-a,a]}\phi\|_{L^2([-a,a])} + \|D_H\phi\|_{L^2}$$

was used (which follows by an explicit computation). Thus (3.4) is shown.

Since the restriction $\phi \mapsto \chi_{[-a,a]}\phi$ is known to be a compact operator from $W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ into $L^2([-a, a])$ by the Rellich embedding theorem, D_H has a closed range and a finite-dimensional kernel by Proposition 3.2.6.

The kernel of D_H consists of those solutions of the differential equation $\phi' = H\phi$ that converge to zero for $t \to \pm\infty$. The solutions of this nonautonomous system with asymptotics (3.2) and invertible limits $H_\pm$ are described by the exponential dichotomy theory. A detailed treatment can be found in [65, Section 3.3] for a differentiable $t \mapsto H_t$ (this can be assumed here because both index and spectral flow do not change if a continuous $t \mapsto H_t$ is approximated by a differentiable one) and for continuous paths in [59]. We hope that the main results of dichotomy theory described next are intuitively clear to the reader. Consider the fundamental solution $\Phi(t, s) \in \mathbb{C}^{N\times N}$, namely

$$\partial_t\Phi(t, s) = H_t\Phi(t, s), \quad \Phi(s, s) = \mathbf{1},$$

which satisfies $\Phi(t, s) = \Phi(t, r)\Phi(r, s)$ for $r, s, t \in \mathbb{R}$. The stable and unstable subspaces are

$$\mathcal{E}^s(t_0) = \Big\{\phi_0 \in \mathbb{C}^N : \lim_{t\to\infty} \Phi(t, t_0)\phi_0 = 0\Big\},$$
$$\mathcal{E}^u(t_0) = \Big\{\phi_0 \in \mathbb{C}^N : \lim_{t\to-\infty} \Phi(t, t_0)\phi_0 = 0\Big\}.$$

Note that $\mathcal{E}^s(t) = \Phi(t, s)\mathcal{E}^s(s)$ and $\mathcal{E}^u(t) = \Phi(t, s)\mathcal{E}^u(s)$ for all $s, t \in \mathbb{R}$ and therefore $t \in \mathbb{R} \mapsto \dim(\mathcal{E}^s(t) \cap \mathcal{E}^u(t))$ is constant. Moreover, $\|\phi(t)\|$ converges to 0 exponentially for $t \to +\infty$ whenever $\phi(t) \in \mathcal{E}^s(t)$, and $\|\phi(t)\|$ converges to ∞ exponentially for $t \to +\infty$ whenever $\phi(t) \notin \mathcal{E}^s(t)$. Similarly, $\|\phi(t)\|$ converges to 0 exponentially for $t \to -\infty$ whenever $\phi(t) \in \mathcal{E}^u(t)$, and $\|\phi(t)\|$ converges to ∞ exponentially for $t \to -\infty$ whenever $\phi(t) \notin \mathcal{E}^u(t)$. Therefore,

$$\mathrm{Ker}(D_H) = \{\phi : \mathbb{R} \to \mathbb{C}^N : \phi' = H\phi,\ \phi(t) \in \mathcal{E}^s(t) \cap \mathcal{E}^u(t)\}$$

and $\dim(\mathrm{Ker}(D_H)) = \dim(\mathcal{E}^s(t) \cap \mathcal{E}^u(t))$ for all $t \in \mathbb{R}$. Next let us examine the cokernel of D_H. Assume that $\psi \in L^2(\mathbb{R}, \mathbb{C}^N)$ is orthogonal to the range of D_H. Then

$$0 = \langle\psi|D_H\phi\rangle = \int_{\mathbb{R}} \big\langle\psi(t)\big|\phi'(t) - H_t\phi(t)\big\rangle dt$$

for all $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$. If there is an $a > 0$ such that $\phi(t) = 0$ for $|t| \geq a$, this implies

$$0 = \int_{-a}^{a} \big\langle\psi(t)\big|\phi'(t)\big\rangle dt - \int_{-a}^{a} \big\langle H_t^*\psi(t)\big|\phi(t)\big\rangle dt.$$

On the other hand, integration by parts shows that

$$0 = \int_{-a}^{a} \langle \tilde{\psi}(t) | \phi(t) \rangle dt + \int_{-a}^{a} \left\langle \int_{-a}^{t} \tilde{\psi}(s) ds \middle| \phi'(t) \right\rangle dt.$$

Using this for $\tilde{\psi} = H^* \psi$ implies

$$0 = \int_{-a}^{a} \langle \psi(t) | \phi'(t) \rangle dt + \int_{-a}^{a} \left\langle \int_{-a}^{t} \tilde{\psi}(s) ds \middle| \phi'(t) \right\rangle dt,$$

for all $\phi \in W^{1,2}(\mathbb{R}, \mathbb{C}^N)$ with support in $(-a, a)$. Therefore

$$\psi(t) + \int_{-\infty}^{t} H_s^* \psi(s) ds = 0.$$

Hence $\psi'(t) = -H_t^* \psi(t)$. The fundamental solution of the latter equation is given by $\tilde{\Phi}(t, s) = \Phi(s, t)^*$ and the associated stable and unstable subspaces are therefore given by $\tilde{\mathcal{E}}^s(t) = \mathcal{E}^s(t)^\perp$ and $\tilde{\mathcal{E}}^u(t) = \mathcal{E}^u(t)^\perp$. Hence

$$\operatorname{Ran}(D_H)^\perp = \{\psi : \mathbb{R} \to \mathbb{C}^N : \psi' = -H^* \psi,\ \psi(t) \in \left(\mathcal{E}^s(t) + \mathcal{E}^u(t)\right)^\perp\}.$$

In particular, the cokernel of D_H is finite dimensional. For $S = S^* \in \mathbb{C}^{N \times N}$, define

$$\mathcal{E}^s(S) = \left\{ v \in \mathbb{C}^N : \lim_{t \to \infty} e^{St} v = 0 \right\} = \operatorname{Ran}(\chi(S < 0))$$

and

$$\mathcal{E}^u(S) = \left\{ v \in \mathbb{C}^N : \lim_{t \to -\infty} e^{St} v = 0 \right\} = \operatorname{Ran}(\chi(S > 0)).$$

Moreover, $\lim_{t \to \infty} \mathcal{E}^s(t) = \mathcal{E}^s(H_{+\infty})$ and $\lim_{t \to -\infty} \mathcal{E}^u(t) = \mathcal{E}^u(H_{-\infty})$ in the standard sense of convergence in the Grassmannian. Therefore,

$$\dim(\mathcal{E}^s(t)) = \dim(\mathcal{E}^s(H_{+\infty})), \quad \dim(\mathcal{E}^u(t)) = \dim(\mathcal{E}^u(H_{-\infty})),$$

and thus

$$\begin{aligned}
\operatorname{Ind}(D_H) &= \dim(\mathcal{E}^s(t) \cap \mathcal{E}^u(t)) - \dim((\mathcal{E}^s(t) + \mathcal{E}^u(t))^\perp) \\
&= \dim(\mathcal{E}^s(t) \cap \mathcal{E}^u(t)) + \dim(\mathcal{E}^s(t) + \mathcal{E}^u(t)) - N \\
&= \dim(\mathcal{E}^s(t)) + \dim(\mathcal{E}^u(t)) - N \\
&= \dim(\mathcal{E}^u(H_{-\infty})) + \dim(\mathcal{E}^s(H_{+\infty})) - N \\
&= \dim(\mathcal{E}^u(H_{-\infty})) - \dim(\mathcal{E}^u(H_{+\infty})).
\end{aligned}$$

As $\dim(\mathcal{E}^u(H_{-\infty})) - \dim(\mathcal{E}^u(H_{+\infty})) = \frac{1}{2}(\operatorname{Sig}(H_{-\infty}) - \operatorname{Sig}(H_{+\infty}))$, this implies the claim. □

3.6 Bounded self-adjoint Fredholm operators

This section introduces the set of bounded self-adjoint Fredholm operators

$$\mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{FB}(\mathcal{H}) : H = H^*\},$$

and collects a few of its basic properties. It is precisely for paths in $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ that the next Chapter 4 then considers the associated spectral flow. First note that by Corollary 3.4.4, one has

$$\mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{B}(\mathcal{H}) : H = H^* \text{ and } 0 \notin \mathrm{spec}_{\mathrm{ess}}(H)\},$$

where $\mathrm{spec}_{\mathrm{ess}}(H) \subset \mathbb{R}$ denotes the essential spectrum. Hence it is natural to introduce the following three subsets of $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$:

$$\mathbb{FB}^{\pm}_{\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}_{\mathrm{ess}}(H) \subset \mathbb{R}_{\pm}\}$$

and

$$\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}_{\mathrm{ess}}(H) \cap \mathbb{R}_{\pm} \neq \emptyset\},$$

where $\mathbb{R}_{\pm} = \{x \in \mathbb{R} : \pm x > 0\}$. The following result goes back to Atiyah and Singer [15]:

Proposition 3.6.1. *With respect to the norm topology,* $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ *has three connected components given by* $\mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$, $\mathbb{FB}^{-}_{\mathrm{sa}}(\mathcal{H})$, *and* $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$. *The components* $\mathbb{FB}^{\pm}_{\mathrm{sa}}(\mathcal{H})$ *are contractible.*

Proof. Clearly, $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) = \mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H}) \cup \mathbb{FB}^{-}_{\mathrm{sa}}(\mathcal{H}) \cup \mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ is a disjoint decomposition. Moreover, the continuity of the essential spectrum implies that $\mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$, $\mathbb{FB}^{-}_{\mathrm{sa}}(\mathcal{H})$, and $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ are open, and hence, due to the above decomposition, also closed. It remains to show that the three sets are (path) connected.

For $H, H' \in \mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$, the linear paths connecting H to $Q = -\chi(H < 0) + \chi(H \geq 0)$ and H' to $Q' = -\chi(H' < 0) + \chi(H' \geq 0)$ lie entirely in $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$. Therefore it is sufficient to show that there is a path in $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ connecting Q to Q'. As the projections $\chi(H < 0)$, $\chi(H \geq 0)$, $\chi(H' < 0)$, and $\chi(H' \geq 0)$ are infinite-dimensional, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ mapping $\mathrm{Ran}(\chi(H' \geq 0))$ onto $\mathrm{Ran}(\chi(H \geq 0))$. Then $Q = UQ'U^*$. By Kuiper's theorem [120], there is a path $t \in [0,1] \mapsto U_t \in \mathbb{U}(\mathcal{H})$ of unitaries connecting $U_0 = U$ to $U_1 = \mathbf{1}$. Then the path $t \in [0,1] \mapsto U_t Q' U_t^*$ lies in $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ and connects Q to Q'.

Next let us show the claimed contractibility which also implies that $\mathbb{FB}^{\pm}_{\mathrm{sa}}(\mathcal{H})$ are connected. For $(H,t) \in \mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H}) \times [0,1]$, let us define $h(H,t) = (1-t)H + t\mathbf{1}$. Then, $\pi(h(H,t)) = (1-t)\pi(H) + t\mathbf{1} > 0$ and, due to $\mathrm{spec}_{\mathrm{ess}}(h(H,t)) = \mathrm{spec}(\pi(h(H,t)))$, by Corollary 3.4.5 one therefore concludes $h(H,t) \in \mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$. Clearly, $h(H,0) = H$ and $h(H,1) = \mathbf{1}$. Thus $\mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$ is contractible to $\mathbf{1}$. Similarly, $\mathbb{FB}^{-}_{\mathrm{sa}}(\mathcal{H})$ is contractible to $-\mathbf{1}$. □

Proposition 3.6.1 determines $\pi_0(\mathbb{FB}_{sa}(\mathcal{H}))$ and shows that the homotopy groups $\pi_k(\mathbb{FB}^{\pm}_{sa}(\mathcal{H}))$ are trivial for $k \geq 1$. The remaining homotopy groups $\pi_k(\mathbb{FB}^*_{sa}(\mathcal{H}))$ will be determined in Section 8.3. It will be shown that some of these groups $\pi_k(\mathbb{FB}^*_{sa}(\mathcal{H}))$ are nontrivial.

In the remainder of this section, let us prove two elementary results that show that the self-adjoint Fredholm operators can be retracted to particularly simple subsets of Fredholm operators without changing the homotopy type. Let us introduce the subsets

$$\mathbb{FB}_{1,sa}(\mathcal{H}) = \mathbb{FB}_{sa}(\mathcal{H}) \cap \mathbb{B}_1(\mathcal{H}), \quad \mathbb{FB}^*_{1,sa}(\mathcal{H}) = \mathbb{FB}^*_{sa}(\mathcal{H}) \cap \mathbb{B}_1(\mathcal{H}),$$

of Fredholm operators lying in the unit ball of bounded operators,

$$\mathbb{B}_1(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

Henceforth we denote by $\mathcal{O}_N$ the norm topology on $\mathbb{B}(\mathcal{H})$.

Proposition 3.6.2. *Space* $(\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FB}_{sa}(\mathcal{H}), \mathcal{O}_N)$. *Moreover,* $(\mathbb{FB}^*_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FB}^*_{sa}(\mathcal{H}), \mathcal{O}_N)$.

Proof. A deformation retraction is given by

$$h : \mathbb{FB}_{sa}(\mathcal{H}) \times [0,1] \to \mathbb{FB}_{sa}(\mathcal{H}), \quad h(H,s) = \left((1-s) + \frac{s}{\max\{1, \|H\|\}}\right)H.$$

Indeed, one has $h(H,0) = H$, $h(H,1) \in \mathbb{FB}_{1,sa}(\mathcal{H})$ for all $H \in \mathbb{FB}_{sa}(\mathcal{H})$, and $h(H,1) = H$ for $H \in \mathbb{FB}_{1,sa}(\mathcal{H})$, and the continuity can readily be checked. This shows the first claim. Because $h(H,s) \in \mathbb{FB}^*_{sa}(\mathcal{H})$ for $H \in \mathbb{FB}^*_{sa}(\mathcal{H})$ and $s \in [0,1]$, the same argument shows that $(\mathbb{FB}^*_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FB}^*_{sa}(\mathcal{H}), \mathcal{O}_N)$. (Note that the same proof shows that $\mathbb{B}(\mathcal{H})$ can be retracted to $\mathbb{B}_1(\mathcal{H})$ in the norm topology.) □

The next result, strengthening Proposition 3.6.2, shows that one can even retract $\mathbb{FB}_{sa}(\mathcal{H})$ to a set of operators that are symmetries, up to compact perturbations:

$$\mathbb{FB}^{C}_{1,sa}(\mathcal{H}) = \{H \in \mathbb{FB}_{sa}(\mathcal{H}) : \|H\| = 1,\ \mathrm{spec}_{ess}(H) \subset \{-1,1\}\}.$$

This set of operators appears naturally in the study of unbounded self-adjoint Fredholm operators, namely it contains the image under the bounded transform of the set of unbounded self-adjoint Fredholm operators with compact resolvent, see Proposition 6.4.4 below. Furthermore, $\mathbb{FB}^*_{sa}(\mathcal{H})$ can be retracted to

$$\mathbb{FB}^{*,C}_{1,sa}(\mathcal{H}) = \{H \in \mathbb{FB}^*_{sa}(\mathcal{H}) : \|H\| = 1,\ \mathrm{spec}_{ess}(H) = \{-1,1\}\}.$$

Proposition 3.6.3. *Space* $(\mathbb{FB}^{C}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FB}_{sa}(\mathcal{H}), \mathcal{O}_N)$. *Moreover,* $(\mathbb{FB}^{*,C}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FB}^*_{sa}(\mathcal{H}), \mathcal{O}_N)$.

Proof. For $H \in \mathbb{FB}_{sa}(\mathcal{H})$, let us define $\delta(H) = \min\{1, \min(\mathrm{spec}_{ess}(H^2))^{\frac{1}{2}}\} > 0$. Then by the spectral radius theorem in the Calkin algebra, it follows that $H \mapsto \delta(H)$ is continuous. For $\delta > 0$, let $f_\delta : \mathbb{R} \to \mathbb{R}$ be the monotone continuous function defined by

$$f_\delta(x) = \chi_{[\delta,\infty)}(x) - \chi_{(-\infty,-\delta]}(x) + \frac{x}{\delta}\chi_{(-\delta,\delta)}(x).$$

Then $\tilde{f} : \mathbb{FB}_{sa}(\mathcal{H}) \to \mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ given by $\tilde{f}(H) = f_{\delta(H)}(H)$ is norm-continuous. Note that $\delta(H) = 1$ for $H \in \mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ and thus $\tilde{f}(H) = H$. Hence the linear homotopy

$$h : \mathbb{FB}_{sa}(\mathcal{H}) \times [0,1] \to \mathbb{FB}_{,sa}(\mathcal{H}), \quad h(H,t) = (1-t)H + t\tilde{f}(H),$$

is a deformation retraction of the space $(\mathbb{FB}_{sa}(\mathcal{H}), \mathcal{O}_N)$ onto $(\mathbb{FB}^{C}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$. As $\tilde{f}$ maps $\mathbb{FB}^{*}_{sa}(\mathcal{H})$ onto $\mathbb{FB}^{*,C}_{1,sa}(\mathcal{H})$, the last claim follows by restricting h to $\mathbb{FB}^{*}_{sa}(\mathcal{H}) \times [0,1]$. □

Remark 3.6.4. For later use, let us note that the proof of Proposition 3.6.3 also implies that $(\mathbb{FB}^{C}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$. Moreover, $(\mathbb{FB}^{*,C}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FB}^{*}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$. ◇

A class of natural elements lying in $\mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ are the symmetries (self-adjoint unitaries) for which we use the notation

$$\mathbb{U}_{sa}(\mathcal{H}) = \mathbb{U}(\mathcal{H}) \cap \mathbb{B}_{sa}(\mathcal{H}).$$

Symmetries lying in $\mathbb{FB}^{*,C}_{1,sa}(\mathcal{H})$ are called proper and the set of proper symmetries is denoted by $\mathbb{U}^{*}_{sa}(\mathcal{H})$. The next result states that operators from the set $\mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ are compact perturbations of symmetries which can be represented in a particular form. Let us stress that this representation does *not* imply that any $H \in \mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ has ± 1 as eigenvalues.

Proposition 3.6.5. *Any $H \in \mathbb{FB}^{C}_{1,sa}(\mathcal{H})$ has a unique representation as*

$$H = Q - K_+ + K_-,$$

with $Q \in \mathbb{U}_{sa}(\mathcal{H})$ and $K_\pm \in \mathbb{K}(\mathcal{H})$ satisfying

$$0 \le K_+ \le \mathbf{1}, \quad 0 \le K_- < \mathbf{1}, \quad K_+K_- = 0, \quad [K_\pm, Q] = 0.$$

In this representation,

$$\mathrm{Ker}(H - \mathbf{1}) = \mathrm{Ker}(Q - \mathbf{1}) \cap \mathrm{Ker}(K_+), \quad \mathrm{Ker}(H + \mathbf{1}) = \mathrm{Ker}(Q + \mathbf{1}) \cap \mathrm{Ker}(K_-).$$

Proof. Given $H \in \mathbb{FB}^{C}_{1,sa}(\mathcal{H})$, let $P_\pm$ denote the spectral projections onto the eigenspaces of ± 1, and let $(\lambda^{\pm}_n)_{n \ge 1}$ be the possibly finite sequences of nonnegative and negative eigenvalues of modulus smaller than 1, ordered such that $0 \leq \lambda_n^+ \leq \lambda_{n+1}^+$ and $0 > \lambda_n^- \geq \lambda_{n+1}^-$. If $\phi_n^\pm$ denote corresponding normalized eigenvectors, then

$$H = P_+ + \sum_{n\geq 1} \lambda_n^+ |\phi_n^+\rangle\langle\phi_n^+| + \sum_{n\geq 1} \lambda_n^- |\phi_n^-\rangle\langle\phi_n^-| - P_- .$$

Then set

$$K_\pm = \sum_{n\geq 1} (1 \mp \lambda_n^\pm) |\phi_n^\pm\rangle\langle\phi_n^\pm| ,$$

and $Q = H + K_+ - K_-$. These operators satisfy all the listed properties.

It remains to prove the uniqueness of the representation. For that purpose, let us set $\mathcal{H}_+ = \chi(H \geq 0)$ and $\mathcal{H}_- = \chi(H < 0)$. Then $[K_\pm, Q] = 0$ implies that $[H, Q] = 0$, and therefore the restrictions $Q_\pm$ of Q to $\mathcal{H}_\pm$ are symmetries (on $\mathcal{H}_\pm$, respectively). In fact, Q_+ is the identity on $\mathcal{H}_+$ and Q_- is minus the identity on $\mathcal{H}_-$. Indeed, let us assume that -1 is an eigenvalue of Q_+, then there is a unit vector $\phi \in \mathcal{H}_+$ such that $Q_+\phi = -\phi$. This implies

$$0 \leq \langle\phi|H\phi\rangle = \big\langle\phi\big|(Q_+ - K_+ + K_-)\phi\big\rangle = -1 - \langle\phi|K_+\phi\rangle + \langle\phi|K_-\phi\rangle$$

and therefore

$$1 \leq 1 + \langle\phi|K_+\phi\rangle \leq \langle\phi|K_-\phi\rangle ,$$

in contradiction to $K_- < \mathbf{1}$. This shows that Q_+ is the identity on $\mathcal{H}_+$. Further assume that 1 is an eigenvalue of Q_-, then there is a unit vector $\psi \in \mathcal{H}_-$ such that $Q_-\psi = \psi$. Then

$$1 - \langle\psi|K_+\psi\rangle + \langle\psi|K_-\psi\rangle = \langle\psi|H\psi\rangle < 0 ,$$

and hence

$$1 \leq 1 + \langle\psi|K_-\psi\rangle < \langle\psi|K_+\psi\rangle ,$$

now in contradiction to $K_+ \leq \mathbf{1}$. Thus Q_- is minus the identity on $\mathcal{H}_-$ and hence the symmetry $Q = \chi(H \geq 0) - \chi(H < 0)$ is uniquely determined. Then $-K_+ + K_- = H - Q$ is uniquely determined. As one, moreover, has $0 \leq K_\pm$ and $K_+K_- = 0$, it can be concluded that $\mp K_\pm = (-K_+ + K_-)\chi(\mp(-K_+ + K_-) \geq 0)$. □

Remark 3.6.6. If one is given a path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{1,\mathrm{sa}}^{\mathbb{C}}(\mathcal{H})$, then Proposition 3.6.5 provides a family $t \in [0,1] \mapsto Q_t$ and $t \in [0,1] \mapsto K_t$ such that $H_t = Q_t + K_t$. In general, though, these families are *not* continuous. If, however, $t \in [0,1] \mapsto H_t$ is invertible, then $P_t = \frac{1}{2}(Q_t - \mathbf{1})$ is given as a Riesz projection via a contour integral and is hence continuous, so that also Q_t and K_t are continuous. ◇

3.7 Essentially gapped unitary operators

Chapter 4 not only studies the spectral flow along paths of bounded self-adjoint Fredholm operators, but also the spectral flow for paths lying in the following set of unitary operators:

$$\mathbb{FU}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U + \mathbf{1} \in \mathbb{FB}(\mathcal{H})\}.$$

This section therefore presents some elementary facts about this set. First of all, due to Corollary 3.4.4, $U + \mathbf{1} \in \mathbb{FB}(\mathcal{H})$ is equivalent to $0 \notin \mathrm{spec}_{\mathrm{ess}}(U + \mathbf{1})$ so that

$$\mathbb{FU}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : -1 \notin \mathrm{spec}_{\mathrm{ess}}(U)\}.$$

Due to this rewriting, unitary operators from $\mathbb{FU}(\mathcal{H})$ will also be called essentially gapped.

Proposition 3.7.1. *The space* $(\mathbb{FU}(\mathcal{H}), \mathcal{O}_N)$ *is connected.*

Proof. We show that for $U \in \mathbb{FU}(\mathcal{H})$ there is a path in $\mathbb{FU}(\mathcal{H})$ connecting U to $\mathbf{1}$. For $t \in [0,1]$, let us define the function $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$f_t(e^{\imath\varphi}) = \begin{cases} e^{\imath\varphi(1-t)}, & \text{for } \varphi \in [0,\pi], \\ e^{\imath(\varphi-2\pi)(1-t)}, & \text{for } \varphi \in (\pi,2\pi). \end{cases}$$

For $U \in \mathbb{FU}(\mathcal{H})$, the continuous path $t \in [0,1] \mapsto f_t(U)$ lies entirely in $\mathbb{FU}(\mathcal{H})$ and connects U to $\mathbf{1}$. □

The set $\mathbb{FU}(\mathcal{H})$ can be retracted to a particularly simple set of unitary operators, namely

$$\mathbb{U}^{\mathrm{C}}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U - \mathbf{1} \in \mathbb{K}(\mathcal{H})\},$$

which can be rewritten as

$$\mathbb{U}^{\mathrm{C}}(\mathcal{H}) = \{\mathbf{1} + K : K \in \mathbb{K}(\mathcal{H}) \text{ with } K + K^* + K^*K = K + K^* + KK^* = 0\}. \tag{3.8}$$

Proposition 3.7.2. *The space* $(\mathbb{U}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FU}(\mathcal{H}), \mathcal{O}_N)$.

Proof. For $U \in \mathbb{FU}(\mathcal{H})$, let us define

$$\delta(U) = \min\{\lambda \in [0,\pi) : \mathrm{spec}_{\mathrm{ess}}(U) \cap \{e^{\imath\varphi} : \varphi \in (\lambda, 2\pi - \lambda)\} = \emptyset\}.$$

Then $U \mapsto \delta(U)$ is continuous by a similar argument as in the proof of Proposition 3.6.3. Moreover, for $U \in \mathbb{U}^{\mathrm{C}}(\mathcal{H})$ one has $\delta(U) = 0$. For $\delta \geq 0$, let $f_\delta : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ be the continuous function defined by

$$f_\delta(e^{\imath\varphi}, t) = \begin{cases} e^{\imath\varphi(1-t)}, & \text{for } \varphi \in [0, \delta), \\ e^{\imath(\varphi(1-t)+t\frac{\pi}{\pi-\delta}(\varphi-\delta))}, & \text{for } \varphi \in [\delta, \pi), \\ e^{\imath((\varphi-2\pi)(1-t)+t\frac{\pi}{\pi-\delta}(\varphi-2\pi+\delta))}, & \text{for } \varphi \in [\pi, 2\pi-\delta), \\ e^{\imath(\varphi-2\pi)(1-t)}, & \text{for } \varphi \in [2\pi-\delta, 2\pi). \end{cases}$$

It can now readily be seen that the homotopy $h : \mathbb{FU}(\mathcal{H}) \times [0,1] \to \mathbb{FU}(\mathcal{H})$ defined by $h(U,t) = f_{\delta(U)}(U,t)$ is the desired deformation retraction of the space $\mathbb{FU}(\mathcal{H})$ onto $\mathbb{U}^{\mathrm{C}}(\mathcal{H})$. □

As explained in Section 8.1, Proposition 3.7.2 readily allows deducing the homotopy type of $\mathbb{FU}(\mathcal{H})$.

4 Spectral flow for bounded self-adjoint Fredholm operators

In this chapter the spectral flow of paths of self-adjoint bounded Fredholm operators is analyzed. In Section 4.1, the spectral flow is defined, essentially as in Phillips' influential work [147], with a minor modification discussed in Remark 4.1.5 below. It is essentially an infinite-dimensional version of the approach already presented in Chapter 1. Then Section 4.2 collects basic properties of the spectral flow such as concatenation, additivity, and homotopy invariance. In Section 4.3, several formulas for the computation of the spectral flow are presented. In the brief Section 4.4, it is sketched how to extend the notion of spectral flow to paths of essentially hyperbolic operators (see [91] for a Banach space generalization). Section 4.5 introduces and studies the spectral flow for paths of unitaries which all do not have −1 in the essential spectrum. Finally, Section 4.6 shows how the spectral flows of paths of bounded self-adjoint Fredholm operators and paths of essentially gapped unitaries are connected.

4.1 The definition of the spectral flow

Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\rm sa}(\mathcal{H})$ be a norm-continuous path, not necessarily closed. For $a \geq 0$, the spectral projections are denoted by

$$P_{a,t} = \chi_{[-a,a]}(H_t). \tag{4.1}$$

The following lemma plays a key role for the definition of the spectral flow.

Lemma 4.1.1. *For $H \in \mathbb{FB}_{\rm sa}(\mathcal{H})$, there are a number $a \geq 0$ and a neighborhood $\mathcal{N}$ of H in $\mathbb{FB}_{\rm sa}(\mathcal{H})$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous, finite-rank projection-valued function on $\mathcal{N}$.*

Proof. Since H is a self-adjoint Fredholm operator, by Corollary 3.4.4 there is an $a \geq 0$ such that $\pm a$ are not in the spectrum of H and $\chi_{[-a,a]}(H)$ is a finite-rank orthogonal projection. Because $\pm a$ are not in the spectrum of H, there exists an $\epsilon > 0$ such that $[-a-\epsilon,-a] \cup [a,a+\epsilon]$ is disjoint from $\mathrm{spec}(H)$. The set

$$\tilde{\mathcal{N}} = \{S \in \mathbb{FB}_{\rm sa}(\mathcal{H}) : ([-a-\epsilon,-a] \cup [a,a+\epsilon]) \cap \mathrm{spec}(S) = \emptyset\}$$

is open and on this set the function $S \mapsto \chi_{[-a,a]}(S)$ is norm-continuous as $\chi_{[-a,a]}$ agrees on $\mathrm{spec}(S)$ with the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \chi_{[-a,a]}(x) - (x-(a+\epsilon))\frac{1}{\epsilon}\chi_{[a,a+\epsilon]}(x) + (x+(a+\epsilon))\frac{1}{\epsilon}\chi_{[-a-\epsilon,-a]}(x).$$

Thus

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$$\mathcal{N} = \{S \in \tilde{\mathcal{N}} : \|\chi_{[-a,a]}(S) - \chi_{[-a,a]}(H)\| < 1\}$$

has the desired properties, as for all $S \in \mathcal{N}$ the dimension of the range of $\chi_{[-a,a]}(S)$ is equal to the dimension of the range of $\chi_{[-a,a]}(H)$, which is finite. □

By compactness and the previous lemma, it is possible to choose a finite partition

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1, \tag{4.2}$$

of $[0,1]$ and $a_m \geq 0$, $m = 1, \ldots, M$, such that

$$t \in [t_{m-1}, t_m] \mapsto P_{a_m,t} \tag{4.3}$$

is norm-continuous with constant finite rank. Furthermore, let us introduce the spectral projections

$$P^{>}_{a,t} = \chi_{(0,a]}(H_t), \quad P^{<}_{a,t} = \chi_{[-a,0)}(H_t).$$

Figure 4.1 shows how a permitted partition looks like. The crucial fact is that the eigenvalues leave the boxes only to the right and left, and never on the top or bottom which would lead to a discontinuity of the dimension of $P_{a_m,t}$.

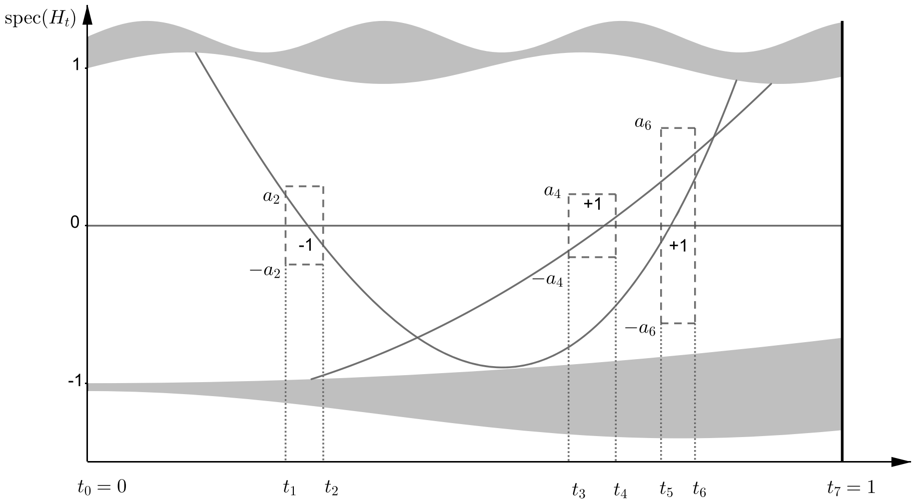

Figure 4.1: Schematic representation of the objects used in Definition 4.1.2 of the spectral flow, as well as in the proof of Theorem 4.1.3. Away from the crossings, it is possible to set $a = 0$.

Definition 4.1.2. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and $a_m \geq 0$, $m = 1, \ldots, M$ as above, the spectral flow of the path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ is defined as

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}). \tag{4.4}$$

Note that all projections involved are finite dimensional so that the trace is finite.

The first basic result about the spectral flow is that it is well defined by the above procedure.

Theorem 4.1.3. *The definition of* $\mathrm{Sf}(t \in [0,1] \mapsto H_t)$ *is independent of the choice of the partition* $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ *of* $[0,1]$ *and values* $a_m \geq 0$ *such that* $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ *is norm-continuous and of constant finite rank.*

Proof. If a point $t_* \in [t_{m-1}, t_m]$ for $m \in \{1, 2, \dots, M\}$ is added to the partition, the number $\mathrm{Tr}(P^{>}_{a_m,t_*} - P^{<}_{a_m,t_*})$ is both added and subtracted, thus $\mathrm{Sf}(t \in [0,1] \mapsto H_t)$ does not change. Therefore the definition of the spectral flow is independent of the choice of the partition.

For $m \in \{1, 2, \dots, M\}$, let us compare a_m to a'_m where $t \in [t_{m-1}, t_m] \mapsto P_{a'_m,t}$ is norm-continuous with constant finite rank. Without loss of generality, one may assume that $a'_m > a_m$ holds. As a_m and a'_m are not in the spectrum of H_t for any $t \in [t_{m-1}, t_m]$, it follows that both $t \in [t_{m-1}, t_m] \mapsto P^{>}_{a'_m,t} - P^{>}_{a_m,t}$ and $t \in [t_{m-1}, t_m] \mapsto P^{<}_{a'_m,t} - P^{<}_{a_m,t}$ are norm-continuous projection-valued functions and hence of constant rank, say $k^{>}$ and $k^{<}$. Thus

$$\begin{aligned}
&\mathrm{Tr}(P^{>}_{a'_m,t_m} - P^{<}_{a'_m,t_m} - P^{>}_{a'_m,t_{m-1}} + P^{<}_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^{>}_{a'_m,t_m}) - \mathrm{Tr}(P^{<}_{a'_m,t_m}) - \mathrm{Tr}(P^{>}_{a'_m,t_{m-1}}) + \mathrm{Tr}(P^{<}_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^{>}_{a_m,t_m}) + k^{>} - \mathrm{Tr}(P^{<}_{a_m,t_m}) - k^{<} - \mathrm{Tr}(P^{>}_{a_m,t_{m-1}}) - k^{>} + \mathrm{Tr}(P^{<}_{a_m,t_{m-1}}) + k^{<} \\
&\quad = \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} - P^{<}_{a_m,t_{m-1}}).
\end{aligned}$$

Therefore the definition of the spectral flow is independent of the choice of the values $a_m > 0$ such that $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ is norm-continuous. □

Remark 4.1.4. Let us note that Definition 4.1.2 is still compatible with the intuitive notion of spectral flow as described in Chapter 1, which also furnishes many examples of paths of finite-dimensional matrices having nontrivial spectral flow. It is also straightforward to provide examples of open paths for infinite-dimensional $\mathcal{H}$; see, e. g., Example 5.7.4. It is more challenging to provide closed paths which have nonvanishing spectral flow. A very explicit construction of such a nontrivial loop is given in Example 8.3.4. ◇

Remark 4.1.5. Let us note that Definition 4.1.2 slightly deviates from Phillips' original definition $\widetilde{\mathrm{Sf}}$ [147]. Indeed, the latter is always integer-valued and the map $\epsilon \mapsto \widetilde{\mathrm{Sf}}(t \in [0,1] \mapsto H_t + \epsilon)$ is right continuous at 0, while here the spectral flow also takes half-integer values and $\epsilon \mapsto \mathrm{Sf}(t \in [0,1] \mapsto H_t + \epsilon)$ is neither left- nor right-continuous. One advantage of this modification is the antisymmetry of Sf under reflection, see item (iv) in Theorem 4.2.1 below. ◇

4.2 Fundamental properties of the spectral flow

In this section some elementary properties of the spectral flow, as well as its homotopy invariance, are collected. All of them are simple generalizations of properties of the spectral flow of paths of self-adjoint matrices shown in Section 1.2. Therefore most of the proofs are omitted.

Theorem 4.2.1. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ be a norm-continuous path.*

(i) *If $t \in [0,1] \mapsto \dim(\mathrm{Ker}(H_t))$ is constant, then* $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = 0$.

(ii) *The spectral flow has a concatenation property, namely if $t \in [1,2] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ is a second norm-continuous path, composable to the first one in the sense that the endpoint of the first path is the initial point of the second path, then*

$$\mathrm{Sf}\big(t \in [0,2] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) + \mathrm{Sf}\big(t \in [1,2] \mapsto H_t\big).$$

(iii) *Changing the orientation of the path leads to a change of the sign of the spectral flow*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = -\,\mathrm{Sf}\big(t \in [0,1] \mapsto H_{1-t}\big).$$

(iv) *The spectral flow has a reflection property, namely*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = -\,\mathrm{Sf}\big(t \in [0,1] \mapsto -H_t\big).$$

(v) *The spectral flow has an additivity property under direct sums, in the sense that if $t \in [0,1] \mapsto S_t \in \mathbb{FB}_{sa}(\mathcal{H}')$ is a second norm-continuous path, then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t \oplus S_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) + \mathrm{Sf}\big(t \in [0,1] \mapsto S_t\big).$$

(vi) *The spectral flow is invariant under conjugation of the path by a norm-continuous path $t \in [0,1] \mapsto U_t \in \mathbb{U}(\mathcal{H})$ of unitaries*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto U_t^* H_t U_t\big).$$

Proof. All items directly follow from the definition of the spectral flow. □

Let us next show that the spectral flow is homotopy invariant.

Theorem 4.2.2. *Let $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t$ be two norm-continuous paths in $\mathbb{FB}_{sa}(\mathcal{H})$ such that $H_0 = H'_0$, $H_1 = H'_1$, and such that there exists a norm-continuous homotopy between the two paths leaving the endpoints fixed. Then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H'_t\big).$$

Proof. Let us first note that for $H_0, H_1 \in \mathbb{FB}_{sa}(\mathcal{H})$ both in the same neighborhood $\mathcal{N}$ of the type given in Lemma 4.1.1 and any path $t \in [0,1] \mapsto H_t$ lying entirely in $\mathcal{N}$, the spectral flow is

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\,\mathrm{Tr}(P^{>}_{a,1} - P^{<}_{a,1} - P^{>}_{a,0} + P^{<}_{a,0}),$$

where $a = a_1$ is chosen as in Lemma 4.1.1 and the partition is trivial, namely $t_0 = 0$ and $t_1 = 1$. Therefore the spectral flow is independent of the path in $\mathcal{N}$ connecting H_0 to H_1.

Let us denote the homotopy between the two paths by $h : [0,1] \times [0,1] \to \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$. Thus h is norm-continuous, $h(t,0) = H_t$ and $H(t,1) = H'_t$ for all $t \in [0,1]$, as well as $h(0,s) = H_0 = H'_0$ and $h(1,s) = H_1 = H'_1$ for all $s \in [0,1]$. By compactness, one can cover the image of h by a finite set $\{\mathcal{N}_1,\dots,\mathcal{N}_k\}$ of neighborhoods as in Lemma 4.1.1. The preimages of these neighborhoods $\{h^{-1}(\mathcal{N}_1),\dots,h^{-1}(\mathcal{N}_k)\}$ form a finite cover of the set $[0,1] \times [0,1]$. For its Lebesgue number $\epsilon_0 > 0$, any subset of $[0,1] \times [0,1]$ of diameter less than ϵ_0 is contained in some element of this finite cover of $[0,1] \times [0,1]$. Thus, if we partition $[0,1] \times [0,1]$ into a grid of squares of diameter less than ϵ_0, then the image of each square will lie entirely within some $\mathcal{N}_l$ for $l \in \{1,\dots,k\}$. By compactness, it is sufficient to show that

$$\mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) = \mathrm{Sf}(t \in [0,1] \mapsto h(t,s''))$$

for $s', s'' \in [0,1]$ with $|s' - s''| < \frac{\epsilon_0}{\sqrt{2}}$. Without loss of generality, one may assume $s' < s''$. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ such that $|t_m - t_{m-1}| < \frac{\epsilon_0}{\sqrt{2}}$ for all $m \in \{1,\dots,M\}$, the image $h([t_{m-1},t_m] \times [s',s''])$ is contained in one of the neighborhoods $\mathcal{N}_l$ for $l \in \{1,\dots,k\}$. Therefore, by the first paragraph of this proof,

$$\begin{aligned}
&\mathrm{Sf}(t \in [t_{m-1},t_m] \mapsto h(t,s')) + \mathrm{Sf}(s \in [s',s''] \mapsto h(t_m,s)) \\
&\quad = \mathrm{Sf}(s \in [s',s''] \mapsto h(t_{m-1},s)) + \mathrm{Sf}(t \in [t_{m-1},t_m] \mapsto h(t,s''))
\end{aligned}$$

for all $m \in \{1,\dots,M\}$. In conclusion,

$$\begin{aligned}
&\mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) \\
&\quad = \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1},t_m] \mapsto h(t,s')) \\
&\quad = \sum_{m=1}^{M} \big(\mathrm{Sf}(s \in [s',s''] \mapsto h(t_{m-1},s)) \\
&\qquad + \mathrm{Sf}(t \in [t_{m-1},t_m] \mapsto h(t,s'')) - \mathrm{Sf}(s \in [s',s''] \mapsto h(t_m,s))\big) \\
&\quad = \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1},t_m] \mapsto h(t,s'')) \\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto h(t,s'')),
\end{aligned}$$

where the third step follows from

$$\mathrm{Sf}(s \in [s',s''] \mapsto h(0,s)) = \mathrm{Sf}(s \in [s',s''] \mapsto h(1,s)) = 0 \tag{4.5}$$

as the considered paths are constant. ☐

Remark 4.2.3. When Theorem 4.2.2 is applied to closed paths, it shows that

$$\mathrm{Sf} : \pi_1\big(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})\big) \to \mathbb{Z}$$

is a well-defined group homomorphism of the fundamental group of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ with $\mathbb{Z}$. In Section 8.3, it will be proved that this is actually an isomorphism. ◇

One can also consider homotopies with varying endpoints, as long as they lie in the invertibles.

Theorem 4.2.4. *Let $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t$ be two norm-continuous paths in $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ such that there exists a norm-continuous homotopy $(t,s) \in [0,1]\times[0,1] \mapsto h(t,s)$ between the two paths with the property that the paths of endpoints $s \in [0,1] \mapsto h(0,s)$ and $s \in [0,1] \mapsto h(1,s)$ both lie in the invertible operators $\mathbb{G}(\mathcal{H})$. Then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H'_t\big).$$

Proof. The proof is a modification of the proof of Theorem 4.2.2. One merely notes that (4.5) remains valid because the appearing paths lie in the invertibles. □

Corollary 4.2.5. *Let $t \in [0,1] \mapsto H_t$ be a norm-continuous path in $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ with invertible endpoints and let $t \in [0,1] \mapsto M_t$ be a norm-continuous path in the invertibles $\mathbb{G}(\mathcal{H})$. Then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto M_t^* H_t M_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big).$$

Proof. Using the polar decomposition $M_t = U_t|M_t|$, one can apply Theorem 4.2.4 to the homotopy $h(t,s) = (U_t|M_t|^s)^* H_t U_t|M_t|^s$ to conclude that

$$\mathrm{Sf}\big(t \in [0,1] \mapsto M_t^* H_t M_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto U_t^* H_t U_t\big).$$

The claim then follows from Theorem 4.2.1(vi). □

Finally, let us prove the monotonicity property of the spectral flow. Often this is proved via crossing forms (e. g., Theorem 7.1 in [146] or Theorem 3.9 in [184]), but here a direct argument based on Loewner's theorem is provided.

Theorem 4.2.6. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ be a norm-continuous and increasing path, namely $H_t \geq H_{t'}$ for $t \geq t'$, then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) \geq 0.$$

Proof. The strategy is to construct an operator monotonic function $f : \mathbb{R} \to \mathbb{R}$ mapping the eigenvalues of H_t close to 0 to the bottom of the spectrum of $f(H_t)$, and then apply the minimax principle to $f(H_t)$. After possibly dividing $[0,1]$ into subintervals, one can assume that there is some $\lambda_c < 0$ such that $\lambda_c \notin \mathrm{spec}(H_t)$ for all $t \in [0,1]$ and

$\text{spec}_{\text{ess}}(H_t) \cap [\lambda_c, 0] = \emptyset$ for all $t \in [0,1]$. Then the function f is simply given by a Möbius transformation of the form

$$f(x) = \frac{x}{x - \lambda_c}.$$

This function satisfies $f(0) = 0$ and $f(\text{spec}_{\text{ess}}(H_t)) \subset (0,\infty)$ for all $t \in [0,1]$. As f is the Möbius transformation with the real matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -\lambda_c \end{smallmatrix}\right)$ which has positive determinant $-\lambda_c$, it is a Herglotz function, namely $\mathfrak{Im}(f(z)) > 0$ for $\mathfrak{Im}(z) > 0$, and by Loewner's theorem (e. g., [182]) f is therefore operator monotone so that, in particular, $f(H_t) \geq f(H_{t'})$ for $t \geq t'$. Now by the spectral mapping theorem,

$$\text{Sf}\big(t \in [0,1] \mapsto H_t\big) = \text{Sf}\big(t \in [0,1] \mapsto f(H_t)\big).$$

By the monotonicity principle (directly following from the minmax principle), the latter spectral flow is nonnegative. □

4.3 Formulas for the spectral flow

This section is about formulas for the spectral flow that generalize the expressions available in the finite-dimensional setting to the infinite-dimensional cases. The first result, Proposition 4.3.1, concerns paths which have only either positive or negative essential spectrum and then the spectral flow only depends on the endpoints, just as in the finite-dimensional setting. Then generalizations of the crossing form computation and integral representation for the spectral flow are proved in Propositions 4.3.6 and 4.3.12, respectively.

The Morse indices $\iota_\pm(H)$ of a self-adjoint Fredholm operator H are defined as

$$\iota_\pm(H) = \text{Tr}\big(\chi(\pm H > 0)\big) \in \mathbb{N}_0 \cup \{\infty\}. \tag{4.6}$$

The terminology slightly deviates from the literature where merely $\iota_-(H)$ is called the Morse index, and $\iota_+(H)$ is called the coindex [86]. If $H \in \mathbb{FB}^\pm_{\text{sa}}(\mathcal{H})$ is a Fredholm operator with only positive/negative essential spectrum, then the Morse index $\iota_\mp(H)$ is finite. The spectral flow of a path of self-adjoint Fredholm operators with only positive/negative essential spectrum can be computed as the difference of the Morse indices of the endpoints of the path.

Proposition 4.3.1. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}^+_{\text{sa}}(\mathcal{H})$ be a norm-continuous path of self-adjoint Fredholm operators with only positive essential spectrum. The spectral flow of this path is*

$$\text{Sf}\big(t \in [0,1] \mapsto H_t\big) = \frac{1}{2}\dim\big(\text{Ker}(H_0)\big) + \iota_-(H_0) - \iota_-(H_1) - \frac{1}{2}\dim\big(\text{Ker}(H_1)\big).$$

Analogously, for a norm-continuous path $t \in [0,1] \mapsto H_t \in \mathbb{FB}^-_{sa}(\mathcal{H})$ *of self-adjoint Fredholm operators with only negative essential spectrum, the spectral flow is*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\dim(\mathrm{Ker}(H_1)) + \iota_+(H_1) - \iota_+(H_0) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)).$$

In particular, the spectral flow of a path in $\mathbb{FB}^{\pm}_{sa}(\mathcal{H})$ *only depends on its endpoints.*

Proof. Let us first focus on a path $t \in [0,1] \mapsto H_t \in \mathbb{FB}^+_{sa}(\mathcal{H})$ with positive essential spectrum. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ as in (4.2) and $a_m \geq 0$ fulfilling (4.3) for all $m = 1, \ldots, M$ as above, by Definition 4.1.2 the spectral flow is

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto H_t),$$

where

$$\mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto H_t) = \frac{1}{2}\mathrm{Tr}(P^>_{a_m,t_m} - P^<_{a_m,t_m} - P^>_{a_m,t_{m-1}} + P^<_{a_m,t_{m-1}}). \tag{4.7}$$

By assumption $t \in [t_{m-1}, t_m] \mapsto \mathrm{Tr}(\chi_{[-a_m,a_m]}(H_t))$ is constant and therefore

$$\begin{aligned}&\mathrm{Tr}(P^<_{a_m,t_m}) + \mathrm{Tr}(P^>_{a_m,t_m}) + \dim(\mathrm{Ker}(H_{t_m}))\\ &\quad = \mathrm{Tr}(P^<_{a_m,t_{m-1}}) + \mathrm{Tr}(P^>_{a_m,t_{m-1}}) + \dim(\mathrm{Ker}(H_{t_{m-1}}))\end{aligned}$$

or equivalently,

$$\begin{aligned}&\mathrm{Tr}(P^>_{a_m,t_m}) - \mathrm{Tr}(P^>_{a_m,t_{m-1}})\\ &\quad = \mathrm{Tr}(P^<_{a_m,t_{m-1}}) + \dim(\mathrm{Ker}(H_{t_{m-1}})) - \mathrm{Tr}(P^<_{a_m,t_m}) - \dim(\mathrm{Ker}(H_{t_m})).\end{aligned}$$

Inserting this into equation (4.7) leads to

$$\begin{aligned}&\mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto H_t)\\ &\quad = \mathrm{Tr}(P^<_{a_m,t_{m-1}}) - \mathrm{Tr}(P^<_{a_m,t_m}) + \frac{1}{2}\dim(\mathrm{Ker}(H_{t_{m-1}})) - \frac{1}{2}\dim(\mathrm{Ker}(H_{t_m})).\end{aligned}$$

As the essential spectrum of H_t is positive, $\chi_{(-\infty,-a_m)}(H_t)$ is traceclass for all $t \in [0,1]$ and, because $-a_m \notin \mathrm{spec}(H_t)$ for $t \in [t_{m-1}, t_m]$, the map $t \in [t_{m-1}, t_m] \mapsto \mathrm{Tr}(\chi_{(-\infty,-a_m)}(H_t))$ is constant. Therefore

$$\begin{aligned}\mathrm{Tr}(P^<_{a_m,t_{m-1}}) - \mathrm{Tr}(P^<_{a_m,t_m}) &= \mathrm{Tr}(P^<_{a_m,t_{m-1}}) + \mathrm{Tr}(\chi_{(-\infty,-a_m)}(H_{t_{m-1}}))\\ &\quad - \mathrm{Tr}(P^<_{a_m,t_m}) - \mathrm{Tr}(\chi_{(-\infty,-a_m)}(H_{t_m}))\\ &= \iota_-(H_{t_{m-1}}) - \iota_-(H_{t_m}).\end{aligned}$$

This implies

$$\begin{aligned}&\mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto H_t)\\ &\quad = \iota_-(H_{t_{m-1}}) - \iota_-(H_{t_m}) + \frac{1}{2}\dim(\mathrm{Ker}(H_{t_{m-1}})) - \frac{1}{2}\dim(\mathrm{Ker}(H_{t_m})).\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathrm{Sf}(t \in [0,1] \mapsto H_t)\\ &= \sum_{m=1}^{M}\Big(\iota_-(H_{t_{m-1}}) - \iota_-(H_{t_m}) + \frac{1}{2}\dim(\mathrm{Ker}(H_{t_{m-1}})) - \frac{1}{2}\dim(\mathrm{Ker}(H_{t_m}))\Big)\\ &= \frac{1}{2}\dim(\mathrm{Ker}(H_0)) + \iota_-(H_0) - \iota_-(H_1) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)),\end{aligned}$$

completing the argument for the case of a family of essentially positive operators. The other case follows by applying this case to the path $t \in [0,1] \mapsto -H_t$. □

Let us stress again that Proposition 4.3.1 implies that the spectral flow of paths in $\mathbb{FB}^{\pm}_{\mathrm{sa}}(\mathcal{H})$ only depends on the difference of the contribution at the endpoints. In particular, for a closed path in $\mathbb{FB}^{\pm}_{\mathrm{sa}}(\mathcal{H})$ the spectral flow vanishes. This is *not* true for paths in $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$. An explicit example of a closed path in $\mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ with nonvanishing spectral flow is given in Example 8.3.4.

From now on, we will also consider paths that do not lie in $\mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$ or $\mathbb{FB}^{-}_{\mathrm{sa}}(\mathcal{H})$. On the other hand, the paths are supposed to have some regularity which can be assured by a small perturbation.

Proposition 4.3.2. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ be a norm-continuous path. For any $\epsilon > 0$ exists a norm-continuous and piecewise real-analytic path $t \in [0,1] \mapsto S_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ with $\|S_t - H_t\| < \epsilon$ uniformly in t such that all eigenvalue crossings are simple and transversal, namely $\dim(\mathrm{Ker}(S_t)) \le 1$ for all $t \in [0,1]$ and $\mathrm{Ker}(S_t) = \{0\}$ except for a discrete set of crossings. For any crossing t_0, there is $\delta > 0$ such that $t \in (t_0 - \delta, t_0 + \delta) \mapsto S_t$ is real analytic and $S'_t|_{\mathrm{Ker}(S_t)} \neq 0$.*

Proof. After a constant shift $t \in [0,1] \mapsto H_t + c\mathbf{1}$ for a small constant $c > 0$, one can assume that the endpoints are invertible. As $t \in [0,1] \mapsto H_t$ is uniformly continuous, there is $\delta' > 0$ such that $\|H_{t'} - H_{t''}\| < \frac{\epsilon}{8}$ for all $t', t'' \in [0,1]$ such that $|t' - t''| < \delta'$. For a partition $0 = t_1 < \cdots < t_M = 1$ such that $|t_m - t_{m-1}| < \delta'$ for $m = 1, \dots, M$, one can replace $t \in [t_{m-1}, t_m] \mapsto H_t$ by the path $t \in [t_{m-1}, t_m] \mapsto \widehat{S}_t = \frac{t - t_m}{t_{m-1} - t_m} H_{t_{m-1}} + \frac{t - t_{m-1}}{t_m - t_{m-1}} H_{t_m}$. Then the path $t \in [0,1] \mapsto \widehat{S}_t$ is continuous, piecewise real-analytic, and $\|\widehat{S}_t - H_t\| < \frac{\epsilon}{4}$ uniformly in t. Moreover, ϵ will be chosen sufficiently small such that the path $t \in [0,1] \mapsto \widehat{S}_t$ remains in $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$. If $[-a, a] \cap \mathrm{spec}_{\mathrm{ess}}(\widehat{S}_t) = \emptyset$ for $a > 0$ and $t \in [t_{m-1}, t_m]$, by Theorem VII.1.8 in [112], one can cover the set $\{(t, \lambda) \in [t_{m-1}, t_m] \times [-a, a] : \lambda \in \mathrm{spec}(\widehat{S}_t)\}$ by finitely many graphs of real-analytic functions λ_j, each possibly defined on some subinterval of $[t_{m-1}, t_m]$ if the eigenvalue leaves $[-a, a]$. In particular, $\mathrm{Ker}(\widehat{S}_t) = \{0\}$ except for finitely many crossings $t \in [t_{m-1}, t_m]$, or $\mathrm{Ker}(\widehat{S}_t) \neq \{0\}$ for all $t \in [t_{m-1}, t_m]$. In the latter case, we replace $\widehat{S}_t$ by $\widetilde{S}_t = \widehat{S}_t + \widetilde{\epsilon}(t-t_{m-1})(t-t_m)\mathbf{1}$ where $0 < \widetilde{\epsilon} < \frac{\epsilon}{4}$ is chosen such that $\mathrm{Ker}(\widetilde{S}_t) = \{0\}$ except for finitely many crossings $t \in [t_{m-1}, t_m]$. Therefore there is a piecewise real-analytic path $t \in [0,1] \mapsto \widetilde{S}_t$ such that $\widetilde{S}_t$ is invertible except for a discrete set of points and such that $\|\widetilde{S}_t - H_t\| < \frac{\epsilon}{2}$ uniformly in t. As $t \in [0,1] \mapsto \widetilde{S}_t$ is uniformly continuous, there is $\delta'' > 0$ such that $\|\widetilde{S}_{t'} - \widetilde{S}_{t''}\| < \frac{\epsilon}{8}$ for all $t', t'' \in [0,1]$ such that $|t' - t''| < \delta''$. If $t_0 \in [0,1]$ is such that $t \mapsto \widetilde{S}_t$ is not analytic in t_0 and such that $\mathrm{Ker}(\widetilde{S}_{t_0}) \neq \{0\}$, there is a $\delta_0' \in (0, \delta'')$ such that $\widetilde{S}_{t_0 \pm \delta_0'}$ is invertible. We then replace $\widetilde{S}_t$ for $t \in [t_0 - \delta_0', t_0 + \delta_0']$ by $\frac{t_0+\delta_0'-t}{2\delta_0'}\widetilde{S}_{t_0-\delta_0'} + \frac{t-t_0+\delta_0'}{2\delta_0'}\widetilde{S}_{t_0+\delta_0'}$. Therefore, one can assume that $\mathrm{Ker}(\widetilde{S}_t) = \{0\}$ except for a discrete set of crossings, and there is $\delta_0 > 0$ such that for each crossing t_0 the path $t \in (t_0 - \delta_0, t_0 + \delta_0) \mapsto \widetilde{S}_t$ is real analytic and such that $\|\widetilde{S}_t - H_t\| < \frac{3\epsilon}{4}$ holds uniformly in t.

For $t_0 \in [0,1]$ such that $\mathrm{Ker}(\widetilde{S}_{t_0}) \neq \{0\}$, there are $a > 0$ and $0 < \delta < \delta_0$ such that $\pm a \notin \mathrm{spec}(\widetilde{S}_t)$ for $t \in (t_0 - \delta, t_0 + \delta)$ and such that $[-a, a] \cap \mathrm{spec}(\widetilde{S}_t)$ consists of finitely many eigenvalues of finite multiplicity for $t \in (t_0 - \delta, t_0 + \delta)$ and such that $\widetilde{S}_{t_0 \pm \delta}$ is invertible. For δ sufficiently small, again by Theorem VII.1.8 in [112], there is a real-analytic path $t \in (t_0 - \delta, t_0 + \delta) \mapsto U_t \in \mathbb{U}(\mathcal{H})$ of unitaries such that one has $U_t\,\mathrm{Ran}(\chi_{[-a,a]}(\widetilde{S}_t)) = \mathrm{Ran}(\chi_{[-a,a]}(\widetilde{S}_{t_0}))$. Then $t \in (t_0 - \delta, t_0 + \delta) \mapsto U_t \widetilde{S}_t U_t^* |_{\mathrm{Ran}(\chi_{[-a,a]}(\widetilde{S}_{t_0}))}$ is a real-analytic path of finite-dimensional operators and, by Theorem II.1.10 and Section II.6.2 in [112], there is a real-analytic path of unitaries $t \in (t_0 - \delta, t_0 + \delta) \mapsto V_t \in \mathbb{B}(\mathrm{Ran}(\chi_{[-a,a]}(\widetilde{S}_{t_0})), \mathbb{C}^M)$ such that $V_t U_t \widetilde{S}_t U_t^* V_t^* = \mathrm{diag}(\lambda_1(t), \dots, \lambda_M(t))$ where $t \mapsto \lambda_k(t)$ are real-analytic functions representing the eigenvalues of $\widetilde{S}_t$. By Sard's theorem, the complement of the set of regular values of the eigenvalues $t \in (t_0 - \delta, t_0 + \delta) \mapsto \lambda_k(t)$, $k = 1, \dots, M$ has measure zero. Therefore there are $\epsilon_1, \dots, \epsilon_M \in (-\min\{\frac{\epsilon}{16}, \|\widetilde{S}_{t_0 \pm \delta}^{-1}\|^{-1}\}, \min\{\frac{\epsilon}{16}, \|\widetilde{S}_{t_0 \pm \delta}^{-1}\|^{-1}\})$ such that 0 is a common regular value of the functions $t \mapsto \lambda_k(t) + \epsilon_k$ for $k = 1, \dots, M$ and such that $\dim(\mathrm{Ker}(\mathrm{diag}(\lambda_1(t) + \epsilon_1, \dots, \lambda_M(t) + \epsilon_M))) \leq 1$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Then $t \in (t_0 - \delta, t_0 + \delta) \mapsto S_t = U_t^* V_t^* \,\mathrm{diag}(\lambda_1(t) + \epsilon_1, \dots, \lambda_N(t) + \epsilon_N) V_t U_t + \widetilde{S}_t(\mathbf{1} - \chi_{[-a,a]}(\widetilde{S}_t))$ is a real-analytic path such that eigenvalue crossings are simple and transversal. Moreover, there is $\widehat{\delta} > 0$ such that $\|\widetilde{S}_t - S_{t_0 - \delta}\| < \frac{\epsilon}{8}$ for all $t \in (t_0 - \delta - \widehat{\delta}, t_0 - \delta)$ and such that $\|\widetilde{S}_t - S_{t_0 + \delta}\| < \frac{\epsilon}{8}$ for all $t \in (t_0 + \delta, t_0 + \delta + \widehat{\delta})$. We then replace $t \in (t_0 - \delta - \widehat{\delta}, t_0 - \delta) \mapsto \widetilde{S}_t$ by the linear path $t \mapsto S_t$ connecting $\widetilde{S}_{t_0 - \delta - \widehat{\delta}}$ to $S_{t_0 - \delta}$ and similar for $t \in (t_0 + \delta, t_0 + \delta + \widehat{\epsilon})$. As $\mathbb{G}(\mathcal{H})$ is open, this linear path lies in the invertibles for $\widehat{\delta}$ sufficiently small. Then setting $S_t = \widetilde{S}_t$ for t not in $(t_0 - \delta - \widehat{\delta}, t_0 + \delta + \widehat{\delta})$ for any eigenvalue crossing t_0, the path $t \in [0,1] \mapsto S_t$ has the desired properties. □

By homotopy invariance, see Theorem 4.2.4, of the spectral flow, one can now compute the spectral flow by the piecewise real-analytic path

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto S_t),$$

provided that the linear path connecting S_0 to H_0 and the linear path connecting S_1 to H_1 are within the invertibles and that $sH_t + (1-s)S_t$ is Fredholm for all $(s,t) \in [0,1] \times [0,1]$. In particular, this is the case for $\epsilon < \min_{t \in [0,1]}\{\|H_0^{-1}\|^{-1}, \|H_1^{-1}\|^{-1}, \|\pi(H_t)^{-1}\|_{\mathbb{Q}}^{-1}\}$. By the above, for any crossing $t_0 \in (0,1)$ there is a real-analytic function $t \in (t_0-\delta, t_0+\delta) \mapsto \lambda(t) \in \mathbb{R}$ with $\lambda(t_0) = 0$ representing an eigenvalue of S_t. Moreover, by Theorem II.5.4 in [112], which can be applied to the path $t \in (t_0-\delta', t_0+\delta') \mapsto S_t\chi_{[-a,a]}(S_t)$ where $a, \delta' > 0$ are chosen such that $t \in (t_0-\delta', t_0+\delta') \mapsto \chi_{[-a,a]}(S_t)$ is of constant finite rank,

$$\lambda'(t_0) = S'_{t_0}|_{\mathrm{Ker}(S_{t_0})} = \langle\phi|S'_{t_0}\phi\rangle$$

for a unit vector $\phi \in \mathrm{Ker}(S_{t_0})$. This leads to another expression for the spectral flow, similar as Proposition 1.4.3 in the finite-dimensional case.

Proposition 4.3.3. *If $t \in [0,1] \mapsto S_t$ is a norm-continuous and piecewise real-analytic path with simple and transversal eigenvalue crossings as given in Proposition* 4.3.2*, then*

$$\mathrm{Sf}(t \in [0,1] \mapsto S_t) = \sum_{\lambda_j(t)=0} \left(1 - \frac{1}{2}\delta_{t,0} - \frac{1}{2}\delta_{t,1}\right) \mathrm{sgn}(\lambda'_j(t)), \tag{4.8}$$

where $\delta_{t,s}$ denotes the Kronecker delta equal to 1 *for $t = s$ and* 0 *otherwise, and the sum runs over pairs (j,t) such that $\lambda_j(t) = 0$.*

Proof. Let us first note that the sum on the right-hand side of (4.8) is finite by the genericity assumption, which also implies that the signs $\mathrm{sgn}(\lambda'_j(t))$ at these points are well defined. Consider $t_0 \in (0,1)$ such that $\mathrm{Ker}(S_{t_0}) \neq \{0\}$. Then choose $a > 0$ such that $\mathrm{spec}(S_{t_0}) \cap [-a,a] = \{0\}$. There is $0 < \epsilon$ such that $\pm a \notin \mathrm{spec}(S_t)$ for $t \in (t_0-\epsilon, t_0+\epsilon)$ and such that $t \in (t_0-\epsilon, t_0+\epsilon) \mapsto S_t$ is real analytic. Let $\lambda : (t_0-\epsilon, t_0+\epsilon) \to (-a,a)$ be the continuously differentiable function representing the eigenvalue of S_t in $[-a,a]$. Because $\lambda'(t_0) \neq 0$, there is $0 < \eta < \frac{\epsilon}{2}$ such that $\lambda(t) \neq 0$ for $t \in (t_0-2\eta, t_0+2\eta) \setminus \{t_0\}$. This implies

$$\mathrm{sgn}(\lambda(t_0+\eta)) = -\,\mathrm{sgn}(\lambda(t_0-\eta)) = \mathrm{sgn}(\lambda'(t_0)),$$

and therefore

$$\begin{aligned}
&\mathrm{sgn}(\lambda'(t_0)) \\
&= \frac{1}{2}\,\mathrm{Tr}\big(\chi_{(0,a]}(S_{t_0+\eta}) - \chi_{[-a,0)}(S_{t_0+\eta}) - \chi_{(0,a]}(S_{t_0-\eta}) + \chi_{[-a,0)}(S_{t_0-\eta})\big) \\
&= \mathrm{Sf}(t \in [t_0-\eta, t_0+\eta] \mapsto S_t).
\end{aligned}$$

The concatenation property of the spectral flow, see item (ii) of Proposition 4.2.1, implies the claim. □

In some situations, one is confronted with paths which are not generic in the above sense, and one would not like to deform them into a generic one as in Proposition 4.3.2. As in Section 1.4 for paths of matrices, under the weaker assumption of so-called regular crossings, it is nevertheless possible to find a generalization of (4.8) which uses the notion of crossing form [160, 200, 84]. Thus we consider a continuously differentiable path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ and establish a connection between the spectral flow of this path and the sum of the signatures of the crossing forms of this path, similar as in Proposition 1.4.5 for a path of matrices. The crossing form for a continuously differentiable path of self-adjoint Fredholm operators is defined as follows.

Definition 4.3.4. Let $t' \in [0,1] \mapsto H_{t'} \in \mathbb{FB}_{sa}(\mathcal{H})$ be a continuously differentiable path. An instant $t \in [0,1]$ is called a crossing for this path if $\mathrm{Ker}(H_t) \neq \{0\}$. Then the crossing form at t is the quadratic form

$$\Gamma_t : \mathrm{Ker}(H_t) \to \mathbb{R}, \quad \Gamma_t(\phi) = \langle \phi | (\partial_t H)_t \phi \rangle .$$

A crossing is called regular, if Γ_t is nondegenerate.

We will freely identify the quadratic form Γ_t with the self-adjoint matrix representing it, hence denoting it also by Γ_t. More precisely, note that by choosing an orthonormal basis, one can identify $\mathrm{Ker}(H_{t_0})$ with $\mathbb{R}^M$, where $M = \dim(\mathrm{Ker}(H_{t_0}))$, namely there is a unitary $U : \mathrm{Ker}(H_{t_0}) \to \mathbb{R}^M$. Then there is a self-adjoint matrix, again denoted by $\Gamma_t \in \mathbb{C}^{M\times M}$ such that $\langle \phi | (\partial_t H)_{t_0} \phi \rangle = \langle \phi | U^* \Gamma_t U \phi \rangle$. As already stressed, this isomorphism will be suppressed. As in the finite-dimensional case, one has the following results.

Proposition 4.3.5. *For a continuously differentiable path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$, there is $\epsilon > 0$ such that*
(i) $t \in [0,1] \mapsto H_t + \delta \mathbf{1}$ *is a path in* $\mathbb{FB}_{sa}(\mathcal{H})$ *for all* $\delta \in (-\epsilon, \epsilon)$,
(ii) $t \in [0,1] \mapsto H_t + \delta \mathbf{1}$ *has only regular crossings for almost every* $\delta \in (-\epsilon, \epsilon)$.

Hence one can always assure to be in a situation where the following result applies:

Proposition 4.3.6. *For a continuously differentiable path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ having only regular crossings*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\,\mathrm{Sig}(\Gamma_0) + \sum_{t\in(0,1)} \mathrm{Sig}(\Gamma_t) + \frac{1}{2}\,\mathrm{Sig}(\Gamma_1). \tag{4.9}$$

Remark 4.3.7. It is worth to point out that these theorems provide the following method for computing the spectral flow of a differentiable path $t \in [0,1] \mapsto H_t$ having invertible endpoints. Since the set of invertible operators on $\mathcal{H}$ is open, there exists $\delta_1 > 0$ such that $H_0 + \delta\mathbf{1}$ and $H_1 + \delta\mathbf{1}$ are invertible for all $-\delta_1 < \delta < \delta_1$. If we assume that δ_1 is less than ϵ in Proposition 4.3.5, then we conclude by the homotopy invariance of the spectral flow that $t \in [0,1] \mapsto H_{\delta,t} = H_t + \delta\mathbf{1}$ and $t \in [0,1] \mapsto H_t$ have the same spectral flow for all these δ. By Proposition 4.3.5, there exists $0 < \delta < \delta_1$ such that $t \in [0,1] \mapsto H_{\delta,t}$ has only regular crossings, thus we can use (4.9) for computing the spectral flow of the original path $t \in [0,1] \mapsto H_t$. Note that in this case the two boundary terms $\frac{1}{2}\,\mathrm{Sig}(\Gamma_0)$ and $\frac{1}{2}\,\mathrm{Sig}(\Gamma_1)$ vanish. ◇

The proof of Proposition 4.3.5 and Proposition 4.3.6 is based on the following lemmas.

Lemma 4.3.8. *As above let* $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ *be a continuously differentiable path. Let* $t_* \in (0,1)$ *and* $a > 0$ *be such that* $\pm a \notin \mathrm{spec}(H_{t_*})$ *and* $\mathrm{spec}(H_{t_*}) \cap [-a,a]$ *consists of finitely many eigenvalues of finite multiplicity. Then there exist* $\epsilon > 0$ *and continuously differentiable functions*

$$f_1, \dots, f_N : (t_* - \epsilon, t_* + \epsilon) \to \mathcal{H}$$

such that $\{f_1(t), \dots, f_N(t)\}$ *is a basis of* $\mathrm{Ran}(\chi_{[-a,a]}(H_t))$ *for all* $t \in (t_* - \epsilon, t_* + \epsilon)$.

Proof. We first recall that there is $\epsilon_0 > 0$ such that $\pm a \notin \mathrm{spec}(H_t)$ and $\mathrm{spec}(H_t) \cap [-a,a]$ consists of finitely many eigenvalues of finite multiplicity for $t \in (t_* - \epsilon_0, t_* + \epsilon_0)$. In particular, $\chi_{[-a,a]}(H_t)$ is defined and finite-dimensional for all $t \in (t_* - \epsilon_0, t_* + \epsilon_0)$, and $t \in (t_* - \epsilon_0, t_* + \epsilon_0) \mapsto \chi_{[-a,a]}(H_t)$ is continuous. Actually, this path is continuously differentiable. Indeed, for an open subset $O \subset \mathbb{C}$ such that $O \cap \mathrm{spec}(H_t) = \emptyset$ for all $t \in (t_* - \epsilon_0, t_* + \epsilon_0)$,

$$(\lambda, t) \in O \times (t_* - \epsilon_0, t_* + \epsilon_0) \mapsto (H_t - \lambda)^{-1}$$

is continuously differentiable. The spectral projections are $\chi_{[-a,a]}(H_t) = \frac{1}{2\pi\imath}\int_\gamma (H_t - \lambda)^{-1} d\lambda$ for $\gamma = \{z \in \mathbb{C} : |z| = a\}$. Differentiation under the integral sign shows

$$\partial_t \chi_{[-a,a]}(H_t) = \frac{1}{2\pi\imath}\int_\gamma \partial_t (H_t - \lambda)^{-1} d\lambda$$

which is continuous. To construct the functions $f_1, \dots, f_N$, we use the operator

$$B_t = \big(\mathbf{1} - \chi_{[-a,a]}(H_{t_*})\big) + \chi_{[-a,a]}(H_t) \in \mathbb{B}(\mathcal{H}).$$

Note that $t \in (t_* - \epsilon_0, t_* + \epsilon_0) \mapsto B_t$ is continuously differentiable and $B_{t_*} = \mathbf{1}$. Therefore there is $\epsilon_0 > \epsilon > 0$ such that B_t is bijective for $t \in (t_* - \epsilon, t_* + \epsilon)$. Moreover, B_t maps $\mathrm{Ran}(\chi_{[-a,a]}(H_{t_*}))$ onto $\mathrm{Ran}(\chi_{[-a,a]}(H_t))$. Then for a basis $\{\phi_1, \dots, \phi_N\}$ of $\mathrm{Ran}(\chi_{[-a,a]}(H_{t_*}))$ we define $f_n(t) = B_t \phi_n$ for $n = 1, \dots, N$. By construction, $t \in (t_* - \epsilon, t_* + \epsilon) \mapsto f_n(t)$ are continuously differentiable functions and $\{f_1(t), \dots, f_N(t)\}$ is a basis of $\mathrm{Ran}(\chi_{[-a,a]}(H_t))$ for all $(t_* - \epsilon, t_* + \epsilon)$. □

In order to allow the reader to appreciate the difficulties (leading to the technical proofs above and further down), let us note that there are continuously differentiable paths $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ such that the eigenvectors of H_t cannot be chosen differentiable in t. More precisely, let $t_* \in (0,1)$ and $a > 0$ be such that one has $\pm a \notin \mathrm{spec}(H_{t_*})$ and $\mathrm{spec}(H_{t_*}) \cap [-a,a]$ consists of finitely many eigenvalues of finite multiplicity. Then by Theorem II.5.4 in [112], there are continuously differentiable functions

$\lambda_k : (t_* - \epsilon, t_* + \epsilon) \to (-a, a)$ for $k = 1, \ldots, \dim(\mathrm{Ran}(\chi_{[-a,a]}(H_{t_*})))$ representing the eigenvalues of H_t in the interval $[-a, a]$. By Example II.5.3 in [112], it may not be possible to chose differentiable functions $\phi_k : (t_* - \epsilon, t_* + \epsilon) \to \mathcal{H} \setminus \{0\}$ of eigenvectors of H_t such that $H_t \phi_k(t) = \lambda_k(t)\phi_k(t)$ holds for all $t \in (t_* - \epsilon, t_* + \epsilon)$ and $k = 1, \ldots, \dim(\mathrm{Ran}(\chi_{[-a,a]}(H_{t_*})))$.

Lemma 4.3.9. *As above let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ be a continuously differentiable path. Let $t_* \in (0,1)$ and $a > 0$ be such that $\pm a \notin \mathrm{spec}(H_{t_*})$ and that the intersection $\mathrm{spec}(H_{t_*}) \cap [-a, a]$ consists only of finitely many eigenvalues of finite multiplicity. Furthermore let $N = \mathrm{Tr}(\chi_{[-a,a]}(H_{t_*}))$. Then there is $\epsilon > 0$ and a continuously differentiable function $t \in (t_* - \epsilon, t_* + \epsilon) \mapsto S_t \in \mathbb{C}^{N \times N}$ of self-adjoint matrices and a continuously differentiable path of unitaries $U_t : \mathrm{Ran}(\chi_{[-a,a]}(H_t)) \to \mathbb{C}^N$ mapping $\mathrm{Ker}(H_t - \delta)$ onto $\mathrm{Ker}(S_t - \delta)$ such that*

$$\Gamma_t(S - \delta\mathbf{1})(\phi) = \Gamma_t(H - \delta\mathbf{1})(U_t^* \phi)$$

for all $\phi \in \mathrm{Ker}(S_t - \delta\mathbf{1})$, $t \in (t_* - \epsilon, t_* + \epsilon)$ and $\delta \in [-a, a]$.

Proof. By Lemma 4.3.8, there exists $\epsilon > 0$ and continuously differentiable functions $f_1, \ldots, f_N : (t_* - \epsilon, t_* + \epsilon) \to \mathcal{H}$ such that $\{f_1(t), \ldots, f_N(t)\}$ is a basis of $\mathrm{Ran}(\chi_{[-a,a]}(H_t))$ for all $(t_* - \epsilon, t_* + \epsilon)$. By using a Gram–Schmidt process, we may assume that these bases are orthonormal. Then define $U_t : \mathrm{Ran}(\chi_{[-a,a]}(H_t)) \to \mathbb{C}^N$ by

$$U_t^* \phi = \sum_{n=1}^{N} \phi_n f_n(t), \quad \phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}.$$

By construction, $t \in (t_* - \epsilon, t_* + \epsilon) \mapsto U_t$ is a continuously differentiable path of unitaries. Therefore the path $t \in (t_* - \epsilon, t_* + \epsilon) \mapsto S_t = U_t H_t U_t^*$ is continuously differentiable. Moreover, U_t maps $\mathrm{Ker}(H_t - \delta\mathbf{1})$ onto $\mathrm{Ker}(S_t - \delta\mathbf{1})$ for all $\delta \in [-a, a]$. For $\phi \in \mathrm{Ker}(S_t - \delta\mathbf{1})$, one has

$$\begin{aligned}
\Gamma_t(S - \delta\mathbf{1})(\phi) &= \langle \phi | (\partial_t S)_t \phi \rangle \\
&= \langle \phi | U_t (\partial_t (H - \delta\mathbf{1}))_t U_t^* \phi \rangle + \langle \phi | (\partial_t U)_t (H_t - \delta\mathbf{1}) U_t^* \phi \rangle \\
&\quad + \langle \phi | U_t (H_t - \delta\mathbf{1}) (\partial_t U^*)_t \phi \rangle \\
&= \langle U_t^* \phi | (\partial_t (H - \delta\mathbf{1}))_t U_t^* \phi \rangle + \langle \phi | (\partial_t U)_t (H_t - \delta\mathbf{1}) U_t^* \phi \rangle \\
&\quad + \langle (H_t - \delta\mathbf{1}) U_t^* \phi | (\partial_t U^*)_t \phi \rangle \\
&= \langle U_t^* \phi | (\partial_t (H - \delta\mathbf{1}))_t U_t^* \phi \rangle \\
&= \Gamma_t(H - \delta\mathbf{1})(U_t^* \phi),
\end{aligned}$$

where the fourth step follows because $(H_t - \delta\mathbf{1}) U_t^* \phi = 0$. □

Remark 4.3.10. Let us note that the eigenvalues $\lambda_j(t)$ of S_t are the eigenvalues of H_t between $-a$ and a. Moreover, by Theorem II.5.4 in [112], the derivatives $\lambda_j'(t)$ for those λ_j with $\lambda_j(t) = \delta$ are the eigenvalues of the crossing operator $\Gamma_t(S - \delta\mathbf{1})$. ◇

Proof of Proposition 4.3.5. We choose as in the definition of the spectral flow a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ of $[0,1]$ and $a_1, \ldots, a_M > 0$ such that $\operatorname{spec}(H_t) \cap [-a_m, a_m]$ consists of eigenvalues of finite multiplicity and $\pm a_m \notin \operatorname{spec}(H_t)$ for all $t \in [t_m, t_{m-1}]$, $m = 1, \ldots, M$. Moreover, we assume that $[t_m, t_{m-1}] \subset (t_* - \epsilon, t_* + \epsilon)$ where $\epsilon > 0$ is chosen as in Lemma 4.3.9 for some $t_* \in [t_m, t_{m-1}]$ and $a = a_m$. By Lemma 4.3.9 and as, by Theorem II.6.8 in [112], the eigenvalues of a continuously differentiable path of self-adjoint matrices are continuously differentiable provided that one chooses the correct branches at level crossings, we can cover the set

$$\bigcup_{m=1}^{M} \{(t, \lambda) \in [t_{m-1}, t_m] \times [-a_m, a_m] : \lambda \in \operatorname{spec}(H_t)\}$$

by finitely many graphs of continuously differentiable functions λ_m^n, each defined on some interval $[t_{m-1}, t_m]$. Because the set of Fredholm operators is open, there exists a positive $\epsilon < \min_{m=1,\ldots,M} a_m$ such that $H_t + \delta\mathbf{1}$ is Fredholm for all $t \in [0,1]$ and $\delta \in (-\epsilon, \epsilon)$. By Sard's theorem, the complement of the set of common regular values of the functions λ_m^n in $(-\epsilon, \epsilon)$ has measure zero. By Lemma 4.3.9 and Remark 4.3.10, $\delta \in (-\epsilon, \epsilon)$ is a common regular value of the functions λ_m^n if and only if $H_t - \delta\mathbf{1}$ has only regular crossings. □

Proof of Proposition 4.3.6. Let us first note that the sum on the right-hand side of (4.9) is finite because the crossings are regular. Consider $t_* \in (0,1)$ such that $\operatorname{Ker}(H_{t_*}) \neq \{0\}$. Choose $a > 0$ such that $\operatorname{spec}(H_{t_*}) \cap [-a, a] = \{0\}$. Then, for $\epsilon > 0$ as in Lemma 4.3.9 and $N = \dim(\operatorname{Ker}(H_{t_*}))$, let

$$\lambda_1, \ldots, \lambda_N : (t_* - \epsilon, t_* + \epsilon) \to (-a, a)$$

be the continuously differentiable functions representing the eigenvalues of H_t in $[-a, a]$ for $t \in (t_* - \epsilon, t_* + \epsilon)$. Because t_* is a regular crossing, $\lambda_n'(t_*) \neq 0$ for $n = 1, \ldots, N$. Thus, there is $0 < \eta < \frac{\epsilon}{2}$ such that $\lambda_n(t) \neq 0$ for $n = 1, \ldots, N$ and $t \in (t_* - 2\eta, t_* + 2\eta) \setminus \{t_*\}$. This implies

$$\operatorname{sgn}(\lambda_n(t_* + \eta)) = -\operatorname{sgn}(\lambda_n(t_* - \eta)) = \operatorname{sgn}(\lambda_n'(t_*)), \quad n = 1, \ldots, N.$$

Taking the sum over all eigenvalues λ_k shows

$$\operatorname{Sig}(\Gamma_{t_*}) = \operatorname{Tr}(\chi_{(0,a]}(H_{t_*+\eta})) - \operatorname{Tr}(\chi_{(0,a]}(H_{t_*-\eta})),$$

where again Theorem II.5.4 in [112] was used, see also Remark 4.3.10. Because the path $t \in (t_* - 2\eta, t_* + 2\eta) \mapsto \chi_{[-a,a]}(H_t)$ is continuous and therefore one can conclude that the path $t \in (t_* - 2\eta, t_* + 2\eta) \mapsto \operatorname{Tr}(\chi_{[-a,a]}(H_t))$ is constant, one has

$$\mathrm{Tr}\big(\chi_{(0,a]}(H_{t_*+\eta})\big) - \mathrm{Tr}\big(\chi_{(0,a]}(H_{t_*-\eta})\big) = \mathrm{Tr}\big(\chi_{[-a,0)}(H_{t_*-\eta})\big) - \mathrm{Tr}\big(\chi_{[-a,0)}(H_{t_*+\eta})\big)$$

and hence

$$\begin{aligned}\mathrm{Sig}(\Gamma_t) &= \frac{1}{2}\big(\mathrm{Tr}\big(\chi_{(0,a]}(H_{t_*+\eta})\big) - \mathrm{Tr}\big(\chi_{[-a,0)}(H_{t_*+\eta})\big) \\ &\quad - \mathrm{Tr}\big(\chi_{(0,a]}(H_{t_*-\eta})\big) + \mathrm{Tr}\big(\chi_{[-a,0)}(H_{t_*-\eta})\big)\big) \\ &= \mathrm{Sf}\big(t \in [t_* - \eta, t_* + \eta] \mapsto H_t\big).\end{aligned}$$

Similarly, for $t_* = 0$ and a, η as above, one has

$$\mathrm{sgn}(\lambda_n(\eta)) = \mathrm{sgn}(\lambda_n'(t_*)), \quad n = 1, \dots, N.$$

Taking the sum over all eigenvalues λ_k shows

$$\mathrm{Sig}(\Gamma_0) = \mathrm{Tr}\big(\chi_{(0,a]}(H_\eta)\big) - \mathrm{Tr}\big(\chi_{[-a,0)}(H_\eta)\big).$$

As $\mathrm{Tr}(\chi_{(0,a]}(H_0)) = \mathrm{Tr}(\chi_{[-a,0)}(H_0)) = 0$, one can conclude

$$\begin{aligned}\frac{1}{2}\,\mathrm{Sig}(\Gamma_0) &= \frac{1}{2}\big(\mathrm{Tr}\big(\chi_{(0,a]}(H_\eta)\big) - \mathrm{Tr}\big(\chi_{[-a,0)}(H_\eta)\big) \\ &\quad - \mathrm{Tr}\big(\chi_{(0,a]}(H_0)\big) + \mathrm{Tr}\big(\chi_{[-a,0)}(H_0)\big)\big) \\ &= \mathrm{Sf}\big(t \in [0, \eta] \mapsto H_t\big).\end{aligned}$$

Analogously, one can show

$$\frac{1}{2}\,\mathrm{Sig}(\Gamma_0) = \mathrm{Sf}\big(t \in [1-\eta, 1] \mapsto H_t\big).$$

The concatenation property of the spectral flow, see item (ii) of Theorem 4.2.1, implies the claim. □

As a first application of the crossing form, let us show how the index of an arbitrary Fredholm operator can be computed as a spectral flow of a suitable path of self-adjoint Fredholm operators. This path is given by a supersymmetric operator constructed from the Fredholm operator and the parameter is then a mass term.

Corollary 4.3.11. *Let $T \in \mathbb{B}(\mathcal{H})$ be a Fredholm operator and set*

$$L_m = \begin{pmatrix} m\mathbf{1} & T^* \\ T & -m\mathbf{1} \end{pmatrix}, \quad m \in \mathbb{R}.$$

Then for all $M > 0$,

$$\mathrm{Ind}(T) = \mathrm{Sf}\big(m \in [-M, M] \mapsto L_m\big).$$

Proof. Note first that the path is clearly differentiable and

$$(L_m)^2 = \begin{pmatrix} m^2\mathbf{1} + T^*T & 0 \\ 0 & TT^* + m^2\mathbf{1} \end{pmatrix} \geq m^2\mathbf{1}_2.$$

Hence there can only be an eigenvalue crossing at $m = 0$ and the crossing form at 0 is

$$\Gamma_0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Moreover,

$$\operatorname{Ker}(L_0) = \operatorname{Ker}(T^*T) \oplus \operatorname{Ker}(TT^*) = \operatorname{Ker}(T) \oplus \operatorname{Ker}(T^*).$$

Hence Proposition 4.3.6 implies the claim. □

The next result provides an integral formula akin to (1.2) in the finite-dimensional case. Proofs of such results can be found in several places, e. g., [55, 194]. For the sake of simplicity, we will assume to be in the generic case where the endpoints are invertible.

Proposition 4.3.12. *Let* $t \in [0,1] \mapsto H_t \in \mathbb{F}\mathbb{B}_{sa}(\mathcal{H})$ *be a continuously differentiable path with invertible endpoints. Moreover, let* $0 < \epsilon < \min\{\|H_0^{-1}\|^{-1}, \|H_1^{-1}\|^{-1}\}$ *be such that* $[-\epsilon, \epsilon] \cap \operatorname{spec}_{ess}(H_t) = \emptyset$ *and let* $g : \mathbb{R} \to [-1,1]$ *be a smooth increasing function which is equal to* -1 *on* $(-\infty, -\epsilon]$ *and equal to* 1 *on* $[\epsilon, \infty)$*. Then*

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2} \int_0^1 dt \operatorname{Tr}(g'(H_t)\partial_t H_t). \tag{4.10}$$

Let us note that there are generalizations of (4.10) to functions g for which $\operatorname{supp}(g')$ touches the essential spectrum of H_t (see Theorem 1.9 in [55]), provided that g is sufficiently regular at these points so that trace class properties can be assured.

Proof. By Lemma 4.3.9 and, as by Theorem II.6.8 in [112], the eigenvalues of a continuously differentiable path of self-adjoint matrices are continuously differentiable provided that one chooses the correct branches at level crossings, we can cover the set $\{(t, \lambda) \in [0,1] \times [-\epsilon, \epsilon] : \lambda \in \operatorname{spec}(H_t)\}$ by finitely many graphs of continuously differentiable functions $\lambda_n : [t_{n,0}, t_{n,1}] \to \mathbb{R}$, $n = 1, \ldots, N$, each defined on some subinterval $[t_{n,0}, t_{n,1}]$ of $[0,1]$. Moreover, $\lambda_n(t_{n,0}) \in \{\pm\epsilon\}$ and $\lambda_n(t_{n,1}) \in \{\pm\epsilon\}$ due to the assumption on ϵ. For $t_0 \in [0,1]$, let $\{\lambda_1(t_0), \ldots, \lambda_{N_{t_0}}(t_0)\}$ be the eigenvalues of H_{t_0} in the interval $[-\epsilon, \epsilon]$. Then, by the remark after Lemma 4.3.9, there is an orthonormal basis of eigenvectors $\phi_1, \ldots, \phi_{N_{t_0}}$ of H_{t_0} corresponding to the eigenvalues $\lambda_1(t_0), \ldots, \lambda_{N_{t_0}}(t_0)$ such that $\langle \phi_n | (\partial_t H)_{t_0} \phi_n \rangle = \lambda_n'(t_0)$. As $\operatorname{Ran}(g'(H_{t_0})) = \operatorname{Ran}(\chi_{[-\epsilon,\epsilon]}(H_{t_0}))$ is spanned by these eigenvectors,

$$\begin{aligned}
\mathrm{Tr}\big(g'(H_{t_0})(\partial_t H)_{t_0}\big) &= \sum_{n=1}^{N_{t_0}} \langle \phi_n | g'(H_{t_0})(\partial_t H)_{t_0} \phi_n \rangle \\
&= \sum_{n=1}^{N_{t_0}} g'(\lambda_n(t_0)) \langle \phi_n | (\partial_t H)_{t_0} \phi_n \rangle = \sum_{n=1}^{N_{t_0}} g'(\lambda_n(t_0)) \lambda_n'(t_0).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2}\int_0^1 dt\, \mathrm{Tr}\big(g'(H_t)(\partial_t H)_t\big) &= \frac{1}{2}\sum_{n=1}^{N} \int_{t_{n,0}}^{t_{n,1}} dt g'(\lambda_n(t))\lambda_n'(t) \\
&= \frac{1}{2}\sum_{n=1}^{N} \big(g(\lambda_n(t_{n,1})) - g(\lambda_n(t_{n,0}))\big).
\end{aligned}$$

Moreover,

$$g(\lambda_n(t_{n,1})) - g(\lambda_n(t_{n,0})) = \begin{cases} 2 & \text{if } \lambda_n(t_{n,1}) = \epsilon \text{ and } \lambda_n(t_{n,0}) = -\epsilon, \\ -2 & \text{if } \lambda_n(t_{n,1}) = -\epsilon \text{ and } \lambda_n(t_{n,0}) = \epsilon, \\ 0 & \text{if } \lambda_n(t_{n,1}) = \lambda_n(t_{n,0}). \end{cases} \tag{4.11}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\int_0^1 dt\, \mathrm{Tr}\big(g'(H_t)(\partial_t H)_t\big) &= \#\{n \in \{1,\dots,N\} : \lambda_n(t_{n,1}) = \epsilon \text{ and } \lambda_n(t_{n,0}) = -\epsilon\} \\
&\quad - \#\{n \in \{1,\dots,N\} : \lambda_n(t_{n,1}) = -\epsilon \text{ and } \lambda_n(t_{n,0}) = \epsilon\}.
\end{aligned}$$

For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and $\epsilon > a_m > 0$ such that $\pm a_m \notin \mathrm{spec}(H_t)$ for $t \in [t_{m-1}, t_m]$ and such that $t \in [t_{m-1}, t_m] \mapsto \chi_{[-a_m, a_m]}(H_t)$ is norm-continuous with constant finite rank for all $m = 1, \dots, M$, the spectral flow is

$$\begin{aligned}
\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) &= \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}\big(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}\big) \\
&= \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}\big(\chi_{(0,a_m]}(H_{t_m}) - \chi_{[-a_m,0)}(H_{t_m}) \\
&\quad - \chi_{(0,a_m]}(H_{t_{m-1}}) + \chi_{[-a_m,0)}(H_{t_{m-1}})\big) \\
&= \frac{1}{2}\sum_{m=1}^{M} \big(\#\{n \in \{1,\dots,N\} : \lambda_n(t_m) \in (0, a_m]\} \\
&\quad - \#\{n \in \{1,\dots,N\} : \lambda_n(t_m) \in [-a_m, 0)\} \\
&\quad - \#\{n \in \{1,\dots,N\} : \lambda_n(t_{m-1}) \in (0, a_m]\} \\
&\quad + \#\{n \in \{1,\dots,N\} : \lambda_n(t_{m-1}) \in [-a_m, 0)\}\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{N}\big(\#\{m \in \{1,\dots,M\} : \lambda_n(t_m) \in (0, a_m]\}\\
&\quad - \#\{m \in \{1,\dots,M\} : \lambda_n(t_m) \in [-a_m, 0)\}\\
&\quad - \#\{m \in \{1,\dots,M\} : \lambda_n(t_{m-1}) \in (0, a_m]\}\\
&\quad + \#\{m \in \{1,\dots,M\} : \lambda_n(t_{m-1}) \in [-a_m, 0)\}\big).
\end{aligned}
$$

As $t \in [t_{m-1}, t_m] \mapsto \lambda_n(t)$ is continuous for all N and $\lambda_n(t) \notin \{\pm a_m\}$ for $t \in [t_{m-1}, t_m]$,

$$
\begin{aligned}
&\#\{m \in \{1,\dots,M\} : \lambda_n(t_m) \in (0, a_m]\}\\
&\quad - \#\{m \in \{1,\dots,M\} : \lambda_n(t_m) \in [-a_m, 0)\}\\
&\quad - \#\{m \in \{1,\dots,M\} : \lambda_n(t_{m-1}) \in (0, a_m]\}\\
&\quad + \#\{m \in \{1,\dots,M\} : \lambda_n(t_{m-1}) \in [-a_m, 0)\}\\
&= \begin{cases} 2 & \text{if } \lambda_n(t_{n,1}) = \epsilon \text{ and } \lambda_n(t_{n,0}) = -\epsilon,\\ -2 & \text{if } \lambda_n(t_{n,1}) = -\epsilon \text{ and } \lambda_n(t_{n,0}) = \epsilon,\\ 0 & \text{if } \lambda_n(t_{n,1}) = \lambda_n(t_{n,0}). \end{cases}
\end{aligned}
\tag{4.12}
$$

Comparing (4.11) and (4.12) and summing over $n = 1, \dots, N$ implies the claim. □

One can now also rewrite the spectral flow as a winding number of a suitable unitary operator, a fact that goes back at least to [194]. The unitary can be interpreted as the image of the path under the K-theoretic exponential map of a suitable exact sequence [171].

Corollary 4.3.13. *Under the same assumptions as in Proposition* 4.3.12*, one has*

$$
\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2\pi\imath}\int_0^1 dt\, \mathrm{Tr}\big(e^{-\imath\pi g(H_t)}\partial_t e^{\imath\pi g(H_t)}\big). \tag{4.13}
$$

Proof. Let us start from DuHamel's formula

$$
\partial_t e^{\imath\pi g(H_t)} = \imath\pi \int_0^1 ds\, e^{(1-s)\imath\pi g(H_t)}\partial_t g(H_t) e^{s\imath\pi g(H_t)}.
$$

Replacing this in the right-hand side of (4.13) and using the cyclicity of the trace, one deduces the claim from the formula in Proposition 4.3.12. □

4.4 Spectral flow for essentially hyperbolic operators

This brief section elaborates on Section 1.6, namely shows how the spectral flow through the imaginary axis can be defined. The suitable Fredholm condition is natural, namely to assume that there is merely discrete spectrum on the imaginary axis. The discrete spectrum consists of isolated eigenvalues of finite algebraic multiplicity where the algebraic multiplicity is the dimension of the Riesz projection of the eigenvalue, see Appendix A.1. Note that for normal operators this coincides with the definition of the discrete spectrum from Section 3.4.

Definition 4.4.1. A bounded operator $A \in \mathbb{B}(\mathcal{H})$ is called essentially hyperbolic if and only if it has only discrete spectrum on the imaginary axis.

Note that a self-adjoint operator is essentially hyperbolic if and only if it is Fredholm. By Theorem IV.5.28 in [112], an operator A is essentially hyperbolic if and only if $A + \imath y\mathbf{1}$ is Fredholm for all $y \in \mathbb{R}$. This implies that if $A \in \mathbb{B}(\mathcal{H})$ is essentially hyperbolic and $K \in \mathbb{K}(\mathcal{H})$ compact, then $A + K$ is also essentially hyperbolic. Moreover, let us stress that an essentially hyperbolic operator has only a finite number of purely imaginary eigenvalues (because it is bounded).

Let now $t \in [0,1] \mapsto A_t$ be a norm-continuous path of essentially hyperbolic operators. Then for $a > 0$ let

$$P_{a,t} = R_{\{x+\imath y: x\in[-a,a], y\in\mathbb{R}\}}(A_t)$$

denote the Riesz projection of A_t on the spectrum with real part in the interval $[-a,a]$. Furthermore, let

$$P^{>}_{a,t} = R_{\{x+\imath y: x\in(0,a], y\in\mathbb{R}\}}(A_t), \quad P^{<}_{a,t} = R_{\{x+\imath y: x\in[-a,0), y\in\mathbb{R}\}}(A_t)$$

be the Riesz projections of A_t on the spectrum with real part in the interval $(0,a]$ and $[-a,0)$, respectively. Note that in general neither of these projections are orthogonal. By adapting the proof of Lemma 4.1.1 and the compactness argument following it, it is possible to choose a finite partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and values $a_m > 0$ for $m = 1,\ldots,M$ such that $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ is continuous and of constant finite rank for all $m = 1,\ldots,M$. Using this partition, the spectral flow of the path $t \in [0,1] \mapsto A_t$ is defined as follows:

Definition 4.4.2. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and values $a_m > 0$ with $m = 1,\ldots,M$ as above, the spectral flow of the path $t \in [0,1] \mapsto A_t$ of essentially hyperbolic operators is defined as

$$\mathrm{Sf}(t \in [0,1] \mapsto A_t) = \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}).$$

Note that all projections involved are finite dimensional so that the trace is finite.

The first task is to verify that this definition is independent of the choice of the partition and the values a_m. This can be shown by following the argument of Theorem 4.1.3. In a similar manner, one can then verify most of the other natural properties of spectral flow, such as homotopy invariance, concatenation, and additivity. Details are not spelled out. Let us note that for a path of self-adjoint Fredholm operators this coincides with Definition 4.1.2. Moreover, if A_t is normal, then by the spectral theorem the real part of an eigenvalue of A_t is given by the eigenvalue of the real part $\mathbb{R}e(A_t) = \frac{1}{2}(A_t + A_t^*)$ of A_t. Therefore, for a path $t \in [0,1] \mapsto A_t$ of normal operators, one has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto A_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto \mathbb{R}e(A_t)\big), \tag{4.14}$$

where the right-hand side is the spectral flow in the sense of Definition 4.1.2. A particular case is that of a path of unitaries $t \in [0,1] \mapsto U_t$ having no essential spectrum at $\imath$ and $-\imath$. Let us stress that the spectral flow (4.14) then does *not* distinguish whether the eigenvalue travels on the upper or lower half of the unit circle, in contradistinction to the spectral flow of essentially gapped unitaries considered in the next section.

4.5 Spectral flow for paths of essentially gapped unitaries

In this section the spectral flow of paths of unitaries not having −1 in the essential spectrum is introduced. By choice of convention, a counterclockwise passage through −1 will be counted as a positive spectral flow, while clockwise passage as a negative spectral flow. Up to extra technical details resulting, the constructions, as well as the properties and proofs, are very similar to those of the spectral flow for paths of unitary matrices as described in Section 1.5. This leads to some repetitions, but as the results are crucial also for the construction of the spectral flow for paths of unbounded Fredholm operators in Section 7.1, we decided to keep full details nevertheless.

If $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ is a norm-continuous path, not necessarily closed, then one can define its spectral flow through −1 as follows: For $a \in [0,\pi)$, the spectral projections are denoted by

$$P_{a,t} = \chi_{\{e^{\imath b}: b\in[\pi-a,\pi+a]\}}(U_t). \tag{4.15}$$

The following lemma is the counterpart to Lemma 4.1.1 for paths of self-adjoint Fredholm operators.

Lemma 4.5.1. *For $U \in \mathbb{FU}(\mathcal{H})$, there are a number $a \in [0,\pi)$ and a neighborhood $\mathcal{N}$ of U in $\mathbb{FU}(\mathcal{H})$ such that $V \mapsto \chi_{\{e^{\imath b}: b\in[\pi-a,\pi+a]\}}(V)$ is a norm-continuous, finite-rank projection-valued function on $\mathcal{N}$.*

Proof. Since −1 is not in the essential spectrum of U, there is an $a \in [0,\pi)$ such that $e^{\imath(\pi\pm a)}$ are not in the spectrum of U and $\chi_{\{e^{\imath b}: b\in[\pi-a,\pi+a]\}}(U)$ is a finite-rank orthogonal projection. Because $e^{\imath(\pi\pm a)}$ are not in the spectrum of U, there exists $\pi - a > \epsilon > 0$ such that $\{e^{\imath(\pi+b)} : b \in [-a-\epsilon,-a] \cup [a,a+\epsilon]\}$ is disjoint from $\mathrm{spec}(U)$. The set

$$\tilde{\mathcal{N}} = \big\{V \in \mathbb{FU}(\mathcal{H}) : \big\{e^{\imath(\pi+b)} : b \in [-a-\epsilon,-a] \cup [a,a+\epsilon]\big\} \cap \mathrm{spec}(V) = \emptyset\big\}$$

is open and on this set the function $V \mapsto \chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(V)$ is norm-continuous as $\chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}$ agrees on $\mathrm{spec}(V)$ with the continuous function $f : \mathbb{S}^1 \to \mathbb{C}$ defined by

$$e^{\imath\varphi} \mapsto \chi_{[\pi-a,\pi+a]}(\varphi) - (\varphi - (\pi + a + \epsilon))\frac{1}{\epsilon}\chi_{[\pi+a,\pi+a+\epsilon]}(\varphi) \\ + (\varphi - (\pi - a - \epsilon))\frac{1}{\epsilon}\chi_{[\pi-a-\epsilon,\pi-a]}(\varphi).$$

Then the subset

$$\mathcal{N} = \{V \in \tilde{\mathcal{N}} : \|\chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(V) - \chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(U)\| < 1\}$$

of $\tilde{\mathcal{N}}$ has the desired properties, as for all unitaries $V \in \mathcal{N}$ the dimension of $\mathrm{Ran}(\chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(V))$ equals $\dim(\mathrm{Ran}(\chi_{\{e^{\imath b}:b\in[\pi-a,\pi+a]\}}(U)))$, which is finite because of the choice of a. □

By compactness and the previous lemma, it is possible to choose a finite partition

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1, \tag{4.16}$$

of $[0, 1]$ and $\pi > a_m > 0$, $m = 1, \ldots, M$, such that

$$t \in [t_{m-1}, t_m] \mapsto P_{a_m,t} \tag{4.17}$$

is norm-continuous with constant finite rank. To define the spectral flow, let us introduce the spectral projections

$$P^{>}_{a,t} = \chi_{\{e^{\imath b}:b\in(\pi,\pi+a]\}}(U_t), \quad P^{<}_{a,t} = \chi_{\{e^{\imath b}:b\in[\pi-a,\pi)\}}(U_t).$$

Definition 4.5.2. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and $a_m \in [0, \pi)$, $m = 1, \ldots, M$ as above, the spectral flow through -1 of the path $t \in [0, 1] \mapsto U_t \in \mathbb{F}\mathbb{U}(\mathcal{H})$ is defined as

$$\mathrm{Sf}(t \in [0, 1] \mapsto U_t) = \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}). \tag{4.18}$$

Note that all appearing spectral projections are finite dimensional so that the trace is finite.

The basic result about the spectral flow is that it is well defined by the above procedure and it is homotopy invariant.

Theorem 4.5.3. *The definition of* $\mathrm{Sf}(t \in [0, 1] \mapsto U_t)$ *is independent of the choice of the partition* $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ *of* $[0, 1]$ *and values* $a_m \in [0, \pi)$ *such that* $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ *is norm-continuous.*

Proof. For each point $t_* \in [t_{m-1}, t_m]$ for $m \in \{1, 2, \dots, M\}$ added to the partition, the number $\mathrm{Tr}(P^>_{a_m,t_*} - P^<_{a_m,t_*})$ is both added and subtracted. Thus $\mathrm{Sf}(t \in [0,1] \mapsto U_t)$ does not change, and therefore the definition of the spectral flow is independent of the choice of the partition.

For $m \in \{1, 2, \dots, M\}$, let us compare a_m to a'_m where $t \in [t_{m-1}, t_m] \mapsto P_{a'_m,t}$ is norm-continuous with constant finite rank. Without loss of generality, one may assume $a'_m > a_m$. As $e^{\imath(\pi \pm a_m)}$ and $e^{\imath(\pi \pm a'_m)}$ are not in the spectrum of U_t for any $t \in [t_{m-1}, t_m]$, it follows that both $t \in [t_{m-1}, t_m] \mapsto P^>_{a'_m,t} - P^>_{a_m,t}$ and $t \in [t_{m-1}, t_m] \mapsto P^<_{a'_m,t} - P^<_{a_m,t}$ are norm-continuous projection-valued functions and hence of constant rank, say $k^>$ and $k^<$. Thus

$$\begin{aligned}
&\mathrm{Tr}(P^>_{a'_m,t_m} - P^<_{a'_m,t_m} - P^>_{a'_m,t_{m-1}} + P^<_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^>_{a'_m,t_m}) - \mathrm{Tr}(P^<_{a'_m,t_m}) - \mathrm{Tr}(P^>_{a'_m,t_{m-1}}) + \mathrm{Tr}(P^<_{a'_m,t_{m-1}}) \\
&\quad = \mathrm{Tr}(P^>_{a_m,t_m}) + k^> - \mathrm{Tr}(P^<_{a_m,t_m}) - k^< - \mathrm{Tr}(P^>_{a_m,t_{m-1}}) - k^> + \mathrm{Tr}(P^<_{a_m,t_{m-1}}) + k^< \\
&\quad = \mathrm{Tr}(P^>_{a_m,t_m} - P^<_{a_m,t_m} - P^>_{a_m,t_{m-1}} + P^<_{a_m,t_{m-1}}).
\end{aligned}$$

Therefore the definition of the spectral flow is independent of the choice of the values $a_m \in [0, \pi)$ such that $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t}$ is norm-continuous. □

The following provides two particularly simple examples of paths of unitaries with nontrivial spectral flow.

Example 4.5.4. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $|n\rangle$, $n \in \mathbb{Z}$. For $k \in \mathbb{Z}$, let us consider the norm-continuous path of unitaries

$$t \in [0,1] \mapsto U_{k,t} = \sum_{n \neq 0} |n\rangle\langle n| + e^{2\pi\imath kt} |0\rangle\langle 0|.$$

Clearly, $\mathrm{Sf}(t \in [0,1] \mapsto U_{k,t}) = k$. For the next example, let $k \geq 1$ and set

$$t \in [0,1] \mapsto U'_{k,t} = \sum_{n \notin \{1,\dots,k\}} |n\rangle\langle n| + \sum_{n \in \{1,\dots,k\}} e^{2\pi\imath t} |n\rangle\langle n|.$$

Then also here $\mathrm{Sf}(t \in [0,1] \mapsto U'_{k,t}) = k$. ◇

Some elementary properties of the spectral flow are collected in the following result.

Theorem 4.5.5. *Let $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ be a norm-continuous path.*

(i) *If* $-1 \notin \mathrm{spec}(U_t)$ *for all* $t \in [0,1]$, *then* $\mathrm{Sf}(t \in [0,1] \mapsto U_t) = 0$.

(ii) *The spectral flow has a concatenation property, namely if $t \in [1,2] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ is a second norm-continuous path, composable to the first one in the sense that the endpoint of the first path is the initial point of the second path, then*

$$\mathrm{Sf}(t \in [0,2] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t) + \mathrm{Sf}(t \in [1,2] \mapsto U_t).$$

(iii) *Changing the orientation of the path leads to a change of the sign of the spectral flow*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto U_{1-t}).$$

(iv) *The spectral flow has a reflection property, namely*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto U_t^*).$$

(v) *The spectral flow has an additivity property under direct sums, namely given a second norm-continuous path $t \in [0,1] \mapsto V_t \in \mathbb{FU}(\mathcal{H}')$,*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t \oplus V_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t) + \mathrm{Sf}(t \in [0,1] \mapsto V_t).$$

(vi) *The spectral flow is invariant under conjugation of the path by another norm-continuous path $t \in [0,1] \mapsto W_t \in \mathbb{U}(\mathcal{H})$ of unitaries*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto W_t U_t W_t^*).$$

Proof. All items follow directly form the definition of the spectral flow. □

The homotopy invariance of the spectral flow of paths in $\mathbb{FU}(\mathcal{H})$ can be proved in the same manner as the homotopy invariance of the spectral flow of paths of self-adjoint Fredholm operators, see Theorem 4.2.2.

Theorem 4.5.6. *Let $t \in [0,1] \mapsto U_t$ and $t \in [0,1] \mapsto U_t'$ be two norm-continuous paths in $\mathbb{FU}(\mathcal{H})$ such that $U_0 = U_0'$, $U_1 = U_1'$ and such that there exists a norm-continuous homotopy between the two paths leaving the endpoints fixed. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \mathrm{Sf}(s \in [0,1] \mapsto U_t').$$

Proof. Let us first note that for $U_0, U_1 \in \mathbb{FU}(\mathcal{H})$ both in the same neighborhood $\mathcal{N}$ of the type given in Lemma 4.5.1 and any path $t \in [0,1] \mapsto U_t$ of unitaries from U_0 to U_1 lying entirely in $\mathcal{N}$, the spectral flow is

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\,\mathrm{Tr}(P_{a,1}^{>} - P_{a,1}^{<} - P_{a,0}^{>} + P_{a,0}^{<}),$$

where $a = a_1$ is chosen as in Lemma 4.5.1 and the partition is trivial, namely $t_0 = 0$ and $t_1 = 1$. Therefore the spectral flow is independent of the path in $\mathcal{N}$ connecting U_0 to U_1.

Let us denote the homotopy between the two paths by $h : [0,1] \times [0,1] \to \mathbb{FU}(\mathcal{H})$, more precisely h is norm-continuous, $h(t,0) = U_t$, $h(t,1) = U_t'$ for all $t \in [0,1]$, as well as $h(0,s) = U_0 = U_0'$ and $h(1,s) = U_1 = U_1'$ for all $s \in [0,1]$. By compactness, one can cover the image of h by a finite set $\{\mathcal{N}_1, \dots, \mathcal{N}_k\}$ of neighborhoods as in Lemma 4.5.1. Then the preimages of these neighborhoods $\{h^{-1}(\mathcal{N}_1), \dots, h^{-1}(\mathcal{N}_k)\}$ form a finite cover of $[0,1] \times [0,1]$. Let $\epsilon_0 > 0$ be its Lebesgue number. Then any subset of $[0,1] \times [0,1]$ of diameter less than ϵ_0 is contained in some element of this finite cover of $[0,1] \times [0,1]$. Thus, if we partition $[0,1] \times [0,1]$ into a grid of squares of diameter less than ϵ_0, then the image of each square will lie entirely within some $\mathcal{N}_l$. By compactness, it is sufficient to show that

$$\mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) = \mathrm{Sf}(t \in [0,1] \mapsto h(t,s''))$$

for $s', s'' \in [0,1]$ with $|s' - s''| < \frac{\epsilon_0}{\sqrt{2}}$. Without loss of generality, one may assume $s' < s''$. For a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ such that $|t_m - t_{m-1}| < \frac{\epsilon_0}{\sqrt{2}}$ for all $m \in \{1, \dots, M\}$, the image $h([t_{m-1}, t_m] \times [s', s''])$ is contained in one of the neighborhoods $\mathcal{N}_l$ for $l \in \{1, \dots, k\}$. Therefore, by the first paragraph of this proof, one has

$$\begin{aligned} &\mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s')) + \mathrm{Sf}(s \in [s', s''] \mapsto h(t_m, s)) \\ &\quad = \mathrm{Sf}(s \in [s', s''] \mapsto h(t_{m-1}, s)) + \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t, s'')) \end{aligned}$$

for all $m \in \{1, \dots, M\}$. In conclusion,

$$\begin{aligned} \mathrm{Sf}(t \in [0,1] \mapsto h(t,s')) &= \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t,s')) \\ &= \sum_{m=1}^{M} \big(\mathrm{Sf}(s \in [s', s''] \mapsto h(t_{m-1}, s)) \\ &\quad + \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t, s'')) \\ &\quad - \mathrm{Sf}(s \in [s', s''] \mapsto h(t_m, s))\big) \\ &= \sum_{m=1}^{M} \mathrm{Sf}(t \in [t_{m-1}, t_m] \mapsto h(t, s'')) \\ &= \mathrm{Sf}(t \in [0,1] \mapsto h(t, s'')), \end{aligned}$$

where the third step follows from $\mathrm{Sf}(s \in [s', s''] \mapsto h(0, s)) = \mathrm{Sf}(s \in [s', s''] \mapsto h(1, s)) = 0$ as the considered paths are constant. □

It is also possible to naturally carry over the concept of crossing form to differentiable paths of essentially unitary operators. This transposes Definition 4.3.4 and Propositions 4.3.5 and 4.3.6 in a suitable manner.

Definition 4.5.7. Let $t' \in [0,1] \mapsto U_{t'} \in \mathbb{FU}(\mathcal{H})$ be a continuously differentiable path. An instant $t \in [0,1]$ is called a crossing for this path if $\mathrm{Ker}(U_t + \mathbf{1}) \neq \{0\}$. Then the crossing form at t is the quadratic form

$$\Gamma_t : \mathrm{Ker}(U_t + \mathbf{1}) \to \mathbb{R}, \quad \Gamma_t(\phi) = -\imath \langle \phi | U_t^* \partial_t U_t \phi \rangle.$$

A crossing is called regular, if Γ_t is nondegenerate.

Note that indeed $(-\imath)U_t^*\partial_t U_t$ is self-adjoint and that the sign is chosen such that counterclockwise passages lead to a positive Γ_t. Again the quadratic form Γ_t will be freely identified with the self-adjoint matrix representing it.

Proposition 4.5.8. *For a continuously differentiable path $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$, there is $\epsilon > 0$ such that*
(i) $t \in [0,1] \mapsto e^{\imath\delta}U_t$ *is a path in* $\mathbb{FU}(\mathcal{H})$ *for all* $\delta \in (-\epsilon,\epsilon)$,
(ii) $t \in [0,1] \mapsto e^{\imath\delta}U_t$ *has only regular crossings for almost every* $\delta \in (-\epsilon,\epsilon)$.

Proposition 4.5.9. *For a continuously differentiable path $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ having only regular crossings,*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2}\,\mathrm{Sig}(\Gamma_0) + \sum_{t\in(0,1)} \mathrm{Sig}(\Gamma_t) + \frac{1}{2}\,\mathrm{Sig}(\Gamma_1). \tag{4.19}$$

The proofs of Propositions 4.5.8 and 4.5.9 are completely analogous to the proofs of Propositions 4.3.5 and 4.3.6 and are therefore not spelled out.

There is also a winding number formula for the spectral flow of loops of essentially gapped unitaries, similar to Corollary 4.3.13.

Proposition 4.5.10. *Let $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ be a closed and continuously differentiable path. Let Σ be an open neighborhood of the joint essential spectrum $\bigcup_{t\in[0,1]} \mathrm{spec}_{\mathrm{ess}}(U_t)$ and $f : \mathbb{S}^1 \to \mathbb{S}^1$ a smooth function homotopic to the identity and such that $f|_\Sigma = 1$, as well as $f(-1) = -1$ and $\imath f'(-1) > 0$. Then $V_t = f(U_t)$ satisfies*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2\pi\imath}\int_0^1 dt\,\mathrm{Tr}(V_t^*\partial_t V_t). \tag{4.20}$$

Proof. By Theorem 4.5.6, $\mathrm{Sf}(t \in [0,1] \mapsto U_t)$ is a homotopy invariant on the set of closed paths in $\mathbb{FU}(\mathcal{H})$. By Proposition 4.5.8, one can deform the path to one with regular crossings so that the spectral flow can be computed by Proposition 4.5.9. These contributions can in turn be obtained as in (the proof of) Proposition 4.3.12 and this leads to the stated formula. Note that by construction $V_t - \mathbf{1} = f(U_t) - \mathbf{1}$ is of finite rank so that the expression is actually the winding number of a finite rank matrix which is homotopy invariant (e. g., Proposition 1.5.12) so that one can deform back from the path with regular crossings to the original one. □

An alternative proof can be given by transforming the spectral flow of essentially gapped unitaries to one of self-adjoint Fredholm operators (by Proposition 4.6.16 below) and then applying Corollary 4.3.13.

Proposition 4.5.11. *Let $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$ be a closed and continuously differentiable path such that $\partial_t U_t$ is trace class. Then*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \frac{1}{2\pi\imath} \int_0^1 dt \, \mathrm{Tr}(U_t^* \partial_t U_t). \tag{4.21}$$

Proof. One can start out with Proposition 4.5.10 and then deform f into the identity by a differentiable homotopy $s \in [0,1] \mapsto f_s$ which is chosen to be

$$f_s(\lambda) = \exp\big((1-s)\,\mathrm{Log}(f(\lambda)) + s\,\mathrm{Log}(\lambda)\big), \quad \lambda \in \mathbb{S}^1,$$

where Log is the principle branch of the logarithm (with cut on $(-\infty, 0]$). Note that $f(-1) = -1$ so that in spite of the discontinuity of Log the map $\lambda \in \mathbb{S}^1 \mapsto f_s(\lambda) \in \mathbb{S}^1$ is continuous. Then define the unitaries $U_{t,s} = f_s(U_t)$ which are continuous (and actually even differentiable) in s and t. Moreover, by functional calculus $U_{t,s} \in \mathbb{F}\mathbb{U}(\mathcal{H})$. Therefore the spectral flow on the left-hand side of (4.21) is constant along this homotopy by Theorem 4.5.6. To show that also the right-hand side does not change, let us first verify that the derivatives $\partial_t U_{t,s}$ are trace class. By DuHamel's formula,

$$\partial_t U_{t,s} = \int_0^1 d\tau (U_{t,s})^{1-\tau}\big((1-s)\partial_t \,\mathrm{Log}(f(U_t)) + s\partial_t \,\mathrm{Log}(U_t)\big)(U_{t,s})^{\tau},$$

which holds as long as U_t has no eigenvalue -1. In case there is such an eigenvalue -1, the associated finite dimensional eigenspace of U_t is separated and only leads to a trace class contribution. From now on, let us hence assume that U_t has no eigenvalue -1. By construction of f, $\partial_t \,\mathrm{Log}(f(U_t))$ is finite dimensional. Moreover, one can prove that $\partial_t \,\mathrm{Log}(U_t)$ is trace class due to the assumption that $\partial_t U_t$ is trace class, by an argument that is now merely sketched: as -1 is not an eigenvalue, one can write $\mathrm{Log}(U_t) = g(U_t)$ for some smooth function g; the smooth function can then be approximated in norm by trigonometric polynomials $p(U_t)$ for which the trace class property of $\partial_t p(U_t)$ is obvious and can be checked to extend to $\partial_t g(U_t)$. A more elegant approach carries out the functional calculus for $g(U_t)$ by a Dynkin–Helffer–Sjorstrand formula for unitaries, e. g., [178]. It states that there is a quasianalytic extension $\widehat{f_s} : \mathbb{C} \to \mathbb{C}$, namely $\widehat{f_s}|_{\mathbb{S}^1} = f_s$ and $(\partial_x + \imath\partial_y)\widehat{f_s}(x + \imath y)|_{\mathbb{S}^1} = 0$, such that

$$U_{t,s} = \frac{1}{2\pi\imath} \int_{\mathbb{R}^2} dx\, dy (\partial_x + \imath\partial_y)\widehat{f_s}(x + \imath y)(x + \imath y - U)^{-1},$$

which then readily allows deducing that $\partial_t g(U_t)$ is trace class. Deriving the above formula with respect to s shows that also $\partial_s\partial_t U_{t,s}$ is trace class and therefore the algebraic computation in the proof of Proposition 1.5.12 shows that also the right-hand side of (4.21) does not change. As both sides are constant along the homotopy, the claim follows from Proposition 4.5.10. □

4.6 Connecting spectral flows of self-adjoints and unitaries

Comparing Sections 4.1 and 4.5, one realizes that the constructions of the spectral flow of paths of bounded self-adjoint Fredholm operators and of paths of essentially gapped unitaries are almost identical. This section shows that, indeed, one can deduce either one from the other, by mapping the bounded self-adjoint Fredholm operators (from $\mathbb{FB}_{sa}(\mathcal{H})$) essentially bijectively onto the essentially gapped unitaries. Many maps with this property might come to mind, e. g., $H \in \mathbb{FB}_{sa}(\mathcal{H}) \mapsto (-1)e^{\imath\pi\|H\|^{-1}H} \in \mathbb{FU}(\mathcal{H})$, but one also realizes that there must be some difficulty involved because $U \in \mathbb{FU}(\mathcal{H})$ has one gap in the essential spectrum, while $H \in \mathbb{FB}_{sa}(\mathcal{H})$ has two of them (one at 0 and one at $\pm\infty$). However, the second of the latter gaps (that at $\pm\infty$) is irrelevant for the spectral flow and should therefore be discarded by a suitable choice of topology. It turns out that this can be achieved in combination with a suitable choice of the map from $\mathbb{FB}_{sa}(\mathcal{H})$ to $\mathbb{FU}(\mathcal{H})$ (which is *not* the one above). Let us note that some of the results of this section also prepare the ground for a definition of the spectral flow of unbounded self-adjoint Fredholm operators (Chapter 7) and, beyond that, for results on the homotopy theory of the set of unbounded self-adjoint operators (Chapter 8).

Let us begin with a preparatory result which normalizes the norm along a given path. Define the map

$$\mathcal{N} : \mathbb{B}(\mathcal{H}) \setminus \{0\} \to \mathbb{B}_1(\mathcal{H}), \quad \mathcal{N}(T) = \frac{1}{\|T\|} T,$$

where for $a > 0$ the closed unit ball of bounded operators is defined by

$$\mathbb{B}_a(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H}) : \|T\| \le a\}.$$

Note that $\mathbb{FB}_{sa}(\mathcal{H}) \subset \mathbb{B}(\mathcal{H}) \setminus \{0\}$ and that, along a given path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$, the norm $\|H_t\|$ is uniformly bounded from below by compactness of the interval. Furthermore, one clearly has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto \mathcal{N}(H_t)\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big).$$

Therefore from now on we only consider paths in

$$\mathbb{FB}_{1,sa}(\mathcal{H}) = \mathbb{FB}_{sa}(\mathcal{H}) \cap \mathbb{B}_1(\mathcal{H}),$$

which is a subset of

$$\mathbb{B}_{1,sa}(\mathcal{H}) = \mathbb{B}_{sa}(\mathcal{H}) \cap \mathbb{B}_1(\mathcal{H}).$$

Let us define a map

$$\mathcal{G} : [-1,1] \to \mathbb{S}^1,$$

by

$$\mathcal{G}(\lambda) = (2\lambda^2 - 1) - 2\imath\lambda(1-\lambda^2)^{\frac{1}{2}}.$$

Alternative expressions are given by

$$\mathcal{G}(\lambda) = \frac{\lambda - \imath(1-\lambda^2)^{\frac{1}{2}}}{\lambda + \imath(1-\lambda^2)^{\frac{1}{2}}}.$$

The map $\mathcal{G}$ is one-to-one except at the boundaries where one has $\mathcal{G}(-1) = \mathcal{G}(1) = 1$. It extends to a continuous map

$$\mathcal{G} : (\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N) \to (\mathbb{U}(\mathcal{H}), \mathcal{O}_N)$$

by

$$\mathcal{G}(H) = 2H^2 - \mathbf{1} - 2\imath H(\mathbf{1} - H^2)^{\frac{1}{2}}. \tag{4.22}$$

Note that $\mathcal{G}(H)$ can also be written as

$$\mathcal{G}(H) = \big(H - \imath(\mathbf{1} - H^2)^{\frac{1}{2}}\big)\big(H + \imath(\mathbf{1} - H^2)^{\frac{1}{2}}\big)^{-1}, \tag{4.23}$$

and both factors are unitary so that $\mathcal{G}(H)$ is indeed in $\mathbb{U}(\mathcal{H})$. Also let us point out that if H has both -1 and 1 as eigenvalues, they are both mapped to 1. In particular, for every symmetry Q (namely, $Q^* = Q$ and $Q^2 = \mathbf{1}$) one has $\mathcal{G}(Q) = \mathbf{1}$. Hence $\mathcal{G}$ is *not* a bijection.

Remark 4.6.1. From (4.23) one can readily check that $\mathcal{G}$ is the continuous extension of the map $\mathcal{C} \circ \mathcal{F}^{-1}$ where $\mathcal{C}$ is the Cayley transform defined in (6.14) and $\mathcal{F}$ the bounded transform which is shown to be invertible in Theorem 6.1.4, see Chapter 6. More precisely, $\mathcal{C} \circ \mathcal{F}^{-1}(H)$ is defined if neither -1 nor 1 is an eigenvalue of H, and in this case $\mathcal{F}^{-1}(H) = H(\mathbf{1} - H^2)^{-\frac{1}{2}}$, which is a (possibly unbounded) self-adjoint operator. ◇

Next note that $\mathcal{G}(0) = -1$, so that $\mathcal{G}$ maps any eigenvalue at 0 to -1. Therefore

$$\mathcal{G}\big(\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})\big) \subset \mathbb{F}\mathbb{U}(\mathcal{H}). \tag{4.24}$$

Restricting to the subset of essential symmetries, one has

$$\mathcal{G}\big(\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})\big) \subset \mathbb{U}^{\mathrm{C}}(\mathcal{H}). \tag{4.25}$$

Now for a norm-continuous path $t \in [0,1] \mapsto H_t \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$, also the associated path $t \in [0,1] \mapsto (\mathcal{G} \circ \mathcal{N})(H_t) \in \mathbb{U}(\mathcal{H})$ is norm-continuous. The following result is thus obvious by the spectral mapping theorem.

Proposition 4.6.2. *For a norm-continuous path $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$, one has*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto (\mathcal{G} \circ \mathcal{N})(H_t)),$$

where the right-hand side is the spectral flow of essentially gapped unitaries.

The same statement also holds if one replaces $\mathcal{G}$ by other maps, for example the map $H \in \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}) \mapsto (-1)e^{\imath\pi H} \in \mathbb{FU}(\mathcal{H})$ already mentioned above. Hence one can easily reduce the spectral flow of a path of self-adjoint Fredholm operators to a spectral flow of essentially gapped unitary operators.

In the following, we will show that the converse is also true, even though this is a more delicate issue because – as noted above – one has to create an extra gap. One of the main tools will be a pseudometric on the set $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ that will be introduced in the next lemma. It will be compared with the standard operator norm topology $\mathcal{O}_N$ induced by the operator norm metric on the bounded operators $\mathbb{B}(\mathcal{H})$ which we denote by

$$d_N(T_0, T_1) = \|T_0 - T_1\|, \quad T_0, T_1 \in \mathbb{B}(\mathcal{H}).$$

Henceforth we use both notations $(\mathbb{B}(\mathcal{H}), d_N)$ and $(\mathbb{B}(\mathcal{H}), \mathcal{O}_N)$ depending on whether we want to stress the metric structure when discussing the continuity of maps on $\mathbb{B}(\mathcal{H})$. Similarly, we will proceed with other spaces below.

Lemma 4.6.3. *On $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ the formula*

$$d_E(H_0, H_1) = \max\{\|H_0^2 - H_1^2\|, \|H_0(\mathbf{1} - H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1} - H_1^2)^{\frac{1}{2}}\|\}$$

defines a pseudometric. The induced topology $\mathcal{O}_E$ is weaker than the norm topology $\mathcal{O}_N$. More precisely,

$$d_E(H_0, H_1) \le 2\sqrt{2} d_N(H_0, H_1)^{\frac{1}{2}}, \quad H_0, H_1 \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}). \tag{4.26}$$

Finally, for $a < 1$ one has

$$d_N(H_0, H_1) \le \frac{1 + \sqrt{2}}{1 - a^2} d_E(H_0, H_1)^{\frac{1}{2}}, \quad H_0, H_1 \in \mathbb{B}_{a,\mathrm{sa}}(\mathcal{H}), \tag{4.27}$$

where

$$\mathbb{B}_{a,\mathrm{sa}}(\mathcal{H}) = \mathbb{B}_a(\mathcal{H}) \cap \mathbb{B}_{\mathrm{sa}}(\mathcal{H}).$$

Hence d_N and d_G induce the same topology on $\mathbb{B}_{a,\mathrm{sa}}(\mathcal{H})$ for all $a < 1$, so that $(\mathbb{B}_{a,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ and $(\mathbb{B}_{a,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$ are the same topological spaces.

Let us note that $d_E(Q_0, Q_1) = 0$ for all symmetries Q_0 and Q_1, so that d_E is indeed degenerate on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ and hence only defines a pseudometric. The topology induced by this pseudometric d_E was introduced by Joachim in [108] who called it the *extended gap topology*, for reasons that will become apparent in Proposition 6.1.7 of Section 6.1. We will follow this terminology.

Proof of Lemma 4.6.3. First of all, let us note that the triangle inequality and the symmetry are indeed satisfied.

Next let us prove (4.26). For the second term in d_E, let us start with

$$\begin{aligned}
&\|H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}}\| \\
&\quad \le \|H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - H_0(\mathbf{1}-H_1^2)^{\frac{1}{2}}\| + \|H_0(\mathbf{1}-H_1^2)^{\frac{1}{2}} - H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}}\| \\
&\quad \le \|(\mathbf{1}-H_0^2)^{\frac{1}{2}} - (\mathbf{1}-H_1^2)^{\frac{1}{2}}\| + \|H_0 - H_1\|.
\end{aligned}$$

For the first summand, recall the fact (Proposition A.2.2) that for two nonnegative operators $A \ge 0$, $B \ge 0$ and $\alpha \in (0,1)$, one has $\|A^\alpha - B^\alpha\| \le \|A-B\|^\alpha$. Hence

$$\|H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}}\| \le \|H_0^2 - H_1^2\|^{\frac{1}{2}} + \|H_0 - H_1\|.$$

As $0 \le H_0^2 \le \mathbf{1}$ and $0 \le H_1^2 \le \mathbf{1}$, one also has $-\mathbf{1} \le H_0^2 - H_1^2 \le \mathbf{1}$ so that $\|H_0^2 - H_1^2\| \le 1$ and thus $\|H_0^2 - H_1^2\| \le \|H_0^2 - H_1^2\|^{\frac{1}{2}}$. Therefore

$$d_E(H_0, H_1) \le \|H_0^2 - H_1^2\|^{\frac{1}{2}} + \|H_0 - H_1\|.$$

Finally,

$$\|H_0^2 - H_1^2\| \le \|H_0(H_0 - H_1)\| + \|(H_0 - H_1)H_1\| \le 2\|H_0 - H_1\|,$$

so that

$$d_E(H_0, H_1) \le \sqrt{2}\|H_0 - H_1\|^{\frac{1}{2}} + \|H_0 - H_1\| \le 2\sqrt{2} d_N(H_0, H_1)^{\frac{1}{2}},$$

because $d_N(H_0, H_1) = \|H_0 - H_1\| \le 2$ for $H_0, H_1 \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$.

Finally, as to the last inequality, let us use $\|(\mathbf{1} - H^2)^{-1}\| \le (1-a^2)^{-1}$ for $H \in \mathbb{B}_{a,\mathrm{sa}}(\mathcal{H})$. Then

$$\begin{aligned}
d_N(H_0, H_1) &\le \|(H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}})(\mathbf{1}-H_0^2)^{-\frac{1}{2}}\| \\
&\quad + \|H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}}((\mathbf{1}-H_0^2)^{-\frac{1}{2}} - (\mathbf{1}-H_1^2)^{-\frac{1}{2}})\| \\
&\le d_E(H_0, H_1)(1-a^2)^{-\frac{1}{2}} + \|(\mathbf{1}-H_0^2)^{-\frac{1}{2}} - (\mathbf{1}-H_1^2)^{-\frac{1}{2}}\| \\
&\le d_E(H_0, H_1)(1-a^2)^{-\frac{1}{2}} + \|(\mathbf{1}-H_0^2)^{-1} - (\mathbf{1}-H_1^2)^{-1}\|^{\frac{1}{2}} \\
&\le d_E(H_0, H_1)(1-a^2)^{-\frac{1}{2}} + (1-a^2)^{-1}\|H_0^2 - H_1^2\|^{\frac{1}{2}}
\end{aligned}$$

$$\leq \left(\sqrt{2}(1-a^2)^{-\frac{1}{2}} + (1-a^2)^{-1}\right) d_E(H_0, H_1)^{\frac{1}{2}},$$

implying the claim. □

As already pointed out, d_E has vanishing distance between symmetries and hence does not distinguish the eigenspaces of eigenvalues −1 and 1. On the other hand, it will follow from Lemma 4.6.6 below (or alternatively from Proposition 6.1.7 which provides an extension to not necessarily self-adjoint operators) that d_E restricted to the subset

$$\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{Ker}(H^2 - \mathbf{1}) = \{0\}\} \tag{4.28}$$

is indeed a metric. Let us note that the set $\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$ later on in Chapter 6 will play a prominent role because it is the image of the unbounded self-adjoint operators under the bounded transform. The upper index 0 indicates that neither 1 nor −1 is an eigenvalue of $H \in \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$. It is, however, also possible to obtain a metric on a larger set of operator classes, namely it is natural to introduce the following equivalence relation.

Definition 4.6.4. Let $H_0, H_1 \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$. Then $H_0 \sim H_1$ if and only if

$$H_0\chi_{(-1,1)}(H_0) = H_1\chi_{(-1,1)}(H_1) \quad \text{and} \quad \chi_{(-1,1)}(H_0) = \chi_{(-1,1)}(H_1). \tag{4.29}$$

The quotient $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})/\sim$ will be denoted by $\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H})$.

Let us stress that $\sim$ is indeed an equivalence relation so that the quotient is well defined. Furthermore, $H_0 \sim H_1$ is equivalent to

$$\mathrm{Ker}(H_0 + \mathbf{1}) \oplus \mathrm{Ker}(H_0 - \mathbf{1}) = \mathrm{Ker}(H_1 + \mathbf{1}) \oplus \mathrm{Ker}(H_1 - \mathbf{1})$$

and that the operators H_0 and H_1 coincide on the orthogonal complement of this subspace. Using spectral calculus, one can immediately reformulate the equivalence relation as follows:

Lemma 4.6.5. *For any $H \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$, there exists a unique orthogonal projection P, as well as unique $H^0 \in \mathbb{B}^0_{1,\mathrm{sa}}(P\mathcal{H})$ and $Q \in \mathbb{U}_{\mathrm{sa}}((\mathbf{1}-P)\mathcal{H})$, such that*

$$H = PH^0 \oplus (\mathbf{1} - P)Q.$$

Then $H_0 = P_0H_0^0 \oplus (\mathbf{1} - P_0)Q_0$ and $H_1 = P_1H_1^0 \oplus (\mathbf{1} - P_1)Q_1$ satisfy $H_0 \sim H_1$ if and only if $P_0 = P_1$ and $H_0^0 = H_1^0$.

It is always possible to choose a representative for a class $[H]_\sim$ from the set

$$\{H \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{Ker}(H + \mathbf{1}) = \{0\}\}. \tag{4.30}$$

While this provides a concrete representation of $\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H})$, it is not helpful when dealing with topological issues. Let us now analyze how the relation $\sim$ is connected to the extended gap metric. By the next lemma, $d_E^{\sim} : \mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H}) \times \mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H}) \to \mathbb{R}$ can be defined by

$$d_E^{\sim}([H_0]_\sim, [H_1]_\sim) = d_E(H_0, H_1),$$

and $d_E^{\sim}$ is actually a metric on $\mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$. The corresponding (quotient) topology on $\mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$ will be denoted by $\mathcal{O}_E^{\sim}$. The tilde on $d_E^{\sim}$ and $\mathcal{O}_E^{\sim}$ will be dropped whenever it is clear from the context.

Lemma 4.6.6. *The relation $\sim$ is the equivalence relation induced by the extended gap metric d_E, namely for $H_0, H_1 \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ one has $H_0 \sim H_1$ if and only if $d_E(H_0, H_1) = 0$. Hence $d_E^{\sim}$ is a metric on $\mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$. Furthermore, if $h : \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ is a class map with respect to $\sim$, namely there exists $\widetilde{h} : \mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H}) \to \mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$ such that $[h(H)]_\sim = \widetilde{h}([H]_\sim)$, then h is continuous with respect to $\mathcal{O}_E$ if and only if $\widetilde{h}$ is such with respect to $\mathcal{O}_E^{\sim}$.*

Proof. Let us first assume that $H_0 \sim H_1$. Then $H_0^2 = H_1^2$ and as

$$\overline{\mathrm{Ran}((\mathbf{1} - H_0^2)^{\frac{1}{2}})} = \mathrm{Ran}(\chi_{(-1,1)}(H_0)) = \mathrm{Ran}(\chi_{(-1,1)}(H_1)) = \overline{\mathrm{Ran}((\mathbf{1} - H_1^2)^{\frac{1}{2}})}$$

and H_0 and H_1 coincide on this subspace by assumption

$$H_0(\mathbf{1} - H_0^2)^{\frac{1}{2}} = H_1(\mathbf{1} - H_1^2)^{\frac{1}{2}}.$$

Therefore $d_E(H_0, H_1) = 0$. Conversely, assume that $d_E(H_0, H_1) = 0$. Then $H_0^2 = H_1^2$ and therefore $\chi_{\{-1,1\}}(H_0) = \chi_{\{1\}}(H_0^2) = \chi_{\{1\}}(H_1^2) = \chi_{\{-1,1\}}(H_1)$ so that also the complements satisfy $\chi_{(-1,1)}(H_0) = \chi_{(-1,1)}(H_1)$. Moreover,

$$\begin{aligned} 0 &= H_0(\mathbf{1} - H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1} - H_1^2)^{\frac{1}{2}} \\ &= H_0(\mathbf{1} - H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1} - H_0^2)^{\frac{1}{2}} \\ &= (H_0 - H_1)(\mathbf{1} - H_0^2)^{\frac{1}{2}} \end{aligned}$$

and therefore H_0 and H_1 coincide on $\mathrm{Ran}(\chi_{(-1,1)}(H_0)) = \overline{\mathrm{Ran}((\mathbf{1} - H_0^2)^{\frac{1}{2}})}$. This shows $H_0 \sim H_1$. Clearly, this implies that $d_E^{\sim}$ is a metric on $\mathbb{B}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$. The last claim is a general fact from topology that is merely noted for later use. □

One can also consider the relation $\sim$ on the subset $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H}) \subset \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$. Clearly, Lemma 4.6.5 applies to this case. One can, moreover, analyze $\sim$ in the representation formula given in Proposition 3.6.5:

Lemma 4.6.7. *Consider $H_0, H_1 \in \mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H})$ given by their representations as in Proposition* 3.6.5*:*

$$H_0 = Q_0 - K_{0,+} + K_{0,-}, \quad H_1 = Q_1 - K_{1,+} + K_{1,-}.$$

Then $H_0 \sim H_1$ *if and only if* $K_{0,\pm} = K_{1,\pm}$ *and* Q_0 *and* Q_1 *differ only on the orthogonal complement of* $\mathrm{Ran}(-K_{0,+} + K_{0,-})$.

Proof. Let us first assume that $H_0 \sim H_1$. Then $\mathrm{Ran}(\chi_{(-1,1)}(H_0)) = \mathrm{Ran}(\chi_{(-1,1)}(H_1))$ and, as $Q_0 = \mathrm{sgn}(H_0) + \chi(H_0 = 0)$ and $Q_1 = \mathrm{sgn}(H_1) + \chi(H_1 = 0)$ by construction (see the proof of Proposition 3.6.5), Q_0 and Q_1 coincide on $\mathrm{Ran}(\chi_{(-1,1)}(H_1))$. Because

$$\begin{aligned}\overline{\mathrm{Ran}(-K_{0,+} + K_{0,-})} &= \mathrm{Ran}\big(\chi_{(-1,1)}(H_0)\big)\\ &= \mathrm{Ran}\big(\chi_{(-1,1)}(H_1)\big) = \overline{\mathrm{Ran}(-K_{1,+} + K_{1,-})},\end{aligned}$$

where the first and last equalities hold by construction of $-K_{j,+} + K_{j,-}$ (see the proof of Proposition 3.6.5), Q_0 and Q_1 differ only on $(\mathrm{Ran}(-K_{0,+} + K_{0,-}))^{\perp}$. By assumption, H_0 and H_1 coincide on

$$\mathrm{Ran}\big(\chi_{(-1,1)}(H_0)\big) = \mathrm{Ker}(-K_{0,+} + K_{0,-})^{\perp} = \mathrm{Ker}(-K_{1,+} + K_{1,-})^{\perp}.$$

Therefore $-K_{0,+} + K_{0,-} = -K_{1,+} + K_{1,-}$ and

$$\begin{aligned}-K_{0,+} &= (-K_{0,+} + K_{0,-})\chi(-K_{0,+} + K_{0,-} \le 0)\\ &= (-K_{1,+} + K_{1,-})\chi(-K_{1,+} + K_{1,-} \le 0) = -K_{1,+}.\end{aligned}$$

Analogously, $K_{0,-} = K_{1,-}$. Conversely, if $K_{0,+} = K_{1,+}$, $K_{0,-} = K_{1,-}$, and Q_0 and Q_1 differ only on the orthogonal complement of $\mathrm{Ran}(-K_{0,+} + K_{0,-})$, then H_0 and H_1 differ only on $\mathrm{Ker}(H_0 + \mathbf{1}) \oplus \mathrm{Ker}(H_0 - \mathbf{1}) = \mathrm{Ker}(H_1 + \mathbf{1}) \oplus \mathrm{Ker}(H_1 - \mathbf{1})$ and therefore $H_0 \sim H_1$. □

Next let us consider the map $\mathcal{G} : \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{U}(\mathcal{H})$ defined by (4.22). As both eigenspaces of -1 and 1 are mapped to 1, it is a class map and therefore descends to $\mathcal{G}^{\sim} : \mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{U}(\mathcal{H})$ defined by

$$\mathcal{G}^{\sim}\big([H]_{\sim}\big) = \mathcal{G}(H).$$

Theorem 4.6.8. *The map* $\mathcal{G}^{\sim}$ *is a bi-Lipshitz-continuous homeomorphism between the metric spaces* $(\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H}), d^{\sim}_{E})$ *and* $(\mathbb{U}(\mathcal{H}), d_N)$.

Proof. Let us first give an explicit expression for the inverse of $\mathcal{G}^{\sim}$. For this purpose, a root $\mathcal{R} : \mathbb{U}(\mathcal{H}) \to \mathbb{U}(\mathcal{H})$ of a unitary is needed. It can be obtained by spectral calculus using the function $r(e^{\imath\varphi}) = e^{\imath\frac{\varphi}{2}}$ where $\varphi \in (0, 2\pi]$ so that $r(1) = -1$. Then $\mathcal{R}(U) = r(U)$. Clearly, the map $U \mapsto \mathcal{R}(U)$ is not continuous on $(\mathbb{U}(\mathcal{H}), \mathcal{O}_N)$. Nevertheless, let us set

$$(\mathcal{G}^{\sim})^{-1}(U) = \left[-\frac{1}{2}(\mathcal{R}(U) + \mathcal{R}(U)^*)\right].$$

Hence on the spectral parameters, $\mathcal{G}^{-1}(e^{\imath\varphi}) = -\cos(\frac{\varphi}{2})$ and $1 - \mathcal{G}^{-1}(e^{\imath\varphi})^2 = \sin(\frac{\varphi}{2})^2$. Thus $2\mathcal{G}^{-1}(e^{\imath\varphi})^2 - 1 = \cos(\varphi)$ and $\mathcal{G}^{-1}(e^{\imath\varphi})(1 - \mathcal{G}^{-1}(e^{\imath\varphi})^2)^{\frac{1}{2}} = -\frac{1}{2}\sin(\varphi)$, and one deduces

$$\begin{aligned}(\mathcal{G}^{\sim}\circ(\mathcal{G}^{\sim})^{-1})(U) &= 2(\mathcal{G}^{\sim})^{-1}(U)^2 - \mathbf{1} - 2\imath(\mathcal{G}^{\sim})^{-1}(U)(\mathbf{1}-(\mathcal{G}^{\sim})^{-1}(U)^2)^{\frac{1}{2}}\\ &= \frac{1}{2}(U+U^*) - 2\imath(-1)\frac{1}{4\imath}(U-U^*) = U,\end{aligned}$$

so indeed $\mathcal{G}^{\sim}\circ(\mathcal{G}^{\sim})^{-1} = \mathrm{id}$. This implies that $\mathcal{G}^{\sim}$ is surjective. As $d_G^{\sim}$ is a metric on $\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H})$, one directly checks that $\mathcal{G}^{\sim}$ is injective and therefore $(\mathcal{G}^{\sim})^{-1}\circ\mathcal{G}^{\sim} = \mathrm{id}$.

To check the Lipshitz-continuity of $(\mathcal{G}^{\sim})^{-1} : (\mathbb{U}(\mathcal{H}), d_N) \to (\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H}), d_E^{\sim})$, it will be used that $d_E^{\sim}([H_0],[H_1]) = d_E(H_0,H_1)$ and thus one can focus on bounding the two contributions in d_E:

$$\begin{aligned}\|\mathcal{G}^{-1}(U_0)^2 - \mathcal{G}^{-1}(U_1)^2\| &= \frac{1}{4}\|(2\mathbf{1}+U_0+U_0^*) - (2\mathbf{1}+U_1+U_1^*)\|\\ &\le \frac{1}{2}d_N(U_0,U_1),\end{aligned}$$

and

$$\begin{aligned}&\|\mathcal{G}^{-1}(U_0)(\mathbf{1}-\mathcal{G}^{-1}(U_0)^2)^{\frac{1}{2}} - \mathcal{G}^{-1}(U_1)(\mathbf{1}-\mathcal{G}^{-1}(U_1)^2)^{\frac{1}{2}}\|\\ &\quad = \frac{1}{4}\|(U_0-U_0^*) - (U_1-U_1^*)\| \le \frac{1}{2}d_N(U_0,U_1).\end{aligned}$$

Therefore $d_E(\mathcal{G}^{-1}(U_0),\mathcal{G}^{-1}(U_1)) \le \frac{1}{2}d_N(U_0,U_1)$ and the Lipshitz constant is $\frac{1}{2}$. Moreover,

$$\begin{aligned}&d_N(\mathcal{G}(H_0),\mathcal{G}(H_1))\\ &\quad = \|2H_0^2 - \mathbf{1} - 2\imath H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - (2H_1^2 - \mathbf{1} - 2\imath H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}})\|\\ &\quad \le 2\|H_0^2 - H_1^2\| + 2\|H_0(\mathbf{1}-H_0^2)^{\frac{1}{2}} - H_1(\mathbf{1}-H_1^2)^{\frac{1}{2}}\|\\ &\quad \le 4d_E(H_0,H_1),\end{aligned}$$

showing the Lipshitz-continuity of $\mathcal{G}^{\sim} : (\mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H}), d_E^{\sim}) \to (\mathbb{U}(\mathcal{H}), d_N)$. □

Remark 4.6.9. The above proof gives an explicit construction of the inverse map $(\mathcal{G}^{\sim})^{-1} : \mathbb{U}(\mathcal{H}) \to \mathbb{B}^{\sim}_{1,\mathrm{sa}}(\mathcal{H})$. Let us here provide another formula for $(\mathcal{G}^{\sim})^{-1}(U)$ for a unitary $U \in \mathbb{U}(\mathcal{H})$. Recall that $\Re e(U) = \frac{1}{2}(U+U^*)$ and $\Im m(U) = \frac{1}{2\imath}(U-U^*)$ are the real and imaginary part of U. Let P denote the projection onto $\mathrm{Ker}(U-\mathbf{1})^{\perp} = \mathrm{Ker}(\Re e(U)-\mathbf{1})^{\perp}$. Then

$$(\mathcal{G}^{\sim})^{-1}(U) = \big[-2^{-\frac{1}{2}}\Im m(U)(\mathbf{1}-\Re e(U))^{-\frac{1}{2}}P + Q(\mathbf{1}-P)\big], \tag{4.31}$$

where $Q \in \mathbb{U}_{\mathrm{sa}}((\mathbf{1}-P)\mathcal{H})$ is an arbitrary symmetry on $\mathrm{Ran}(\mathbf{1}-P)$, see the representation in Lemma 4.6.5. Note that (4.31) is well defined because by construction $\mathbf{1}-\Re e(U)$ is invertible on the range of P. To verify this formula, simply note that $\Re e(U) = 2H^2 - \mathbf{1}$ if $H = \mathcal{G}^{-1}(U)$ so that $\mathbf{1}-\Re e(U) = 2(\mathbf{1}-H^2)$ and thus $(\mathbf{1}-H^2)^{\frac{1}{2}} = 2^{-\frac{1}{2}}(\mathbf{1}-\Re e(U))^{\frac{1}{2}}$. Replacing this in $\Im m(U) = -2H(\mathbf{1}-H^2)^{\frac{1}{2}}$ shows $\Im m(U) = -2^{\frac{1}{2}}H(\mathbf{1}-\Re e(U))^{\frac{1}{2}}$ which in turn specifies

H on the range of P to be $HP = -2^{-\frac{1}{2}}\mathfrak{Im}(U)(\mathbf{1}-\mathfrak{Re}(U))^{-\frac{1}{2}}P$. The complement $\mathrm{Ker}(U-\mathbf{1})$ is mapped to the eigenspace of H with eigenvalues ± 1, leading to Q. ◇

Let us note several immediate corollaries of Theorem 4.6.8. The first concerns the set

$$\mathbb{U}^0(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : \mathrm{Ker}(U-\mathbf{1}) = \{0\}\}.$$

It appears in Section 6.3 as the image of the unbounded self-adjoint operators under the Cayley transform. Furthermore, recall the definition (4.28) of the set $\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$. For each of its elements $H \in \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$, the equivalence class $[H]$ contains only one point and thus one can naturally identify $\mathcal{G}^\sim$ with $\mathcal{G}$ on this set.

Corollary 4.6.10. *The map $\mathcal{G}$ defined by (4.22) is a bi-Lipshitz-continuous homeomorphism between the metric spaces $(\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), d_E)$ and $(\mathbb{U}^0(\mathcal{H}), d_N)$.*

It is also possible to restrict the homeomorphism $\mathcal{G}^\sim$ to the subset

$$\mathbb{FB}^\sim_{1,\mathrm{sa}}(\mathcal{H}) = \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})/\sim.$$

As in (4.30), one can concretely identify $\mathbb{FB}^\sim_{1,\mathrm{sa}}(\mathcal{H})$ with the set

$$\{H \in \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{Ker}(H+\mathbf{1}) = \{0\}\}.$$

Due to (4.24), one then deduces a result that will be of relevance in Chapter 6.

Corollary 4.6.11. *The map $\mathcal{G}^\sim$ is a bi-Lipshitz-continuous homeomorphism between the metric spaces $(\mathbb{FB}^\sim_{1,\mathrm{sa}}(\mathcal{H}), d^\sim_E)$ and $(\mathbb{FU}(\mathcal{H}), d_N)$.*

Just as Theorem 4.6.8 implies Corollary 4.6.10, one deduces the following fact from Corollary 4.6.11 upon restriction to the subset

$$\mathbb{FU}^0(\mathcal{H}) = \{U \in \mathbb{FU}(\mathcal{H}) : \mathrm{Ker}(U-\mathbf{1}) = \{0\}\} = \mathbb{FU}(\mathcal{H}) \cap \mathbb{U}^0(\mathcal{H}).$$

Corollary 4.6.12. *The map $\mathcal{G}$ defined by (4.22) is a bi-Lipshitz-continuous homeomorphism between the metric spaces $(\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), d_E)$ and $(\mathbb{FU}^0(\mathcal{H}), d_N)$.*

Also the following subset of $\mathbb{FB}^\sim_{1,\mathrm{sa}}(\mathcal{H})$ will be of relevance:

$$\mathbb{FB}^{\mathrm{C},\sim}_{1,\mathrm{sa}}(\mathcal{H}) = \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})/\sim.$$

Corollary 4.6.13. *The map $\mathcal{G}^\sim$ is a bi-Lipshitz-continuous homeomorphism between the metric spaces $(\mathbb{FB}^{\mathrm{C},\sim}_{1,\mathrm{sa}}(\mathcal{H}), d^\sim_E)$ and $(\mathbb{U}^{\mathrm{C}}(\mathcal{H}), d_N)$.*

Remark 4.6.14. The formula (4.31) for the inverse of $\mathcal{G}^\sim$ can be further rewritten in the case of Corollary 4.6.13. If $U = \mathbf{1} + K \in \mathbb{U}^{\mathrm{C}}(\mathcal{H})$ in the representation (3.8), then

$$(\mathcal{G}^{\sim})^{-1}(\mathbf{1}+K) = \left[-\mathfrak{Im}(K|K|^{-1})P + (\mathbf{1}-P)Q\right], \tag{4.32}$$

where as above P is the projection onto $\mathrm{Ker}(U-\mathbf{1})^{\perp} = \mathrm{Ker}(K^*K)^{\perp}$ and $Q \in \mathbb{U}_{\mathrm{sa}}((\mathbf{1}-P)\mathcal{H})$. This follows from (4.31) by a direct computation using the relations in (3.8). ◇

Now let us consider

$$\mathbb{FB}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H}) = \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H}) \cap \mathbb{FB}^{0}_{1,\mathrm{sa}}(\mathcal{H}),$$
$$\mathbb{U}^{\mathrm{C},0}(\mathcal{H}) = \mathbb{U}^{\mathrm{C}}(\mathcal{H}) \cap \mathbb{U}^{0}(\mathcal{H}).$$

The set $\mathbb{FB}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H})$ can be seen as a subset of $\mathbb{FB}^{\mathrm{C},\sim}_{1,\mathrm{sa}}(\mathcal{H})$ because the classes of $[H]_{\sim}$ of $H \in \mathbb{FB}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H})$ only contain one element. Hence again one can identify $\mathcal{G}^{\sim}$ with $\mathcal{G}$.

Corollary 4.6.15. *The map $\mathcal{G}$ is a bi-Lipshitz-continuous homeomorphism between the metric spaces $(\mathbb{FB}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H}), d_E)$ and $(\mathbb{U}^{\mathrm{C},0}(\mathcal{H}), d_N)$.*

Corollary 4.6.11 allows proving a counterpart of Proposition 4.6.2. For a norm-continuous path $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$, the path $t \in [0,1] \mapsto (\mathcal{G}^{\sim})^{-1}(U_t)$ is continuous in $(\mathbb{FB}^{\sim}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$. One can choose representatives $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$, namely $[H_t] = (\mathcal{G}^{\sim})^{-1}(U_t)$, but the map $t \in [0,1] \mapsto H_t$ need not to be norm-continuous. Nevertheless, the low-lying spectrum of H_t is continuous and this is sufficient to define the spectral flow.

Proposition 4.6.16. *For a norm-continuous path $t \in [0,1] \mapsto U_t \in \mathbb{FU}(\mathcal{H})$, one has*

$$\mathrm{Sf}(t \in [0,1] \mapsto U_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t),$$

where $[H_t] = (\mathcal{G}^{\sim})^{-1}(U_t)$ and the spectral flow on the right-hand side is independent of the choice of representative of $(\mathcal{G}^{\sim})^{-1}(U_t)$.

Proof. The path $t \in [0,1] \mapsto \mathcal{G}^{-1}(U_t)$ is continuous with respect to $\mathcal{O}^{\sim}_E$ by Corollary 4.6.11. Next let us note that the operator $H_t^2 \in \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$ is independent of the choice of the representative H_t. By definition of d_E, its square $t \in [0,1] \mapsto H_t^2$ is then norm continuous. Therefore, for $t_0 \in [0,1]$ and $a \geq 0$ sufficiently small and such that $a^2 \notin \mathrm{spec}(H_{t_0}^2)$, the finite-dimensional projections $\chi_{[0,a^2]}(H_t^2)$ are norm-continuous in t and of constant finite rank on an open subinterval of $[0,1]$ containing t_0. Hence also the path $t \mapsto H_t\chi_{[0,a^2]}(H_t^2)$ is independent of the representative H_t and continuous with respect to d_E. Therefore, by Lemma 4.6.3, it is also norm-continuous on this subinterval and thus also the eigenvalues are continuous. This allows constructing the spectral flow as in Section 4.1.1, even though there may not exist a norm-continuous path $t \in [0,1] \mapsto H_t$ of representatives. That this spectral flow coincides with $\mathrm{Sf}(t \in [0,1] \mapsto U_t)$ directly follows from the spectral mapping theorem. □

5 Fredholm pairs and their index

This chapter is about Fredholm pairs of projections and their index, a concept introduced by Kato [112], and independently also by Brown, Douglas, and Fillmore [42] where the index is called essential codimension. Section 5.2 gives different characterizations of Fredholm pairs of projections and collects basic facts about them, to a large extend following the influential work by Avron, Seiler, and Simon [18]. It avoids to use the orthogonality of the projections, and supplementary aspects linked to self-adjointness are then regrouped in Section 5.3. Section 5.4 then accesses the same Fredholm concept from the point of view of symmetry operators which provides yet another formula for the index which readily allows connecting it to the spectral flow later on. Section 5.5 focusses on a special type of Fredholm pairs where one projection is unitary conjugate to the other. Sections 5.6, 5.7, and 5.8 provide several formulas connecting the spectral flow to the index of a Fredholm pair of projections. In particular, the spectral flow of a path of self-adjoint Fredholm operators is expressed as the sum of indices of pairs of projections. The chapter concludes by introducing the relative Morse index in Section 5.9 and giving a formula for the spectral flow as sum of relative Morse indices, as in the work of Fitzpatrick, Pejsachowicz, and Recht [84].

5.1 Projections and orthogonal projections

This short section merely reviews some well-known basic definitions and facts about projections, frames, and the action of invertible operators thereon.

Definition 5.1.1. Let $P \in \mathbb{B}(\mathcal{H})$.
(i) P is called a projection if $P^2 = P$.
(ii) A projection P is called orthogonal if, moreover, $P = P^*$.
(iii) A projection P is called finite or finite dimensional if $\dim(\mathrm{Ran}(P)) < \infty$.
(iv) A projection P is called proper if $\dim(\mathrm{Ker}(P)) = \dim(\mathrm{Ran}(P)) = \infty$.
(v) The complementary projection of a projection P is $\mathbf{1}-P$ and it is denoted by $P^\perp = \mathbf{1}-P$.

The set of all proper orthogonal projections on $\mathcal{H}$ is denoted by $\mathbb{P}(\mathcal{H})$.

In a large part but not nearly all of the literature, projections are called idempotent (as all powers are the same) and orthogonal projections are called projections. We hope that the reader can get accustomed to Definition 5.1.1. From $P = P^2$ one gets $\|P\| \leq \|P\|^2$ so that $\|P\| \geq 1$ for every projection $P \neq 0$. However, nonvanishing orthogonal projections always have norm 1.

There is a tight connection between closed subspaces $\mathcal{E} \subset \mathcal{H}$ of $\mathcal{H}$ and orthogonal projections. In fact, for any $P \in \mathbb{P}(\mathcal{H})$ the range $\mathrm{Ran}(P) = \mathrm{Ker}(\mathbf{1} - P)$ is a closed subspace, and given a closed subspace, there is always an associated orthogonal projection. For this reason, $\mathbb{P}(\mathcal{H})$ is also called the (closed proper) Grassmannian of $\mathcal{H}$. Furthermore,

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given a projection P (not necessarily orthogonal), one can always construct two naturally associated orthogonal projections: the range projection P_{R} onto $\mathrm{Ran}(P) = \mathrm{Ker}(\mathbf{1}-P)$ and the kernel projection P_{K} onto $\mathrm{Ker}(P) = \mathrm{Ran}(\mathbf{1}-P)$.

Proposition 5.1.2. *The range and kernel projection associated to a projection P satisfy*

$$\mathrm{Ran}(P_{\mathrm{K}}) \cap \mathrm{Ran}(P_{\mathrm{R}}) = \{0\}, \quad \mathrm{Ran}(P_{\mathrm{K}}) + \mathrm{Ran}(P_{\mathrm{R}}) = \mathcal{H}, \tag{5.1}$$

and are given by

$$P_{\mathrm{R}} = P(P^*P)^{-1}P^*, \quad P_{\mathrm{K}} = P^{\perp}((P^{\perp})^*P^{\perp})^{-1}(P^{\perp})^*. \tag{5.2}$$

Then one has

$$P = P_{\mathrm{R}}(P_{\mathrm{K}}^{\perp}P_{\mathrm{R}})^{-1}P_{\mathrm{K}}^{\perp}. \tag{5.3}$$

Inversely, given two orthogonal projections P_{R} and P_{K} satisfying (5.1)*, formula* (5.3) *defines a projection with range projection P_{R} and kernel projection P_{K}.*

Proof. Both claims in (5.1) follow from the well-known fact that each vector $\phi \in \mathcal{H}$ can be uniquely decomposed into $\phi = \phi_{\mathrm{R}}+\phi_{\mathrm{K}}$ with $P\phi_{\mathrm{R}} = \phi_{\mathrm{R}}$ and $P\phi_{\mathrm{K}} = 0$. In the first formula of (5.2), note that P^*P is not an invertible operator, however, it maps $\mathrm{Ker}(P)^{\perp} = \mathrm{Ran}(P^*)$ bijectively onto $\mathrm{Ran}(P^*)$. Hence $P^*P : \mathrm{Ran}(P^*) \to \mathrm{Ran}(P^*)$ is an invertible operator by the inverse mapping theorem. Thus $P(P^*P)^{-1}P^*$ is well defined, and one readily sees that it is indeed an orthogonal projection, with range given by $\mathrm{Ran}(P)$. The formula for P_{K} can be verified in the same manner. To check (5.3), one notes that

$$P_{\mathrm{K}}^{\perp}P_{\mathrm{R}} : \mathrm{Ker}(P_{\mathrm{R}})^{\perp} = \mathrm{Ran}(P_{\mathrm{R}}) \to \mathrm{Ran}(P_{\mathrm{K}}^{\perp}) = \mathrm{Ker}(P_{\mathrm{K}})$$

is a bijection. Indeed, if $\phi \in \mathrm{Ran}(P_{\mathrm{R}})$, then $P_{\mathrm{R}}\phi = \phi$ so that $0 = P_{\mathrm{K}}^{\perp}P_{\mathrm{R}}\phi = P_{\mathrm{K}}^{\perp}\phi$ implies $\phi \in \mathrm{Ker}(P_{\mathrm{K}}^{\perp}) = \mathrm{Ran}(P_{\mathrm{K}})$, and hence $\phi = 0$ by (5.1); moreover, if $\psi = P_{\mathrm{K}}^{\perp}\psi \in \mathrm{Ran}(P_{\mathrm{K}}^{\perp})$, then by (5.1) one can decompose uniquely $\psi = \psi^{\perp} + P_{\mathrm{R}}\phi$ with $\psi^{\perp} \in \mathrm{Ran}(P_{\mathrm{K}}) = \mathrm{Ker}(P_{\mathrm{K}}^{\perp})$ and some $\phi = P_{\mathrm{R}}\phi \in \mathrm{Ran}(P_{\mathrm{R}})$, so that $\psi = P_{\mathrm{K}}^{\perp}(\psi^{\perp} + P_{\mathrm{R}}\phi) = P_{\mathrm{K}}^{\perp}P_{\mathrm{R}}\phi$. Again the inverse mapping theorem implies that $(P_{\mathrm{K}}^{\perp}P_{\mathrm{R}})^{-1} : \mathrm{Ran}(P_{\mathrm{K}}^{\perp}) \to \mathrm{Ran}(P_{\mathrm{R}})$ is well defined, and then one can check that (5.3) holds. The last claim follows from the above argument. □

Remark 5.1.3. There is an alternative way to write out the range projection, namely it will be checked that

$$P_{\mathrm{R}} = P(\mathbf{1} - (P - P^*)^2)^{-1}P^*.$$

Note that $-(P - P^*)^2 = (P - P^*)^*(P - P^*) \geq 0$, which implies that the inverse exists. Moreover, an explicit computation shows that P commutes with $\mathbf{1} - (P - P^*)^2$ and thus so does P^*. Furthermore, $PP^*P = P(\mathbf{1} - (P - P^*)^2)$. Now let P_K' denote the right-hand side

$P(\mathbf{1}-(P-P^*)^2)^{-1}P^*$. Combining the above facts allows to check $(P'_{\mathrm{R}})^2 = P'_{\mathrm{R}}$ and, clearly, also $(P'_{\mathrm{R}})^* = P'_{\mathrm{R}}$. As $\mathrm{Ran}(P'_{\mathrm{R}}) = \mathrm{Ran}(P)$, this implies that $P'_{\mathrm{R}} = P_{\mathrm{R}}$. Similarly,

$$P_{\mathrm{K}} = (\mathbf{1}-P)\big(\mathbf{1}-(P-P^*)^2\big)^{-1}(\mathbf{1}-P)^*,$$

which follows from the above applied to $\mathbf{1}-P$, or can be checked in the same manner. ◇

Corollary 5.1.4. *Every projection can be connected to its range projection within the set of projections.*

Proof. Note that $P_{\mathrm{R}} = P(P^*P)^{-1}P^*$ satisfies $P_{\mathrm{R}}P = P$ and $PP_{\mathrm{R}} = P_{\mathrm{R}}$. Therefore one readily checks that

$$t \in [0,1] \mapsto P_t = (1-t)P + tP_{\mathrm{R}}$$

is indeed a path of projections connecting P to P_{R}. □

Next let us introduce the concept of a frame. While this was already used in Chapter 2, let us here give a precise definition for the case of infinite-dimensional Hilbert spaces.

Definition 5.1.5. A frame is a bounded injective linear map $\Phi : \mathfrak{h} \to \mathcal{H}$ with closed range, from an auxiliary Hilbert space $\mathfrak{h}$ into $\mathcal{H}$. The frame is called normalized if $\Phi^*\Phi = \mathbf{1}_{\mathfrak{h}}$. Furthermore, $\Phi^\perp : \mathfrak{h}' \to \mathcal{H}$ denotes a frame with $\mathrm{Ran}(\Phi^\perp) = \mathrm{Ran}(\Phi)^\perp$.

Given a frame Φ, one can always associate an orthogonal projection onto its range by

$$P = \Phi\big(\Phi^*\Phi\big)^{-1}\Phi^*. \tag{5.4}$$

Note that this is well defined because $\Phi^*\Phi : \mathfrak{h} \to \mathfrak{h}$ is invertible. Let us then also say that Φ is a frame for P. If, moreover, Φ is normalized, the formula reduces to $P = \Phi\Phi^*$. One particular frame for P is always given by choosing $\mathfrak{h} = \mathrm{Ran}(P)$ and Φ the embedding. Another standard way to construct normalized frames, say for an infinite-dimensional projection P, is to choose an orthonormal basis $(\phi_n)_{n\geq 1}$ of $\mathrm{Ran}(P)$ and then set $\mathfrak{h} = \ell^2(\mathbb{N})$ and

$$\Phi = \sum_{n\geq 1} |\phi_n\rangle\langle n|.$$

Note, however, that there are many frames for a given P. Indeed, given a frame Φ for P and any invertible map $a \in \mathbb{B}(\mathfrak{h})$, also Φa is a frame for P. Furthermore, if Φ is normalized and $u \in \mathbb{B}(\mathfrak{h})$ is unitary, also Φu is normalized. Let us also note that, clearly, $\Phi^*\Phi^\perp = 0$. Finally, $(\Phi, \Phi^\perp) : \mathfrak{h} \oplus \mathfrak{h}' \to \mathcal{H}$ is an isomorphism which is unitary if both Φ

and $\Phi^\perp$ are normalized. Now one can also use frames to write out an arbitrary (not necessarily orthogonal) projection, analogous to Proposition 5.1.2. The proof is essentially the same and therefore skipped.

Proposition 5.1.6. *Let P be a projection and Φ_{R} and Φ_{K} be frames for P_{R} and P_{K}. Then*

$$P = \Phi_{\mathrm{R}}\big((\Phi_{\mathrm{K}}^\perp)^*\Phi_{\mathrm{R}}\big)^{-1}(\Phi_{\mathrm{K}}^\perp)^*. \tag{5.5}$$

Inversely, given two frames Φ_{R} and Φ_{K} satisfying

$$\mathrm{Ran}(\Phi_{\mathrm{K}}) \cap \mathrm{Ran}(\Phi_{\mathrm{R}}) = \{0\}, \quad \mathrm{Ran}(\Phi_{\mathrm{K}}) + \mathrm{Ran}(\Phi_{\mathrm{R}}) = \mathcal{H}, \tag{5.6}$$

formula (5.5) *defines a projection with range and kernel projection given as in* (5.4).

To illustrate the use of frames, let us prove a result that will be used several times later on.

Proposition 5.1.7. *If P_0 and P_1 are proper orthogonal projections, then there exists a unitary U such that $P_1 = U^*P_0U$.*

Proof. Let Φ_0 and Φ_1 be normalized frames for P_0 and P_1, respectively. Then

$$V = \Phi_1\Phi_0^*$$

is a partial isometry from $\mathrm{Ran}(P_0)$ to $\mathrm{Ran}(P_1)$, namely $V^*V = P_0$ and $VV^* = P_1$. Similarly, let W be a partial isometry satisfying $W^*W = \mathbf{1} - P_0$ and $WW^* = \mathbf{1} - P_1$. Multiplying two of these identities shows $VV^*WW^* = 0$ so that $V^*W = 0$ and $V^*VW^*W = 0$ so that $VW^* = 0$. Hence $U = V^* + W^*$ is a unitary because $UU^* = V^*V + W^*W = P_0 + \mathbf{1} - P_0 = \mathbf{1}$ and $U^*U = \mathbf{1}$. By construction, $P_1 = U^*P_0U$. □

In the remainder of this section, the action of an invertible operator $T \in \mathbb{G}(\mathcal{H})$ on projections will be introduced and studied. Let us first begin with the action on an orthogonal projection P. Then the formula

$$T \cdot P = (TPT^*)(TPT^*)^{-2}(TPT^*) \tag{5.7}$$

is well defined because $TPT^* : \mathrm{Ran}(TP) \to \mathrm{Ran}(TP)$ is invertible (even though TPT^* is not invertible as an operator on all $\mathcal{H}$). Clearly, $T \cdot P$ is the orthogonal projection onto

$$T\,\mathrm{Ran}(P) = \mathrm{Ran}(T \cdot P),$$

and one has

$$\begin{aligned}\mathrm{Ker}(T \cdot P) &= \{\phi \in \mathcal{H} : PT^*\phi = 0\}\\ &= (T^*)^{-1}\{T^*\phi \in \mathcal{H} : PT^*\phi = 0\}\end{aligned}$$

$$= (T^*)^{-1} \operatorname{Ker}(P).$$

Moreover, (5.7) defines a group action of the group $\mathbb{G}(\mathcal{H})$ on the set of orthogonal projections, namely one has $S\cdot(T\cdot P) = (ST)\cdot P$ for $S, T \in \mathbb{G}(\mathcal{H})$. Let us also note that for the subgroup $\mathbb{U}(\mathcal{H}) \subset \mathbb{G}(\mathcal{H})$ of unitary operators, the action reduces to $U \cdot P = UPU^* = UPU^{-1}$. Another property worth mentioning is that

$$(T \cdot P)^\perp = (T^*)^{-1} \cdot P^\perp. \tag{5.8}$$

Indeed, both sides are orthogonal projections, and one has

$$\operatorname{Ran}((T \cdot P)^\perp) = \operatorname{Ker}(T \cdot P) = (T^*)^{-1} \operatorname{Ker}(P) = (T^*)^{-1} \operatorname{Ran}(P^\perp).$$

Furthermore, if $P = \Phi(\Phi^*\Phi)^{-1}\Phi^*$ is given in terms of a frame as in (5.4), then $T\Phi$ is a frame for $T \cdot P$ and therefore

$$T \cdot P = T\Phi(\Phi^* T^* T\Phi)^{-1}\Phi^* T^*. \tag{5.9}$$

Based on this, there is an alternative way to verify (5.8) by checking that $T\cdot P$ is orthogonal to $(T^*)^{-1} \cdot P^\perp$.

While it is not possible to extend the action (5.7) to projections that are not orthogonal, one can define another group action of $\mathbb{G}(\mathcal{H})$ by $(T, P) \mapsto TPT^{-1}$. One readily checks that this is indeed well defined and is a group action on all projections. When restricted to the unitary group $\mathbb{U}(\mathcal{H}) \subset \mathbb{G}(\mathcal{H})$, this action coincides with (5.7). In general, however, it does not conserve the orthogonality of projections. This second action will be used at several instances below, e. g., Proposition 5.2.9.

5.2 Characterization of Fredholm pairs of projections

The definition of Fredholm pairs of projections and many of the results of this section and the next sections are due to Kato [112, Chapter IV.4] and Avron, Seiler, and Simon [18], see also [3].

Definition 5.2.1. Let (P_0, P_1) be a pair of projections and consider the operator

$$A : \operatorname{Ran}(P_0) \to \operatorname{Ran}(P_1)$$

defined by

$$A\phi = P_1 P_0 \phi, \quad \phi \in \operatorname{Ran}(P_0).$$

Then (P_0, P_1) is a Fredholm pair of projections if and only if A is a Fredholm operator. The index of a Fredholm pair (P_0, P_1) of projections is defined by

$$\operatorname{Ind}(P_0, P_1) = \operatorname{Ind}(A).$$

For the case of two orthogonal projections, it will be shown in Proposition 5.3.2 below that for a Fredholm pair the projections P_0 and $\mathbf{1}-P_1$ are complementary up to finite-dimensional defects, in the sense that $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1)$ has finite codimension and $\mathrm{Ran}(P_0) \cap \mathrm{Ran}(\mathbf{1} - P_1)$ is finite dimensional. Of course, in interesting cases both $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ are infinite dimensional. If they are both finite dimensional, then the index is simply given by the difference of the dimensions of the ranges, as shown next.

Proposition 5.2.2. *Let P_0 and P_1 be two finite-dimensional projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair of projections with index*

$$\mathrm{Ind}(P_0, P_1) = \dim(\mathrm{Ran}(P_0)) - \dim(\mathrm{Ran}(P_1)).$$

Proof. Consider the linear operator $A = P_1 P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$. By the rank theorem,

$$\dim(\mathrm{Ran}(P_0)) = \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ran}(A)).$$

Moreover,

$$\dim(\mathrm{Ran}(A)) + \dim(\mathrm{Ran}(A)^\perp) = \dim(\mathrm{Ran}(P_1)),$$

where the orthogonal complement is taken in the Hilbert space $\mathrm{Ran}(P_1)$. Hence from the definition of the index,

$$\begin{aligned}
\mathrm{Ind}(A) &= \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*)) \\
&= \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ran}(A)^\perp) \\
&= \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ran}(A)) - (\dim(\mathrm{Ran}(A)^\perp) + \dim(\mathrm{Ran}(A))) \\
&= \dim(\mathrm{Ker}(A)) - (\dim(\mathrm{Ran}(P_1)) - \dim(\mathrm{Ran}(A))) \\
&= \dim(\mathrm{Ran}(P_0)) - \dim(\mathrm{Ran}(P_1)),
\end{aligned}$$

concluding the proof. □

Remark 5.2.3. Let us suppose, just for this remark, that $\mathcal{H} = \mathbb{C}^{2N}$ is finite dimensional with Krein quadratic form $J = \mathrm{diag}(\mathbf{1}_N, -\mathbf{1}_N)$ and that P_0 and P_1 project on two J-Lagrangian subspaces, as defined in Chapter 2. Then

$$\dim(\mathrm{Ran}(P_0)) = N = \dim(\mathrm{Ran}(P_1))$$

and hence $\mathrm{Ind}(P_0, P_1) = 0$ by Proposition 5.2.2. This remains true in the infinite-dimensional setting, see Proposition 9.4.7. ◇

The most elementary example of a Fredholm pair arises as follows:

Proposition 5.2.4. *Let P_0 and P_1 be two projections such that $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$ is compact. Then (P_0, P_1) and (P_1, P_0) are both Fredholm pairs of projections.*

Proof. Set $A_0 = P_0P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_0)$ and $A_1 = P_1P_0P_1 : \mathrm{Ran}(P_1) \to \mathrm{Ran}(P_1)$. Then

$$A_0 = P_0 + P_0(P_1 - P_0)P_0 = \mathbf{1}_{\mathrm{Ran}(P_0)} + P_0(P_1 - P_0)P_0$$

is a compact perturbation of the identity on $\mathrm{Ran}(P_0)$ and hence a Fredholm operator (with vanishing index). Hence $\mathrm{Ker}(A) \subset \mathrm{Ker}(A_0)$ is finite dimensional. Similarly, A_1 is a Fredholm operator so that also $\mathrm{Ran}(A) \supset \mathrm{Ran}(A_1)$ has finite codimension. Hence (P_0, P_1) is a Fredholm pair. For (P_1, P_0), one argues in the same way, namely exchanges P_0 and P_1 in the above. □

Remark 5.2.5. In general, it is not true that the Fredholm property of (P_0, P_1) implies that also (P_1, P_0) is a Fredholm pair. Let us illustrate this with an example on an infinite-dimensional Hilbert space of the form $\mathcal{H} \oplus \mathcal{H}$. Two projections are given by

$$P_0 = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix}, \quad P_1 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}\Big|_{\mathrm{Ran}(P_0)}$$

is surjective, namely $\mathrm{Ran}(A) = \mathrm{Ran}(P_1)$. As $\mathrm{Ker}(A) = \{0\}$, A is Fredholm and therefore (P_0, P_1) is a Fredholm pair (with vanishing index). On the other hand, $P_0P_1 = 0$ and therefore (P_1, P_0) is *not* a Fredholm pair. ◇

Next let us come to some basic properties of Fredholm pairs of projections. First of all, Fredholm pairs have a natural transformation property under invertible linear maps, namely if (P_0, P_1) is a Fredholm pair of projections and $T \in \mathbb{G}(\mathcal{H})$ an invertible operator, then (TP_0T^{-1}, TP_1T^{-1}) is a Fredholm pair of projections and

$$\mathrm{Ind}(TP_0T^{-1}, TP_1T^{-1}) = \mathrm{Ind}(P_0, P_1). \tag{5.10}$$

Secondly, one has the following concatenation formula for the index of Fredholm pairs.

Proposition 5.2.6. *Suppose given three projections P_0, P_1 and P_2 such that (P_0, P_1) is a Fredholm pair and $P_2 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact (or vice versa). Then also (P_1, P_2) and (P_0, P_2) are Fredholm pairs and*

$$\mathrm{Ind}(P_0, P_2) = \mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2). \tag{5.11}$$

Proof. By Proposition 5.2.4, (P_1, P_2) is a Fredholm pair. Now consider the equality

$$P_2P_0 = (P_2)^2P_0 = P_2P_1P_0 + P_2(P_2 - P_1)P_0 = (P_2P_1)(P_1P_0) + P_2(P_2 - P_1)P_0,$$

as operators from $\mathrm{Ran}(P_0)$ to $\mathrm{Ran}(P_2)$. Then $(P_2P_1)(P_1P_0) : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is a concatenation of two Fredholm operators with index given by $\mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2)$. As

$P_2(P_2 - P_1)P_0$ is compact by hypothesis, also $P_2P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is a Fredholm operator with the same index by Theorem 3.3.4. □

Next let us show a stability result for the index of Fredholm pairs.

Proposition 5.2.7. *Let $t \in [0,1] \mapsto P_0(t)$ and $t \in [0,1] \mapsto P_1(t)$ be norm-continuous paths of projections such that $(P_0(t), P_1(t))$ is a Fredholm pair for every $t \in [0,1]$. Then $t \in [0,1] \mapsto \mathrm{Ind}(P_0(t), P_1(t))$ is constant.*

Proof. It is clearly sufficient to prove local constancy of the index. Hence let us fix some $t_0 \in [0,1]$ and consider the paths

$$B_j(t) = \mathbf{1} - P_j(t) + P_j(t_0), \quad t \in [0,1], \quad j = 0,1.$$

As the set of invertibles is open in $\mathbb{B}(\mathcal{H})$, there exists a neighborhood N of t_0 such that $B_j(t)$ is invertible for $t \in N$. Consequently, the restrictions $C_j(t) = B_j(t)|_{\mathrm{Ran}(P_j(t))}$ map $\mathrm{Ran}(P_j(t))$ bijectively onto $\mathrm{Ran}(P_j(t_0))$. Thus

$$C_1(t) \circ P_1(t) \circ C_0(t)^{-1} : \mathrm{Ran}\big(P_0(t_0)\big) \to \mathrm{Ran}\big(P_1(t_0)\big),$$

are Fredholm operators with index

$$\mathrm{Ind}\big(C_1(t) \circ P_1(t) \circ C_0(t)^{-1}\big) = \mathrm{Ind}\big(P_1(t)|_{\mathrm{Ran}(P_0(t))}\big) = \mathrm{Ind}\big(P_0(t), P_1(t)\big),$$

where the last step is the definition. On the left-hand side, one has the index of a path of Fredholm operators on the same Hilbert space, which is constant by Theorem 3.3.4. Thus also the index on the right-hand side is constant in t. □

If two projections with compact difference are sufficiently close to each other, then one can actually construct a path of Fredholm pairs connecting the pair to a trivial pair.

Proposition 5.2.8. *Let $P_0, P_1 \in \mathbb{B}(\mathcal{H})$ be projections with $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ satisfying*

$$\|P_0 - P_1\| < \|\mathbf{1} - 2P_1\|^{-1}.$$

Then there is a path $t \in [0,1] \mapsto (P_0, P_1(t))$ of Fredholm pairs with $P_1(1) = P_1$ and $P_1(0) = P_0$. Along this path the index vanishes.

Proof. The path is constructed just as in Proposition 4.3.2 in [23]. Let us set

$$M = \frac{1}{2}(\mathbf{1} - 2P_0)(\mathbf{1} - 2P_1) + \frac{1}{2}\mathbf{1}.$$

Then $\mathbf{1} - M = (\mathbf{1} - 2P_0)(P_1 - P_0) = (P_0 - P_1)(\mathbf{1} - 2P_1)$ is a compact operator. Moreover, by hypothesis this operator satisfies $\|\mathbf{1} - M\| < 1$. Therefore $M = \mathbf{1} - (\mathbf{1} - M)$ is invertible with inverse given by the Neumann series. Furthermore, one has $P_0M = P_0P_1 = MP_1$ so that $P_0 = MP_1M^{-1}$. Now set

$$M_t = (1-t)M + t\mathbf{1} = \mathbf{1} - (1-t)(\mathbf{1} - M), \quad t \in [0,1],$$

and $P_1(t) = M_t P_1 M_t^{-1}$. This is a path of projections connecting $P_1(1) = P_1$ to $P_1(0) = P_0$, and one has that $P_0 - P_1(t) = (P_0 - P_1) - (P_1(t) - P_1) \in \mathbb{K}(\mathcal{H})$ because $\mathbf{1} - M_t \in \mathbb{K}(\mathcal{H})$. By Proposition 5.2.4, one concludes that indeed $(P_0, P_1(t))$ is a Fredholm pair. The last claim follows from Proposition 5.2.7. □

The construction in the proof of Proposition 5.2.8 leads to another important result on the lifting of paths of idempotents that is at the root of numerous arguments later on. It does not pend on Fredholm properties.

Proposition 5.2.9. *Let $t \in [0,1] \mapsto P_t$ be a path of projections. Then there exists a path $t \in [0,1] \mapsto M_t$ of invertibles such that*

$$P_t = M_t P_0 M_t^{-1}.$$

Proof. Let us begin by setting

$$M_t = \frac{1}{2}(\mathbf{1} - 2P_0)(\mathbf{1} - 2P_t) + \frac{1}{2}\mathbf{1}.$$

As above, $\mathbf{1} - M_t = (P_0 - P_t)(\mathbf{1} - 2P_t)$ so that $\|\mathbf{1} - M_t\| \le \|P_0 - P_t\|\|\mathbf{1} - 2P_t\|$. As $t \mapsto P_t$ is norm continuous, this implies that M_t is invertible for t sufficiently small, say $t \in [0, t_1]$. Therefore $t \in [0, t_1] \mapsto M_t$ and $t \in [0, t_1] \mapsto M_t^{-1}$ are both continuous and, as in the proof of Proposition 5.2.8, one has $P_t = M_t P_0 M_t^{-1}$. Also note that $M_0 = \mathbf{1}$. Next one can start out with the path $t \in [t_1, 1] \mapsto P_t$ and construct in the same manner a $t_2 > t_1$ and a path $t \in [t_1, t_2] \mapsto M'_t$ such that $P_t = M'_t P_{t_1} (M'_t)^{-1}$ and $M'_{t_1} = \mathbf{1}$. By replacing, one gets $P_t = M'_t M_{t_1} P_0 (M'_t M_{t_1})^{-1}$. Thus setting $M_t = M'_t M_{t_1}$ for $t \in [t_1, t_2]$ completes the construction on the interval $[0, t_2]$. Iterating the procedure a final number of times completes the proof. □

Next let us turn to formulas for the index of a Fredholm pair of projections. The index of a Fredholm operator can be computed by the Calderon–Fedosov formula given in Theorem 3.3.7, provided that some trace class conditions hold. The following statement spells this out for a Fredholm pair of projections.

Proposition 5.2.10. *Let $P_0, P_1 \in \mathbb{B}(\mathcal{H})$ be projections and $n \in \mathbb{N}$ such that*

$$P_0 - P_0 P_1 P_0 \in \mathcal{L}^n(\mathrm{Ran}(P_0)), \quad P_1 - P_1 P_0 P_1 \in \mathcal{L}^n(\mathrm{Ran}(P_1)).$$

Then (P_0, P_1) is a Fredholm pair of projections, and for all $m \ge n$ one has

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}\big((P_0 - P_0 P_1 P_0)^m\big) - \mathrm{Tr}\big((P_1 - P_1 P_0 P_1)^m\big).$$

Proof. Let us apply Theorem 3.3.7 to the operator $A = P_1P_0|_{\mathrm{Ran}(P_0)} : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ with pseudoinverse $B = P_0P_1|_{\mathrm{Ran}(P_1)} : \mathrm{Ran}(P_1) \to \mathrm{Ran}(P_0)$. Due to the hypothesis, Theorem 3.3.7 then implies that A is Fredholm with index

$$\mathrm{Ind}(A) = \mathrm{Tr}\big((P_0 - P_0P_1P_0)^m\big) - \mathrm{Tr}\big((P_1 - P_1P_0P_1)^m\big)$$

for all $m \geq n$. By Definition 5.2.1, this implies that (P_0, P_1) is a Fredholm pair with the same index. □

One way to reformulate Proposition 5.2.2 is to state that, for finite-rank projections P_0 and P_1,

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}(P_0) - \mathrm{Tr}(P_1) = \mathrm{Tr}(P_0 - P_1).$$

The right-hand side not only makes sense if P_0 and P_1 are finite dimensional, but also if $P_0 - P_1$ is a trace class operator. The following result shows that then $\mathrm{Tr}(P_0 - P_1)$ is indeed equal to the index, actually under the even weaker assumption that some power of $P_0 - P_1$ is trace class. This provides yet another formula for the index of a Fredholm pair of projections.

Theorem 5.2.11. *Let (P_0, P_1) be a Fredholm pair of projections. If the operator $(P_0 - P_1)^{2n+1}$ is trace class for some integer $n \geq 0$, then for all $k \geq n$,*

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}\big((P_0 - P_1)^{2k+1}\big).$$

Proof. First let us note the following algebraic identities:

$$P_0 - P_0P_1P_0 = P_0(P_0 - P_1)P_0 = P_0(P_0 - P_1)^2P_0 = P_0(P_0 - P_1)^2 = (P_0 - P_1)^2P_0.$$

Therefore,

$$\begin{aligned}(P_0 - P_0P_1P_0)^{k+1} &= (P_0 - P_0P_1P_0)^k(P_0 - P_0P_1P_0)\\ &= \big(P_0(P_0 - P_1)^2P_0\big)^k\big(P_0(P_0 - P_1)P_0\big)\\ &= P_0(P_0 - P_1)^{2k}(P_0 - P_1)P_0\\ &= P_0(P_0 - P_1)^{2k+1}P_0.\end{aligned}$$

In particular, the trace class condition on $(P_0 - P_1)^{2k+1}$ implies that $(P_0 - P_0P_1P_0)^{k+1}$ is trace class. This holds for all $k \geq n$. Similarly, one can deduce

$$(P_1 - P_1P_0P_1)^{k+1} = P_1(P_1 - P_0)^{2k+1}P_1,$$

and verify the trace class property of $(P_1 - P_1P_0P_1)^{k+1}$. Now by Proposition 5.2.10 and the cyclicity of the trace, one has

$$\begin{aligned}\operatorname{Ind}(P_0,P_1) &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1}P_0\big) - \operatorname{Tr}\big(P_1(P_1-P_0)^{2k+1}P_1\big)\\ &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1} - P_1(P_1-P_0)^{2k+1}\big)\\ &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1} + P_1(P_0-P_1)^{2k+1}\big)\\ &= \operatorname{Tr}\big((P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k}\big).\end{aligned}$$

It remains to show

$$\operatorname{Tr}\big((P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k}\big) = \operatorname{Tr}\big((P_0-P_1)^{2k+1}\big). \tag{5.12}$$

Note that

$$\begin{aligned}&(P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k} - (P_0-P_1)^{2k+1}\\ &\quad= (P_1P_0 - P_0P_1)(P_0-P_1)^{2k}\\ &\quad= \big(P_1(P_0-P_1) - (P_0-P_1)P_1\big)(P_0-P_1)^{2k}\\ &\quad= P_1(P_0-P_1)^{2k+1} - (P_0-P_1)P_1(P_0-P_1)^{2k}.\end{aligned}$$

As in the last line both summands are trace class, (5.12) now follows from the cyclicity of the trace. ☐

Based on Theorem 5.2.11, one can derive integral formulas for the index of a pair of projections which is due to Phillips [148]. They directly lead to formulas for the spectral flow in Section 5.6.

Theorem 5.2.12. *Let (P_0,P_1) be a Fredholm pair of projections on $\mathcal{H}$ such that $(P_0-P_1)^{2n+1}$ is trace class for some integer $n \geq 0$. For $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ consider the linear path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$. Then for all integers $k \geq n$,*

$$\operatorname{Ind}(P_0,P_1) = \frac{1}{C_k}\int_0^1 dt\,\operatorname{Tr}\big((\partial_t Q)_t(1-Q_t^2)^k\big),$$

where, with $(2k+1)!! = (2k+1)(2k-1)\cdots 3\cdot 1$,

$$C_k = \int_{-1}^{1} dt(1-t^2)^k = \frac{k!\,2^{k+1}}{(2k+1)!!}. \tag{5.13}$$

Proof. One directly checks

$$(\partial_t Q)_t = 2(P_0 - P_1)$$

and

$$\mathbf{1} - Q_t^2 = t(1-t)(Q_0-Q_1)^2 = 4t(1-t)(P_0-P_1)^2.$$

Hence $(\partial_t Q)_t(\mathbf{1} - Q_t^2)^k$ is trace class by assumption, and thus

$$\int_0^1 dt\,\mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1} - Q_t^2)^k\big) = \int_0^1 dt\,\mathrm{Tr}\big(2(P_0 - P_1)((4(t-t^2))(P_0 - P_1)^2)^k\big)$$
$$= 2\cdot 4^k \int_0^1 dt(t - t^2)^k\,\mathrm{Ind}(P_0, P_1)$$
$$= \int_{-1}^1 ds(1 - s^2)^k\,\mathrm{Ind}(P_0, P_1),$$

where the last step follows after the change of variables $s = 2t - 1$. The value of the integral can be computed and gives the constant C_k. □

5.3 Fredholm pairs of orthogonal projections

In this section, unless otherwise stated, all projections are supposed to be orthogonal, namely to be self-adjoint idempotents. Let us begin by proving two results that reformulate the definition and give a geometric interpretation of the index of a Fredholm pair of orthogonal projections.

Proposition 5.3.1. *Let P_0 and P_1 be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*

$$P_0P_1P_0 + \mathbf{1} - P_0 \quad \textit{and} \quad P_1P_0P_1 + \mathbf{1} - P_1$$

are Fredholm operators on $\mathcal{H}$. If (P_0, P_1) is a Fredholm pair, then

$$\mathrm{Ind}(P_0, P_1) = \dim\big(\mathrm{Ker}(P_0P_1P_0 + \mathbf{1} - P_0)\big) - \dim\big(\mathrm{Ker}(P_1P_0P_1 + \mathbf{1} - P_1)\big).$$

Proof. This follows directly from Theorem 3.4.1 applied to the operator A of Definition 5.2.1, after complementing A^*A and AA^* to operators on all of $\mathcal{H}$. □

Proposition 5.3.2. *Let P_0 and P_1 be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*
(i) *the linear span* $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1) = \mathrm{Ran}(P_0) + \mathrm{Ker}(P_1)$ *is a closed subspace;*
(ii) $\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$ *is finite dimensional;*
(iii) $\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)$ *is finite dimensional.*

The index $\mathrm{Ind}(P_0, P_1)$ *of the Fredholm pair is then given by*

$$\mathrm{Ind}(P_0, P_1) = \dim\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big) - \dim\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big).$$

Proof. First of all, let us note that $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ are closed subspaces and thus Hilbert spaces. Now by Definition 3.2.1, the operator $A = P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ is a Fredholm operator if and only if

$$\mathrm{Ker}(A) = \mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$$

is finite dimensional,

$$\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp = \mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)$$

is finite dimensional, and $\mathrm{Ran}(A) = \mathrm{Ran}(P_1P_0)$ is closed. Now

$$\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1) = \mathrm{Ran}(\mathbf{1} - P_1) \oplus \mathrm{Ran}(P_1P_0),$$

where $\oplus$ denotes the orthogonal sum. Thus $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1)$ is closed if and only if $\mathrm{Ran}(A) = \mathrm{Ran}(P_1P_0)$ is closed by Lemma 5.3.3 below. Therefore A is indeed a Fredholm operator if and only if (i), (ii), and (iii) hold. Furthermore, by definition, the index $\mathrm{Ind}(P_0, P_1) = \mathrm{Ind}(A) = \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*))$ is given by the formula claimed. □

Lemma 5.3.3. *Let P be a projection (not necessarily orthogonal) and $\mathcal{E} \subset \mathrm{Ker}(P)$ as well as $\mathcal{F} \subset \mathrm{Ran}(P)$ subspaces. Then $\mathcal{E} + \mathcal{F}$ is closed if and only if $\mathcal{E}$ and $\mathcal{F}$ are closed.*

Proof. Suppose that $\mathcal{E} + \mathcal{F}$ is closed. Let $(\phi_n)_{n\geq 1}$ be a convergent sequence in $\mathcal{E}$ with limit $\phi \in \mathcal{H}$. It is then also convergent in $\mathcal{E} + \mathcal{F}$ and therefore $\phi \in \mathcal{E} + \mathcal{F}$ as $\mathcal{E} + \mathcal{F}$ is closed. But $P\phi = \lim P\phi_n = 0$ so that $\phi \in \mathrm{Ker}(P)$ and thus $\phi \in \mathcal{E}$. Similarly, one checks that $\mathcal{F}$ is closed. For the converse, let $(\phi_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{E} + \mathcal{F}$. Then $(P\phi_n)_{n\in\mathbb{N}}$ and $((\mathbf{1} - P)\phi_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{F}$ and $\mathcal{E}$, respectively. As $\mathcal{F}$ and $\mathcal{E}$ are closed, $(P\phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ and $((\mathbf{1} - P)\phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{E}$ and hence also the sequence $\phi_n = P\phi_n + (\mathbf{1} - P)\phi_n$ converges in $\mathcal{E} + \mathcal{F}$. □

It follows directly from Definition 5.2.1 that for a Fredholm pair (P_0, P_1) of orthogonal projections also the pair (P_1, P_0) is Fredholm (because then the corresponding Fredholm operators are A and its adjoint A^*, respectively) and that one has

$$\mathrm{Ind}(P_0, P_1) = -\mathrm{Ind}(P_1, P_0).$$

Moreover, by Proposition 5.3.2, $(\mathbf{1} - P_0, \mathbf{1} - P_1)$ is Fredholm if and only if (P_0, P_1) is Fredholm and then

$$\mathrm{Ind}(\mathbf{1} - P_0, \mathbf{1} - P_1) = \mathrm{Ind}(P_1, P_0).$$

Finally, it follows from Proposition 5.3.2, or alternatively from (5.10), that for every Fredholm pair (P_0, P_1) of orthogonal projections and any unitary operator U, also (UP_0U^*, UP_1U^*) is a Fredholm pair of orthogonal projections with index

$$\mathrm{Ind}\big(UP_0U^*, UP_1U^*\big) = \mathrm{Ind}(P_0, P_1). \tag{5.14}$$

Generalizing the unitary conjugation, one can also consider the natural action (5.7) of invertibles on orthogonal projections.

Proposition 5.3.4. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections and furthermore let $T \in \mathbb{G}(\mathcal{H})$ be invertible. Then $(T \cdot P_0, (T^{-1})^* \cdot P_1)$ is a Fredholm pair of orthogonal projections with the same index. Moreover,*

$$\dim\big(\mathrm{Ran}(T \cdot P_0) \cap \mathrm{Ker}\big((T^{-1})^* \cdot P_1\big)\big) = \dim\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big)$$

and

$$\dim\big(\mathrm{Ran}\big((T^{-1})^* \cdot P_1\big) \cap \mathrm{Ker}(T \cdot P_0)\big) = \dim\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big).$$

Proof. For any orthogonal projection P, one deduces from the definition of $T \cdot P$ that

$$\mathrm{Ran}(T \cdot P) = T\,\mathrm{Ran}(P), \quad \mathrm{Ker}(T \cdot P) = \big(T^{-1}\big)^*\,\mathrm{Ker}(P),$$

see the argument after equation (5.7). The Fredholm property of $(T \cdot P_0, (T^{-1})^* \cdot P_1)$ is checked by verifying the three conditions (i)–(iii) stated in Proposition 5.3.2. One has

$$\begin{aligned}
\mathrm{Ran}(T \cdot P_0) + \mathrm{Ran}\big(\mathbf{1} - \big(T^{-1}\big)^* \cdot P_1\big) &= \mathrm{Ran}(T \cdot P_0) + \mathrm{Ker}\big(\big(T^{-1}\big)^* \cdot P_1\big) \\
&= T\,\mathrm{Ran}(P_0) + T\,\mathrm{Ker}(P_1) \\
&= T\big(\mathrm{Ran}(P_0) + \mathrm{Ker}(P_1)\big),
\end{aligned}$$

showing that this is a closed subspace because T is invertible. Moreover,

$$\begin{aligned}
\mathrm{Ran}(T \cdot P_0) \cap \mathrm{Ker}\big(\big(T^{-1}\big)^* \cdot P_1\big) &= \big(T\,\mathrm{Ran}(P_0)\big) \cap \big(T\,\mathrm{Ker}(P_1)\big) \\
&= T\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big),
\end{aligned}$$

which has the same finite dimension as $\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$. In the same way,

$$\mathrm{Ran}\big(\big(T^{-1}\big)^* \cdot P_1\big) \cap \mathrm{Ker}(T \cdot P_0) = \big(T^{-1}\big)^*\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big),$$

implying all remaining claims. □

One may wonder if for a Fredholm pair (P_0, P_1) of orthogonal projections and an invertible $T \in \mathbb{G}(\mathcal{H})$ the pair $(T \cdot P_0, T \cdot P_1)$ is Fredholm. In general, however, this is not true as is shown by the next example.

Example 5.3.5. For a fixed grading $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$ where $\mathcal{H}'$ is an infinite-dimensional separable Hilbert space, let us set

$$P_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_1 = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \quad T = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\operatorname{Ker}(P_1P_0) = \operatorname{Ker}(P_0)$ and $\operatorname{Ran}(P_1P_0) = \operatorname{Ran}(P_1)$, and therefore (P_0, P_1) is a Fredholm pair. Moreover,

$$\operatorname{Ran}(T \cdot P_0) = \operatorname{span}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \operatorname{Ran}(T \cdot P_1) = \operatorname{span}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

This shows $T \cdot P_1 = \mathbf{1} - T \cdot P_0$ and therefore $(T \cdot P_0, T \cdot P_1)$ is not a Fredholm pair. ◇

The following aim is to give a spectral theoretic approach to Fredholm pairs of orthogonal projections. As a preparation for the proofs, let us present a set of algebraic relations satisfied by two projections (which need not be orthogonal). They can be traced back to Kato [112], see also [49] and [18].

Lemma 5.3.6. *Let P_0 and P_1 be projections. Set*

$$R_0 = \mathbf{1} - P_0 - P_1, \quad R_1 = P_0 - P_1.$$

Then the following identities hold:

$$R_0^2 + R_1^2 = \mathbf{1}, \quad R_0R_1 = -R_1R_0. \tag{5.15}$$

Moreover,

$$\begin{aligned} R_0P_0 &= P_1R_0, & R_0P_1 &= P_0R_0, \\ R_1(\mathbf{1} - P_0) &= P_1R_1, & R_1(\mathbf{1} - P_1) &= P_0R_1, \\ R_0^2P_0 &= P_0R_0^2, & R_0R_1(\mathbf{1} - P_0) &= P_0R_0R_1. \end{aligned}$$

Proof. Multiplying out, one finds

$$R_0^2 = \mathbf{1} - P_0 - P_1 + P_0P_1 + P_1P_0,$$

and similarly

$$R_1^2 = P_0 + P_1 - P_0P_1 - P_1P_0.$$

Adding this up, leads to the first identity. The others are also verified by straightforward algebraic computations. □

The identities of Lemma 5.3.6 lead to interesting spectral information of $P_1 - P_0$ and $P_1 + P_0$, stated in terms of R_0 and R_1.

Proposition 5.3.7. *Let P_0 and P_1 be orthogonal projections and R_0 and R_1 as in Lemma* 5.3.6. *Then for $j = 0, 1$, the spectrum* $\mathrm{spec}(R_j)$ *of R_j lies in* $[-1, 1]$ *and satisfies*

$$\mathrm{spec}(R_j) \setminus \{-1, 1\} = -\big(\mathrm{spec}(R_j) \setminus \{-1, 1\}\big).$$

Moreover, for any $\lambda \notin \{-1, 1\}$,

$$\dim\big(\mathrm{Ker}(R_j - \lambda\mathbf{1})\big) = \dim\big(\mathrm{Ker}(R_j + \lambda\mathbf{1})\big).$$

Proof. Let us focus on $j = 1$. The proof for the case $j = 0$ is the same as only the relations (5.15) are used and they are symmetric in the indices. The inclusion $\mathrm{spec}(R_1) \subset [-1, 1]$ follows from $R_1^2 = \mathbf{1} - R_0^2 \leq \mathbf{1}$. The symmetry of the spectrum can be shown using Weyl sequences, namely if $(R_1 - \lambda\mathbf{1})\phi_n \to 0$ for $\lambda \in (-1, 1)$ and a sequence of unit vectors $(\phi_n)_{n\geq 1}$, then, by the identity $R_0R_1 = -R_1R_0$ of Lemma 5.3.6, one has $(R_1 + \lambda\mathbf{1})R_0\phi_n \to 0$. As $R_0^2\phi_n = (1-\lambda^2)\phi_n - (R_1^2 - \lambda^2\mathbf{1})\phi_n$ by the first relation of (5.15) and $(R_1^2 - \lambda^2\mathbf{1})\phi_n \to 0$, it follows from $|\lambda| < 1$ that $\|R_0\phi_n\| \geq c$ for some constant $c > 0$ and n sufficiently large. Hence $(R_1 + \lambda\mathbf{1})\frac{R_0\phi_n}{\|R_0\phi_n\|} \to 0$ and $(\frac{R_0\phi_n}{\|R_0\phi_n\|})_{n\geq 1}$ is a Weyl sequence for $-\lambda$. Finally, set $\mathcal{H}_\lambda = \mathrm{Ker}(R_1 - \lambda\mathbf{1})$. Then by the same identity $R_0(\mathcal{H}_\lambda) \subset \mathcal{H}_{-\lambda}$ and $R_0(\mathcal{H}_{-\lambda}) \subset \mathcal{H}_\lambda$. As $R_0^2|_{\mathcal{H}_\lambda} = (1 - \lambda^2)\mathbf{1}|_{\mathcal{H}_\lambda}$ by the identity $R_0^2 = \mathbf{1} - R_1^2$ in (5.15), it follows that R_0 is an isomorphism from $\mathcal{H}_\lambda$ to $\mathcal{H}_{-\lambda}$ for any value $\lambda \notin \{-1, 1\}$. □

Now the main spectral theoretic result for the index of a Fredholm pair of orthogonal projections can be stated and proved.

Theorem 5.3.8. *Two orthogonal projections P_0 and P_1 form a Fredholm pair if and only if* ± 1 *are not in the essential spectrum of the operator $P_0 - P_1$. Then*

$$\mathrm{Ind}(P_0, P_1) = \dim\big(\mathrm{Ker}(P_0 - P_1 - \mathbf{1})\big) - \dim\big(\mathrm{Ker}(P_0 - P_1 + \mathbf{1})\big). \tag{5.16}$$

Proof. Recall that (P_0, P_1) is a Fredholm pair if and only if $A : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ defined by $A\phi = P_1P_0\phi$ for $\phi \in \mathrm{Ran}(P_0)$ is a Fredholm operator.

Let 1 be in the essential spectrum of $P_0 - P_1$. By Proposition 3.4.7, there is a singular Weyl sequence $(\phi_n)_{n\geq 1}$ such that $(P_0 - P_1 - \mathbf{1})\phi_n \to 0$. Then $\langle\phi_n|(P_0 - P_1)\phi_n\rangle \to 1$, thus $\|P_0\phi_n\| \to 1$ and $\|P_1\phi_n\| \to 0$. Therefore $\psi_n = \frac{P_0\phi_n}{\|P_0\phi_n\|}$ has norm 1, converges weakly to 0, and $P_0P_1P_0\psi_n \to 0$, which shows that $0 \in \mathrm{spec}_{\mathrm{ess}}(A^*A)$ by Proposition 3.4.7. By Theorem 3.4.1, this is a contradiction to the Fredholm property of A. Therefore (P_0, P_1) is no Fredholm pair. Similarly, $-1 \in \mathrm{spec}_{\mathrm{ess}}(P_0 - P_1)$ implies $0 \in \mathrm{spec}_{\mathrm{ess}}(AA^*)$ and, again by Theorem 3.4.1, this is a contradiction to the Fredholm property of A, thus (P_0, P_1) is no Fredholm pair.

Conversely, let ± 1 be not in the essential spectrum of the operator $P_0 - P_1$. By the spectral radius theorem, one has $P_0 - P_1 = B + F$ where F is of finite rank and, moreover, $(\epsilon - 1)\mathbf{1} \leq B \leq (1 - \epsilon)\mathbf{1}$ for some $\epsilon > 0$. As

$$P_0P_1P_0 = P_0\big(\mathbf{1} - (P_0 - P_1)\big)P_0$$

$$= -P_0FP_0 + P_0(\mathbf{1} - B)P_0$$
$$\geq -P_0FP_0 + \epsilon P_0,$$

this implies $0 \notin \operatorname{spec}_{\mathrm{ess}}(A^*A)$. Analogously, one has $P_1P_0P_1 \geq P_1FP_1 + \epsilon P_1$ and consequently $0 \notin \operatorname{spec}_{\mathrm{ess}}(AA^*)$. By Theorem 3.4.1, this implies that A is Fredholm and (P_0, P_1) is a Fredholm pair.

It remains to show (5.16) if (P_0, P_1) is a Fredholm pair. The kernel of a sum of two nonnegative operators is the intersection of their kernels. Therefore

$$\begin{aligned}\operatorname{Ker}(P_0 - P_1 - \mathbf{1}) &= \operatorname{Ker}\big(P_1 + (\mathbf{1} - P_0)\big)\\ &= \operatorname{Ker}(P_1) \cap \operatorname{Ker}(\mathbf{1} - P_0)\\ &= \operatorname{Ker}(P_1) \cap \operatorname{Ran}(P_0).\end{aligned}$$

Similarly, $\operatorname{Ker}(P_0 - P_1 + \mathbf{1}) = \operatorname{Ker}(P_0) \cap \operatorname{Ran}(P_1)$, and this implies the claimed identity due to Proposition 5.3.2. □

Remark 5.3.9. Proposition 5.3.7 and Theorem 5.3.8 allow giving an alternative proof of Theorem 5.2.11 for orthogonal projections. Under the hypothesis that $(P_0-P_1)^{2k+1}$ is trace-class, the spectrum of $(P_0 - P_1)^{2k+1} = R_1^{2k+1}$ consists of eigenvalues accumulating only at 0. By Proposition 5.3.7 and because $2k + 1$ is odd, this spectrum is symmetric and the eigenspaces $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$ have the same dimension for $\lambda \notin \{-1, 0, 1\}$ which, moreover, is finite. Thus by Lidskii's theorem,

$$\begin{aligned}\operatorname{Tr}\big((P_0 - P_1)^{2k+1}\big) &= \sum_{\lambda\in\operatorname{spec}(P_0-P_1)} \lambda^{2k+1} \dim(\mathcal{H}_\lambda)\\ &= \sum_{\substack{\lambda\in\operatorname{spec}(P_0-P_1),\\ \lambda>0}} \lambda^{2k+1}\big(\dim(\mathcal{H}_\lambda) - \dim(\mathcal{H}_{-\lambda})\big)\\ &= \dim(\mathcal{H}_1) - \dim(\mathcal{H}_{-1})\\ &= \operatorname{Ind}(P_0, P_1),\end{aligned}$$

where the last equality follows from Theorem 5.3.8. ◇

Let us also provide a slight generalization of Theorem 5.2.11 going back to [56].

Proposition 5.3.10. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections such that $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact and let $f : [-1, 1] \to \mathbb{R}$ be a continuous odd function such that $f(1) = 1$ and such that $f(P_0 - P_1)$ is trace class, then*

$$\operatorname{Ind}(P_0, P_1) = \operatorname{Tr}\big(f(P_0 - P_1)\big).$$

Proof. Recall from Proposition 5.3.7 that

$$\operatorname{spec}(P_0 - P_1) \setminus \{-1, 1\} = -\big(\operatorname{spec}(P_0 - P_1) \setminus \{-1, 1\}\big)$$

and

$$\dim\big(\mathrm{Ker}(P_0 - P_1 - \lambda\mathbf{1})\big) = \dim\big(\mathrm{Ker}(P_0 - P_1 + \lambda\mathbf{1})\big)$$

for any $\lambda \notin \{-1, 1\}$. Therefore, by the same argument as in Remark 5.3.9,

$$\begin{aligned}\mathrm{Tr}\big(f(P_0 - P_1)\big) &= \big(\dim\big(\mathrm{Ker}(P_0 - P_1 - \mathbf{1})\big) - \dim\big(\mathrm{Ker}(P_0 - P_1 + \mathbf{1})\big)\big)\\ &= \mathrm{Ind}(P_0, P_1),\end{aligned}$$

where the last step follows from Theorem 5.3.8. □

Based on Proposition 5.3.10, one can also express the index of a pair of projections as an integral similar as in Theorem 5.2.12, but under weaker hypothesis. Combined with the results of Section 5.6, this leads to integral formulas for the spectral flow of paths between Fredholm pairs of symmetries. In the following, functions $f : [0, \infty) \to [0, \infty)$ of the form $f(x) = x^{-r} e^{-x^{-\sigma}}$ for $r \geq 0$ and $\sigma \geq 1$ are considered. These functions are defined to be 0 at $x = 0$.

Proposition 5.3.11. *Let $P_0, P_1 \in \mathbb{P}(\mathcal{H})$ be orthogonal projections such that the operator $\exp(-((P_0 - P_1)^2)^{-\frac{1}{q}})$ is trace class for some $0 < q \leq 1$. Then (P_0, P_1) is a Fredholm pair of projections and for $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ the linear path*

$$t \in [0, 1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$$

is within the Fredholm operators. Moreover,

$$\mathrm{Ind}(P_0, P_1) = \frac{1}{C_{r,q}} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (\mathbf{1} - Q_t^2)^{-r} e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}}\big)$$

for $r \geq 0$, where

$$C_{r,q} = \int_{-1}^{1} du (1 - u^2)^{-r} e^{-(1-u^2)^{-\frac{1}{q}}}. \tag{5.17}$$

Proof. First of all, let us note that $e^{-((P_0 - P_1)^2)^{-\frac{1}{q}}}$ is trace class and, in particular, compact so that $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact. Thus (P_0, P_1) is a Fredholm pair and Q_t is Fredholm for all $t \in [0, 1]$. Now recall from the proof of Theorem 5.2.12 that $(\partial_t Q)_t = 2(P_0 - P_1)$ and $\mathbf{1} - Q_t^2 = 4t(1 - t)(P_0 - P_1)^2$. Thus

$$e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}} = \big(e^{-((P_0 - P_1)^2)^{-\frac{1}{q}}}\big)^{(4t(1-t))^{-\frac{1}{q}}}$$

is trace class as $(4t(1 - t))^{-\frac{1}{q}} \geq 1$ for $t \in (0, 1)$ while it trivially is trace class for $t \in \{0, 1\}$. One obtains

$$\begin{aligned}
&\int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1}-Q_t^2)^{-r} e^{-(\mathbf{1}-Q_t^2)^{-\frac{1}{q}}}\big)\\
&\quad = \int_0^1 dt\, \mathrm{Tr}\big(2(P_0-P_1)(4(t-t^2)(P_0-P_1)^2)^{-r} e^{-(4(t-t^2)(P_0-P_1)^2)^{-\frac{1}{q}}}\big).
\end{aligned}$$

For $t \in (0,1)$, the function $f_t : \mathbb{R} \to \mathbb{R}$ defined by

$$f_t(x) = 2x(4(t-t^2)x^2)^{-r} e^{-(4(t-t^2)x^2)^{-\frac{1}{q}}}$$

is odd and $f_t(P_0 - P_1)$ is trace class. Thus by Proposition 5.3.10,

$$\mathrm{Tr}(f_t(P_0-P_1)) = f_t(1)\,\mathrm{Ind}(P_0,P_1)$$

and therefore

$$\begin{aligned}
&\int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1}-Q_t^2)^{-r} e^{-(\mathbf{1}-Q_t^2)^{-\frac{1}{q}}}\big)\\
&\quad = \int_0^1 dt\, f_t(1)\,\mathrm{Ind}(P_0,P_1)\\
&\quad = \mathrm{Ind}(P_0,P_1)\int_0^1 dt\, 2(4(t-t^2))^{-r} e^{-(4(t-t^2))^{-\frac{1}{q}}}\\
&\quad = \mathrm{Ind}(P_0,P_1)\int_{-1}^1 du (1-u^2)^{-r} e^{-(1-u^2)^{-\frac{1}{q}}},
\end{aligned}$$

where the last step follows from the change of variables $u = 2t-1$. Dividing by $C_{r,q}$ shows the claim. □

Theorem 5.3.8 has several other consequences. The first gives an important criterion for a pair of projections to be a Fredholm pair with vanishing index.

Proposition 5.3.12. *Let P_0 and P_1 be a pair of orthogonal projections on $\mathcal{H}$. If*

$$\|P_0 - P_1\| < 1,$$

then (P_0, P_1) is a Fredholm pair and $\mathrm{Ind}(P_0,P_1) = 0$.

Proof. This follows immediately from Theorem 5.3.8 because the hypothesis implies that ± 1 are not in the spectrum of $P_0 - P_1$. □

One can go beyond Proposition 5.3.12 and show that $\|P_0 - P_1\| < 1$ implies that there exists a unitary V such that $VP_0V^* = P_1$ and $VP_1V^* = P_0$, see Proposition 5.5.6. The next consequence is a characterization of the Fredholmness of a pair of orthogonal projections that is often used as the definition of a Fredholm pair.

Corollary 5.3.13. *Two orthogonal projections P_0 and P_1 form a Fredholm pair if and only if the norm of their difference in the Calkin algebra is less than* 1,

$$\|\pi(P_1 - P_0)\|_{\mathbb{Q}} < 1.$$

The following characterization of the Fredholmness of a pair of orthogonal projections is another direct consequence of Theorem 5.3.8.

Corollary 5.3.14. *A pair of orthogonal projections (P_0, P_1) is a Fredholm pair if and only if*

$$P_0 - P_1 = B + F,$$

where B, F are self-adjoint operators on $\mathcal{H}$, $\|B\| < 1$ and F is of finite rank.

Proof. If $P_0 - P_1 = B + F$, then (P_0, P_1) is a Fredholm pair by Theorem 5.3.8.

For the converse, set $P_\pm = \chi_{\{\pm 1\}}(P_0 - P_1)$. Then define $F = P_+ - P_-$ which is of finite rank and $B = (\mathbf{1} - P_+ - P_-)(P_0 - P_1)(\mathbf{1} - P_+ - P_-)$ for which $\|B\| < 1$. One directly checks that $P_0 - P_1 = B + F$. □

Next let us strengthen Proposition 5.2.6 on the concatenation of Fredholm pairs.

Proposition 5.3.15. *Suppose given three orthogonal projections P_0, P_1 and P_2 such that $\|\pi(P_0) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_2)\|_{\mathbb{Q}} < 1$. Then (P_0, P_1), (P_1, P_2) and (P_0, P_2) are all Fredholm pairs and*

$$\mathrm{Ind}(P_0, P_2) = \mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2). \tag{5.18}$$

Proof. By Corollary 5.3.13, (P_0, P_1), (P_1, P_2), and (P_0, P_2) are Fredholm pairs. Therefore, by definition $P_iP_j : \mathrm{Ran}(P_j) \to \mathrm{Ran}(P_i)$ is Fredholm for $i, j \in \{0, 1, 2\}$ with $i > j$ and $\mathrm{Ind}(P_j, P_i) = \mathrm{Ind}(P_iP_j)$. Thus, by item (iii) of Theorem 3.3.4,

$$\mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2) = \mathrm{Ind}(P_2P_1P_0),$$

where $P_2P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is Fredholm by item (i) of Corollary 3.3.2. Then (5.18) is equivalent to

$$\mathrm{Ind}(P_2P_1P_0) = \mathrm{Ind}(P_2P_0),$$

which is, again by Corollary 3.3.2, equivalent to

$$\mathrm{Ind}(P_0P_2P_1P_0) = \mathrm{Ind}\big((P_2P_0)^*(P_2P_1P_0)\big) = 0.$$

As

$$\begin{aligned}\|\pi(P_0P_2P_1P_0) - \pi(P_0)\|_{\mathbb{Q}} &\le \|\pi(P_2P_1) - \pi(P_0)\|_{\mathbb{Q}}\\ &\le \|\pi(P_2P_1) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_0)\|_{\mathbb{Q}}\\ &\le \|\pi(P_0) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_2)\|_{\mathbb{Q}} < 1,\end{aligned}$$

there is a compact operator $K : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_0)$ such that

$$\|P_0P_2P_1P_0 + K - P_0\| < 1.$$

This implies that $P_0P_2P_1P_0 + K - P_0 + P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_0)$ is invertible and therefore $\mathrm{Ind}(P_0P_2P_1P_0) = \mathrm{Ind}(P_0P_2P_1P_0 + K) = 0$. □

Remark 5.3.16. It is not sufficient to suppose that (P_0, P_1) and (P_1, P_2) are Fredholm pairs, because then (P_0, P_2) is not necessarily a Fredholm pair. Indeed, let us set

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \mathbf{1} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \mathbf{1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

acting on $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. One directly checks $\|P_0 - P_1\| = \|P_1 - P_2\| = \frac{1}{\sqrt{2}} < 1$, thus (P_0, P_1) and (P_1, P_2) are Fredholm pairs by Corollary 5.3.13. But $\|\pi(P_0 - P_2)\|_{\mathbb{Q}} = 1$ and therefore (P_0, P_2) does not form a Fredholm pair, again by Corollary 5.3.13. ◇

Even though a Fredholm pair (P_0, P_1) with compact difference $P_1 - P_0$ is only a special case, it nevertheless appears often, as in the following situation:

Proposition 5.3.17. *Let $H_0, H_1 \in \mathbb{FB}_{sa}(\mathcal{H})$ be two self-adjoint bounded Fredholm operators such that the difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ is compact. Then the spectral projections $P_0 = \chi(H_0 < 0)$ and $P_1 = \chi(H_1 < 0)$ form a Fredholm pair with compact difference $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$.*

Proof. Because H_0 and H_1 are Fredholm, and therefore $0 \notin \mathrm{spec}_{ess}(H_0) \cup \mathrm{spec}_{ess}(H_1)$ by Corollary 3.4.4, 0 is not an accumulation point of $\Sigma = \mathrm{spec}(H_0) \cup \mathrm{spec}(H_1)$. Therefore $\chi_{(-\infty,0)}|_\Sigma : \Sigma \to \{0, 1\}$ is a continuous function on the compact domain Σ and one has $P_1 - P_0 = \chi_{(-\infty,0)}|_\Sigma(H_1) - \chi_{(-\infty,0)}|_\Sigma(H_0)$. As

$$H_1^n - H_0^n = H_1\big(H_1^{n-1} - H_0^{n-1}\big) + (H_1 - H_0)H_0^{n-1}, \quad n \ge 2,$$

$p(H_1) - p(H_0)$ is compact for any polynomial $p : \mathbb{C} \to \mathbb{C}$. As the set of compact operators $\mathbb{K}(\mathcal{H})$ is a closed subset of the set of bounded operators $\mathbb{B}(\mathcal{H})$ and the polynomials are dense in set of continuous functions on compact domains, we see that $h(H_1) - h(H_0)$ is compact for every continuous function $h : \Sigma \to \mathbb{C}$. In conclusion, $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$ is compact and therefore (P_0, P_1) is a Fredholm pair by Corollary 5.3.13. □

In Remark 5.3.16, the orthogonal projection P_1 is obtained from P_0 by a rotation of less than a right angle. The following result states that, inversely, one can always rotate one of the orthogonal projections of a Fredholm pair to attain a Fredholm pair with compact difference.

Proposition 5.3.18. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections. Then there exists a path $t \in [0,1] \mapsto P_1(t)$ of orthogonal projections such that $(P_0, P_1(t))$ is a Fredholm pair for all $t \in [0,1]$, $P_1(1) = P_1$ and $P_0 - P_1(0)$ is compact.*

Note that by Proposition 5.2.7, $t \in [0,1] \mapsto \mathrm{Ind}(P_0, P_1(t))$ is constant along this path. The proof of Proposition 5.3.18 starts out with a special case.

Proposition 5.3.19. *Let (P_0, P_1) be a pair of orthogonal projections satisfying the bound $\|P_0 - P_1\| < 1$. Then there exists a path $t \in [0,1] \mapsto P_t$ of orthogonal projections connecting P_0 with P_1 such that (P_0, P_t) is a Fredholm pair for all $t \in [0,1]$.*

Proof. (This uses the construction after Proposition 4.6.6 in [23].) Let $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ be the two associated symmetries and then set

$$R = Q_0 Q_1 + Q_1 Q_0 = 2\mathbf{1} - 4(P_0 - P_1)^2.$$

Then one has $[R, Q_0] = 0 = [R, Q_1]$. Let $a = \|P_0 - P_1\| < 1$. Then $-2\mathbf{1} < (2 - 4a^2)\mathbf{1} \le R \le 2\mathbf{1}$ so that $\mathbf{1} + \lambda R > 0$ uniformly in $\lambda \in [0, \frac{1}{2}]$. Therefore one can set

$$Q_t = \left(\mathbf{1} + R\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}}\left(Q_0\cos\left(\frac{\pi}{2}t\right) + Q_1\sin\left(\frac{\pi}{2}t\right)\right), \quad t \in [0,1].$$

Clearly, $Q_t^* = Q_t$ and computing the square shows $Q_t^2 = \mathbf{1}$, so this is a path of symmetries which, indeed, connects Q_0 and Q_1. Set $P_t = \frac{1}{2}(\mathbf{1} - Q_t)$. To verify the Fredholm property along this path, let us compute

$$(P_t - P_0)^2 = \frac{1}{2}\mathbf{1} - \frac{1}{4}\left(\mathbf{1} + R\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}}\left(2\cos\left(\frac{\pi}{2}t\right)\mathbf{1} + R\sin\left(\frac{\pi}{2}t\right)\right).$$

The right-hand side is merely a function of the self-adjoint operator R. Hence the norm is bounded by the maximum of the function

$$f(t,r) = \frac{1}{2} - \frac{1}{4}\left(1 + r\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}}\left(2\cos\left(\frac{\pi}{2}t\right) + r\sin\left(\frac{\pi}{2}t\right)\right)$$

on the rectangle $[0,1] \times [2 - 4a^2, 2]$. One finds

$$\sup_{t\in[0,1]} f(t,r) = f(1,r) = \frac{1}{2} - \frac{r}{4} \le a^2,$$

so that $\|P_t - P_0\| \le a$ uniformly in $t \in [0,1]$. By Proposition 5.3.12, this implies that (P_0, P_t) is a Fredholm pair for all $t \in [0,1]$. □

Proof of Proposition 5.3.18. Let us set $K_0 = \chi_{\{1\}}(P_0 - P_1)$ and $K_1 = \chi_{\{-1\}}(P_0 - P_1)$ which are finite-dimensional orthogonal projections satisfying $K_0 K_1 = 0$. For $\phi \in \mathrm{Ran}(K_0)$, one has $P_0\phi = \phi$ and $P_1\phi = 0$ so that $\mathrm{Ran}(K_0)$ is left invariant by both P_0 and P_1. The same holds for $\mathrm{Ran}(K_1)$. Then consider $\mathcal{H}' = \mathcal{H} \ominus (\mathrm{Ran}(K_0) \oplus \mathrm{Ran}(K_1))$ and the restrictions $P_0' = P_0|_{\mathcal{H}'}$ and $P_1' = P_1|_{\mathcal{H}'}$. By construction, P_0' and P_1' are orthogonal projections on $\mathcal{H}'$ satisfying $\|P_0' - P_1'\| < 1$. Let $P_1'(t)$ be the path of orthogonal projections on $\mathcal{H}'$ given by Proposition 5.3.19. Finally, set $P_1(t) = P_1'(t) \oplus K_1$ which is an orthogonal projection on $\mathcal{H}$. The pair $(P_0, P_1(t))$ is Fredholm and satisfies the claim. □

The next aim is to lift the path of Proposition 5.3.19 by generalizing Proposition 5.2.9 in the following manner.

Proposition 5.3.20. *Let $t \in [0,1] \mapsto P_t$ be a path of orthogonal projections. Then there exists a path $t \in [0,1] \mapsto U_t$ of unitaries such that*

$$P_t = U_t^* P_0 U_t.$$

Proof. The operator M_t used in Proposition 5.2.9 satisfies $M_t P_0 = P_t M_t$ so that also $P_0 M_t^* = M_t^* P_t$. Therefore $P_t = M_t P_0 (M_t)^{-1}$ and $P_0 = M_t^* P_t (M_t^*)^{-1}$ so that upon replacing also

$$P_t = \left(M_t M_t^*\right)^{-1} P_t (M_t M_t^*).$$

This implies $P_t = (M_t M_t^*)^{-\frac{1}{2}} P_t (M_t M_t^*)^{\frac{1}{2}}$. Now set

$$U_t = M_t^* \left(M_t M_t^*\right)^{-\frac{1}{2}}.$$

This is indeed unitary and satisfies the claim. □

Remark 5.3.21. If the path $t \mapsto P_t$ is differentiable, then there is another standard way to obtain the path $t \mapsto U_t$ as the solution to Kato's adiabatic time-evolution:

$$\imath \partial_t U_t = U_t \imath [P_t, \partial_t P_t], \quad U_0 = \mathbf{1}.$$

Note that $\imath[P_t, \partial_t P_t]$ is self-adjoint so that indeed U_t is unitary. Furthermore, one has

$$\begin{aligned}\partial_t\left(U_t P_t U_t^*\right) &= (\partial_t U_t) P_t U_t^* + U_t (\partial_t P_t) U_t^* - U_t P_t U_t^* (\partial_t U_t) U_t^* \\ &= U_t [P_t, \partial_t P_t] P_t U_t^* + U_t (\partial_t P_t) U_t^* - U_t P_t [P_t, \partial_t P_t] U_t^* \\ &= 0,\end{aligned}$$

the latter because $\partial_t P_t = \partial_t P_t^2 = \partial_t P_t P_t + P_t \partial_t P_t$ and $P_t \partial_t P_t P_t = 0$ for any differentiable path of projections. Hence the initial condition implies indeed that $P_t = U_t^* P_0 U_t$. This argument can be modified to show that there are many possible choices for the path $t \mapsto U_t$. More precisely, one can modify the adiabatic equation to

$$\imath\partial_t U_t = U_t\big(\imath[P_t, \partial_t P_t] - H_t\big), \quad U_0 = \mathbf{1},$$

where $t \mapsto H_t$ is an arbitrary path of self-adjoints satisfying $[H_t, P_t] = 0$, without spoiling the conjugacy relation $P_t = U_t^* P_0 U_t$. ◇

Combining Proposition 5.3.20 with Proposition 5.3.19 one obtains the following:

Corollary 5.3.22. *Let P_0 and P_1 be a pair of orthogonal projections satisfying the bound $\|P_0 - P_1\| < 1$. Then exists a path $t \in [0,1] \mapsto U_t$ of unitaries such that for $P_1(t) = U_t^* P_0 U_t$ one has $P_1(1) = P_1$ and $P_1(0) = P_0$, and $(P_0, P_1(t))$ is a Fredholm pair for all $t \in [0,1]$.*

As an application of Proposition 5.3.18 let us prove a statement on the connected components of Fredholm pairs of proper orthogonal projections:

$$\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{H}) = \big\{(P_0, P_1) \text{ Fredholm pair} : \dim(P_j) = \dim(\mathbf{1} - P_j) = \infty\big\}. \tag{5.19}$$

The result is the equivalent of Theorem 3.3.5 for Fredholm operators.

Proposition 5.3.23. *With respect to the norm topology on $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$, the set*

$$\mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H}) = \big\{(P_0, P_1) \textit{ Fredholm} : \operatorname{Ind}(P_0, P_1) = n,\ \dim(P_j) = \dim(\mathbf{1} - P_j) = \infty\big\}$$

is connected.

Proof. Let $(P_{0,\text{ref}}, P_{1,\text{ref}}) \in \mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H})$ be a fixed Fredholm pair with index n such that $P_{0,\text{ref}} \geq P_{1,\text{ref}}$ if $n > 0$, $P_{0,\text{ref}} \leq P_{1,\text{ref}}$ if $n < 0$ and $P_{0,\text{ref}} = P_{1,\text{ref}}$ if $n = 0$. It will be shown that for $(P_0, P_1) \in \mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H})$ there is a norm-continuous path of Fredholm pairs connecting (P_0, P_1) to $(P_{0,\text{ref}}, P_{1,\text{ref}})$. First recall from the proof of Proposition 5.3.18 that there is a norm-continuous path of Fredholm pairs connecting (P_0, P_1) to (P_0, P_1') where with respect to the grading

$$\mathcal{H} = \operatorname{Ran}\big(\chi_{\{1\}}(P_0 - P_1)\big) \oplus \operatorname{Ran}\big(\chi_{\{-1\}}(P_0 - P_1)\big) \oplus \operatorname{Ran}\big(\chi_{\{-1,1\}}(P_0 - P_1)\big)^{\perp}$$

one has $P_1' = 0 \oplus \mathbf{1} \oplus \widetilde{P}_0$ and $\widetilde{P}_0 = P_0\chi_{(-1,1)}(P_0 - P_1)$. In this grading, P_0 is of the form $P_0 = \mathbf{1} \oplus 0 \oplus \widetilde{P}_0$. Moreover, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ acting nontrivially only on $\operatorname{Ran}(\chi_{\{1\}}(P_0 - P_1)) \oplus \operatorname{Ran}(\chi_{\{-1\}}(P_0 - P_1))$ such that $P_1'' = UP_1'U^*$ fulfils $P_0 \geq P_1''$ if $n > 0$, $P_0 \leq P_1''$ if $n < 0$ and $P_0 = P_1''$ if $n = 0$. As $\mathbf{1} - U \in \mathbb{K}(\mathcal{H})$ is a compact operator, $t \in [0,1] \mapsto (P_0, U^t P_1'(U^t)^*)$ is a continuous path of Fredholm pairs connecting (P_0, P_1') to (P_0, P_1''). Finally, there is a unitary $V \in \mathbb{U}(\mathcal{H})$ such that $VP_0V^* = P_{0,\text{ref}}$ and $VP_1''V^* = P_{1,\text{ref}}$. Indeed, say for $n \geq 0$, one can first rotate P_0 to $P_{0,\text{ref}}$ via a unitary $\hat{V}$, namely $\hat{V}P_0\hat{V}^* = P_{0,\text{ref}}$; then $\hat{V}P_1''\hat{V}^* \leq P_{0,\text{ref}}$; thus one can choose $\tilde{V}$ commuting with $P_{0,\text{ref}}$ so that $\tilde{V}\hat{V}P_1''\hat{V}^*\tilde{V}^* = P_{1,\text{ref}}$; finally, set $V = \tilde{V}\hat{V}$. Then $t \in [0,1] \mapsto (V^t P_0 (V^t)^*, V^t P_1''(V^t)^*)$ is a norm-continuous path of Fredholm pairs connecting (P_0, P_1'') to $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}})$. Concatenation of these paths leads to a path of Fredholm pairs connecting (P_0, P_1) to $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}})$ (as the index of Fredholm pairs is locally constant by Proposition 5.2.7 the path lies in $\mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H})$). As the Fredholm pair $(P_0, P_1) \in \mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H})$ was arbitrary, this shows that $\mathbb{F}_n\mathbb{P}\mathbb{P}(\mathcal{H})$ is connected. □

Corollary 5.3.24. *The path-connected components of* $\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{H})$ *are labeled by the index map* $\mathrm{Ind} : \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{H}) \to \mathbb{Z}$.

Remark 5.3.25. Given an arbitrary pair (P_0, P_1) of orthogonal projections, it is always possible to find a path $t \in [0,1] \mapsto P_t$ connecting them. Indeed, there always exists a unitary U such that $P_1 = UP_0U^*$ (see Proposition 5.1.7) and then one can simply set $P_t = U^tP_0(U^t)^*$ where U^t is the $\frac{1}{t}$th root of U defined by spectral calculus. However, along this path, the Fredholm property is in general violated. ◇

5.4 Fredholm pairs of symmetries

Associated to an orthogonal projection P is always a symmetry, that is, a self-adjoint unitary, by the formula

$$Q = \mathbf{1} - 2P.$$

Definition 5.2.1 therefore naturally leads to the following:

Definition 5.4.1. Two symmetries Q_0 and Q_1 form a Fredholm pair of symmetries if and only if $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$ and $P_1 = \frac{1}{2}(\mathbf{1} - Q_1)$ are a Fredholm pair of orthogonal projections. Then the index of the Fredholm pair of symmetries is given by

$$\mathrm{Ind}(Q_0, Q_1) = \mathrm{Ind}(P_0, P_1).$$

Of course, Fredholm pairs of symmetries are merely a reformulation of Fredholm pairs of orthogonal projections, but in some instances below this leads to nicer formulas. The first result shows that a pair of symmetries is Fredholm if and only if the sum of this symmetries is Fredholm.

Proposition 5.4.2. *A pair* (P_0, P_1) *of orthogonal projections is Fredholm if and only the operator if* $Q_0 + Q_1$ *is Fredholm, where* $Q_0 = \mathbf{1} - 2P_0$ *and* $Q_1 = \mathbf{1} - 2P_1$.

Proof. As

$$Q_0 + Q_1 = 2(\mathbf{1} - P_0 - P_1),$$

$Q_0 + Q_1$ is Fredholm if and only if $\mathbf{1} - P_0 - P_1$ is Fredholm. Moreover,

$$(\mathbf{1} - P_0 - P_1)^2 = \mathbf{1} - P_0 - P_1 + P_0P_1 + P_1P_0 = (\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0).$$

By Theorem 5.3.8, (P_0, P_1) is a Fredholm pair if and only if $\mathbf{1} - P_0 + P_1$ and $\mathbf{1} - P_1 + P_0$ are Fredholm. Therefore $Q_0 + Q_1$ is Fredholm if (P_0, P_1) is a Fredholm pair. Conversely, if $Q_0 + Q_1$ is Fredholm, $(\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0) = (\mathbf{1} - P_1 + P_0)(\mathbf{1} - P_0 + P_1)$ is Fredholm by the above. We show that $(\mathbf{1} - P_0 + P_1)$ and $(\mathbf{1} - P_1 + P_0)$ are Fredholm. First

$$\mathrm{Ker}(\mathbf{1} - P_0 + P_1) \subset \mathrm{Ker}\big((\mathbf{1} - P_1 + P_0)(\mathbf{1} - P_0 + P_1)\big)$$

is finite dimensional. Analogously, $\mathrm{Ker}(\mathbf{1} - P_1 + P_0)$ is finite dimensional. The range of $\mathbf{1} - P_0 + P_1$ can be decomposed into a direct sum of two subspaces

$$\mathrm{Ran}(\mathbf{1} - P_0 + P_1) = \mathrm{Ran}\big((\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0)\big) \oplus \mathrm{Ran}\big((\mathbf{1} - P_0 + P_1)|_{\mathrm{Ker}(\mathbf{1} - P_1 + P_0)}\big).$$

The first summand is closed by the Fredholm property of $(\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0)$, the second is finite dimensional. Thus $\mathrm{Ran}(\mathbf{1} - P_0 + P_1)$ is closed, and one concludes that $(\mathbf{1} - P_0 + P_1)$ is Fredholm. Analogously, $(\mathbf{1} - P_1 + P_0)$ is Fredholm. Theorem 5.3.8 allows concluding that (P_0, P_1) is a Fredholm pair. □

Lemma 5.4.3. *For symmetries Q_0 and Q_1, one has*

$$\mathrm{Ker}(Q_0 + Q_1) = \big(\mathrm{Ker}(Q_0 - \mathbf{1}) \cap \mathrm{Ker}(Q_1 + \mathbf{1})\big) \oplus \big(\mathrm{Ker}(Q_0 + \mathbf{1}) \cap \mathrm{Ker}(Q_1 - \mathbf{1})\big).$$

Proof. If Q_0 and Q_1 are expressed in terms of orthogonal projections P_0 and P_1, then

$$\mathrm{Ker}(Q_0 + Q_1) = \mathrm{Ker}(\mathbf{1} - P_0 - P_1).$$

For some vector $\phi = \phi_1 + \phi_2$ in this kernel such that $P_0\phi_1 = \phi_1$ and $P_0\phi_2 = 0$, one has

$$\begin{aligned}(\mathbf{1} - P_0 - P_1)\phi = 0 \quad &\Longleftrightarrow \quad \phi_2 - P_1\phi_1 - P_1\phi_2 = 0 \\ &\Longleftrightarrow \quad (\mathbf{1} - P_1)\phi_2 = P_1\phi_1.\end{aligned}$$

Hence $(\mathbf{1} - P_1)\phi_2 = 0 = P_1\phi_1$ and therefore

$$\mathrm{Ker}(Q_0 + Q_1) \subset \Big(\big(\mathrm{Ker}(Q_0 - \mathbf{1}) \cap \mathrm{Ker}(Q_1 + \mathbf{1})\big) \oplus \big(\mathrm{Ker}(Q_0 + \mathbf{1}) \cap \mathrm{Ker}(Q_1 - \mathbf{1})\big)\Big)$$

As the reverse inclusion is obvious, this implies the claim. □

If Q_0 and Q_1 are expressed in terms of P_0 and P_1, then the operators R_0 and R_1 defined in Lemma 5.3.6 are given by

$$R_0 = \mathbf{1} - (P_0 + P_1) = \frac{1}{2}(Q_0 + Q_1), \quad R_1 = P_0 - P_1 = \frac{1}{2}(Q_1 - Q_0).$$

Then the second set of identities of Lemma 5.3.6 becomes

$$\begin{aligned} R_0Q_0 &= Q_1R_0, \quad & R_1Q_0 &= -Q_1R_1, \quad & R_0^2Q_0 &= Q_0R_0^2, \\ R_0Q_1 &= Q_0R_0, \quad & R_1Q_1 &= -Q_0R_1, \quad & R_0R_1Q_0 &= -Q_0R_0R_1. \end{aligned}$$

Replacing R_1 in the formula in Theorem 5.3.8, one finds for a Fredholm pair (Q_0, Q_1) of symmetries

$$\mathrm{Ind}(Q_0, Q_1) = \dim(\mathrm{Ker}(R_1 - \mathbf{1})) - \dim(\mathrm{Ker}(R_1 + \mathbf{1})). \tag{5.20}$$

This leads to the following further formula for the index of Fredholm pairs.

Proposition 5.4.4. *For a Fredholm pair* (Q_0, Q_1) *of symmetries,*

$$\mathrm{Ind}(Q_0, Q_1) = \mathrm{Sig}\big((Q_1 - Q_0)|_{\mathrm{Ker}(Q_0+Q_1)}\big).$$

Proof. First of all, let us note that $\mathrm{Ker}(R_0)$ is an invariant subspace for R_1. Indeed, if $\phi \in \mathrm{Ker}(R_0)$, then, exploring the second identity in (5.15), one finds $R_0R_1\phi = -R_1R_0\phi = 0$. Moreover, the first identity $R_0^2 + R_1^2 = \mathbf{1}$ implies that $R_1|_{\mathrm{Ker}(R_0)}$ is nondegenerate. Hence the signature of this finite-dimensional operator is well defined. More precisely, one has

$$(R_1|_{\mathrm{Ker}(R_0)})^2 = \mathbf{1}|_{\mathrm{Ker}(R_0)},$$

namely $R_1|_{\mathrm{Ker}(R_0)}$ is a symmetry on $\mathrm{Ker}(R_0)$. Using again that on the spectral subspaces $\chi_{\{\pm 1\}}(R_1)$ of R_1 the projections P_0 and P_1 are either the identity or the zero map, one obtains

$$\mathrm{Ker}(R_1 - \mathbf{1}) = \big(\mathrm{Ker}(Q_1 - \mathbf{1}) \cap \mathrm{Ker}(Q_0 + \mathbf{1})\big)$$

and

$$\mathrm{Ker}(R_1 + \mathbf{1}) = \big(\mathrm{Ker}(Q_1 + \mathbf{1}) \cap \mathrm{Ker}(Q_0 - \mathbf{1})\big).$$

By Lemma 5.4.3,

$$\mathrm{Ker}(R_0) = \mathrm{Ker}(R_1 - \mathbf{1}) \oplus \mathrm{Ker}(R_1 + \mathbf{1}).$$

Thus $\mathrm{Sig}((Q_1 - Q_0)|_{\mathrm{Ker}(Q_0+Q_1)})$ is given by the difference of dimension on the right-hand side of (5.20). □

5.5 Fredholm pairs of unitary conjugate projections

In many applications Fredholm pairs are explicitly given by pairs of unitary conjugate orthogonal projections, namely given in the form $(P_0, P_1) = (P, U^*PU)$ with a unitary operator U. Conversely, if P_0 and P_1 are both proper, namely have infinite-dimensional range and kernel, then they are always unitarily equivalent, as Proposition 5.1.7 shows. Hence many of the results of the last two sections transfer to this case, but sometimes take a slightly different form worth noting, in particular, for the context of applications.

Let us begin rewriting the Fredholm condition in this situation, which follows directly from Proposition 5.3.1.

Corollary 5.5.1. *Let P_0 and $P_1 = U^* P_0 U$ be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*

$$P_0 U^* P_0 U P_0 + \mathbf{1} - P_0 \quad \textit{and} \quad P_0 U P_0 U^* P_0 + \mathbf{1} - P_0$$

are Fredholm operators on $\mathcal{H}$. If (P_0, P_1) is a Fredholm pair, then

$$\operatorname{Ind}(P_0, P_1) = \dim(\operatorname{Ker}(P_0 U^* P_0 U P_0 + \mathbf{1} - P_0)) - \dim(\operatorname{Ker}(P_0 U P_0 U^* P_0 + \mathbf{1} - P_0)).$$

Note that the Fredholm property of $P_0 U^* P_0 U P_0 + \mathbf{1} - P_0$ is not sufficient for $(P_0, U^* P_0 U)$ to be a Fredholm pair. This can be shown by considering $\mathcal{H} = \mathcal{H}' \otimes \mathbb{C}^3$ for an infinite-dimensional separable Hilbert space $\mathcal{H}'$ and setting

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by Proposition 5.1.7, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that

$$P_1 = U^* P_0 U = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One directly checks that $P_0 U^* P_0 U P_0 + \mathbf{1} - P_0 = \mathbf{1}$ is Fredholm but $P_1 P_0 P_1 + \mathbf{1} - P_1 = P_0 + \mathbf{1} - P_1$ is not Fredholm. Therefore by Proposition 5.3.1, $(P_0, U^* P_0 U)$ is not a Fredholm pair.

In many situations one has the property that $[P, U]$ is compact. This does, however, not necessary hold for every Fredholm pair $(P, U^* P U)$, as shows the following remark.

Remark 5.5.2. This elaborates on Remark 5.3.16. Let

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \mathbf{1} \otimes \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

act on $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. Then $P_1 = U^* P_0 U$ for the unitary operator

$$U = \mathbf{1} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and, by Remark 5.3.16, (P_0, P_1) is a Fredholm pair. On the other hand, neither the commutator $[U, P_0]$ nor $P_0 - P_1$ is compact. This generalizes as follows: Let P_0 and P_1 be two orthogonal projections such that $P_1 = U^* P_0 U$ for a unitary U. As

$$P_0 - P_1 = P_0 - U^* P_0 U = U^* [U, P_0],$$

the difference $P_0 - P_1$ is compact if and only if $[U, P_0]$ is compact. ◇

Proposition 5.5.3. *Let P be an orthogonal projection and $U \in \mathbb{U}(\mathcal{H})$ unitary. Then (P, U^*PU) is a Fredholm pair if and only if $PUP + \mathbf{1} - P$ is Fredholm and in this case*

$$\mathrm{Ind}(P, U^*PU) = \mathrm{Ind}(PUP + \mathbf{1} - P).$$

Proof. Let us set $T = PUP + \mathbf{1} - P$. Then, by Corollary 5.5.1, (P, U^*PU) is a Fredholm pair if and only if $PU^*PUP + \mathbf{1} - P = T^*T$ and $PUPU^*P + \mathbf{1} - P = TT^*$ are Fredholm, which is, by Theorem 3.4.1, equivalent to the Fredholm property of T. Moreover, by the expression for the index given in by Corollary 5.5.1,

$$\begin{aligned}
&\mathrm{Ind}(P, U^*PU) \\
&\quad = \dim(\mathrm{Ker}(PU^*PUP + \mathbf{1} - P)) - \dim(\mathrm{Ker}(PUPU^*P + \mathbf{1} - P)) \\
&\quad = \dim(\mathrm{Ker}(T^*T)) - \dim(\mathrm{Ker}(TT^*)).
\end{aligned}$$

As $\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T^*T)) - \dim(\mathrm{Ker}(TT^*))$, this implies the claim. □

The following corollary is a consequence of Proposition 5.3.15.

Corollary 5.5.4. (i) *Suppose given an orthogonal projection P and furthermore two unitaries $U_1, U_2 \in \mathbb{U}(\mathcal{H})$ such that $\|\pi([U_1, P])\|_{\mathbb{Q}} + \|\pi([U_2, U_1^*PU_1])\|_{\mathbb{Q}} < 1$ holds. Then $(P, U_1^*PU_1)$, $(U_1^*PU_1, (U_1U_2)^*PU_1U_2)$, and $(P, (U_1U_2)^*PU_1U_2)$ are all Fredholm pairs and*

$$\mathrm{Ind}(P, (U_1U_2)^*PU_1U_2) = \mathrm{Ind}(P, U_1^*PU_1) + \mathrm{Ind}(U_1^*PU_1, (U_1U_2)^*PU_1U_2).$$

(ii) *Let $P \in \mathbb{B}(\mathcal{H})$ be an orthogonal projection and $U \in \mathbb{U}(\mathcal{H})$ unitary such that for some $n \in \mathbb{N}$ and all $k \in \{0, \ldots, n-1\}$ one has $\|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, P])\|_{\mathbb{Q}} < 1$. Then (P, U^*PU), $(P, (U^*)^nPU^n)$, and $(P, U^nP(U^*)^n)$ are Fredholm pairs with index*

$$\mathrm{Ind}(P, (U^*)^nPU^n) = n\,\mathrm{Ind}(P, U^*PU) = -\mathrm{Ind}(P, U^nP(U^*)^n).$$

Proof. To show (i), note that $P - U^*PU = U^*[U, P]$ and therefore

$$\|\pi(P - U^*PU)\|_{\mathbb{Q}} = \|\pi([U, P])\|_{\mathbb{Q}}$$

for any orthogonal projection $P \in \mathbb{B}(\mathcal{H})$ and unitary $U \in \mathbb{U}(\mathcal{H})$. Hence

$$\|\pi(P - U_1^*PU_1)\|_{\mathbb{Q}} + \|\pi(U_1^*PU_1 - (U_1U_2)^*PU_1U_2)\|_{\mathbb{Q}} < 1,$$

and the claim follows from Proposition 5.3.15.

Because $\|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, U^*PU])\|_{\mathbb{Q}} = \|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, P])\|_{\mathbb{Q}} < 1$, the first part of this corollary implies that (P, U^*PU), $(U^*PU, (U^*)^{k+1}PU^{k+1})$, and $(P, (U^*)^{k+1}PU^{k+1})$ are Fredholm pairs and

$$\mathrm{Ind}(P, (U^*)^{k+1}PU^{k+1}) = \mathrm{Ind}(P, U^*PU) + \mathrm{Ind}(U^*PU, (U^*)^{k+1}PU^{k+1}).$$

As $\mathrm{Ind}(U^*PU, (U^*)^{k+1}PU^{k+1}) = \mathrm{Ind}(P, (U^*)^k PU^k)$, it follows iteratively that

$$\mathrm{Ind}(P, (U^*)^{k+1}PU^{k+1}) = (k+1)\,\mathrm{Ind}(P, U^*PU),$$

which implies the first claim. The claim on $\mathrm{Ind}(P, U^nP(U^*)^n)$ follows by exchanging the roles of U and U^*. □

The following is merely a reformulation of Proposition 5.2.10 and Theorem 5.2.11.

Proposition 5.5.5. *Let $P \in \mathbb{B}(\mathcal{H})$ be an orthogonal projection, $U \in \mathbb{U}(\mathcal{H})$ unitary, and $n \in \mathbb{N}$ be such that*

$$P - PU^*PUP \in \mathcal{L}^n(\mathrm{Ran}(P)), \quad P - PUPU^*P \in \mathcal{L}^n(\mathrm{Ran}(P)).$$

Then (P, U^*PU) is a Fredholm pair of orthogonal projections, and for all $m \geq n$ one has

$$\mathrm{Ind}(P, U^*PU) = \mathrm{Tr}((P - PU^*PUP)^m) - \mathrm{Tr}((P - PUPU^*P)^m).$$

If (P, U^*PU) is a Fredholm pair of orthogonal projections and $(P - U^*PU)^{2n'+1}$ is trace class for some integer $n' \geq 0$, then for all $m' \geq n'$,

$$\mathrm{Ind}(P, U^*PU) = \mathrm{Tr}((P - U^*PU)^{2m'+1}).$$

Proof. As the property $P - PUPU^*P \in \mathcal{L}^n(\mathrm{Ran}(P))$ is equivalent to $U^*PU - U^*PUPU^*PU$ lying in $\mathcal{L}^n(\mathrm{Ran}(U^*PU))$ and

$$\mathrm{Tr}((P - PUPU^*P)^m) = \mathrm{Tr}((U^*PU - U^*PUPU^*PU)^m),$$

Proposition 5.2.10 implies the first claim. The second claim directly follows from Theorem 5.2.11. □

From the formula in Theorem 5.3.8, one can directly deduce the next result (taken from [18]).

Proposition 5.5.6. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections. There exists a unitary $V \in \mathbb{U}(\mathcal{H})$ such that*

$$VP_0V^* = P_1 \quad \text{and} \quad VP_1V^* = P_0$$

if and only if $\mathrm{Ind}(P_0, P_1) = 0$.

Proof. If such a V exists, then $V(P_0 - P_1)V^* = P_1 - P_0$ and thus, by Theorem 5.3.8, $\mathrm{Ind}(P_0, P_1) = 0$.

Conversely, let (P_0, P_1) be a Fredholm pair with vanishing index. As above, define $P_\pm = \chi_{\{\pm1\}}(P_0 - P_1)$. As $\mathrm{Ind}(P_0, P_1) = 0$, $\mathrm{Ran}(P_+)$ and $\mathrm{Ran}(P_-)$ have the same dimensions by Theorem 5.3.8, there is a unitary operator $U_0 : \mathrm{Ran}(P_+) \to \mathrm{Ran}(P_-)$. Then the operator

$V_0 : \mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-) \to \mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-)$ defined by $V_0(\phi_+ + \phi_-) = U_0^*\phi_- + U_0\phi_+$ for $\phi_+ \in \mathrm{Ran}(P_+)$ and $\phi_- \in \mathrm{Ran}(P_-)$ is unitary. Let $\tilde{V}$ denote the partial isometry in the polar decomposition $\mathbf{1} - P_0 - P_1 = \tilde{V}|\mathbf{1} - P_0 - P_1|$ of the operator $\mathbf{1} - P_0 - P_1$. Then also the restriction $V_1 = \tilde{V}|_{\mathrm{Ran}(\mathbf{1}-P_0-P_1)} : \mathrm{Ran}(\mathbf{1} - P_0 - P_1) \to \mathrm{Ran}(\mathbf{1} - P_0 - P_1)$ is unitary. Note that, by Lemma 5.4.3, $\mathrm{Ran}(\mathbf{1} - P_0 - P_1) = \mathcal{H} \ominus (\mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-))$. As $\mathbf{1} - P_0 - P_1$ and $P_0 - P_1$ anticommute,

$$\tilde{V}(P_0 - P_1) = (P_1 - P_0)\tilde{V} \quad \text{and} \quad \tilde{V}(P_0 + P_1) = (P_0 + P_1)\tilde{V},$$

thus

$$\tilde{V}P_0 = P_1\tilde{V} \quad \text{and} \quad \tilde{V}P_1 = P_0\tilde{V}.$$

One directly checks that

$$VP_0V^* = P_1 \quad \text{and} \quad VP_1V^* = P_0$$

hold for $V = V_0 \oplus \tilde{V}$. □

5.6 Spectral flow of linear paths between Fredholm pairs

This section collects several formulas connecting the index of Fredholm pairs of orthogonal projections to a spectral flow. Let us begin with an expression of the spectral flow of the linear path connecting two symmetries that form a Fredholm pair by the index of this Fredholm pair.

Theorem 5.6.1. *For any Fredholm pair of symmetries Q_0, Q_1 on $\mathcal{H}$, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big) = \mathrm{Ind}(Q_0, Q_1). \tag{5.21}$$

Proof. The operators $H_t = (1-t)Q_0 + tQ_1$ are Fredholm. Indeed, $H_t = Q_0 + t(Q_1 - Q_0)$ is for $t \in [0, \frac{1}{2}]$ a perturbation of an operator Q_0 with spectrum $\{-1, 1\}$. As the Fredholm condition of the pair (Q_0, Q_1) is equivalent to $\|\pi(Q_0 - Q_1)\|_{\mathcal{Q}} < 2$ by Corollary 5.3.13, it follows that H_t has its essential spectrum bounded away from 0 for $t \in [0, \frac{1}{2}]$. Furthermore, for $t \in [\frac{1}{2}, 1]$ one can write $H_t = Q_1 + (1-t)(Q_0 - Q_1)$ so that the same argument applies. Moreover, H_t is invertible except possibly at $t = \frac{1}{2}$. The derivative at this point is

$$\partial_t H_t|_{t=\frac{1}{2}} = Q_1 - Q_0.$$

Hence the crossing form at $t = \frac{1}{2}$ is

$$\Gamma_{\frac{1}{2}} : \mathrm{Ker}(H_{\frac{1}{2}}) \to \mathrm{Ker}(H_{\frac{1}{2}}), \quad \Gamma_{\frac{1}{2}}(\phi) = \big\langle \phi \big| (Q_1 - Q_0)\phi \big\rangle,$$

and its signature is equal to the spectral flow by Proposition 4.3.6 which applies because the crossing was shown to be regular in the proof of Proposition 5.4.4. But, by Proposition 5.4.4, this signature is precisely the index of the Fredholm pair (Q_0, Q_1) of symmetries. □

Let us stress that in the earlier works [207, 148, 84], the equality (5.21) was only shown under the hypothesis that $Q_0 - Q_1$ is compact (or equivalently that the associated orthogonal projections have a compact difference $P_0 - P_1$). Next recall from Proposition 5.1.7 that, given two proper symmetries Q_0 and Q_1, it is always possible to find a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $Q_1 = U^* Q_0 U$. For this situation, one thus obtains from Proposition 5.5.3:

Corollary 5.6.2. *For a Fredholm pair of symmetries $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ on $\mathcal{H}$ and a unitary U such that $Q_1 = U^* Q_0 U$, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big) = \mathrm{Ind}(P_0 U P_0 + \mathbf{1} - P_0).$$

Proof. As $P_1 = U^* P_0 U$, the claim directly follows from Theorem 5.6.1 and Proposition 5.5.3. □

Combined with Theorem 5.2.12, one also deduces the following formula for the spectral flow which is similar in spirit to Proposition 4.3.12.

Corollary 5.6.3. *Let (Q_0, Q_1) be a Fredholm pair of symmetries such that $(Q_0 - Q_1)^{2n+1}$ is trace class for some integer $n \geq 0$. Then for the linear path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$ and any $k \geq n$,*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) = \frac{1}{C_k} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (1 - Q_t^2)^k\big), \tag{5.22}$$

with C_k given in (5.13).

Similarly, also Proposition 5.3.11 leads to a formula for the spectral flow, see [56].

Corollary 5.6.4. *Let (Q_0, Q_1) be a Fredholm pair of symmetries on $\mathcal{H}$ such that the operator* $\exp(-((Q_0 - Q_1)^2)^{-\frac{1}{q}})$ *is trace class for some* $0 < q \leq 1$. *Then the path*

$$t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$$

satisfies

$$\mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) = \frac{1}{C_{r,q}} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (\mathbf{1} - Q_t^2)^{-r} e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}}\big)$$

for $r \geq 0$ and where $C_{r,q}$ is given in (5.17).

5.7 Spectral flow formulas for paths with compact difference

Section 4.3 already presented a quite diverse selection of formulas for the spectral flow. Here further formulas are provided, all based on the results of the last Section 5.6. First let us generalize Theorem 5.6.1 to linear paths connecting two invertible self-adjoint operators (instead of symmetries) with compact difference.

Corollary 5.7.1. *For self-adjoint invertible operators $H_0, H_1 \in \mathbb{B}(\mathcal{H})$ such that the difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ is compact, one has*

$$\begin{aligned}\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big) &= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big) \\ &= \mathrm{Ind}(Q_0, Q_1),\end{aligned}$$

where $Q_i = H_i|H_i|^{-1}$ is the unitary phase of H_i for $i = 0, 1$.

Proof. (Some elements are similar to the proof of Proposition 5.3.17.) As $H_1 - H_0$ is compact, $(1-t)H_0 + tH_1 = H_0 + t(H_1 - H_0)$ is Fredholm for all $t \in [0,1]$ by Corollary 3.2.3. Moreover, $h(H_1) - h(H_0)$ is compact for every continuous function $h : \Sigma \to \mathbb{C}$ where $\Sigma = \mathrm{spec}(H_0) \cup \mathrm{spec}(H_1)$. As

$$H_1^n - H_0^n = H_1(H_1^{n-1} - H_0^{n-1}) + (H_1 - H_0)H_0^{n-1}, \quad n \geq 2,$$

$p(H_1) - p(H_0)$ is compact for any polynomial $p : \mathbb{C} \to \mathbb{C}$. As the set of compact operators $\mathbb{K}(\mathcal{H})$ is a closed subset of the set of bounded operators $\mathbb{B}(\mathcal{H})$ and the polynomials are dense in the set of continuous functions on compact domains, we see that $h(H_1) - h(H_0)$ is compact for every continuous function $h : \mathrm{spec}(H_0) \cup \mathrm{spec}(H_1) \to \mathbb{C}$. Therefore $(t, s) \in [0,1] \times [0,1] \mapsto (1-t)H_0|H_0|^{-s} + tH_1|H_1|^{-s}$ is a continuous homotopy of Fredholm operators. By Theorem 4.2.2,

$$\begin{aligned}&\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big) \\ &\quad = \mathrm{Sf}\big(s \in [0,1] \mapsto H_0|H_0|^{-s}\big) \\ &\qquad + \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0|H_0|^{-1} + tH_1|H_1|^{-1}\big) \\ &\qquad - \mathrm{Sf}\big(s \in [0,1] \mapsto H_1|H_1|^{-s}\big).\end{aligned}$$

As

$$s \in [0,1] \mapsto H_0|H_0|^{-s} \quad \text{and} \quad s \in [0,1] \mapsto H_1|H_1|^{-s}$$

are paths of invertibles and therefore the spectral flow along these paths vanishes, and $H_0|H_0|^{-1} = Q_0$ and $H_1|H_1|^{-1} = Q_1$, one has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big).$$

The remaining claim follows from Theorem 5.6.1. □

Similar as Corollary 5.6.2 is a special case of Theorem 5.6.1, one can now state Corollary 5.7.1 for the special case of paths with unitary equivalent endpoints.

Corollary 5.7.2. *For a self-adjoint invertible operator $H \in \mathbb{B}(\mathcal{H})$ and a unitary $U \in \mathbb{U}(\mathcal{H})$ such that the commutator $[H, U] \in \mathbb{K}(\mathcal{H})$ is compact, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H + tU^*HU\big) = \mathrm{Ind}(PUP + \mathbf{1} - P),$$

where $P = \chi(H \le 0)$ is the orthogonal projection onto the negative spectrum of H.

Proof. As $H - U^*HU = [H, U^*]U$ is compact,

$$\begin{aligned}\mathrm{Sf}\big(t \in [0,1] &\mapsto (1-t)H + tU^*HU\big)\\ &= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)(\mathbf{1}-2P) + tU^*(\mathbf{1}-2P)U\big),\end{aligned}$$

by Corollary 5.7.1. Now the claim follows from Corollary 5.6.2. □

The next result is the starting point for many applications, e. g., all of Chapter 10. It also considers a situation similar to Corollary 5.7.2, namely paths with unitary conjugate endpoints, but does not require the paths to be linear. The result goes back to the work of Phillips [148] with precursors like Wojciechowski [207], see also [70].

Theorem 5.7.3. *Let $t \in [0,1] \mapsto H_t$ be a norm-continuous path of self-adjoint operators with invertible endpoints H_0 and H_1 such that $H_t - H_0$ is compact for all $t \in [0,1]$ and $H_1 = U^*H_0U$. If $P = \chi(H_0 \le 0)$, then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Ind}(PUP + \mathbf{1} - P).$$

In particular, one has for the linear path connecting $\mathbf{1} - 2P$ and $U^*(\mathbf{1} - 2P)U$,

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)(\mathbf{1}-2P) + tU^*(\mathbf{1}-2P)U\big) = \mathrm{Ind}(PUP + \mathbf{1} - P).$$

Proof. First $H_1 - H_0 = U^*H_0U - H_0 = U^*[H_0, U] \in \mathbb{K}(\mathcal{H})$ is compact by assumption. Thus $[H_0, U] \in \mathbb{K}(\mathcal{H})$ is compact and, by Corollary 5.7.2,

$$\mathrm{Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tU^*H_0U\big). \tag{5.23}$$

The homotopy $h : [0,1] \times [0,1] \to \mathbb{B}(\mathcal{H})$,

$$h(t,s) = (1-s)H_t + s\big((1-t)H_0 + tH_1\big),$$

is within the Fredholm operators as $h(t,s) = H_0 + (1-s)(H_t - H_0) + st(H_1 - H_0)$ and $(1-s)(H_t - H_0) + st(H_1 - H_0)$ is compact for all $(t,s) \in [0,1] \times [0,1]$. As $h(0,s) = H_0$ and $h(1,s) = H_1$ for all $s \in [0,1]$, Theorem 4.2.2 implies

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto h(t,0)\big)$$

$$= \mathrm{Sf}\big(t \in [0,1] \mapsto h(t,1)\big)$$
$$= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tU^* H_0 U\big),$$

which by (5.23) implies the claim. □

Let us give an elementary example illustrating some of the above facts.

Example 5.7.4. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $|n\rangle$, $n \in \mathbb{Z}$. Introduce the symmetry

$$Q_0 = \sum_{n\geq 0} |n\rangle\langle n| - \sum_{n<0} |n\rangle\langle n|.$$

Furthermore, let S be the left-shift on $\mathcal{H}$ given by $S|n\rangle = |n-1\rangle$. For $k \in \mathbb{N}$, choose $U = (S^k)^*$ and set $Q_k = (S^k)^* Q_0 S^k$. Now, roughly stated, Q_k has k less positive eigenvalues than Q_0. This difference between infinities is taken into account by the spectral flow. Calculating the spectrum on the straight line path $H_t = (1-t)Q_0 + tQ_k \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ explicitly shows

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = -k.$$

Alternatively, $(\partial_t H)_t = Q_k - Q_0 = -2P_k$ where P_k is the finite-dimensional orthogonal projection on the span of $|n\rangle$, $n = 0, \dots, k-1$. Then by Proposition 4.3.12,

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \frac{1}{2}\int_0^1 dt\, \mathrm{Tr}\big(g'(H_t)(\partial_t H)_t\big) = -k,$$

where g is a smooth nonnegative function of integral 1 which is supported in the gap of the essential spectrum of H_t for all $t \in [0,1]$. ◇

The Fredholm operators in Theorem 5.7.3 often appear as the result of an even index pairing between a K_0-class specified by P (e. g., of the C^*-algebra generated by H_0) and a graded Dirac operator, see Section 10.1 for a detailed description. Section 10.1 also describes odd index pairings and the following result can be interpreted as an odd (or dual) analogue to Theorem 5.7.3, namely an index formula for paths of unitaries with conjugate endpoints by a self-adjoint conjugation operator. It will use the notion of spectral flow of paths of normal operators as given in (4.14).

Theorem 5.7.5. *Let $t \in [0,1] \mapsto U_t$ be a path of unitaries such that $U_t - U_0$ is compact. Suppose that there is a self-adjoint unitary G such that $U_1 = GU_0G$. If $E = \chi(G \geq 0)$, then $EU_0E + \mathbf{1} - E$ is a Fredholm operator with index given by*

$$\mathrm{Ind}(EU_0E + \mathbf{1} - E) = \mathrm{Sf}\big(t \in [0,1] \mapsto \mathbb{R}e(W_t)\big), \tag{5.24}$$

where $W_t = GU_tU_0^*$ is unitary.

Proof. To show that the spectral flow is well defined, first note that $U_t U_0^* - \mathbf{1} = (U_t - U_0)U_0^*$ is compact by hypothesis, so that $W_t - G$ is compact and so is $\mathbb{R}e(W_t) - G$. Moreover, $W_0 = G = \mathbb{R}e(W_0)$ and $W_1 = U_0 G U_0^* = \mathbb{R}e(W_1)$ are both self-adjoint, and one has $\mathbb{R}e(W_1) = U_0 \mathbb{R}e(W_0) U_0^*$. Thus Theorem 5.7.3 can be applied to $t \in [0,1] \mapsto H_t = \mathbb{R}e(W_t)$. There are now two sign changes in the index pairing involved, one because E is the spectral projection onto the positive spectrum of G and one because U_0 is on the left-hand side in $\mathbb{R}e(W_1) = U_0 \mathbb{R}e(W_0) U_0^*$ (while P is the negative spectral projection of H_0 and $H_1 = U^* H_0 U$ in Theorem 5.7.3). This concludes the proof. □

The spectral flow of unitaries appearing on the right-hand side of (5.24) inherits natural homotopy invariance properties. For example, choosing $U_0 U_t^* G$ instead of W_t is another natural choice giving a different path connecting G and $U_0 G U_0^*$. The choices $G U_t^* U_0$ and $U_0^* U_t G$ reverse the path and thus the sign of the spectral flow. A standard choice of $t \in [0,1] \mapsto U_t$ leading to a path $t \in [0,1] \mapsto G U_t U_0^*$ from G to $U_0 G U_0^*$, expressed merely in terms of U_0 and G, is given by

$$U_t = U_0 \exp\left(\frac{\imath\pi}{2} (G - \mathbf{1} + t U_0^* [G, U_0]) \right) G.$$

As explained in [70], this path leads to a K-theoretic interpretation of Theorem 5.7.5.

The next set of results generalizes the formula given in Corollary 5.6.3. The proofs are based on Singer's idea to use closed 1-forms [183] which in this context was further developed in the work of Getzler [96] and more thoroughly in the works of Carey and Phillips [55, 56]. The latter two papers contain more general versions of the next results. More precisely, these works require fewer summability assumptions and also deal with the case of semifinite spectral flow discussed in Chapter 11.

Proposition 5.7.6. *Let (Q_0, Q_1) be a Fredholm pair of symmetries on $\mathcal{H}$ connected by a path $t \in [0,1] \mapsto Q_t \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ such that $(Q_t - Q_0)^{2n+1}$ is trace class for some integer $n \geq 0$ and the path is continuously $\mathcal{L}^{2n+1}$-differentiable. Then one has for any $k \geq n$,*

$$\mathrm{Sf}(t \in [0,1] \mapsto Q_t) = \frac{1}{C_k} \int_0^1 dt \, \mathrm{Tr}((\partial_t Q)_t (\mathbf{1} - Q_t^2)^k), \tag{5.25}$$

with C_k given in (5.13).

Proof. Let $\mathcal{L}_{\mathrm{sa}}^{2n+1}(\mathcal{H})$ denote the set of self-adjoint operators in the $(2n+1)$th Schatten ideal $\mathcal{L}^{2n+1}(\mathcal{H})$. Then consider the set $\mathcal{M} = Q_0 + \mathcal{L}_{\mathrm{sa}}^{2n+1}(\mathcal{H})$ as a manifold with tangent space $T\mathcal{M} = \mathcal{L}_{\mathrm{sa}}^{2n+1}(\mathcal{H})$. By assumption the path $t \in [0,1] \mapsto Q_t$ lies in $\mathcal{M}$ and is differentiable with derivatives $(\partial_t Q)_t$ lying in $\mathcal{L}_{\mathrm{sa}}^{2n+1}(\mathcal{H})$. Let us introduce a 1-form α_k on $\mathcal{M}$ by setting

$$\alpha_{k,Q}(X) = \frac{1}{C_k} \mathrm{Tr}(X(\mathbf{1} - Q^2)^k), \quad X \in T\mathcal{M}, \ Q \in \mathcal{M}. \tag{5.26}$$

Note that $\alpha_{k,Q}(X)$ is real because it is given by the trace of a product of two self-adjoint operators. The integral on the right-hand side of (5.25) is by definition the integral of α_k over the path $t \in [0,1] \mapsto Q_t$. To show the claim (5.25), it will be verified that this integral of α_k over a path is invariant under changes of the path inside $\mathcal{M}$ with fixed endpoints, or alternatively that it vanishes on closed curves. This will follow by adapting a standard argument. Let us first show that the form α_k is closed, namely that one has

$$\partial_s|_{s=0}\alpha_{k,Q+sY}(X) = \partial_s|_{s=0}\alpha_{k,Q+sX}(Y), \quad X, Y \in T\mathcal{M}.$$

This follows from a computation based on the Leibniz rule, $\partial_s|_{s=0}(Q+sY)^2 = QY + YQ$ and the cyclicity of the trace:

$$\begin{aligned} C_k\partial_s|_{s=0}\alpha_{k,Q+sY}(X) &= \partial_s|_{s=0}\operatorname{Tr}\big(X(\mathbf{1}-(Q+sY)^2)^k\big) \\ &= -\sum_{l=0}^{k-1}\operatorname{Tr}\big(X(\mathbf{1}-Q^2)^l(QY+YQ)(\mathbf{1}-Q^2)^{k-1-l}\big) \\ &= -\sum_{l=0}^{k-1}\operatorname{Tr}\big(Y(\mathbf{1}-Q^2)^{k-1-l}(QX+XQ)(\mathbf{1}-Q^2)^l\big) \\ &= C_k\partial_s|_{s=0}\alpha_{k,Q+sX}(Y). \end{aligned}$$

Now given that α_k is closed, one can deduce that the integral of α_k over a closed curve vanishes. This can first be shown for rectangles lying in a two-dimensional plane spanned by two vectors $X, Y \in T\mathcal{M}$, by transposing Pirkheimer's proof of the Goursat lemma. Then one can deduce it by the usual approximation arguments for an arbitrary differentiable curve in $\mathcal{M}$. Let us stress that the argument only requires that the derivatives of α_k exist (and neither their continuity nor the exactness of the form which in the present situation is given, but not when the above argument is applied in the proof of Theorem 7.2.2 later on). Therefore the right-hand side of (5.25) equals the integral of α_k over the linear path connecting Q_0 to Q_1 which, by Corollary 5.6.3, equals the spectral flow of the linear path connecting Q_0 to Q_1. As $Q_0 - Q_t$ is compact for all $t \in [0,1]$, the claim follows from the homotopy invariance of the spectral flow. □

Remark 5.7.7. The 1-form defined in (5.26) satisfies $\alpha_k = d\beta_{k,F'}$ where, for an arbitrary fixed point $F' \in \mathcal{M} = Q_0 + \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$, the 0-form $\beta_{k,F'} : \mathcal{M} \to \mathbb{C}$ is defined by

$$\beta_{k,F'}(F) = \frac{1}{C_k}\int_0^1 dt\,\operatorname{Tr}\big((\partial_t F)_t(\mathbf{1}-F_t^2)^k\big),$$

where $F_t = F' + t(F - F')$ is the linear path between F' and F. This is merely the Poincaré lemma for the 1-form α_k which holds globally in all of $\mathcal{M}$. This can be verified by an explicit computation as the one in the proof of Proposition 5.7.6 which we provide for the convenience of the reader. The claim $\alpha_k = d\beta_{k,F'}$ explicitly means

$$\partial_s|_{s=0}\beta_{k,F'}(F+sX)=\alpha_{k,F}(X),\quad X\in T\mathcal{M}.$$

To verify this, let us set

$$F_r(s)=(1-r)F'+r(F+sX),$$

so that $\partial_s F_r(s)=rX$ and $\partial_r F_r(s)=F+sX-F'$, as well as $F_r=F_r(0)$. Then

$$\begin{aligned}
&C_k\partial_s|_{s=0}\beta_{k,F'}(F+sX)\\
&\quad=\partial_s|_{s=0}\int_0^1 dr\,\mathrm{Tr}\big((F+sX-F')(1-F_r(s)^2)^k\big)\\
&\quad=\int_0^1 dr\big[\mathrm{Tr}\big(X(1-F_r^2)^k\big)+\partial_s|_{s=0}\,\mathrm{Tr}\big((F-F')(1-F_r(s)^2)^k\big)\big].
\end{aligned}$$

The derivative is computed as above, using the Leibniz rule and the cyclicity of the trace,

$$\begin{aligned}
&\partial_s|_{s=0}\,\mathrm{Tr}\big((F-F')(1-F_r(s)^2)^k\big)\\
&\quad=-\sum_{l=0}^{k-1}\mathrm{Tr}\big((F-F')(1-F_r^2)^l\partial_s|_{s=0}F_r(s)^2(1-F_r^2)^{k-1-l}\big)\\
&\quad=-r\sum_{l=0}^{k-1}\mathrm{Tr}\big((F-F')(1-F_r^2)^l(XF_r+F_rX)(1-F_r^2)^{k-1-l}\big)\\
&\quad=-r\sum_{l=0}^{k-1}\mathrm{Tr}\big(X(1-F_r^2)^{k-1-l}(F_r(F-F')+(F-F')F_r)(1-F_r^2)^l\big)\\
&\quad=r\partial_r\,\mathrm{Tr}\big(X(1-F_r^2)^k\big).
\end{aligned}$$

By replacing, one finds

$$\begin{aligned}
C_k\partial_s|_{s=0}\beta_{k,F'}(F+sX)&=\int_0^1 dr\big[\mathrm{Tr}\big(X(1-F_r^2)^k\big)+r\partial_r\,\mathrm{Tr}\big(X(1-F_r^2)^k\big)\big]\\
&=\int_0^1 dr\partial_r\big[r\,\mathrm{Tr}\big(X(1-F_r^2)^k\big)\big]\\
&=\mathrm{Tr}\big(X(1-F_1^2)^k\big)\\
&=\mathrm{Tr}\big(X(1-F^2)^k\big)=C_k\alpha_{k,F}(X),
\end{aligned}$$

which shows the claim. ◇

In the following, Proposition 5.7.6 will be further generalized to paths for which the endpoints are not necessarily symmetries. The following object is needed.

Definition 5.7.8. Let $F_0 \in \mathbb{B}_{\rm sa}(\mathcal{H})$ be a base point satisfying $F_0^2 - \mathbf{1} \in \mathcal{L}^{2n+1}_{\rm sa}(\mathcal{H})$ for some $n \in \mathbb{N}$. Then for an invertible $F \in \mathcal{M} = F_0 + \mathcal{L}^{2n+1}_{\rm sa}(\mathcal{H})$ with phase $Q = \mathrm{sgn}(F)$ set

$$\beta_k(F) = \frac{1}{C_k} \int_0^1 dt \, \mathrm{Tr}\big((\partial_t F)_t (\mathbf{1} - F_t^2)^k\big),$$

where $t \mapsto F_t = F + t(Q - F)$ is the linear path from F to Q and $k \geq n$.

Theorem 5.7.9. *Let $t \in [0,1] \mapsto F_t \in \mathbb{B}_{\rm sa}(\mathcal{H})$ be such that*
(i) $F_0^2 - \mathbf{1} \in \mathcal{L}^{2n+1}_{\rm sa}(\mathcal{H})$,
(ii) $F_t - F_0 \in \mathcal{L}^{2n+1}_{\rm sa}(\mathcal{H})$,
(iii) *the path $t \mapsto F_t$ is continuously $\mathcal{L}^{2n+1}$-differentiable.*

Then one has for any $k \geq n$,

$$\mathrm{Sf}\big(t \in [0,1] \mapsto F_t\big) = \beta_k(F_1) - \beta_k(F_0) + \frac{1}{C_k} \int_0^1 dt \, \mathrm{Tr}\big((\partial_t F)_t (\mathbf{1} - F_t^2)^k\big).$$

Proof. First of all, note that

$$\beta_k(F_0) = \int_{[F_0, Q_0]} \alpha_k, \quad \beta_k(F_1) = \int_{[F_1, Q_1]} \alpha_k,$$

where $[F_j, Q_j]$ denotes the straight-line path from F_j to $Q_j = \mathrm{sgn}(F_j)$ for $j = 0, 1$ (these paths lie in $\mathcal{M}$). Moreover, let $[Q_0, Q_1]$ denote the path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$ from Q_0 to Q_1 (attention: Q_t is *not* equal to $\mathrm{sgn}(F_t)$). Then the path $t \in [0,1] \mapsto F_t$ is homotopic to $[F_0, Q_0] * [Q_0, Q_1] * (-[F_1, Q_1])$ where $-[F_1, Q_1]$ denotes the reversed path of $[F_1, Q_1]$. As the paths $[F_0, Q_0]$ and $[F_1, Q_1]$ lie in the invertibles, there is no spectral flow along them. Hence by the homotopy invariance of the spectral flow and Proposition 5.7.6,

$$\begin{aligned} \mathrm{Sf}\big(t \in [0,1] \mapsto F_t\big) &= \mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) \\ &= \int_{[Q_0, Q_1]} \alpha_k \\ &= \int_{[Q_0, F_0]} \alpha_k + \int_{[t \in [0,1] \mapsto F_t]} \alpha_k + \int_{[F_1, Q_1]} \alpha_k, \end{aligned}$$

where in the last step the closedness of the 1-form α_k was used in order to deform the integration path. The middle term is precisely the integral in the statement, which is hence verified. □

Remark 5.7.10. The essential ingredient of the proof of Theorem 5.7.9 is Corollary 5.6.3. It is possible to carry out a similar reasoning based on Corollary 5.6.4. This is carried out in the work of Carey and Phillips [56]. If the statement is then applied to the bounded transform of paths of self-adjoint Fredholm operators with compact resolvent, one obtains, after the change of variables connected to the bounded transform, a proof of Theorem 7.2.2, namely the equivalents of the boundary terms $\beta_k(F)$ become the η-invariants. ◇

5.8 Spectral flow as sum of indices of Fredholm pairs

In this section, it is shown how the spectral flow of an arbitrary norm-continuous path $t \in [0,1] \mapsto H_t$ of bounded self-adjoint Fredholm operators can be expressed as a sum of indices of Fredholm pairs of orthogonal projections. The outcome is Proposition 5.8.2 below. As a preparation for the statement, the following lemma is needed.

Lemma 5.8.1. *For every $H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ there is $a > 0$ and a neighborhood $\mathcal{N}'_{H,a} \subset \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous finite-rank projection-valued function on $\mathcal{N}'_{H,a}$, $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair, where $P^{\geq}(A) = \chi_{[0,\infty)}(A)$ for every self-adjoint Fredholm operator A, and*

$$\mathrm{Ind}(P^{\geq}(S), P^{\geq}(H)) = \mathrm{Ind}(\chi_{[0,a]}(S), \chi_{[0,a]}(H))$$

for all $S \in \mathcal{N}'_{H,a}$.

Proof. By Lemma 4.1.1, there is a neighborhood $\mathcal{N}$ of H and $a > 0$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous finite-rank projection-valued function on $\mathcal{N}$. By construction (see the proof of Lemma 4.1.1), the function $S \mapsto \chi_{(a,\infty)}(S)$ is norm-continuous on $\mathcal{N}$. Define $\tilde{\mathcal{N}}'_{H,a}$ as

$$\tilde{\mathcal{N}}'_{H,a} = \{S \in \mathcal{N} : \|\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S)\| < 1\}.$$

As $S \mapsto \chi_{(a,\infty)}(S)$ is norm-continuous on $\mathcal{N}$, this is a neighborhood of H. Then define $\mathcal{N}'_{H,a}$ as the connected component of $\tilde{\mathcal{N}}_{H,a}$ containing H. It remains to show that $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair with index $\mathrm{Ind}(\chi_{[0,a]}(S), \chi_{[0,a]}(H))$ for all $S \in \mathcal{N}'_{H,a}$. As $P^{\geq}(H) = \chi_{[0,a]}(H) + \chi_{(a,\infty)}(H)$ and, similarly, $P^{\geq}(S) = \chi_{[0,a]}(S) + \chi_{(a,\infty)}(S)$, where $\chi_{[0,a]}(H)$ and $\chi_{[0,a]}(S)$ are compact,

$$\begin{aligned}\|\pi(P^{\geq}(H) - P^{\geq}(S))\|_{\mathcal{Q}} &= \|\pi(\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S))\|_{\mathcal{Q}} \\ &\leq \|\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S)\| < 1.\end{aligned}$$

Therefore, by Corollary 5.3.13, $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair. Its index equals the index of $P^{\geq}(H)P^{\geq}(S)|_{\mathrm{Ran}(P^{\geq}(S))} : \mathrm{Ran}(P^{\geq}(S)) \to \mathrm{Ran}(P^{\geq}(H))$ by Definition 5.2.1. Thus

$$\begin{aligned}\mathrm{Ind}(P^{\geq}(S), P^{\geq}(H)) &= \mathrm{Ind}((\chi_{[0,a]}(H) + \chi_{(a,\infty)}(H))(\chi_{[0,a]}(S) + \chi_{(a,\infty)}(S))) \\ &= \mathrm{Ind}(\chi_{[0,a]}(H)\chi_{[0,a]}(S) + \chi_{[0,a]}(H)\chi_{(a,\infty)}(S)\end{aligned}$$

$$+\chi_{(a,\infty)}(H)\chi_{[0,a]}(S)+\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big).$$

As the second and third summands in the last expression are compact, Theorem 3.3.4 implies

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(H)\chi_{[0,a]}(S)+\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big).$$

By Corollary 3.3.2, this implies

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(H)\chi_{[0,a]}(S)\big)+\operatorname{Ind}\big(\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big),$$

where $\chi_{[0,a]}(H)\chi_{[0,a]}(S) : \operatorname{Ran}(\chi_{[0,a]}(S)) \to \operatorname{Ran}(\chi_{[0,a]}(H))$ and $\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S) : \operatorname{Ran}(\chi_{(a,\infty)}(S)) \to \operatorname{Ran}(\chi_{(a,\infty)}(H))$ are Fredholm operators. Again by Definition 5.2.1,

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(S),\chi_{[0,a]}(H)\big)+\operatorname{Ind}\big(\chi_{(a,\infty)}(S),\chi_{(a,\infty)}(H)\big)$$

follows. By definition of $\mathcal{N}'_{H,a}$, $\|\chi_{(a,\infty)}(H)-\chi_{(a,\infty)}(S)\|<1$ which, by Proposition 5.3.12, implies

$$\operatorname{Ind}\big(\chi_{(a,\infty)}(S),\chi_{(a,\infty)}(H)\big)=0.$$

Therefore $\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}(\chi_{[0,a]}(S),\chi_{[0,a]}(H))$, finishing the proof. □

By compactness, it is possible to choose a finite partition

$$0=t_0<t_1<\cdots<t_{M-1}<t_M=1, \tag{5.27}$$

of $[0,1]$ and $a_m>0$, $m=1,\ldots,M$, such that

$$t\in[t_{m-1},t_m]\mapsto H_t$$

lies entirely in the neighborhood $\mathcal{N}'_{H_{t_m},a_m}$ of H_{t_m} defined in Lemma 5.8.1.

Proposition 5.8.2. *For a partition* $0=t_0<t_1<\cdots<t_{M-1}<t_M=1$ *as above, one has*

$$\begin{aligned}&\operatorname{Sf}(t\in[0,1]\mapsto H_t)\\&\quad=\frac{1}{2}\dim(\operatorname{Ker}(H_0))+\sum_{m=1}^{M}\operatorname{Ind}(P^{\geq}(H_{t_m}),P^{\geq}(H_{t_{m-1}}))-\frac{1}{2}\dim(\operatorname{Ker}(H_1)).\end{aligned}$$

Proof. By Lemma 5.8.1,

$$\operatorname{Ind}(P^{\geq}(H_{t_m}),P^{\geq}(H_{t_{m-1}}))=\operatorname{Ind}(P^{\geq}_{a_m,t_m},P^{\geq}_{a_m,t_{m-1}})$$

for all $m=1,2,\ldots,M$ where $P^{\geq}_{a_m,t}=\chi_{[0,a_m]}(H_t)$. As $P^{\geq}_{a_m,t_m}$ and $P^{\geq}_{a_m,t_{m-1}}$ are finite-dimensional,

$$\begin{aligned}\mathrm{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) &= \dim(\mathrm{Ran}(P^{\geq}_{a_m,t_m})) - \dim(\mathrm{Ran}(P^{\geq}_{a_m,t_{m-1}}))\\ &= \mathrm{Tr}(P^{\geq}_{a_m,t_m} - P^{\geq}_{a_m,t_{m-1}})\\ &= \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}}) + \dim(\mathrm{Ker}(H_{t_m})) - \dim(\mathrm{Ker}(H_{t_{m-1}}))\end{aligned} \tag{5.28}$$

by Proposition 5.2.2. By Definition 4.1.2,

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}).$$

As $t \in [t_{m-1}, t_m] \mapsto H_t$ lies entirely in the neighborhood $\mathcal{N}'_{H_{t_m},a_m}$ of H_{t_m}, one concludes that the path $t \in [t_{m-1}, t_m] \mapsto \mathrm{Tr}(\chi_{[-a_m,a_m]}(H_t))$ is constant. Therefore

$$\begin{aligned}&\mathrm{Tr}(P^{<}_{a_m,t_{m-1}}) - \mathrm{Tr}(P^{<}_{a_m,t_m})\\ &\quad= \mathrm{Tr}(P^{>}_{a_m,t_m}) + \dim(\mathrm{Ker}(H_{t_m})) - \mathrm{Tr}(P^{>}_{a_m,t_{m-1}}) - \dim(\mathrm{Ker}(H_{t_{m-1}}))\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Sf}(t \in [0,1] \mapsto H_t)\\ &= \frac{1}{2}\sum_{m=1}^{M}(\mathrm{Tr}(2P^{>}_{a_m,t_m} - 2P^{>}_{a_m,t_{m-1}}) + \dim(\mathrm{Ker}(H_{t_m})) - \dim(\mathrm{Ker}(H_{t_{m-1}})))\\ &= \sum_{m=1}^{M} \mathrm{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) - \dim(\mathrm{Ker}(H_{t_m})) + \dim(\mathrm{Ker}(H_{t_{m-1}}))\\ &\quad+ \frac{1}{2}\dim(\mathrm{Ker}(H_{t_m})) - \frac{1}{2}\dim(\mathrm{Ker}(H_{t_{m-1}}))\\ &= \sum_{m=1}^{M} \mathrm{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) - \frac{1}{2}\dim(\mathrm{Ker}(H_{t_m})) + \frac{1}{2}\dim(\mathrm{Ker}(H_{t_{m-1}}))\\ &= \frac{1}{2}\dim(\mathrm{Ker}(H_0)) + \sum_{m=1}^{M} \mathrm{Ind}(P^{\geq}(H_{t_m}), P^{\geq}(H_{t_{m-1}})) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)),\end{aligned}$$

where the second step follows from (5.28). □

5.9 Relative Morse indices and spectral flow

The Morse index of an invertible self-adjoint matrix is defined as the number of negative eigenvalues. It is a standard object in Morse and stability theories as it is used to determine the qualitative behavior of flow lines of gradient flows on Riemannian manifolds close to rest points. It is possible to define the Morse index for self-adjoint Fredholm operators $H \in \mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$ with positive essential spectrum as the same object. However, for a self-adjoint Fredholm operator $H \in \mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ having both positive and negative essential spectrum, there is no interesting definition of the Morse index itself. It is, however, possible to define a relative Morse index for a pair $H_0, H_1 \in \mathbb{FB}^*_{sa}(\mathcal{H})$ with compact difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ (namely, H_0 and H_1 are Calkin equivalent). Indeed, due to Proposition 5.3.17, the following definition is justified.

Definition 5.9.1. For self-adjoint bounded Fredholm operators $H_0, H_1 \in \mathbb{FB}_{sa}(\mathcal{H})$ with compact difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$, the *relative Morse index* is defined by

$$\mu_{\mathrm{rel}}(H_0, H_1) = \mathrm{Ind}(P_0, P_1),$$

where $P_0 = \chi(H_0 < 0)$ and $P_1 = \chi(H_1 < 0)$.

Let us list the basic properties of the relative Morse index which are all directly inherited from properties of Fredholm pairs and their index. Hence even though there is little extra mathematical content, this allows summarizing all these properties in a compact form (moreover, in the language of relative Morse indices that may be more familiar to some readers).

Proposition 5.9.2. *Let $H_0, H_1, H_2 \in \mathbb{FB}_{sa}(\mathcal{H})$ be such that the differences $H_1 - H_0$ and $H_2 - H_1$ are compact.*

(i) *One has $\mu_{\mathrm{rel}}(H_0, H_1) = -\mu_{\mathrm{rel}}(H_1, H_0)$.*

(ii) *The relative Morse index is additive in the sense that*

$$\mu_{\mathrm{rel}}(H_0, H_2) = \mu_{\mathrm{rel}}(H_0, H_1) + \mu_{\mathrm{rel}}(H_1, H_2).$$

(iii) *Let $R \in \mathbb{B}(\mathcal{H})$ be invertible, then*

$$\mu_{\mathrm{rel}}(H_0, H_1) = \mu_{\mathrm{rel}}\big(R^* H_0 R, R^* H_1 R\big).$$

(iv) *If H_1 is positive semidefinite, then*

$$\mu_{\mathrm{rel}}(H_0, H_1) = \iota_-(H_0),$$

where the Morse index $\iota_-(H_1)$ is defined in (4.6).

(v) *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ and $t \in [0,1] \mapsto H'_t \in \mathbb{FB}_{sa}(\mathcal{H})$ be norm-continuous paths of invertibles such that $H_t - H'_t \in \mathbb{K}(\mathcal{H})$ is compact for all $t \in [0,1]$. Then $t \in [0,1] \mapsto \mu_{\mathrm{rel}}(H'_t, H_t)$ is constant.*

Proof. The first claim follows from the remark after Lemma 5.3.3. Item (ii) is a direct consequence of Proposition 5.3.15 and (iv) follows from Definition 5.2.1. Claim (v) is implied by Proposition 5.2.7 because $t \in [0,1] \mapsto \chi(H_t < 0)$ and $t \in [0,1] \mapsto \chi(H'_t < 0)$ are norm-continuous paths of orthogonal projections with compact difference. It remains to show (iii). Let us set $R_t = U|R|^{1-t}$ where $U = R|R|^{-1}$ is the unitary phase of R, then $R_0 = R, R_1 = U$ and, moreover, $R_t^* H_0 R_t$ and $R_t^* H_1 R_t$ are Calkin equivalent for all $t \in [0,1]$.

Because $t \in [0,1] \mapsto \dim(\mathrm{Ker}(R_t^* H_0 R_t))$ and $t \in [0,1] \mapsto \dim(\mathrm{Ker}(R_t^* H_0 R_t))$ are constant $t \in [0,1] \mapsto \chi(R_t^* H_0 R_t < 0)$ and $t \in [0,1] \mapsto \chi(R_t^* H_1 R_t < 0)$ are norm-continuous paths of orthogonal projections with compact difference. Hence, by Proposition 5.2.7,

$$\mu_{\mathrm{rel}}(R^* H_0 R, R^* H_1 R) = \mu_{\mathrm{rel}}\big(U^* H_0 U, U^* H_1 U\big) = \mu_{\mathrm{rel}}(H_0, H_1),$$

where the last equality follows from the fact that the relative Morse index is invariant under conjugation by unitary operators by (5.14). □

The relative Morse index can be used to give an alternative description of the spectral flow in Theorem 5.9.6 below, as put forward in [84]. It is based on the following fact for which we provide an alternative proof.

Theorem 5.9.3. *Associated to $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ are norm-continuous paths of invertibles $t \in [0,1] \mapsto M_t \in \mathbb{B}(\mathcal{H})$ and self-adjoint compacts $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ such that*

$$M_t^* H_t M_t = Q + K_t, \tag{5.29}$$

where Q is a symmetry. If H_0 is invertible, one can choose $K_0 = 0$.

Proof. Suppose that the partition (4.2) and $a_m > 0$ are chosen such that the spectral projections $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t} = \chi_{[-a_m,a_m]}(H_t)$ are norm-continuous and finite dimensional, see (4.3). Then let us set

$$H_{t,-} = P_{-\infty,-a_m,t} H_t, \quad H_{t,0} = P_{a_m,t} H_t, \quad H_{t,+} = P_{a_m,\infty,t} H_t,$$

where $P_{-\infty,-a_m,t} = \chi_{(-\infty,-a_m)}(H_t)$ and $P_{a_m,\infty,t} = \chi_{(a_m,\infty)}(H_t)$ are spectral projections of H_t. Let us note that all of these operators are not necessarily continuous at $t_1, \dots, t_M$, as there may be jumps in the dimension of the finite-dimensional projection. Nevertheless, for each $t \in [t_{m-1}, t_m]$, let us set

$$S_t = (H_{t,+})^{-\frac{1}{2}} + P_{a_m,t} + (H_{t,-})^{-\frac{1}{2}}.$$

Here $H_{t,\pm}$ are understood as invertible operators on their range. By construction, $t \in [t_{m-1}, t_m] \mapsto S_t$ is norm-continuous, self-adjoint, and invertible. Moreover, for $t \in [t_{m-1}, t_m]$,

$$S_t H_t S_t = -P_{-\infty,-a_m,t} + H_{t,0} + P_{a_m,\infty,t}.$$

Each summand on the right-hand side is continuous in $t \in [t_{m-1}, t_m]$. Moreover, the operator $-P_{-\infty,-a_m,t} + P_{a_m,\infty,t}$ differs from a symmetry only by a compact operator, which will be chosen to be $P_{a_m,t}$, notably let us set

$$Q_t = -P_{-\infty,-a_m,t} + P_{a_m,t} + P_{a_m,\infty,t}.$$

These symmetries can be continuously deformed by a path of unitaries U_t into a given one, say $Q_m = Q_{t_{m-1}}$. Hence there exists a continuous path $t \in [t_{m-1}, t_m] \mapsto N_t$ of invertible operators such that

$$N_t^* H_t N_t = Q_m + K_{m,t},$$

where $t \in [t_{m-1}, t_m] \mapsto K_{m,t}$ is norm-continuous and compact. This proves the statement locally in t. It remains to join the pieces in such a manner that the Q_m can be chosen to be equal. This will be achieved inductively in m after a finite number of steps. Hence let us assume that (5.29) already holds for $t \le t_{m-1}$. At t_{m-1}, one then has

$$\begin{aligned} H_{t_{m-1}} &= (M_{t_{m-1}}^*)^{-1}(Q + K_{t_{m-1}})(M_{t_{m-1}})^{-1} \\ &= (N_{t_{m-1}}^*)^{-1}(Q_m + K_{m,t_{m-1}})(N_{t_{m-1}})^{-1}. \end{aligned}$$

Thus set $A = (N_{t_{m-1}})^{-1} M_{t_{m-1}}$ and $M_t = N_t A$ for $t \in [t_{m-1}, t_m]$. It now follows that

$$A^* Q_m A = Q + K_{t_{m-1}} - M_{t_{m-1}}^* (N_{t_{m-1}}^*)^{-1} K_{m,t_{m-1}} (N_{t_{m-1}})^{-1} M_{t_{m-1}},$$

and so $A^* Q_m A = Q + K$ for a compact self-adjoint operator K. Hence, for $t \in [t_{m-1}, t_m]$,

$$\begin{aligned} M_t^* H_t M_t &= A^* N_t^* H_t N_t A \\ &= A^* (Q_m + K_{m,t}) A \\ &= Q + K + A^* K_{m,t} A \\ &= Q + K_t, \end{aligned}$$

for the compact self-adjoint operators $K_t = K + A^* K_{m,t} A$. This finishes the proof. □

Remark 5.9.4. It is possible to reformulate Theorem 5.9.3. Because M_t is invertible, one can set $\hat{H}_t = (M_t^{-1})^* Q M_t^{-1}$ and $\hat{K}_t = (M_t^{-1})^* K_t M_t^{-1}$ and obtains

$$H_t = \hat{H}_t + \hat{K}_t. \tag{5.30}$$

Hence the path $t \mapsto H_t$ can be decomposed into a path $t \mapsto \hat{H}_t$ of invertibles and a compact perturbation $t \mapsto \hat{K}_t$ thereof. Let us stress that if $t \in [0,1] \mapsto H_t$ is a loop, namely $H_0 = H_1$, the two paths $t \mapsto \hat{H}_t$ and $t \mapsto \hat{K}_t$ are in general *not* closed.

Provided that H_0 is invertible (so that $K_0 = 0$), one can homotopically deform the time parameter in the two summands on the right-hand side of (5.30) to deduce the following: the nontrivial loop $t \in [0,1] \mapsto H_t$ is homotopic to the concatenation of two paths

$$\big(t \in [0,1] \mapsto \hat{H}_t\big) * \big(t \in [0,1] \mapsto \hat{H}_1 + \hat{K}_t\big).$$

The first of these paths is within the invertible operators and hence has no spectral flow, while the second is merely a compact perturbation of $\hat{H}_1 = H_0 - \hat{K}_1$. On this second part though, there is possibly a spectral flow given by

$$\mathrm{Sf}\big(t \in [0,1] \mapsto \hat{H}_1 + \hat{K}_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big).$$

Let us note that a particular case of this is the following: given two symmetries Q_0 and Q_1 with compact difference $Q_0 - Q_1$ and a given index $\mathrm{Ind}(Q_0, Q_1)$, first rotate Q_0 into Q_1 by a path of unitaries, then use the straight-line path to complete a nontrivial loop rooted in Q_0 (playing the role of H_0 in the above). Such a loop is constructed explicitly in Example 8.3.4, which is based on Example 5.7.4. In order to be even closer to this Example 8.3.4, the next result further specializes (5.30) to the case where H_t is a proper symmetry up to a compact perturbation. ◇

Corollary 5.9.5. *Let $t \in [0,1] \mapsto H_t$ be a path of essential proper symmetries, namely lying in the set*

$$\mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}) = \big\{H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}_{\mathrm{ess}}(H) = \{-1, 1\}\big\}.$$

Then there are norm-continuous paths of unitaries $t \in [0,1] \mapsto U_t \in \mathbb{U}(\mathcal{H})$ and self-adjoint compacts $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ such that

$$U_t^* H_t U_t = Q + K_t, \tag{5.31}$$

for some proper symmetry Q.

Proof. Let us start out from (5.30). As $H_t \in \mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ and $\hat{K}_t \in \mathbb{K}(\mathcal{H})$, it follows that also $\hat{H}_t \in \mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$, due to the compact stability of the essential spectrum. As $\hat{H}_t$ is invertible, the proof of Proposition 3.6.5 implies that it can be decomposed as $\hat{H}_t = Q_t + \tilde{K}_t$ into a symmetry Q_t and a compact $\tilde{K}_t$. Moreover, this decomposition is continuous, see Remark 3.6.6. Then

$$H_t = Q_t + \hat{K}_t + \tilde{K}_t.$$

By Proposition 5.3.20, one can write $Q_t = U_t Q_0 U_t^*$ for some path of unitaries. Setting $Q = Q_0$ and $K_t = U_t^*(\hat{K}_t + \tilde{K}_t)U_t$ concludes the proof. □

Based on Theorem 5.9.3, one has the following formula for the spectral flow as Morse index.

Theorem 5.9.6. *For paths $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ and $t \in [0,1] \mapsto M_t^* H_t M_t = Q + K_t$ where as above Q is a symmetry, $t \in [0,1] \mapsto M_t \in \mathbb{B}(\mathcal{H})$ is a path of invertibles and $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ is a path of compacts, the spectral flow of the path $t \in [0,1] \mapsto H_t$ satisfies*

$$\begin{aligned}\mathrm{Sf}\big(t\in[0,1]\mapsto H_t\big) &= \mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)\\ &= \frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\mu_{\mathrm{rel}}(Q+K_0,Q+K_1)-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big). \end{aligned}\tag{5.32}$$

Proof. For $t\in[0,1]$, let $U_t^* = M_t^*|M_t^*|^{-1}$ be the unitary phase of M_t^*. Then let us consider the continuous homotopy $h:[0,1]\times[0,1]\to\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ defined by

$$h(t,s)=U_t^*|M_t^*|^sH_t|M_t^*|^sU_t.$$

By Theorem 4.2.2, one has

$$\begin{aligned}&\mathrm{Sf}\big(t\in[0,1]\mapsto h(t,1)\big)\\ &\quad= -\mathrm{Sf}\big(s\in[0,1]\mapsto h(0,s)\big)+\mathrm{Sf}\big(t\in[0,1]\mapsto h(t,0)\big)+\mathrm{Sf}\big(s\in[0,1]\mapsto h(1,s)\big).\end{aligned}$$

As $s\in[0,1]\mapsto\dim(\mathrm{Ker}(h(0,s)))$ and $s\in[0,1]\mapsto\dim(\mathrm{Ker}(h(1,s)))$ are constant, item (i) of Theorem 4.2.1 implies

$$\mathrm{Sf}\big(s\in[0,1]\mapsto h(0,s)\big)=\mathrm{Sf}\big(s\in[0,1]\mapsto h(1,s)\big)=0.$$

Therefore

$$\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)=\mathrm{Sf}\big(t\in[0,1]\mapsto U_t^*H_tU_t\big)=\mathrm{Sf}\big(t\in[0,1]\mapsto H_t\big),$$

where the last step follows from item (vi) of Theorem 4.2.1. This implies the first claim. The second holds because

$$\begin{aligned}\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big) = {}&\frac{1}{2}\dim\big(\mathrm{Ker}(Q+K_0)\big)\\ &+\sum_{m=1}^{M}\mathrm{Ind}\big(P^{\geq}(Q+K_{t_m}),P^{\geq}(Q+K_{t_{m-1}})\big)\\ &-\frac{1}{2}\dim\big(\mathrm{Ker}(Q+K_1)\big)\end{aligned}$$

for a partition $0=t_0<t_1<\cdots<t_{M-1}<t_M=1$ as in Proposition 5.8.2. By definition,

$$\begin{aligned}\mathrm{Ind}\big(P^{\geq}(Q+K_{t_m}),P^{\geq}(Q+K_{t_{m-1}})\big) &= -\mu_{\mathrm{rel}}(Q+K_{t_m},Q+K_{t_{m-1}})\\ &= \mu_{\mathrm{rel}}(Q+K_{t_{m-1}},Q+K_{t_m}).\end{aligned}$$

Therefore and as $\dim(\mathrm{Ker}(Q+K_0))=\dim(\mathrm{Ker}(H_0))$ and $\dim(\mathrm{Ker}(Q+K_1))=\dim(\mathrm{Ker}(H_1))$,

$$\begin{aligned}&\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)\\ &\quad=\frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\sum_{m=1}^{M}\mu_{\mathrm{rel}}(Q+K_{t_{m-1}},Q+K_{t_m})-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big)\\ &\quad=\frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\mu_{\mathrm{rel}}(Q+K_0,Q+K_1)-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big),\end{aligned}$$

where the last step follows from item (ii) in Proposition 5.9.2. □

6 Unbounded Fredholm operators

This chapter offers a detailed introduction to various subsets of the unbounded Fredholm operators, with a particular focus on natural topologies thereon. This is a necessary preparation for the definition of spectral flow of unbounded self-adjoint Fredholm operators given in the next Chapter 7. First, Section 6.1 reviews various topologies on the set of closed operators. Section 6.2 recalls some fundamentals about unbounded Fredholm operators that can be found in numerous books, e. g., [80, 99, 165]. Then, following the works of Booß-Bavnbek, Lesch, and Phillips [31], as well as Lesch [126], the set of unbounded self-adjoint Fredholm operators and its topology is studied in detail in Section 6.3. Section 6.4 considers the important subclass of self-adjoint Fredholm operators with compact resolvent and proves numerous topological results.

6.1 Topologies on closed and densely defined operators

Let us first recall that an unbounded operator is a linear map $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ where $\mathcal{D}(T)$ is a linear subspace of some Hilbert space $\mathcal{H}$, called the domain of T. It is called closed if its graph $\{(\phi, T\phi) : \phi \in \mathcal{D}(T)\}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}'$. Let us introduce a notation for the set of closed densely defined (also called regular) operators:

$$\mathbb{L}(\mathcal{H}, \mathcal{H}') = \{T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}' \text{ closed and densely defined}\}.$$

In the case $\mathcal{H}' = \mathcal{H}$, we also use the notation $\mathbb{L}(\mathcal{H}) = \mathbb{L}(\mathcal{H}, \mathcal{H})$. For any $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, the adjoint operator $T^* \in \mathbb{L}(\mathcal{H}', \mathcal{H})$ is defined by $\langle T^*\phi|\psi\rangle = \langle \phi|T\psi\rangle$ for $\psi \in \mathcal{D}(T)$ and ϕ in $\mathcal{D}(T^*) = \{\phi \in \mathcal{H}' : \psi \in \mathcal{D}(T) \mapsto \langle \phi|T\psi\rangle \text{ bounded}\}$. Then T is called symmetric if $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T^*|_{\mathcal{D}(T)} = T$, and furthermore T is called self-adjoint whenever one has $T = T^*$ which includes $\mathcal{D}(T) = \mathcal{D}(T^*)$. As a preparation for the constructions below, some rather standard facts are needed that are included for the convenience of the reader.

Lemma 6.1.1. *Let T be a closed and densely defined operator. Then T^*T is self-adjoint with domain $\mathcal{D}(T^*T) = \{\phi \in \mathcal{H} : \phi \in \mathcal{D}(T),\ T\phi \in \mathcal{D}(T^*)\}$.*

Proof. (See, e. g., Korollar VII.2.13 in [204].) Clearly, T^*T is well defined and symmetric on $\mathcal{D}(T^*T)$. It remains to show that it is densely defined and self-adjoint. For that purpose, let us equip $\mathcal{D}(T)$ with the scalar product

$$\langle \phi|\psi\rangle_T = \langle T\phi|T\psi\rangle + \langle \phi|\psi\rangle.$$

Because T is closed, $(\mathcal{D}(T), \langle\cdot|\cdot\rangle_T)$ is a Hilbert space which will be denoted by $\hat{\mathcal{H}}$. Let $I \in \mathbb{B}(\hat{\mathcal{H}}, \mathcal{H})$ denote the natural embedding and $I^* \in \mathbb{B}(\mathcal{H}, \hat{\mathcal{H}})$ its adjoint. Then II^* is self-adjoint and has a trivial kernel because $\langle \phi|I^*\psi\rangle_T = \langle I\phi|\psi\rangle = \langle \phi|\psi\rangle$ and T is densely defined. Thus $\overline{\mathrm{Ran}(II^*)} = \mathrm{Ker}(II^*)^\perp = \mathcal{H}$ and II^* has dense range. It will next be shown

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that $\operatorname{Ran}(II^*) \subset \mathcal{D}(T^*T)$. Indeed, let $\phi \in \mathcal{H}$ and $\psi = II^*\phi = I^*\phi \in \operatorname{Ran}(II^*)$ so that $\psi \in \hat{\mathcal{H}}$ and for any $\eta \in \mathcal{D}(T)$,

$$\begin{aligned}\langle T\eta|T\psi\rangle &= \langle \eta|\psi\rangle_T - \langle \eta|\psi\rangle \\ &= \langle \eta|I^*\phi\rangle_T - \langle \eta|\psi\rangle \\ &= \langle I\eta|\phi\rangle - \langle \eta|\psi\rangle \\ &= \langle \eta|\phi - \psi\rangle.\end{aligned}$$

Hence $\eta \in \mathcal{D}(T) \mapsto \langle T\eta|T\psi\rangle$ is continuous and thus $T\psi \in \mathcal{D}(T^*)$, so that $\psi \in \mathcal{D}(T^*T)$. It follows that $\mathcal{D}(T^*T)$ is dense. As T^*T is bounded from below and symmetric, it has a self-adjoint extension with domain $\operatorname{Ran}(II^*)$, given by the Friedrich extension (this is the only nonnegative self-adjoint extension). Hence one must have $\operatorname{Ran}(II^*) = \mathcal{D}(T^*T)$, and T^*T is self-adjoint. □

Lemma 6.1.2. *For any regular operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, the domain $\mathcal{D}(T^*T)$ is a core for T, namely T is the closure of $T|_{\mathcal{D}(T^*T)}$ which in turn is also given by the double adjoint of $T|_{\mathcal{D}(T^*T)}$. Moreover, $T(\mathbf{1} + T^*T)^{-1}$ and $T(\mathbf{1} + T^*T)^{-\frac{1}{2}}$ are bounded operators, both with norm bounded by* 1.

Proof. (See, e. g., Lemma 9.2 in [121].) Let us first show that

$$\mathbf{1} + T^*T : \mathcal{D}(\mathbf{1} + T^*T) = \mathcal{D}(T^*T) \to \mathcal{H}$$

is a bijection. For $\phi \in \mathcal{D}(T^*T)$, one has

$$\langle \phi|(\mathbf{1} + T^*T)\phi\rangle = \langle \phi|\phi\rangle + \langle T\phi|T\phi\rangle \geq \langle \phi|\phi\rangle,$$

and therefore $\|(\mathbf{1} + T^*T)\phi\| \geq \|\phi\|$. This implies that $\mathbf{1} + T^*T$ is injective. Furthermore, if $(\phi_n)_{n\geq 1}$ is a Cauchy sequence in $\operatorname{Ran}(\mathbf{1} + T^*T)$ and $\phi_n = (\mathbf{1} + T^*T)\psi_n$, then also $(\psi_n)_{n\geq 1}$ is a Cauchy sequence converging to ψ, and then the closedness of $\mathbf{1} + T^*T$ implies that $\psi \in \mathcal{D}(\mathbf{1} + T^*T)$ and $(\mathbf{1} + T^*T)\psi = \lim \phi_n$. Thus $\operatorname{Ran}(\mathbf{1} + T^*T)$ is closed and therefore equal to $\mathcal{H}$. Moreover, it follows that the inverse $(\mathbf{1} + T^*T)^{-1} : \mathcal{H} \to \mathcal{H}$ is bounded with norm $\|(\mathbf{1} + T^*T)^{-1}\| \leq 1$ and its range is $\operatorname{Ran}((\mathbf{1} + T^*T)^{-1}) = \mathcal{D}(T^*T)$. Let us note that, in particular, the range of the operator $(\mathbf{1}+T^*T)^{-1}$ is dense in $\mathcal{H}$. As, clearly, $(\mathbf{1}+T^*T)^{-1} > 0$, its square root $(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \to \mathcal{H}$ is well defined and has a dense range. Then for $\phi \in \mathcal{H}$, one has

$$\begin{aligned}\langle T(\mathbf{1} + T^*T)^{-1}\phi|T(\mathbf{1} + T^*T)^{-1}\phi\rangle &= \langle (\mathbf{1} + T^*T)^{-1}\phi|T^*T(\mathbf{1} + T^*T)^{-1}\phi\rangle \\ &\leq \langle (\mathbf{1} + T^*T)^{-1}\phi|(\mathbf{1} + T^*T)(\mathbf{1} + T^*T)^{-1}\phi\rangle \\ &= \langle (\mathbf{1} + T^*T)^{-1}\phi|\phi\rangle \\ &= \langle (\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi|(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi\rangle,\end{aligned}$$

and therefore $\|T(\mathbf{1}+T^*T)^{-\frac{1}{2}}(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi\| \le \|(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi\|$. This implies that the operator $T(\mathbf{1}+T^*T)^{-\frac{1}{2}} : \mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}}) \to \mathcal{H}'$ is bounded with norm bounded by 1 and therefore has an extension from $\mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}})$ to all of $\mathcal{H}$ which is also bounded with norm bounded by 1. Next is shown that $\mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ such that this extension is given by $T(\mathbf{1}+T^*T)^{-\frac{1}{2}} : \mathcal{H} \to \mathcal{H}'$. Let $\phi \in \mathcal{H}$. As $\mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}}(\mathbf{1}+T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T^*T)$ is dense, there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ in the range of $((\mathbf{1}+T^*T)^{-\frac{1}{2}})^2$ converging to ϕ. Then, as $(\mathbf{1}+T^*T)^{-\frac{1}{2}}$ is bounded, $\lim_{n\to\infty}(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi_n = (\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi$. Because the operator $T(\mathbf{1}+T^*T)^{-\frac{1}{2}} : \mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}}) \to \mathcal{H}'$ is bounded, $(T(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore converges to some $\psi = \lim_{n\to\infty} T(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi_n \in \mathcal{H}'$. As $\lim_{n\to\infty}((\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi_n, T(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi_n) = ((\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi, \psi)$ in $\mathcal{H}\times\mathcal{H}'$ and T is closed, $(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi$ is in the domain of T and $T(\mathbf{1}+T^*T)^{-\frac{1}{2}}\phi = \psi$. Conversely, assume $\phi \in \mathcal{D}(T)$. Then as $(\mathbf{1}+T^*T)^{-\frac{1}{2}}T^* \subset (T(\mathbf{1}+T^*T)^{-\frac{1}{2}})^*$ is bounded, one has

$$\begin{aligned}\phi &= ((\mathbf{1}+T^*T)^{-\frac{1}{2}}(\mathbf{1}+T^*T)^{-\frac{1}{2}}T^*T + (\mathbf{1}+T^*T)^{-1})\phi \\ &= (\mathbf{1}+T^*T)^{-\frac{1}{2}}((\mathbf{1}+T^*T)^{-\frac{1}{2}}T^*T + (\mathbf{1}+T^*T)^{-\frac{1}{2}})\phi \in \mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}}).\end{aligned}$$

This implies $\mathcal{D}(T) = \mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}})$. Thus, for $\phi \in \mathcal{D}(T)$ there is $\psi \in \mathcal{H}$ such that $\phi = (\mathbf{1}+T^*T)^{-\frac{1}{2}}\psi$. As $\mathrm{Ran}((\mathbf{1}+T^*T)^{-1})$ is dense in $\mathcal{H}$, there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ in $\mathrm{Ran}((\mathbf{1}+T^*T)^{-1})$ such that $\lim_{n\to\infty}(\mathbf{1}+T^*T)^{-\frac{1}{2}}\theta_n = \psi$. Then

$$\lim_{n\to\infty}(\mathbf{1}+T^*T)^{-1}\theta_n = \phi$$

and

$$\begin{aligned}\lim_{n\to\infty} T(\mathbf{1}+T^*T)^{-1}\theta_n &= \lim_{n\to\infty} T(\mathbf{1}+T^*T)^{-\frac{1}{2}}(\mathbf{1}+T^*T)^{-\frac{1}{2}}\theta_n \\ &= T(\mathbf{1}+T^*T)^{-\frac{1}{2}}\psi \\ &= T\phi.\end{aligned}$$

One concludes that $\lim_{n\to\infty}((\mathbf{1}+T^*T)^{-1}\theta_n, T(\mathbf{1}+T^*T)^{-1}\theta_n) = (\phi, T\phi)$ and therefore $\mathcal{D}(T^*T)$ is a core for T because $(\mathbf{1}+T^*T)^{-1}\theta_n \in \mathcal{D}(T^*T)$ for all $n \in \mathbb{N}$. □

In this section two topologies on $\mathbb{L}(\mathcal{H},\mathcal{H}')$ are studied, as well as naturally associated topologies on the image of $\mathbb{L}(\mathcal{H},\mathcal{H}')$ under the bounded transform that will be introduced in (6.3) below. Let us begin with the gap topology. As $T \in \mathbb{L}(\mathcal{H},\mathcal{H}')$ is closed, the orthogonal projection $P_T \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H}')$ onto the graph of T is bounded. Then the *gap metric* on $\mathbb{L}(\mathcal{H},\mathcal{H}')$ is defined by

$$d_G(T_0,T_1) = \|P_{T_0} - P_{T_1}\|, \quad T_0,T_1 \in \mathbb{L}(\mathcal{H},\mathcal{H}'). \tag{6.1}$$

The topology $\mathcal{O}_G$ on $\mathbb{L}(\mathcal{H},\mathcal{H}')$ induced by d_G is called the gap topology. In order to get a better grip on it, let us write out the explicit form of the graph projections.

Proposition 6.1.3. *For $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, let us set*

$$R_T = (\mathbf{1} + T^*T)^{-1}.$$

Then the orthogonal projection onto the graph of T is

$$P_T = \begin{pmatrix} R_T & T^*R_{T^*} \\ TR_T & \mathbf{1} - R_{T^*} \end{pmatrix}.$$

The gap metric is equivalent to the metric defined by

$$d'_G(T_0, T_1) = \|R_{T_0} - R_{T_1}\| + \|R_{T_0^*} - R_{T_1^*}\| + \|T_0R_{T_0} - T_1R_{T_1}\|.$$

Proof. By Lemma 6.1.2, TR_T is bounded, and thus also $T^*R_{T^*}$ is bounded. Let us first check that

$$R_{T^*}T \subset TR_T, \quad R_TT^* \subset (TR_T)^* = T^*R_{T^*}. \tag{6.2}$$

For the first equality, let $\phi \in \mathcal{D}(T)$. Then $\psi = R_T\phi \in \operatorname{Ran}(R_T) = \mathcal{D}(T^*T) \subset \mathcal{D}(T)$. As then $(\mathbf{1} + T^*T)\psi = \phi$, one has

$$T\phi = T(\mathbf{1} + T^*T)\psi = (\mathbf{1} + TT^*)T\psi = (\mathbf{1} + TT^*)TR_T\phi,$$

and multiplying by $(\mathbf{1} + TT^*)^{-1}$ shows the first inclusion of (6.2). The second inclusion in (6.2) follows from general principles. Indeed, for $\phi \in \mathcal{D}(T^*)$, one concludes that $(TR_T)^*\phi = R_TT^*\phi = T^*R_{T^*}\phi$ where the last equality follows from the first inclusion of (6.2). As $\mathcal{D}(T^*)$ is dense, this implies the last equality in (6.2). Using (6.2), an algebraic computation shows that P_T is an orthogonal projection. Moreover, one readily verifies

$$P_T\begin{pmatrix} \phi \\ T\phi \end{pmatrix} = \begin{pmatrix} (\mathbf{1} + T^*T)^{-1}\phi + T^*(\mathbf{1} + TT^*)^{-1}T\phi \\ T(\mathbf{1} + T^*T)^{-1}\phi + (\mathbf{1} - (\mathbf{1} + TT^*)^{-1})T\phi \end{pmatrix} = \begin{pmatrix} \phi \\ T\phi \end{pmatrix}$$

for all $\phi \in \mathcal{D}(T)$, due to $(\mathbf{1} + TT^*)^{-1}T = R_{T^*}T \subset TR_T = T(\mathbf{1} + T^*T)^{-1}$. Note that the set $\{\binom{T^*\psi}{-\psi} : \psi \in \mathcal{D}(T^*)\}$ is the orthogonal complement of the graph of T in $\mathcal{H} \oplus \mathcal{H}'$. One checks that for $\psi \in \mathcal{D}(T^*)$,

$$P_T\begin{pmatrix} T^*\psi \\ -\psi \end{pmatrix} = \begin{pmatrix} (\mathbf{1} + T^*T)^{-1}T^*\psi - T^*(\mathbf{1} + TT^*)^{-1}\psi \\ T(\mathbf{1} + T^*T)^{-1}T^*\psi - (\mathbf{1} - (\mathbf{1} + TT^*)^{-1})\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $(\mathbf{1} + T^*T)^{-1}T^* = R_TT^* \subset T^*R_{T^*} = T^*(\mathbf{1} + TT^*)^{-1}$ was used. Hence P_T is the orthogonal projection onto the graph of T. Replacing the formula for P_T twice in definition (6.1), one readily deduces the equivalence of d_G and d'_G. □

The key element for the definition of the Riesz topology on $\mathbb{L}(\mathcal{H},\mathcal{H}')$ is the bounded transform (sometimes also called Riesz transform due to the work of Riesz and Lorch, on which it is elaborated in the textbook [158]; note though that there is no square root in these works)

$$\mathcal{F}(T) = T(\mathbf{1} + T^*T)^{-\frac{1}{2}} \in \mathbb{B}(\mathcal{H},\mathcal{H}') \tag{6.3}$$

of $T \in \mathbb{L}(\mathcal{H},\mathcal{H}')$. By Lemma 6.1.2, the operator $\mathcal{F}(T)$ is well defined and bounded so that also the map $\mathcal{F} : \mathbb{L}(\mathcal{H},\mathcal{H}') \to \mathbb{B}(\mathcal{H},\mathcal{H}')$ given by (6.3) is well defined. To analyze its mapping properties, let us introduce the ball of bounded operators of radius $a > 0$,

$$\mathbb{B}_a(\mathcal{H},\mathcal{H}') = \{F \in \mathbb{B}(\mathcal{H},\mathcal{H}') : \|F\| \le a\},$$

as well as the following subset of the unit ball:

$$\mathbb{B}_1^0(\mathcal{H},\mathcal{H}') = \{F \in \mathbb{B}(\mathcal{H},\mathcal{H}') : \|F\| \le 1, \operatorname{Ker}(\mathbf{1} - F^*F) = \{0\}\}.$$

This notation fits with that of Section 4.6, namely the lower index 1 indicates that the norm is bounded by 1 and the upper index 0 denotes that 1 is not a singular value of F.

Theorem 6.1.4. *The bounded transform establishes a bijection*

$$\mathcal{F} : \mathbb{L}(\mathcal{H},\mathcal{H}') \to \mathbb{B}_1^0(\mathcal{H},\mathcal{H}').$$

Moreover, $\mathcal{F}(T)^* = \mathcal{F}(T^*)$.

Proof. (See, e. g., Theorem 10.4 in [121].) In the proof of Lemma 6.1.2, it was shown that $(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \to \mathcal{H}$ is well defined and bounded with norm bounded by 1. Moreover, $\operatorname{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ and $\mathcal{F}(T) : \mathcal{H} \to \mathcal{H}$ is well defined and bounded with norm $\|\mathcal{F}(T)\| \le 1$, see the proof of Lemma 6.1.2.

Clearly,

$$(\mathbf{1} + T^*T)^{-\frac{1}{2}}T^* \subset \mathcal{F}(T)^*, \tag{6.4}$$

and therefore one has for $\phi \in \mathcal{H}$,

$$\begin{aligned}\mathcal{F}(T)^*\mathcal{F}(T)(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi &= (\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*T(\mathbf{1} + T^*T)^{-1}\phi \\ &= (\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T - \mathbf{1})(\mathbf{1} + T^*T)^{-1}\phi \\ &= (\mathbf{1} - (\mathbf{1} + T^*T)^{-1})(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi.\end{aligned}$$

As $\operatorname{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ is dense in $\mathcal{H}$, this implies

$$\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T) = (\mathbf{1} + T^*T)^{-1}. \tag{6.5}$$

Thus as $\mathrm{Ker}((\mathbf{1}+T^*T)^{-\frac{1}{2}}) = \mathrm{Ker}((\mathbf{1}+T^*T)^{-1}) = \{0\}$, the kernel of $\mathbf{1}-\mathcal{F}(T)^*\mathcal{F}(T)$ is trivial. This shows that the map $\mathcal{F}$ is well defined.

Let us next show that the map $\mathcal{F}$ is surjective. Let $F \in \mathbb{B}(\mathcal{H},\mathcal{H}')$ be such that $\|F\| \le 1$ and $\mathrm{Ker}(\mathbf{1}-F^*F) = \{0\}$. As $\mathrm{Ker}(\mathbf{1}-F^*F)$ is trivial, it follows that $\mathcal{D} = \mathrm{Ran}((\mathbf{1}-F^*F)^{\frac{1}{2}}) \subset \mathcal{H}$ is dense. Then an unbounded operator $T : \mathcal{D} \to \mathcal{H}'$ is defined by $T(\mathbf{1}-F^*F)^{\frac{1}{2}}\phi = F\phi$ for $\phi \in \mathcal{H}$. As $\mathbf{1}-F^*F$ is injective, this is well defined and

$$T = F\big(\mathbf{1}-F^*F\big)^{-\frac{1}{2}}. \tag{6.6}$$

Clearly, T is densely defined and it remains to show that it is closed and $\mathcal{F}(T) = F$. We next show that the kernel of $\mathbf{1}-FF^*$ is trivial. Suppose to the contrary, namely that there is $\phi \in \mathcal{H}'$ with $\|\phi\| = 1$ such that $FF^*\phi = \phi$. This implies that $FF^*FF^*\phi = \phi$ and therefore

$$1 = \langle \phi | FF^*FF^*\phi\rangle = \langle F^*\phi | F^*F(F^*\phi)\rangle.$$

As $\|F^*\phi\| \le 1$, this implies by the Cauchy–Schwarz inequality that $\|F^*\phi\| = 1$ and that $F^*F(F^*\phi) = F^*\phi$, which is a contradiction to $\mathrm{Ker}(\mathbf{1}-F^*F) = \{0\}$. Thus $\mathrm{Ker}(\mathbf{1}-FF^*) = \{0\}$ and the range $\mathcal{D}^* = \mathrm{Ran}((\mathbf{1}-FF^*)^{\frac{1}{2}}) \subset \mathcal{H}'$ is dense. Then $S : \mathcal{D}^* \to \mathcal{H}$, defined by $S(\mathbf{1}-FF^*)^{\frac{1}{2}}\phi = F^*\phi$ for $\phi \in \mathcal{H}'$, is well defined and $S = F^*(\mathbf{1}-FF^*)^{-\frac{1}{2}}$. Next setting $\phi = (\mathbf{1}-F^*F)^{\frac{1}{2}}\phi' \in \mathcal{D}$ and $\psi = (\mathbf{1}-FF^*)^{\frac{1}{2}}\psi' \in \mathcal{D}^*$, one has

$$\begin{aligned}
\langle T\phi|\psi\rangle &= \big\langle F\phi' \big| (\mathbf{1}-FF^*)^{\frac{1}{2}}\psi'\big\rangle \\
&= \big\langle (\mathbf{1}-FF^*)^{\frac{1}{2}}F\phi' \big| \psi'\big\rangle \\
&= \big\langle F(\mathbf{1}-F^*F)^{\frac{1}{2}}\phi' \big| \psi'\big\rangle \\
&= \big\langle (\mathbf{1}-F^*F)^{\frac{1}{2}}\phi' \big| F^*\psi'\big\rangle \\
&= \langle \phi|S\psi\rangle.
\end{aligned}$$

This implies $S \subset T^*$ and, in particular, T^* is densely defined (and T is closable). One directly checks that

$$P = \begin{pmatrix} \mathbf{1}-F^*F & (\mathbf{1}-F^*F)^{\frac{1}{2}}F^* \\ F(\mathbf{1}-F^*F)^{\frac{1}{2}} & FF^* \end{pmatrix} \in \mathbb{B}\big(\mathcal{H}\oplus\mathcal{H}'\big)$$

is an orthogonal projection. An explicit computation shows that the graph of T is $\mathrm{Ran}(P)$ and therefore T is closed. Moreover, $\{(-S\psi,\psi) : \psi \in \mathcal{D}^*\} = \mathrm{Ker}(P)$ and, because one has $\{(\phi, T\phi) : \phi \in \mathcal{D}\}^\perp = \{(-T^*\psi,\psi) : \psi \in \mathcal{D}(T^*)\}$, this implies $\mathcal{D}^* = \mathcal{D}(T^*)$ and $S = T^*$. Next let us verify that $F = \mathcal{F}(T)$. By Lemma 6.1.2, $\mathcal{D}(T) \subset \mathrm{Ran}((\mathbf{1}+T^*T)^{-\frac{1}{2}})$ and therefore

$$\mathcal{F}(T)\big(\mathbf{1}-\mathcal{F}(T)^*\mathcal{F}(T)\big)^{-\frac{1}{2}} = T\big(\mathbf{1}+T^*T\big)^{-\frac{1}{2}}\big(\mathbf{1}+T^*T\big)^{\frac{1}{2}} = T.$$

This implies that T is given by (6.6) for $F = \mathcal{F}(T)$ and the map $\mathcal{F}$ is injective. Let T be as in (6.6), then $T^* = F^*(\mathbf{1} - FF^*)^{-\frac{1}{2}}$ and

$$\begin{aligned}\mathcal{F}(T) &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}\big(\mathbf{1} + F^*(\mathbf{1} - FF^*)^{-\frac{1}{2}}F(\mathbf{1} - F^*F)^{-\frac{1}{2}}\big)^{-\frac{1}{2}} \\ &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}\big(\mathbf{1} + F^*F(\mathbf{1} - F^*F)^{-1}\big)^{-\frac{1}{2}} \\ &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}\big((\mathbf{1} - F^*F + F^*F)(\mathbf{1} - F^*F)^{-1}\big)^{-\frac{1}{2}} \\ &= F.\end{aligned}$$

Therefore $\mathcal{F}$ is bijective and $\mathcal{F}(T)^* = \mathcal{F}(T^*)$. □

The so-called *Riesz metric* on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is defined by

$$d_R(T_0, T_1) = \|\mathcal{F}(T_0) - \mathcal{F}(T_1)\|, \quad T_0, T_1 \in \mathbb{L}(\mathcal{H}, \mathcal{H}').$$

Using Theorem 6.1.4, one checks the nondegeneracy assumption for d_R. The triangle inequality and symmetry are obvious. The topology $\mathcal{O}_R$ on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ induced by the Riesz metric is also called the Riesz topology. Henceforth we use both notations $(\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_R)$ and $(\mathbb{L}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R)$ depending on whether we want to stress the metric structure when discussing the continuity of maps on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. Similarly, we will proceed with other spaces below.

As d_R is naturally associated to the bounded transform, the following holds:

Proposition 6.1.5. *The bounded transform*

$$\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_R) \to (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_N)$$

is a homeomorphism. As above, $d_N(T_0, T_1) = \|T_0 - T_1\|$ is here the norm distance.

Proof. By Theorem 6.1.4, $\mathcal{F} : \mathbb{L}(\mathcal{H}, \mathcal{H}') \to \mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$ is bijective and, by the very definition of the Riesz metric, it is a homeomorphism. □

Proposition 6.1.6. *An operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is bounded if and only if its bounded transform has norm less than* 1, *namely $\|\mathcal{F}(T)\| < 1$.*

Proof. Let us first suppose that $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is bounded. Then it is sufficient to show that $\|\mathcal{F}(T)^*\mathcal{F}(T)\| = \|\mathcal{F}(T)\|^2 < 1$. As

$$\mathcal{F}(T)^*\mathcal{F}(T) = (\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*T(\mathbf{1} + T^*T)^{-\frac{1}{2}} = T^*T(\mathbf{1} + T^*T)^{-1},$$

by the spectral radius theorem one has

$$\begin{aligned}\|\mathcal{F}(T)^*\mathcal{F}(T)\| &= \sup(\operatorname{spec}(\mathcal{F}(T)^*\mathcal{F}(T))) \\ &= \sup\{\lambda(1 + \lambda)^{-1} : \lambda \in \operatorname{spec}(T^*T)\} < 1,\end{aligned}$$

where the spectral mapping theorem was used. Conversely, assume that $\|\mathcal{F}(T)\| < 1$, then $\|\mathcal{F}(T)^*\mathcal{F}(T)\| < 1$ and therefore $\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T)$ is invertible with bounded inverse. This implies that

$$T = \mathcal{F}(T)\big(\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T)\big)^{-\frac{1}{2}}$$

is bounded. □

Next let us introduce a pseudometric on the unit ball $\mathbb{B}_1(\mathcal{H}, \mathcal{H}')$ by setting

$$\begin{aligned} d_E(F_0, F_1) \\ = \max\Big\{\|F_0^*F_0 - F_1^*F_1\|, \|F_0F_0^* - F_1F_1^*\|, \big\|F_0(\mathbf{1} - F_0^*F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^*F_1)^{\frac{1}{2}}\big\|\Big\}. \end{aligned}$$

Clearly, d_E satisfies the triangle inequality and is symmetric. Note that this is an extension of the pseudometric introduced in Lemma 4.6.3 to operators which are not self-adjoint any more. As discussed after Lemma 4.6.3, it goes back to [108] and is called the extended gap metric, and the topology is then called the extended gap topology. The next result justifies this terminology, namely the extended gap metric is just the push-forward of the gap metric under the bounded transform.

Proposition 6.1.7. *The bounded transform*

$$\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_G) \to (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_E)$$

is a bi-Lipshitz-continuous homeomorphism. In particular, d_E defines a metric on $\mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$.

Proof. In the proof of Theorem 6.1.4, it is shown that for $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$,

$$P_T = \begin{pmatrix} \mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T) & (\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T))^{\frac{1}{2}}\mathcal{F}(T)^* \\ \mathcal{F}(T)(\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T))^{\frac{1}{2}} & \mathcal{F}(T)\mathcal{F}(T)^* \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$$

is the projection onto the graph of T. Comparing this to the definition of d_E leads to

$$d_E\big(\mathcal{F}(T_0), \mathcal{F}(T_1)\big) \le d_G(T_0, T_1) \le \sqrt{2}\, d_E\big(\mathcal{F}(T_0), \mathcal{F}(T_1)\big).$$

This implies all statements. □

The next result extends the applicability of Lemma 4.6.3.

Lemma 6.1.8. *The extended gap topology on $\mathbb{B}_1(\mathcal{H}, \mathcal{H}')$ is weaker than the norm topology. More precisely,*

$$d_E(F_0, F_1) \le 2\sqrt{2}\, d_N(F_0, F_1)^{\frac{1}{2}}, \quad F_0, F_1 \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}'). \tag{6.7}$$

Moreover, for $a < 1$,

$$d_N(F_0, F_1) \le \frac{1+\sqrt{2}}{1-a^2} d_E(F_0, F_1)^{\frac{1}{2}}, \quad F_0, F_1 \in \mathbb{B}_a(\mathcal{H}, \mathcal{H}').$$

Proof. For the third term in d_E, let us start with

$$\begin{aligned}
&\left\| F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} \right\| \\
&\quad \le \| F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_0(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} \| + \| F_0(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} \| \\
&\quad \le \| (\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - (\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} \| + \| F_0 - F_1 \|.
\end{aligned}$$

For the first summand, recall the fact (Proposition A.2.2) that for two nonnegative operators $A \ge 0$, $B \ge 0$ and $\alpha \in (0,1)$, one has $\|A^\alpha - B^\alpha\| \le \|A - B\|^\alpha$. Hence

$$\| F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} \| \le \| F_0^* F_0 - F_1^* F_1 \|^{\frac{1}{2}} + \| F_0 - F_1 \|.$$

Now

$$\| F_0^* F_0 - F_1^* F_1 \| \le \left\| (F_0 - F_1)^* F_0 \right\| + \left\| F_1^* (F_0 - F_1) \right\| \le 2 \| F_0 - F_1 \|,$$

and similarly

$$\| F_0 F_0^* - F_1 F_1^* \| \le 2 \| F_0 - F_1 \|.$$

Therefore

$$d_E(F_0, F_1) \le \sqrt{2} \| F_0 - F_1 \|^{\frac{1}{2}} + \| F_0 - F_1 \|,$$

so that

$$d_E(F_0, F_1) \le 2\sqrt{2} d_N(F_0, F_1)^{\frac{1}{2}},$$

because $d_N(F_0, F_1) = \|F_0 - F_1\| \le 2$ for $F_0, F_1 \in \mathbb{B}_1(\mathcal{H})$. The proof of the other bound (6.7) is as in Lemma 4.6.3, upon replacing H^2 by $F^* F$. □

Next comes an extension of a result of Nicolaescu [139] showing that the gap topology is weaker than the Riesz topology.

Proposition 6.1.9. *The gap topology on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is strictly weaker than the Riesz topology.*

Proof. The fact that the gap topology is weaker than the Riesz topology on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ directly follows from the first part of Lemma 6.1.8 combined with Propositions 6.1.5 and 6.1.7.

To show that the Riesz topology is different form the gap topology, we choose an orthonormal basis $\{\phi_k : k \in \mathbb{N}\}$ of $\mathcal{H}$ and define the linear operator

$$H : \mathcal{D}(H) \to \mathcal{H}, \quad \sum_{k\in\mathbb{N}} a_k\phi_k \mapsto \sum_{k\in\mathbb{N}} ka_k\phi_k$$

with domain $\mathcal{D}(H) = \{\sum_{k\in\mathbb{N}} a_k\phi_k : \sum_{k\in\mathbb{N}} k^2|a_k|^2 < \infty\}$. Clearly, H is self-adjoint and therefore in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. For $n \in \mathbb{N}$, let us define

$$H_n : \mathcal{D}(H_n) \to \mathcal{H}, \quad \sum_{k\in\mathbb{N}} a_k\phi_k \mapsto \sum_{k\in\mathbb{N}} ka_k\phi_k - 2na_n\phi_n$$

with domain $\mathcal{D}(H_n) = \mathcal{D}(H)$. Then H_n is self-adjoint and therefore in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. As $H_n^2 = H^2$ for all n and thus $R_{H_n} = R_H$ and

$$\lim_{n\to\infty} \|H_nR_{H_n} - HR_H\| = \lim_{n\to\infty} \|H_nR_{H_n}\phi_n - HR_H\phi_n\| = \lim_{n\to\infty} \|2n(1+n^2)^{-1}\| = 0,$$

the sequence $(H_n)_{n\in\mathbb{N}}$ converges to H with respect to the gap topology. For the Riesz topology, one has

$$\lim_{n\to\infty} \|\mathcal{F}(H_n)\phi_n - \mathcal{F}(H)\phi_n\| = \lim_{n\to\infty} \left\| -\frac{n}{\sqrt{1+n^2}}\phi_n - \frac{n}{\sqrt{1+n^2}}\phi_n \right\| = \lim_{n\to\infty} \frac{2n}{\sqrt{1+n^2}} = 2.$$

Therefore $(H_n)_{n\in\mathbb{N}}$ does not converge to H with respect to the Riesz topology and the gap topology is strictly weaker than the Riesz topology. □

Proposition 6.1.9 directly implies that the bounded transform $\mathcal{F}$ is *not* continuous as a map $\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_G) \to (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_N)$. In other words, there are not enough open sets in the gap topology to assure continuity of $\mathcal{F}$ in this sense.

The following is due to Cordes and Labrousse, see the addendum to [66]. However, the proof presented here is considerably simpler.

Theorem 6.1.10. *On the space of bounded operators* $\mathbb{B}(\mathcal{H}, \mathcal{H}')$*, the topologies induced by* d_G *and* d_R *coincide with the norm topology. Moreover, with respect to both the gap and Riesz topologies,* $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ *is open and dense in* $\mathbb{L}(\mathcal{H}, \mathcal{H}')$.

Proof. Let us introduce the set

$$\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}') = \{F \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}') : \|F\| < 1\}.$$

Then $\mathcal{F}(\mathbb{B}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$ by Proposition 6.1.6 and, furthermore, by the definition of the bounded transform,

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_N) \to (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_N)$$

is a homeomorphism. On the other hand, the two maps,

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_G) \to (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_E)$$

and

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_R) \to (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_N),$$

are also homeomorphisms by Propositions 6.1.7 and 6.1.5, respectively.

But Lemma 6.1.8 implies that the metrics d_E and d_N induce the same topologies on $\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$, showing the first claim.

By Proposition 6.1.6, the image of $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ under the bounded transform is dense and open in $\mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}'))$ with respect to the norm topology. By Proposition 6.1.5, this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is dense and open in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the Riesz topology. As the gap topology is weaker than the Riesz topology by Proposition 6.1.9, this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is also dense in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the gap topology. Furthermore, $\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$ is open in $(\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_E)$. Combined with Proposition 6.1.7 this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the gap topology. □

6.2 Basic properties of unbounded Fredholm operators

This section introduces unbounded Fredholm operators. As for bounded Fredholm operators, we recall several basic facts about them which can also be found in the literature, e. g., [99, 165]. Most of the results presented here are similar to the properties of bounded Fredholm operators studied in Section 3.2. However, as several modifications are necessary, the proofs are provided with full details, even though this leads to some repetitions.

Let us first recall that the quotient $\mathcal{H}/\mathcal{E}$ of $\mathcal{H}$ with respect to a subspace $\mathcal{E} \subset \mathcal{H}$ is the set of equivalence classes of the relation $\phi \sim \psi \Longleftrightarrow \phi - \psi \in \mathcal{E}$.

Definition 6.2.1. A linear operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is a Fredholm operator if and only if

(i) T is regular,
(ii) $\dim(\operatorname{Ker}(T)) < \infty$,
(iii) $\dim(\mathcal{H}'/\operatorname{Ran}(T)) < \infty$.

The set of Fredholm operators is denoted by $\mathbb{F}(\mathcal{H}, \mathcal{H}')$ and simply by $\mathbb{F}(\mathcal{H}) = \mathbb{F}(\mathcal{H}, \mathcal{H})$ whenever $\mathcal{H}' = \mathcal{H}$.

For a closed operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$, the linear space $\mathcal{D}(T)$ equipped with the T-norm $\|\phi\|_T = (\|\phi\|_{\mathcal{H}}^2 + \|T\phi\|_{\mathcal{H}'}^2)^{\frac{1}{2}}$ is a Hilbert space. Associated with T there is a bounded operator $\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}'$ defined by $\widetilde{T}\phi = T\phi$.

Proposition 6.2.2. *A closed operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is Fredholm if and only if the associated bounded operator $\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}'$ is Fredholm.*

Proof. As $\mathrm{Ker}(T) = \mathrm{Ker}(\widetilde{T})$ and $\mathrm{Ran}(T) = \mathrm{Ran}(\widetilde{T})$, the claim directly follows from item (ii) of Theorem 3.2.2. □

As for bounded Fredholm operators, one has the following characterization.

Lemma 6.2.3. *A linear operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is Fredholm if and only if*
(i) *T is regular,*
(ii) $\dim(\mathrm{Ker}(T)) < \infty$,
(iii) $\dim(\mathrm{Ker}(T^*)) < \infty$,
(iv) $\mathrm{Ran}(T)$ *is closed in* $\mathcal{H}'$.

Proof. Let us first assume that T is Fredholm. Then, by Proposition 6.2.2, the associated bounded operator $\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}'$ is Fredholm and $\mathrm{Ran}(\widetilde{T}) = \mathrm{Ran}(T)$ is closed. Therefore

$$\dim(\mathcal{H}'/\,\mathrm{Ran}(T)) = \dim(\mathrm{Ran}(T)^{\perp}) = \dim(\mathrm{Ker}(T^*))$$

is finite. Conversely, if $\mathrm{Ran}(T)$ is closed then $\mathcal{H}'/\,\mathrm{Ran}(T)$ is known to be a Hilbert space of dimension $\dim(\mathcal{H}'/\,\mathrm{Ran}(T)) = \dim(\mathrm{Ker}(T^*))$. Thus the equivalence is shown. □

The following extends Theorem 3.2.2 to unbounded operators.

Theorem 6.2.4. *For a regular operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$, the following are equivalent:*
(i) *T is a Fredholm operator.*
(ii) *There exists a unique $S_0 \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ such that*

$$\mathrm{Ker}(S_0) = \mathrm{Ran}(T)^{\perp}, \quad \mathrm{Ker}(S_0^*) = \mathrm{Ker}(T),$$

and such that $S_0 T$ can be continuously extended to the orthogonal projection onto $\mathrm{Ker}(T)^{\perp}$. *Moreover, TS_0 is the orthogonal projection onto* $\mathrm{Ran}(T)$ *and*

$$\dim(\mathrm{Ran}(\mathbf{1} - S_0 T)) < \infty, \quad \dim(\mathrm{Ran}(\mathbf{1} - TS_0)) < \infty.$$

(iii) *There exists a so-called pseudoinverse $S \in \mathbb{B}(\mathcal{H}', \mathcal{H})$ such that $TS - \mathbf{1}$ and $ST - \mathbf{1}$ can be extended to compact operators on $\mathcal{H}$ and $\mathcal{H}'$, respectively.*

Proof. (i) ⟹ (ii). First note that $T|_{\mathrm{Ker}(T)^{\perp}} : \mathcal{D}(T) \cap \mathrm{Ker}(T)^{\perp} \to \mathrm{Ran}(T)$ is bijective and the graph of its inverse $\{(T\phi, \phi) : \phi \in \mathcal{D}(T) \cap \mathrm{Ker}(T)^{\perp}\}$ is closed as T is closed. Now, as $\mathrm{Ran}(T)$ is closed and therefore a Hilbert space, the closed graph theorem shows that the inverse $S_0 : \mathrm{Ran}(T) \to \mathrm{Ker}(T)^{\perp}$ is bounded. It can be extended to all of $\mathcal{H}'$ by $S_0\psi = 0$ for $\psi \in \mathrm{Ran}(T)^{\perp}$. Then by construction TS_0 is the projection in $\mathcal{H}'$ onto $\mathrm{Ran}(T)$ and $S_0 T$ is bounded and can be extended to the projection in $\mathcal{H}$ onto $\mathrm{Ker}(T)^{\perp}$. This implies all the stated properties. Uniqueness is obvious.

(ii) ⟹ (iii). This is obvious.

(iii) $\Longrightarrow$ (i). Suppose that $(\psi_n)_{n\geq 1}$ is an infinite orthonormal basis of $\operatorname{Ker}(T)$. As these vectors are all eigenvectors of the compact operator $K = ST - \mathbf{1}$ for the eigenvalue 1, this is a contradiction to Riesz' theorem (Theorem 3.1.6). Suppose that $(\phi_n)_{n\geq 1}$ is an infinite orthonormal basis of $\operatorname{Ran}(T)^\perp$. Consequently, one has $\|(TS - \mathbf{1})\phi_n\| = \|TS\phi_n - \phi_n\| \geq 1$ as $TS\phi_n \perp \phi_n$, a contradiction to the compactness of $TS - \mathbf{1}$. It remains to show that $\operatorname{Ran}(T)$ is closed. Let K be the compact extension of $ST - \mathbf{1}$. Choose $L \in \mathbb{K}(\mathcal{H})$ with a finite-dimensional range and such that

$$\|K - L\| \leq \frac{1}{2}.$$

Then for all $\phi \in \operatorname{Ker}(L) \cap \mathcal{D}(T)$:

$$\begin{aligned}\|S\|\|T\phi\| &\geq \|ST\phi\| \\ &= \big\|(\mathbf{1} + K)\phi\big\| \\ &\geq \|\phi\| - \|K\phi\| \\ &\geq \|\phi\| - \big\|(K - L)\phi\big\| - \|L\phi\| \\ &\geq \frac{1}{2}\|\phi\|.\end{aligned}$$

Thus $\|\phi\| \leq 2\|S\|\|T\phi\|$ for all $\phi \in \operatorname{Ker}(L) \cap \mathcal{D}(T)$. This implies that $T(\operatorname{Ker}(L) \cap \mathcal{D}(T))$ is closed. Indeed, given a convergent sequence $(T\phi_n)_{n\geq 1}$ with $\phi_n \in \operatorname{Ker}(L) \cap \mathcal{D}(T)$, one can set $\psi = \lim_n T\phi_n$. Then

$$\|\phi_n - \phi_m\| \leq 2\|S\|\|T\phi_n - T\phi_m\|.$$

Thus $(\phi_n)_{n\geq 1}$ is a Cauchy sequence and hence has a limit point $\phi = \lim \phi_n \in \mathcal{H}$. As T is closed, one has $\psi = T\phi \in T(\operatorname{Ker}(L) \cap \mathcal{D}(T))$. On the other hand,

$$T\big(\operatorname{Ker}(L)^\perp \cap \mathcal{D}(T)\big) = T\big(\operatorname{Ran}(L^*) \cap \mathcal{D}(T)\big).$$

As L^* also has a finite-dimensional image, it follows that $T(\operatorname{Ker}(L)^\perp \cap \mathcal{D}(T))$ is finite dimensional. Thus $\operatorname{Ran}(T) = T(\operatorname{Ker}(L) \cap \mathcal{D}(T)) + T(\operatorname{Ker}(L)^\perp \cap \mathcal{D}(T))$ is closed. □

The following two propositions present criteria for regular operators to be Fredholm. They are the analogues of Lemma 3.4.2 and Proposition 3.2.6 for bounded operators.

Proposition 6.2.5. *For a regular operator $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$, the following are equivalent:*

(i) $\dim(\operatorname{Ker}(T)) < \infty$ *and* $\operatorname{Ran}(T)$ *is closed.*
(ii) $\dim(\operatorname{Ker}(T)) < \infty$ *and there is a constant $c > 0$ such that $\|T\phi\| > c\|\phi\|$ for all vectors* $\phi \in \mathcal{D}(T) \cap \operatorname{Ker}(T)^\perp$.

(iii) *If* $(\phi_n)_{n\geq 1}$ *is a bounded sequence in* $\mathcal{D}(T)$ *such that* $(T\phi_n)_{n\geq 1}$ *is convergent, then there is a convergent subsequence of* $(\phi_n)_{n\geq 1}$.

Proof. (i) $\Longrightarrow$ (ii). The restriction T_r of T is a bijection from $\mathcal{D}(T)\cap\mathrm{Ker}(T)^\perp$ to $\mathrm{Ran}(T)$. The graph of its inverse T_r^{-1} is $\{(T\phi,\phi) : \phi \in \mathcal{D}(T)\cap\mathrm{Ker}(T)^\perp\}$. As T is a closed operator, also its restriction T_r can be seen to be closed, so that the graph of $T_r^{-1} : \mathrm{Ran}(T) \to \mathcal{H}$ is closed. As $\mathrm{Ran}(T_r)$ is closed and therefore a Hilbert space, the closed graph theorem shows that T_r^{-1} is bounded. Therefore $\|\phi\| = \|T_r^{-1}T\phi\| \leq \|T_r^{-1}\|\|T\phi\|$ holds for all $\phi \in \mathcal{D}(T)\cap\mathrm{Ker}(T)^\perp$.

(ii) $\Longrightarrow$ (i). Let $(\psi_n)_{n\geq 1}$ be a sequence in $\mathrm{Ran}(T)$ converging to $\psi \in \mathcal{H}'$. Then there are $\phi_n \in \mathrm{Ker}(T)^\perp\cap\mathcal{D}(T)$ with $T\phi_n = \psi_n$. By (ii), one has $\|\phi_n - \phi_m\| < \frac{1}{c}\|\psi_n - \psi_m\|$ so that $(\phi_n)_{n\geq 1}$ is Cauchy and thus converges to some $\phi \in \mathcal{H}$. As $(\phi_n, T\phi_n)$ converges to (ϕ,ψ) and T is closed, one has $T\psi = \phi$ so that $\phi \in \mathrm{Ran}(T)$ and $\mathrm{Ran}(T)$ is closed.

(ii) $\Longrightarrow$ (iii). Let $(\phi_n)_{n\geq 1}$ be a bounded sequence in $\mathcal{D}(T)$ such that $(T\phi_n)_{n\geq 1}$ is convergent. One has $\phi_n = \theta_n + \psi_n$ with $\theta_n \in \mathrm{Ker}(T)$ and $\psi_n \in \mathcal{D}(T)\cap\mathrm{Ker}(T)^\perp$. Because $\|\psi_n - \psi_m\| < \frac{1}{c}\|T\phi_n - T\phi_m\|$ by (ii), $(\psi_n)_{n\geq 1}$ is Cauchy and therefore convergent. As $(\phi_n)_{n\geq 1}$ and $(\psi_n)_{n\geq 1}$ are bounded, also $(\theta_n)_{n\geq 1}$ is bounded. Because the dimension of the kernel of T is finite, $(\theta_n)_{n\geq 1}$ and therefore $(\phi_n)_{n\geq 1}$ has a convergent subsequence.

(iii) $\Longrightarrow$ (ii). Suppose that the kernel of T is infinite dimensional and that $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of it. Then $(\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{H}$ such that $T\phi_n$ is constant (equal to 0) and therefore convergent. As there is no convergent subsequence of $(\phi_n)_{n\in\mathbb{N}}$, this is a contradiction to (ii). Thus $\mathrm{Ker}(T)$ is finite dimensional. Moreover, there is a constant $c > 0$ such that $\|\phi\| \leq c\|T\phi\|$ for all $\phi \in \mathrm{Ker}(T)^\perp\cap\mathcal{D}(T)$, because otherwise there is a sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathrm{Ker}(T)^\perp\cap\mathcal{D}(T)$ such that $\|\phi_n\| = 1$ for all $n \in \mathbb{N}$ and $\|T\phi_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. As $(T\phi_n)_{n\in\mathbb{N}}$ is convergent, by assumption there is a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ converging to some vector $\phi \in \mathrm{Ker}(T)^\perp$ with $\|\phi\| = 1$. As $(\phi_{n_k}, T\phi_{n_k})$ converges to $(\phi, 0)$ and T is closed, one has $\phi \in \mathcal{D}(T)$ and $T\phi = 0$. This is a contradiction to $\phi \in \mathrm{Ker}(T)^\perp$. □

Proposition 6.2.6. *Let* $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ *be a regular operator. If there is a compact operator* $K \in \mathbb{K}(\mathcal{H},\mathcal{H}'')$ *and a constant* $c > 0$ *such that*

$$\|\phi\| \leq c\big(\|T\phi\| + \|K\phi\|\big)$$

for all $\phi \in \mathcal{D}(T)$*, then* T *has a closed range and a finite-dimensional kernel.*

Proof. Let $(\phi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{D}(T)$ such that $T\phi_n$ is convergent, namely there is a $\psi \in \mathcal{H}'$ such that $\lim_{n\to\infty} T\phi_n = \psi$. As K is compact, there is a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ such that $K\phi_{n_k}$ is convergent. Then $(K\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and as $\lim_{k\to\infty} T\phi_{n_k} = \psi$, also $(T\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence. Therefore for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\max\{\|T\phi_{n_k} - T\phi_{n_m}\|, \|K\phi_{n_k} - K\phi_{n_m}\|\} < \frac{\epsilon}{2c}$ for all $k, m > N$. Thus

$$\|\phi_{n_k} - \phi_{n_m}\| \leq c\big(\|T\phi_{n_k} - T\phi_{n_m}\| + \|K\phi_{n_k} - K\phi_{n_m}\|\big) < \epsilon$$

or all $k, m > N$, which shows that $(\phi_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent. Finally, item (iii) of Proposition 6.2.5 shows the assertion. □

Definition 6.2.7. The index of a Fredholm operator $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ is

$$\text{Ind}(T) = \dim(\text{Ker}(T)) - \dim(\mathcal{H}'/\,\text{Ran}(T)).$$

Next let us generalize Corollary 3.3.2 to unbounded Fredholm operators.

Corollary 6.2.8. (i) *For* $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, $T' \in \mathbb{F}(\mathcal{H}'', \mathcal{H})$, *also* $TT' \in \mathbb{F}(\mathcal{H}'', \mathcal{H}')$.
(ii) *If* $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, *then* $T^* \in \mathbb{F}(\mathcal{H}', \mathcal{H})$. *Moreover,*

$$\text{Ind}(T) = \dim(\text{Ker}(T)) - \dim(\text{Ker}(T^*))$$

and

$$\text{Ind}(T^*) = -\,\text{Ind}(T).$$

(iii) *If* $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, *then*

$$\text{Ind}(T) = \dim(\text{Ker}(T^*T)) - \dim(\text{Ker}(TT^*)).$$

(iv) *For* $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ *and* $T' \in \mathbb{F}(\mathcal{H}'', \mathcal{H}''')$, *one has* $T \oplus T' \in \mathbb{F}(\mathcal{H} \oplus \mathcal{H}'', \mathcal{H}' \oplus \mathcal{H}''')$ *and*

$$\text{Ind}(T \oplus T') = \text{Ind}(T) + \text{Ind}(T').$$

Proof. For the proof of (i), let us first show that TT' is densely defined, namely that $\mathcal{D}(TT') = \{\phi \in \mathcal{D}(T') : T'\phi \in \mathcal{D}(T)\}$ is dense in $\mathcal{H}''$. First, it is checked that $\mathcal{D}(T)\cap\text{Ran}(T')$ is dense in $\text{Ran}(T')$. As T' is Fredholm, $\text{Ran}(T')^\perp$ is finite dimensional. Let $\{\psi_1, \dots, \psi_n\}$ be an orthonormal basis of $\text{Ran}(T')^\perp$. Let $\epsilon_1 > 0$. Because $\mathcal{D}(T)$ is dense in $\mathcal{H}$, there are $\theta_i \in \mathcal{D}(T)$ for $i = 1, \dots, n$ such that $\|\phi_i - \theta_i\| < \epsilon_1$. Then $\mathcal{E} = \text{span}(\{\theta_1, \dots, \theta_n\})$ is a subspace of $\mathcal{D}(T)$ and, for ϵ_1 sufficiently small, $\text{Ran}(T') \cap \mathcal{E} = \{0\}$ and $\text{Ran}(T') \oplus \mathcal{E} = \mathcal{H}$. By Proposition 5.1.6, there is a projection $P \in \mathbb{B}(\mathcal{H})$ with $\text{Ran}(P) = \text{Ran}(T')$ and $\text{Ker}(P) = \mathcal{E}$. Because $\mathcal{D}(T) \subset \mathcal{H}$ is dense, for any vector $\psi \in \text{Ran}(T')$ and $\epsilon_2 > 0$ there is $\psi' \in \mathcal{D}(T)$ such that $\|\psi - \psi'\| < \epsilon_2$. Then $P\psi' = \psi' - (\mathbf{1} - P)\psi' \in \mathcal{D}(T) \cap \text{Ran}(T')$ and

$$\begin{aligned}\|\psi - P\psi'\| &\le \|\psi - \psi'\| + \|(\mathbf{1} - P)\psi'\| \\ &< \epsilon_2 + \|(\mathbf{1} - P)(\psi' - \psi)\| \\ &\le \epsilon_2(1 + \|\mathbf{1} - P\|),\end{aligned}$$

where the second step follows as $\psi \in \text{Ran}(T') = \text{Ker}(\mathbf{1}-P)$. This shows that $\mathcal{D}(T)\cap\text{Ran}(T')$ is dense in $\text{Ran}(T')$. To show that $\mathcal{D}(TT')$ is dense in $\mathcal{H}''$, it is sufficient to show that for $\epsilon > 0$ and $\phi \in \mathcal{D}(T')$ there is $\widetilde{\phi} \in \mathcal{D}(TT')$ such that $\|\phi - \widetilde{\phi}\| < \epsilon$ (because $\mathcal{D}(T') \subset \mathcal{H}''$ is dense). For $\phi \in \mathcal{D}(T')$, there is $\phi' \in \mathcal{D}(T') \cap \text{Ker}(T')^\perp$ such that $T'\phi' = T'\phi$ and thus $\phi - \phi' \in \operatorname{Ker}(T) \subset \mathcal{D}(TT')$. By the above, there is $\psi \in \operatorname{Ran}(T') \cap \mathcal{D}(T)$ such that $\|\psi - T'\phi\| < \epsilon c$ for $c > 0$ as in item (ii) of Proposition 6.2.5 applied to T'. Then there is $\theta \in \mathcal{D}(T') \cap \operatorname{Ker}(T')^\perp$ such that $\psi = T'\theta$ and therefore $\theta \in \mathcal{D}(TT')$. Thus one concluldes that $\phi' - \theta \in \mathcal{D}(T') \cap \operatorname{Ker}(T')^\perp$ and, by Proposition 6.2.5,

$$\|\phi' - \theta\| < \frac{1}{c}\|T'(\phi' - \theta)\| = \frac{1}{c}\|T'\phi - \psi\| < \epsilon.$$

By construction, $\widetilde{\phi} = \phi - \phi' + \theta \in \mathcal{D}(TT')$ fulfills

$$\|\phi - \widetilde{\phi}\| = \|\phi' - \theta\| < \epsilon.$$

This shows that TT' is densely defined.

To show that TT' is closed, let us choose a sequence $(\phi_n)_{n\geq 1}$ in $\mathcal{D}(TT')$ such that $(\phi_n, TT'\phi_n)$ converges to (ϕ, θ). For $\psi_n = T'\phi_n$, there are $\psi_n' \in \mathcal{D}(T) \cap \operatorname{Ker}(T)^\perp$ and $\psi_n'' \in \operatorname{Ker}(T)$ such that $\psi_n = \psi_n' + \psi_n''$. Then $(\psi_n')_{n\geq 1}$ is Cauchy, as, by Proposition 6.2.5, there is a constant $c > 0$ such that

$$\|\psi_n' - \psi_m'\| < c\|T\psi_n' - T\psi_m'\| = c\|TT'\phi_n - TT'\phi_m\|$$

and $(TT'\phi_n)_{n\geq 1}$ is Cauchy by assumption. Therefore $(\psi_n')_{n\geq 1}$ is convergent, and one can define $\psi = \lim_{n\to\infty} \psi_n' \in \mathcal{H}$. As $(\psi_n', T\psi_n') = (\psi_n', TT'\phi_n)$ converges to (ψ, θ) and T is closed, one has $\psi \in \mathcal{D}(T)$ and $T\psi = \theta$. We show that $(\psi_n'')_{n\geq 1}$ is bounded. Suppose that $(\psi_n'')_{n\geq 1}$ is unbounded, then there is a subsequence, again denoted by $(\psi_n'')_{n\geq 1}$, such that $\lim_{n\to\infty} \|\psi_n''\| = \infty$. Then $(\frac{\psi_n''}{\|\psi_n''\|})_{n\geq 1}$ is a bounded sequence in the finite-dimensional kernel of T. Again by choosing a subsequence, without loss of generality one can assume that $(\frac{\psi_n''}{\|\psi_n''\|})_{n\geq 1}$ converges to $\widetilde{\psi} \in \operatorname{Ker}(T)$ with $\|\widetilde{\psi}\| = 1$. As $(\phi_n)_{n\geq 1}$ is bounded, one has $\lim_{n\to\infty} \frac{\phi_n}{\|\psi_n''\|} = 0$ and $T' \frac{\phi_n}{\|\psi_n''\|} = \frac{\psi_n' + \psi_n''}{\|\psi_n''\|}$ converges to $\widetilde{\psi}$. As T' is closed, this implies $\widetilde{\psi} = 0$, which is a contradiction. Therefore the sequence $(\psi_n'')_{n\geq 1}$ is bounded. As the dimension of $\operatorname{Ker}(T)$ is finite, there is a convergent subsequence $(\psi_{n_j}'')_{j\geq 1}$ of $(\psi_n'')_{n\geq 1}$. Setting $\psi'' = \lim_{j\to\infty} \psi_{n_j}'' \in \operatorname{Ker}(T)$, one has $\lim_{j\to\infty} \psi_{n_j} = \psi + \psi''$. As $(\phi_{n_j})_{j\geq 1}$ converges to ϕ and $(T\phi_{n_j})_{j\geq 1} = (\psi_{n_j})_{j\geq 1}$ converges to $\psi + \psi''$, one has $\phi \in \mathcal{D}(T')$ and $T'\phi = \psi + \psi''$. As $T\psi = \theta$ and $\psi'' \in \operatorname{Ker}(T)$, one has $\theta = T(\psi + \psi'') = TT'\phi$. In conclusion, (ϕ, θ) is an element of the graph of TT' and therefore TT' is closed.

We next use Proposition 6.2.5 to show that $\operatorname{Ran}(TT')$ is closed and that the dimension of the kernel of TT' is finite. Let $(\phi_n)_{n\geq 1}$ in $\mathcal{D}(TT')$ be a bounded sequence such that $(TT'\phi_n)_{n\geq 1}$ is convergent. For $\psi_n = T'\phi_n$, there are $\psi_n' \in \mathcal{D}(T) \cap \operatorname{Ker}(T)^\perp$ and $\psi_n'' \in \operatorname{Ker}(T)$ such that $\psi_n = \psi_n' + \psi_n''$. Then $(\psi_n')_{n\geq 1}$ is Cauchy, as, by Proposition 6.2.5, there is a constant $c > 0$ such that

$$\|\psi_n' - \psi_m'\| < c\|T\psi_n' - T\psi_m'\| = c\|TT'\phi_n - TT'\phi_m\|$$

and $(TT'\phi_n)_{n\geq 1}$ is Cauchy by assumption. Therefore $(\psi'_n)_{n\geq 1}$ is convergent and one can set $\psi = \lim_{n\to\infty}\psi'_n \in \mathcal{H}$. As above one can show that $(\psi''_n)_{n\geq 1}$ is bounded. As the dimension of $\mathrm{Ker}(T)$ is finite, there is a convergent subsequence $(\psi''_{n_j})_{j\geq 1}$ of $(\psi''_n)_{n\geq 1}$. Next setting $\psi'' = \lim_{j\to\infty}\psi''_{n_j} \in \mathrm{Ker}(T)$, one has $\lim_{j\to\infty}\psi_{n_j} = \psi + \psi''$. Thus $(T'\phi_{n_j})_{j\geq 1} = (\psi_{n_j})_{j\geq 1}$ converges to $\psi+\psi''$. By item (iii) of Proposition 6.2.5 applied to the Fredholm operator T', there is a convergent subsequence of $(\phi_{n_j})_{j\geq 1}$. Thus $\mathrm{Ran}(TT')$ is closed and $\dim(\mathrm{Ker}(TT'))$ is finite. To show that $\mathrm{Ran}(TT')^\perp$ is finite dimensional, note that $\dim(\mathrm{Ran}(T')^\perp)$ is finite and thus the dimension of $T(\mathrm{Ran}(T')^\perp)$ is finite. As

$$\mathrm{Ran}(T) = T(\mathrm{Ran}(T')) + T(\mathrm{Ran}(T')^\perp),$$

one has

$$\mathrm{Ran}(T)^\perp = \big(T(\mathrm{Ran}(T'))\big)^\perp \cap \big(T(\mathrm{Ran}(T')^\perp)\big)^\perp.$$

As $\mathrm{Ran}(T)^\perp$ and $T(\mathrm{Ran}(T')^\perp)$ are finite dimensional, this implies that the dimension of $(T(\mathrm{Ran}(T')))^\perp = \mathrm{Ran}(TT')^\perp$ is finite.

In order to show (ii), let us note that T^* is regular and $\mathrm{Ker}(T^*) = \mathrm{Ran}(T)^\perp$ and $\mathrm{Ran}(T^*)^\perp = \mathrm{Ker}(T)$ are finite dimensional. It remains to show that $\mathrm{Ran}(T^*)$ is closed. This follows from Proposition 6.2.5, because $\mathrm{Ker}(T^*)^\perp = \mathrm{Ran}(T)$ as $\mathrm{Ran}(T)$ is closed. Therefore for $\theta \in \mathcal{D}(T^*)\cap\mathrm{Ker}(T^*)^\perp$ there is $\phi \in \mathcal{D}(T)\cap\mathrm{Ker}(T)^\perp$ such that $T\phi = \theta$. Then

$$\|\phi\|\|T^*\theta\| \geq \langle T^*\theta|\phi\rangle = \langle T\phi|T\phi\rangle = \|T\phi\|^2 \geq c\|\phi\|\|T\phi\| = c\|\phi\|\|\theta\|$$

for a constant $c > 0$ by Proposition 6.2.5. Thus $\|T^*\theta\| \geq c\|\theta\|$ for all $\theta \in \mathcal{D}(T^*)\cap\mathrm{Ker}(T^*)^\perp$ and $\mathrm{Ran}(T^*)$ is closed, again by Proposition 6.2.5. The claim about the index of T follows directly from Definition 6.2.7.

As $\mathrm{Ker}(T) = \mathrm{Ker}(T^*T)$ and $\mathrm{Ker}(T^*) = \mathrm{Ker}(TT^*)$, item (iii) is a direct consequence of (ii).

The last claim follows from the obvious identities $\mathrm{Ker}(T\oplus T') = \mathrm{Ker}(T)\oplus\mathrm{Ker}(T')$ and $\mathrm{Ran}(T\oplus T') = \mathrm{Ran}(T)\oplus\mathrm{Ran}(T')$. □

Proposition 6.2.9. *If $T \in \mathbb{F}(\mathcal{H},\mathcal{H}')$ and $T' \in \mathbb{F}(\mathcal{H}'',\mathcal{H})$, then the index of the Fredholm operator $TT' \in \mathbb{F}(\mathcal{H}'',\mathcal{H}')$ is given by*

$$\mathrm{Ind}(TT') = \mathrm{Ind}(T) + \mathrm{Ind}(T').$$

Proof. Recall that TT' is Fredholm by Corollary 6.2.8. One has

$$\dim(\mathrm{Ker}(TT')) = \dim(\mathrm{Ker}(T')) + \dim(\mathrm{Ker}(T)\cap\mathrm{Ran}(T')).$$

Setting $\mathcal{N}_1 = \mathrm{Ker}(T)\cap\mathrm{Ran}(T')$, there is a finite-dimensional subspace $\mathcal{N}_2 \subset \mathcal{H}$ such that

$$\mathrm{Ker}(T) = \mathcal{N}_1 \oplus \mathcal{N}_2.$$

Note that $\text{Ran}(T') \cap \mathcal{N}_2 = \{0\}$ and $\text{Ran}(T') \oplus \mathcal{N}_2$ is closed. Next it is shown that there is a finite-dimensional subspace $\mathcal{N}_3 \subset \mathcal{D}(T)$ such that

$$\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 = \mathcal{H}.$$

Because $(\text{Ran}(T') \oplus \mathcal{N}_2)^\perp$ is a subspace of the finite-dimensional space $\text{Ran}(T')^\perp$, it is finite dimensional. If $\dim((\text{Ran}(T') \oplus \mathcal{N}_2)^\perp) = 0$, the claim holds for $\mathcal{N}_3 = \{0\}$. Therefore, without loss of generality, one can assume $\dim((\text{Ran}(T') \oplus \mathcal{N}_2)^\perp) = l \in \mathbb{N}$. Next since $\text{Ran}(T') \oplus \mathcal{N}_2$ is closed and T is densely defined, there is a vector $\phi_1 \in \mathcal{D}(T) \backslash (\text{Ran}(T') \oplus \mathcal{N}_2)$. Then $\mathcal{H}_1 = \text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\})$ is closed and $\dim((\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\}))^\perp) = l - 1$. If $l \geq 2$, there is a vector $\phi_2 \in \mathcal{D}(T) \backslash (\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\}))$. Repeating this procedure l times, one finds vectors $\phi_1, \ldots, \phi_l \in \mathcal{D}(T)$ such that $\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1, \ldots, \phi_l\}) = \mathcal{H}$. Then the claim holds for $\mathcal{N}_3 = \text{span}(\{\phi_1, \ldots, \phi_l\})$.

The restriction $T|_{\mathcal{N}_3}$ is injective and

$$\text{Ran}(T) = \text{Ran}(TT') \oplus T\mathcal{N}_3.$$

The last claim holds as $\text{Ran}(T) = \text{Ran}(TT') + T\mathcal{N}_3$ by construction and since, for vectors $\phi \in \text{Ran}(T')$ and $\psi \in \mathcal{N}_3$ such that $T\phi = T\psi \in \text{Ran}(TT')$, one has $\psi \in \text{Ran}(T') + \text{Ker}(T)$ and therefore $\psi = 0$ by definition of $\mathcal{N}_3$. Thus

$$\dim\big(\text{Ran}(TT')^\perp\big) = \dim\big(\text{Ran}(T)^\perp\big) + \dim(\mathcal{N}_3).$$

One can conclude that

$$\begin{aligned}
\text{Ind}(TT') &= \dim(\text{Ker}(TT')) - \dim(\mathcal{H}' / \text{Ran}(TT')) \\
&= \dim(\text{Ker}(T')) + \dim(\mathcal{N}_1) - \dim(\text{Ran}(T)^\perp) - \dim(\mathcal{N}_3) \\
&= \dim(\text{Ker}(T')) + \dim(\mathcal{N}_1) + \dim(\mathcal{N}_2) \\
&\quad - \dim(\text{Ran}(T)^\perp) - \dim(\mathcal{N}_3) - \dim(\mathcal{N}_2) \\
&= \dim(\text{Ker}(T')) + \dim(\text{Ker}(T)) - \dim(\text{Ran}(T)^\perp) - \dim\big(\text{Ran}(T')^\perp\big) \\
&= \text{Ind}(T) + \text{Ind}(T'),
\end{aligned}$$

by definition of $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$. □

The next aim is to show that the Fredholm property and that the index is invariant under small or compact perturbations. Therefore we introduce the notion of relatively bounded and relatively compact operators.

Definition 6.2.10. Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ be a closed linear operator. Another operator $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}'$ with $\mathcal{D}(T) \subset \mathcal{D}(S)$ is called relatively bounded with respect to T (or T-bounded) if the restriction $S|_{\mathcal{D}(T)}$ is bounded as operator $S : \mathcal{D}(T) \to \mathcal{H}'$ where $\mathcal{D}(T)$ is equipped with the T-norm $\|\cdot\|_T$. Analogously, S is called relatively compact with respect to T (or T-compact) if the restriction $S|_{\mathcal{D}(T)} : \mathcal{D}(T) \to \mathcal{H}'$ is compact, where again $\mathcal{D}(T)$ is equipped with the T-norm.

Note that $\phi \mapsto \|T\phi\| + \|\phi\|$ defines a norm on $\mathcal{D}(T)$ that is equivalent to the T-norm. Therefore an operator $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}'$ is relatively bounded with respect to T if and only if there are constants $c_1, c_2 > 0$ such that

$$\|S\phi\| \le c_1\|T\phi\| + c_2\|\phi\| \tag{6.8}$$

for all $\phi \in \mathcal{D}(T)$. In particular, every bounded operator $S : \mathcal{H} \to \mathcal{H}'$ is T-bounded and every compact operator $S : \mathcal{H} \to \mathcal{H}'$ is T-compact.

Lemma 6.2.11. *If $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is a closed operator and $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}'$ is relatively bounded with respect to T and the relative bound c_1 in* (6.8) *is less than* 1*, then $T + S : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is a closed operator.*

Proof. Equation (6.8) with $c_1, c_2 > 0$ implies

$$\|(T + S)\phi\| \le (1 + c_1)\|T\phi\| + c_2\|\phi\| \tag{6.9}$$

and

$$\|(T + S)\phi\| \ge \|T\phi\| - \|S\phi\| \ge (1 - c_1)\|T\phi\| - c_2\|\phi\|.$$

As $c_1 < 1$, the last inequality is equivalent to

$$\|T\phi\| \le \frac{1}{1 - c_1}\big(\|(T + S)\phi\| + c_2\|\phi\|\big). \tag{6.10}$$

Let $(\phi_n)_{n\ge1}$ be a sequence in $\mathcal{D}(T)$ such that $(\phi_n, (T + S)\phi_n)$ converges to (ϕ, θ). By (6.10),

$$\|T\phi_n - T\phi_m\| \le \frac{1}{1 - c_1}\big(\|(T + S)(\phi_n - \phi_m)\| + c_2\|\phi_n - \phi_m\|\big)$$

and therefore $(T\phi_n)_{n\ge1}$ is Cauchy and thus convergent. Setting $\psi = \lim_{n\to\infty} T\phi_n$, this implies that $(\phi_n, T\phi_n)_{n\ge1}$ converges to (ϕ, ψ). As T is closed, ϕ is in $\mathcal{D}(T) = \mathcal{D}(T + S)$ and $T\phi = \psi$. Moreover, by (6.9),

$$\|(T + S)(\phi - \phi_n)\| \le (1 + c_1)\|T(\phi - \phi_n)\| + c_2\|\phi - \phi_n\|$$

converges to 0. Therefore $(T + S)\phi = \lim_{n\to\infty}(T + S)\phi_n = \theta$ and the graph of $T + S$ is closed. □

A similar result holds for relatively compact operators.

Lemma 6.2.12. *If $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is a Fredholm operator and $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}'$ is relatively compact with respect to T, then $T + S : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ is a closed operator.*

Proof. Let $\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}$ be the bounded operator associated with T. Similarly, define $\widetilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}$ by $\widetilde{S}\phi = S\phi$. Then by Proposition 6.2.2, $\widetilde{T}$ and therefore $\widetilde{T}+\widetilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}$ are bounded Fredholm operators. Let us define the embedding $I : \mathcal{D}(T) \subset (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \to (\mathcal{D}(T), \|\cdot\|_T)$ by $I\phi = \phi$. Then I is invertible and I^{-1} is bounded and therefore closed. Thus also I is closed and as $\operatorname{Ker}(I) = \{0\}$ and $\operatorname{Ran}(I) = \mathcal{D}(T)$, I is a Fredholm operator. Therefore, by item (i) of Corollary 6.2.8, $T + S = (\widetilde{T} + \widetilde{S})I$ is Fredholm and, in particular, closed. □

After these preparations, we can now show that the Fredholm property is invariant under small or compact perturbations.

Proposition 6.2.13. *Let $T : \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}'$ be a Fredholm operator and furthermore let $S : \mathcal{D}(S) \subset \mathcal{H} \to \mathcal{H}'$ be relatively compact with respect to T or relatively bounded with respect to T such that the constants c_1, c_2 in* (6.8) *are sufficiently small, then the operator $T + S : \mathcal{D}(T) \to \mathcal{H}'$ is Fredholm and*

$$\operatorname{Ind}(T + S) = \operatorname{Ind}(T).$$

Proof. By the above lemmata, where $c_1 < 1$ is assumed, operator $T + S$ is closed. Let $\widetilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}$ be the operator associated with T and again let $\widetilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \to \mathcal{H}$ be given by $\widetilde{S}\phi = S\phi$. Then by Proposition 6.2.2, $T + S$ is Fredholm if and only if $\widetilde{T} + \widetilde{S}$ is Fredholm. If S is relatively compact with respect to T, $\widetilde{T} + \widetilde{S}$ and therefore $T + S$ are Fredholm by Theorem 3.3.4. Moreover, $\operatorname{Ind}(T + S) = \operatorname{Ind}(\widetilde{T} + \widetilde{S}) = \operatorname{Ind}(\widetilde{T}) = \operatorname{Ind}(T)$ again by Theorem 3.3.4. If $\widetilde{T}$ is Fredholm, as the set of bounded Fredholm operators is open, see Theorem 3.3.5, there is a constant $c > 0$ such that $\widetilde{T} + A$ is Fredholm for all $A \in \mathbb{B}((\mathcal{D}(T), \|\cdot\|_T), \mathcal{H}')$ such that $\|A\| < c$. If S is relatively bounded with respect to T, then $\widetilde{S} \in \mathbb{B}((\mathcal{D}(T), \|\cdot\|_T), \mathcal{H}')$ has norm less than c provided the constants c_1 and c_2 in (6.8) are sufficiently small. Then, by the above, $T+S$ is Fredholm with an index satisfying $\operatorname{Ind}(T + S) = \operatorname{Ind}(\widetilde{T} + \widetilde{S}) = \operatorname{Ind}(\widetilde{T}) = \operatorname{Ind}(T)$. □

As in the bounded case for self-adjoint operators, there is another characterization using the notion of essential spectrum. The essential spectrum of a self-adjoint operator $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ is defined as in Section 3.4 for bounded self-adjoint operators, namely $\operatorname{spec}_{\text{ess}}(H) = \operatorname{spec}(H) \setminus \operatorname{spec}_{\text{dis}}(H)$ where the discrete spectrum $\operatorname{spec}_{\text{dis}}(H)$ consists of all isolated eigenvalues of H of finite multiplicity.

Theorem 6.2.14. *A self-adjoint operator $H = H^* \in \mathbb{L}(\mathcal{H})$ is Fredholm if and only if one has* $0 \notin \operatorname{spec}_{\text{ess}}(H)$.

Proof. Let us first assume that H is Fredholm. As $\operatorname{Ran}(H) = \operatorname{Ker}(H)^{\perp}$, then either H is invertible with a bounded inverse, by the Hellinger–Toeplitz theorem, or 0 is an eigenvalue of finite multiplicity. It remains to show that there exists $\epsilon > 0$ such that one has $\operatorname{spec}(H) \cap (-\epsilon, \epsilon) \setminus \{0\} = \emptyset$. The restriction H' of H to $\mathcal{D}(H) \cap \operatorname{Ker}(H)^{\perp}$ is a bijection onto its range, which is a Hilbert space. Its graph $\{(\phi, H\phi) : \phi \in \mathcal{D}(H) \cap \operatorname{Ker}(H)^{\perp}\}$ is closed because H is a closed operator. Therefore the closed graph theorem shows that $(H')^{-1}$ is bounded and therefore 0 lies in the resolvent set of H'. Thus there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \text{spec}(H') = \emptyset$. Furthermore, $H + \delta\mathbf{1} : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ is a Fredholm operator of all $\delta \in (-\epsilon, \epsilon)$. Then $\text{Ran}(H - \delta\mathbf{1}) = \text{Ker}(H - \delta\mathbf{1})^{\perp}$ and therefore δ is an eigenvalue of H or $H - \delta\mathbf{1}$ is invertible with bounded inverse so that $\delta \notin \text{spec}(H)$. If δ is an eigenvalue of H, there is $\phi = \phi_1 + \phi_2 \in \mathcal{D}(H)$ with $\phi_1 \in \text{Ker}(H)$ and $\phi_2 \in \mathcal{D}(H) \cap \text{Ker}(H)^{\perp}$ such that $H\phi = H\phi_2 = \delta\phi = \delta\phi_1 + \delta\phi_2$. Therefore $\delta\phi_1 = (H - \delta)\phi_2$ and, as $\phi_1 \in \text{Ker}(H)$ and $(H - \delta)\phi_2 \in \text{Ker}(H)^{\perp}$, this implies $\phi_1 = 0$. Therefore $H'\phi_2 = H\phi_2 = \delta\phi_2$, which is a contradiction.

Conversely assume that $0 \notin \text{spec}_{\text{ess}}(H)$. Then $\dim(\text{Ker}(H)) < \infty$ and $\|H\phi\| \geq c\|\phi\|$ for some $c > 0$ for all $\phi \in \mathcal{D}(H) \cap \text{Ker}(H)^{\perp}$, which is, by Proposition 6.2.5, equivalent to the Fredholm property of H. □

The following generalizes Theorem 3.4.1 to unbounded operators.

Theorem 6.2.15. *A regular operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is Fredholm if and only if one has $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and $0 \notin \text{spec}_{\text{ess}}(TT^*)$.*

Proof. Let us first suppose that T is Fredholm. Then by Corollary 6.2.8, T^* is Fredholm and therefore T^*T and TT^* are Fredholm. As T^*T and also TT^* are self-adjoint by Lemma 6.1.1 (note also that $(T^*)^* = \overline{T} = T$), this implies $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and furthermore $0 \notin \text{spec}_{\text{ess}}(TT^*)$ by Theorem 6.2.14.

Conversely assume that $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and $0 \notin \text{spec}_{\text{ess}}(TT^*)$. Then by Theorem 6.2.14 and Lemma 6.1.1, T^*T and TT^* are Fredholm. Therefore the dimensions of $\text{Ker}(T) = \text{Ker}(T^*T)$ and $\text{Ker}(T^*) = \text{Ker}(TT^*)$ are finite. Moreover, Lemma 5.3.3 implies that $\text{Ran}(T) = \text{Ran}(TT^*) \oplus (\text{Ran}(T) \cap \text{Ran}(TT^*)^{\perp})$ is closed. This implies by Lemma 6.2.3 that T is a Fredholm operator. □

As for bounded Fredholm operators, there is another characterization of the index of a Fredholm operator $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ using the operator $L : \mathcal{D}(T) \oplus \mathcal{D}(T^*) \to \mathcal{H} \oplus \mathcal{H}'$ defined by

$$L = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}. \tag{6.11}$$

Note that the square L^2 commutes with $J = \text{diag}(\mathbf{1}, -\mathbf{1})$ and therefore $\text{Ker}(L) = \text{Ker}(L^2)$ is invariant under J. Now $\text{Ind}(T)$ can be calculated as follows.

Proposition 6.2.16. *Let $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ be a Fredholm operator. Then the operator L defined by* (6.11) *is self-adjoint. Moreover, the index of T is equal to the signature of the operator $J = \mathbf{1} \oplus -\mathbf{1} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$ restricted to the kernel of L, namely*

$$\text{Ind}(T) = \text{Sig}(J|_{\text{Ker}(L)}).$$

Proof. One directly checks that L is symmetric. Therefore it is sufficient to show that $\mathcal{D}(L^*) \subset \mathcal{D}(L)$. As $\mathrm{Ker}(L) = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T^*)$ and

$$\mathrm{Ran}(L) = \mathrm{Ran}(T^*) \oplus \mathrm{Ran}(T) = \mathrm{Ker}(T)^\perp \oplus \mathrm{Ker}(T^*)^\perp = \mathrm{Ker}(L)^\perp,$$

one has $\mathrm{Ran}(L^*) \subset \mathrm{Ker}(L)^\perp = \mathrm{Ran}(L)$ and $\mathrm{Ker}(L^*) = \mathrm{Ran}(L)^\perp = \mathrm{Ker}(L)$. Now let be given $\psi \in \mathcal{D}(L^*) \setminus \mathcal{D}(L)$. Then $L^*\psi \in \mathrm{Ran}(L^*) \subset \mathrm{Ran}(L)$ so that there is a $\phi \in \mathcal{D}(L)$ with $L^*\psi = L\phi = L^*\phi$. Hence $0 = L^*(\psi - \phi) = L(\psi - \phi)$ as $\mathrm{Ker}(L) = \mathrm{Ker}(L^*)$, and one concludes $\psi \in \mathcal{D}(L)$, in contradiction to the assumption. Hence $\mathcal{D}(L^*) \subset \mathcal{D}(L)$ and L is self-adjoint. As $\mathrm{Ker}(L) = \mathrm{Ker}(T) \oplus \mathrm{Ker}(T^*)$, one concludes that

$$\mathrm{Sig}(J|_{\mathrm{Ker}(L)}) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ker}(T^*)) = \mathrm{Ind}(T),$$

completing the proof. □

As the final topic of this section, let us examine the image of Fredholm operators under the bounded transform $\mathcal{F}$, namely let us restrict the bounded transform $\mathcal{F}$ to the subset $\mathbb{F}(\mathcal{H}, \mathcal{H}') \subset \mathbb{L}(\mathcal{H}, \mathcal{H}')$. Combining Theorems 6.1.4 and 6.2.15 and using the identity $\mathcal{F}(T)^*\mathcal{F}(T) = T^*T(\mathbf{1} + T^*T)^{-1}$ where $(\mathbf{1} + T^*T)^{-1} : \mathcal{H} \to \mathcal{D}(T^*T)$ is a bijection so that $\mathcal{F}(T)^*\mathcal{F}(T)$ is Fredholm if and only if T^*T is Fredholm, one obtains

$$\begin{aligned}&\mathcal{F}(\mathbb{F}(\mathcal{H}, \mathcal{H}')) \\ &\quad = \{F \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}') : \mathrm{Ker}(\mathbf{1} - F^*F) = \{0\},\ 0 \notin \mathrm{spec}_{\mathrm{ess}}(F^*F) \cup \mathrm{spec}_{\mathrm{ess}}(FF^*)\},\end{aligned}$$

so that, by Theorem 6.2.15,

$$\mathcal{F}(\mathbb{F}(\mathcal{H}, \mathcal{H}')) = \mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \tag{6.12}$$

where $\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}') = \mathbb{FB}(\mathcal{H}, \mathcal{H}') \cap \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$. Moreover, one has

$$\mathrm{Ind}(\mathcal{F}(T)) = \mathrm{Ind}(T), \quad T \in \mathbb{F}(\mathcal{H}, \mathcal{H}').$$

Now Propositions 6.1.5 and 6.1.7 immediately imply the following

Proposition 6.2.17. *The bounded transform provides two homeomorphisms:*

$$\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R) \to (\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N)$$

and

$$\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_G) \to (\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \mathcal{O}_E).$$

Proposition 6.2.17 leads to the following result that will be used in Section 8.2 for the computation of the homotopy groups of $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$.

Proposition 6.2.18. *The inclusion* $i : (\mathbb{FB}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N) \to (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R)$ *is a homotopy equivalence with homotopy inverse* $\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R) \to (\mathbb{FB}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N)$.

Proof. (Modification of the proof of Theorem 5.10 in [126].) Let us first show that the composition $\mathcal{F} \circ i : \mathbb{FB}(\mathcal{H}, \mathcal{H}') \to \mathbb{FB}(\mathcal{H}, \mathcal{H}')$ is a homotopic to the identity. Consider the norm-continuous homotopy $h : \mathbb{FB}(\mathcal{H}, \mathcal{H}') \times [0, \frac{1}{2}] \to \mathbb{FB}(\mathcal{H}, \mathcal{H}')$ defined by

$$h(T, t) = T(\mathbf{1} + T^* T)^{-t}.$$

Then, clearly, $h(T, 0) = T$ and $h(T, \frac{1}{2}) = (\mathcal{F} \circ i)(T)$ for all $T \in \mathbb{FB}(\mathcal{H}, \mathcal{H}')$. By Proposition 6.2.17, this implies that $i \circ \mathcal{F} = \mathcal{F}^{-1} \circ (\mathcal{F} \circ i) \circ \mathcal{F}$ is also homotopic to the identity. Putting these facts together, one concludes that i is a homotopy equivalence. □

6.3 Unbounded self-adjoint Fredholm operators

This section analyzes the set $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ of unbounded self-adjoint Fredholm operators on $\mathcal{H}$. As a subset of the set $\mathbb{L}(\mathcal{H}) = \mathbb{L}(\mathcal{H}, \mathcal{H})$ of closed densely defined operators, it inherits two natural metrics, namely the Riesz metric d_R and the gap metric d_G. The induced topologies will still be called Riesz and gap topologies, respectively. Let us begin by analyzing the image of $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ under the bounded transform. Recall from Section 4.6 the notations

$$\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H}) : \|H\| \leq 1, \mathrm{Ker}(H^2 - \mathbf{1}) = \{0\}\}$$

and

$$\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}) = \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}) \cap \mathbb{FB}(\mathcal{H}).$$

Proposition 6.3.1. *The bounded transform $\mathcal{F}$ maps $\mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ and $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ bijectively onto $\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$ and $\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$, respectively.*

Proof. By Theorem 6.1.4, one has $\mathcal{F}(T^*) = \mathcal{F}(T)^*$ for all $T \in \mathbb{L}(\mathcal{H})$. Therefore T is self-adjoint if and only if $\mathcal{F}(T)$ is self-adjoint. Moreover, as $(\mathbf{1} + T^* T)^{-\frac{1}{2}} : \mathcal{H} \to \mathcal{D}(T)$ is bijective, $\mathrm{Ran}(T) = \mathrm{Ran}(\mathcal{F}(T))$ and $\dim(\mathrm{Ker}(T)) = \dim(\mathrm{Ker}(\mathcal{F}(T)))$. This implies that T is Fredholm if and only if $\mathcal{F}(T)$ is Fredholm. Theorem 6.1.4 implies the claim. □

Even though it is not the main focus of this section, let us begin by studying the Riesz metric. As it is obtained (by definition) via the bounded transform from the norm on the bounded linear operators on $\mathcal{H}$, the following is natural and actually directly follows by combining Propositions 6.3.1 and 6.1.5.

Corollary 6.3.2. *The bounded transform*

$$\mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$$

is a homeomorphism.

Corollary 6.3.2 allows deducing the next result which later on allows us to determine the homotopy groups of $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), d_R)$, see Theorem 8.6.1. By repeating the proof of Proposition 6.2.18 for self-adjoint operators, one obtains

Proposition 6.3.3. *The inclusion* $i : (\mathbb{FB}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N) \to (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ *is a homotopy equivalence with homotopy inverse* $\mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{FB}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$.

The remainder of this section concerns the gap topology. First, let us combine Proposition 6.3.1 with Proposition 6.1.7 which concerns the continuity properties of $\mathcal{F}$ when the gap metric d_G and the extended gap metric d_E are used. One immediately deduces

Corollary 6.3.4. *The bounded transform* $\mathcal{F}$ *provides two bi-Lipshitz-continuous homeomorphisms*

$$\mathcal{F} : (\mathbb{L}_{\mathrm{sa}}(\mathcal{H}), d_G) \to (\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), d_E), \quad \mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), d_G) \to (\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), d_E).$$

The metric space $(\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), d_E)$ was already analyzed in Section 4.6. In particular, Corollary 4.6.10 showed that $\mathcal{G} : (\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E) \to (\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$ with $\mathcal{G}$ defined by (4.22) and

$$\mathbb{U}^0(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : \mathrm{Ker}(U - \mathbf{1}) = \{0\}\} \tag{6.13}$$

is a homeomorphism. Moreover, Corollary 4.6.12 already stated that also the map $\mathcal{G} : (\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E) \to (\mathbb{FU}^0(\mathcal{H}), \mathcal{O}_N)$ is a homeomorphism. Combining this with Corollary 6.3.4, one immediately obtains a central result of this section.

Theorem 6.3.5. *The maps*

$$\mathcal{G} \circ \mathcal{F} : (\mathbb{L}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$$

and

$$\mathcal{G} \circ \mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{FU}^0(\mathcal{H}), \mathcal{O}_N)$$

are homeomorphisms.

Based on Theorem 6.3.5, one can then define the spectral flow of gap-continuous paths in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ as the spectral flow of essentially gapped unitaries introduced in Section 4.5. This will be carried out in detail in Section 7.1 below.

Let us next compute the map $\mathcal{G}\circ\mathcal{F}$. Using spectral calculus of the self-adjoint operator $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$, one has

$$\begin{aligned}
\mathcal{G} \circ \mathcal{F}(H) &= 2H^2(\mathbf{1} + H^2)^{-1} - \mathbf{1} - 2\imath H(\mathbf{1} + H^2)^{-\frac{1}{2}}(\mathbf{1} - H^2(\mathbf{1} + H^2)^{-1})^{\frac{1}{2}} \\
&= 2H^2(\mathbf{1} + H^2)^{-1} - \mathbf{1} - 2\imath H(\mathbf{1} + H^2)^{-1} \\
&= 2H(H - \imath\mathbf{1})(\mathbf{1} + H^2)^{-1} - \mathbf{1}
\end{aligned}$$

$$= (H - \imath\mathbf{1})(H + \imath\mathbf{1})^{-1}.$$

This shows that $\mathcal{G} \circ \mathcal{F} = \mathcal{C}$, where the Cayley transform is defined by

$$\mathcal{C} : \mathbb{R} \to \mathbb{S}^1 \setminus \{1\}, \quad x \mapsto \frac{x - \imath}{x + \imath}. \tag{6.14}$$

Then the operator

$$\mathcal{C}(H) = (H - \imath\mathbf{1})(H + \imath\mathbf{1})^{-1} = \mathbf{1} - 2\imath(H + \imath\mathbf{1})^{-1} \tag{6.15}$$

is called the *Cayley transform* of H. It is a unitary operator $\mathcal{C}(H) \in \mathbb{U}(\mathcal{H})$ by the spectral theorem (this will also be proved more directly below). The mapping properties in the first formula for $\mathcal{C}(H)$ in (6.15) are given by $(H + \imath\mathbf{1})^{-1} : \mathcal{H} \to \operatorname{Ran}((H + \imath\mathbf{1})^{-1}) = \mathcal{D}(H)$ and afterwards $H - \imath\mathbf{1} : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$.

Theorem 6.3.5 was deduced from the results of $\mathcal{G}$ as given in Section 4.6 combined with those on $\mathcal{F}$ given in Section 6.1. While this is clearly sufficient to go on to the definition of the spectral flow in Section 7.1, we will provide also a direct proof of Theorem 6.3.5 along the works [31, 126]. This also provides several useful metrics that are equivalent to the gap metric d_G. Moreover, these direct arguments are useful in other contexts, e. g., [38]. Let us start by analyzing the mapping properties of the Cayley transform and its inverse.

Proposition 6.3.6. *If $U \in \mathbb{U}(\mathcal{H})$ and $U - \mathbf{1}$ is injective, then $H = \imath(\mathbf{1} + U)(\mathbf{1} - U)^{-1}$ is self-adjoint on $\mathcal{D}(H) = \operatorname{Ran}(\mathbf{1} - U)$. Moreover, $H = \imath(\mathbf{1} - U)^{-1}(\mathbf{1} + U)$.*

Proof. Since U is normal, $\operatorname{Ker}(\mathbf{1} - U^*) = \operatorname{Ker}(\mathbf{1} - U)$ and thus

$$\overline{\operatorname{Ran}(\mathbf{1} - U)} = \operatorname{Ker}(\mathbf{1} - U^*)^{\perp} = \operatorname{Ker}(\mathbf{1} - U)^{\perp} = \mathcal{H},$$

as $\mathbf{1} - U$ is injective. Consequently, $\mathcal{D}(H) = \operatorname{Ran}(\mathbf{1} - U)$ is dense in $\mathcal{H}$. From

$$(\mathbf{1} - U)(\mathbf{1} + U) = \mathbf{1} - U^2 = (\mathbf{1} + U)(\mathbf{1} - U), \tag{6.16}$$

it follows that

$$\begin{aligned}(\mathbf{1} + U)(\mathbf{1} - U)^{-1} &= (\mathbf{1} - U)^{-1}(\mathbf{1} - U)(\mathbf{1} + U)(\mathbf{1} - U)^{-1} \\ &= (\mathbf{1} - U)^{-1}(\mathbf{1} + U)|_{\operatorname{Ran}(\mathbf{1}-U)}. \end{aligned} \tag{6.17}$$

On the other hand, if $\psi \in \mathcal{D}((\mathbf{1}-U)^{-1}(\mathbf{1}+U))$, then $(\mathbf{1}+U)\psi \in \mathcal{D}((\mathbf{1}-U)^{-1}) = \operatorname{Ran}(\mathbf{1}-U)$ and accordingly there exists $\phi \in \mathcal{H}$ such that $(\mathbf{1}+U)\psi = (\mathbf{1}-U)\phi$. Thus $\psi = (\mathbf{1}-U)\phi + (\mathbf{1}-U)\psi - \psi$ and hence

$$\psi = \frac{1}{2}(\mathbf{1} - U)(\psi + \phi) \in \mathcal{D}\big((\mathbf{1} + U)(\mathbf{1} - U)^{-1}\big).$$

It follows from (6.17) that

$$H = \imath(\mathbf{1} + U)(\mathbf{1} - U)^{-1} = \imath(\mathbf{1} - U)^{-1}(\mathbf{1} + U).$$

Next, let us show that H is symmetric. If $\psi, \phi \in \mathcal{D}(H) = \mathrm{Ran}(\mathbf{1} - U)$, then there exist $\phi', \psi' \in \mathcal{H}$ such that $\psi = \psi' - U\psi'$ and $\phi = \phi' - U\phi'$ and therefore $H\psi = \imath(\psi' + U\psi')$ and $H\phi = \imath(\phi' + U\phi')$. One gets

$$\begin{aligned}
\langle\phi|H\psi\rangle &= \imath\langle\phi' - U\phi'|\psi' + U\psi'\rangle \\
&= \imath(\langle\phi'|\psi'\rangle - \langle U\phi'|\psi'\rangle + \langle\phi'|U\psi'\rangle - \langle U\phi'|U\psi'\rangle) \\
&= -\imath\langle U\phi'|\psi'\rangle + \imath\langle\phi'|U\psi'\rangle \\
&= \langle\imath(\phi' + U\phi')|\psi' - U\psi'\rangle \\
&= \langle H\phi|\psi\rangle.
\end{aligned}$$

Hence H is symmetric and

$$H \subset H^* = -\imath(\mathbf{1} - U^*)^{-1}(\mathbf{1} + U^*). \tag{6.18}$$

As $U^* - \mathbf{1}$ is injective, arguing as above one gets

$$H^* = -\imath(\mathbf{1} - U^*)^{-1}(\mathbf{1} + U^*) = -\imath(\mathbf{1} + U^*)(\mathbf{1} - U^*)^{-1},$$

thus exchanging U and U^* shows that H^* is symmetric. Hence

$$H^* \subset H^{**} = \imath(\mathbf{1} - U)^{-1}(\mathbf{1} + U) = H$$

and it follows from (6.18) that $H = H^*$. □

Corollary 6.3.7. *If U and H are as in Proposition* 6.3.6*, then $\mathcal{C}(H) = U$. Moreover, the Cayley transform $\mathcal{C} : \mathbb{L}_{\mathrm{sa}}(\mathcal{H}) \to \mathbb{U}^0(\mathcal{H})$ is a bijection.*

Proof. By Proposition 6.3.6, $H = \imath(\mathbf{1} + U)(\mathbf{1} - U)^{-1}$. Hence

$$H + \imath\mathbf{1} = \imath(\mathbf{1} + U)(\mathbf{1} - U)^{-1} + \imath(\mathbf{1} - U)(\mathbf{1} - U)^{-1} = 2\imath(\mathbf{1} - U)^{-1},$$

and thus

$$(H + \imath\mathbf{1})^{-1} = \frac{1}{2\imath}(\mathbf{1} - U).$$

Analogously,

$$H - \imath\mathbf{1} = \imath(\mathbf{1} + U)(\mathbf{1} - U)^{-1} - \imath(\mathbf{1} - U)(\mathbf{1} - U)^{-1} = 2\imath U(\mathbf{1} - U)^{-1},$$

and one obtains

$$\mathcal{C}(H) = (H - \imath\mathbf{1})(H + \imath\mathbf{1})^{-1} = U(\mathbf{1} - U)^{-1}(\mathbf{1} - U) = U.$$

To prove the second claim, one only has to show that $U = \mathcal{C}(H)$ is unitary and $\mathbf{1} - \mathcal{C}(H)$ injective for all $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$. It is clear that U is surjective. For $\phi \in \mathcal{D}(H)$,

$$\begin{aligned}\|H\phi + \imath\phi\|^2 &= \langle H\phi + \imath\phi | H\phi + \imath\phi\rangle \\ &= \|H\phi\|^2 + \|\phi\|^2 - \imath\langle\phi|H\phi\rangle + \imath\langle H\phi|\phi\rangle \\ &= \|H\phi\|^2 + \|\phi\|^2 \\ &= \|H\phi - \imath\phi\|^2\end{aligned}$$

and, since $U(H\phi + \imath\phi) = H\phi - \imath\phi$, it follows that $\|U\psi\| = \|\psi\|$ for all $\psi \in \mathcal{H}$. Hence U is a surjective isometry defined on all of $\mathcal{H}$, and consequently it is a unitary operator. Now let us assume that $\psi \in \mathcal{H}$ is such that $\mathcal{C}(H)\psi = \psi$. Then one obtains from (6.15)

$$\psi = \mathcal{C}(H)\psi = \psi - 2\imath(H + \imath\mathbf{1})^{-1}\psi,$$

and hence $(H + \imath\mathbf{1})^{-1}\psi = 0$ which implies that $\psi = 0$. □

The following connection of the spectrum of $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ to the spectrum of its image $\mathcal{C}(H) \in \mathbb{U}(\mathcal{H})$ follows from the spectral mapping theorem, but again a direct proof is provided due to its importance for the definition of the spectral flow of paths of unbounded self-adjoint Fredholm operators in Section 7.1.

Corollary 6.3.8. *If* $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ *and* $\lambda \in \mathbb{R}$, *then*
(i) $\mathrm{Ker}(\lambda\mathbf{1} - H) = \mathrm{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$;
(ii) $\mathrm{Ran}(\lambda\mathbf{1} - H) = \mathrm{Ran}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$;
(iii) $\lambda \in \mathrm{spec}(H) \Longleftrightarrow \mathcal{C}(\lambda) \in \mathrm{spec}(\mathcal{C}(H))$;
(iv) $\lambda \in \mathrm{spec}_p(H) \Longleftrightarrow \mathcal{C}(\lambda) \in \mathrm{spec}_p(\mathcal{C}(H))$;
(v) $\lambda \in \mathrm{spec}_{\mathrm{ess}}(H) \Longleftrightarrow \mathcal{C}(\lambda) \in \mathrm{spec}_{\mathrm{ess}}(\mathcal{C}(H))$.

The proof is based on the following lemma.

Lemma 6.3.9. *For* $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ *and* $\lambda \in \mathbb{R}$, *one has*

$$\lambda\mathbf{1} - H = (\lambda + \imath)\big(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H)\big)\big(\mathbf{1} - \mathcal{C}(H)\big)^{-1}.$$

Proof. The equality

$$\begin{aligned}\lambda\mathbf{1} - H &= \lambda\mathbf{1} - \imath\big(\mathbf{1} + \mathcal{C}(H)\big)\big(\mathbf{1} - \mathcal{C}(H)\big)^{-1} \\ &= \big(\lambda(\mathbf{1} - \mathcal{C}(H)) - \imath(\mathbf{1} + \mathcal{C}(H))\big)\big(\mathbf{1} - \mathcal{C}(H)\big)^{-1} \\ &= \big(\lambda\mathbf{1} - \lambda\mathcal{C}(H) - \imath\mathbf{1} - \imath\mathcal{C}(H)\big)\big(\mathbf{1} - \mathcal{C}(H)\big)^{-1} \\ &= \big((\lambda - \imath)\mathbf{1} - (\lambda + \imath)\mathcal{C}(H)\big)\big(\mathbf{1} - \mathcal{C}(H)\big)^{-1}\end{aligned}$$

$$= (\lambda + \imath)\big((\lambda - \imath)(\lambda + \imath)^{-1}\mathbf{1} - \mathcal{C}(H)\big)(\mathbf{1} - \mathcal{C}(H))^{-1}$$
$$= (\lambda + \imath)\big(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H)\big)(\mathbf{1} - \mathcal{C}(H))^{-1}$$

implies the claim. □

Proof of Corollary 6.3.8. First of all, let us note that $(\mathbf{1} - \mathcal{C}(H))^{-1}$ maps $\mathcal{D}(H)$ bijectively onto $\mathcal{H}$ as $\mathbf{1} - \mathcal{C}(H) = 2\imath(H + \imath\mathbf{1})^{-1}$. Thus by the previous Lemma 6.3.9,

$$\begin{aligned}\operatorname{Ker}(\lambda\mathbf{1} - H) &= (\mathbf{1} - \mathcal{C}(H))\big(\operatorname{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))\big)\\ &= \operatorname{Ker}\big(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H)\big),\end{aligned}$$

where the second equality follows from the fact that $\operatorname{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$ is invariant under $\mathcal{C}(H)$. This implies the assertion (i). As $(\mathbf{1} - \mathcal{C}(H))^{-1} : \mathcal{D}(H) \to \mathcal{H}$ is a bijection, Lemma 6.3.9 directly implies (ii). All other claims are immediate consequences of (i) and (ii). □

Let us recall that for an operator $H \in \mathbb{L}_{sa}(\mathcal{H})$ that is bounded, the spectrum of its image $\mathcal{C}(H) \in \mathbb{U}(\mathcal{H})$ does not contain 1. This is made more precise in the following statement.

Lemma 6.3.10. *For $H \in \mathbb{L}_{sa}(\mathcal{H})$, one has*
(i) $1 \notin \operatorname{spec}(\mathcal{C}(H)) \Longleftrightarrow \mathcal{D}(H) = \mathcal{H}$, *and this is true if and only if H is bounded.*
(ii) $1 \in \operatorname{spec}_{ess}(\mathcal{C}(H)) \Longleftrightarrow \mathcal{D}(H) \neq \mathcal{H}$, *and this is true if and only if H is unbounded.*

Proof. The assertions regarding the boundedness and unboundedness of H follow as any self-adjoint operator $H : \mathcal{D}(H) \subset \mathcal{H} \to \mathcal{H}$ is bounded if and only if $\mathcal{D}(H) = \mathcal{H}$. By (6.15), one has

$$\mathbf{1} - \mathcal{C}(H) = 2\imath(H + \imath\mathbf{1})^{-1} \in \mathbb{B}(\mathcal{H})$$

mapping $\mathcal{H}$ bijectively onto $\mathcal{D}(H)$. Accordingly, if 1 is in the resolvent set of $\mathcal{C}(H)$, one infers $\mathcal{H} = \operatorname{Ran}(\mathbf{1} - \mathcal{C}(H)) = \mathcal{D}(H)$. Conversely, if $\mathcal{D}(H) = \mathcal{H}$, then $\mathbf{1} - \mathcal{C}(H)$ maps $\mathcal{H}$ bijectively onto $\mathcal{H}$, showing that 1 is in the resolvent set of $\mathcal{C}(H)$. Hence assertion (i) is proved.

In order to show (ii), we note at first that by (i), $1 \in \operatorname{spec}(\mathcal{C}(H))$ if and only if $\mathcal{D}(H) \neq \mathcal{H}$. Now it remains to show that if $1 \in \operatorname{spec}(\mathcal{C}(H))$, then we actually have $1 \in \operatorname{spec}_{ess}(\mathcal{C}(H))$. But, if $\mathcal{D}(H) \neq \mathcal{H}$, we see that $\operatorname{Ran}(\mathbf{1} - \mathcal{C}(H)) = \mathcal{D}(H)$ is a proper dense subspace of $\mathcal{H}$ and hence in particular not closed. Accordingly, $\mathbf{1} - \mathcal{C}(H)$ is not a Fredholm operator and, by Corollary 3.4.4, $1 \in \operatorname{spec}_{ess}(\mathcal{C}(H))$. □

Corollary 6.3.8 implies:

Corollary 6.3.11. *If $H \in \mathbb{L}_{sa}(\mathcal{H})$, then*
(i) $\mathcal{C}(\operatorname{spec}(H)) = \operatorname{spec}(\mathcal{C}(H))$ *if H is bounded.*
(ii) $\mathcal{C}(\operatorname{spec}(H)) \cup \{1\} = \operatorname{spec}(\mathcal{C}(H))$ *if H is unbounded.*

Now all is prepared to state and prove the result that is essentially already contained in Theorem 6.3.5. However, as already stressed above, the result also feature a metric d''_G on $\mathbb{L}_{sa}(\mathcal{H})$ defined by

$$d''_G(H_0, H_1) = \|\mathcal{C}(H_0) - \mathcal{C}(H_1)\|, \quad H_0, H_1 \in \mathbb{L}_{sa}(\mathcal{H}).$$

Due to (6.15), one then has

$$d''_G(H_0, H_1) = 2\|(H_0 + \imath\mathbf{1})^{-1} - (H_1 + \imath\mathbf{1})^{-1}\|, \quad H_0, H_1 \in \mathbb{L}_{sa}(\mathcal{H}). \tag{6.19}$$

Hence the following theorem shows that the gap topology can be obtained form the Cayley transform, similarly as the Riesz topology is obtained from the bounded transform in Proposition 6.1.5.

Theorem 6.3.12. *On $\mathbb{L}_{sa}(\mathcal{H})$ the gap metric d_G is equivalent to the metric d''_G. The Cayley transform*

$$\mathcal{C} : (\mathbb{L}_{sa}(\mathcal{H}), d_G) \to (\mathbb{U}^0(\mathcal{H}), d_N)$$

is a Lipshitz-continuous homeomorphism.

Proof. Recall from Proposition 6.1.3 that d_G is equivalent to

$$d'_G(H_0, H_1) = 2\|R_{H_0} - R_{H_1}\| + \|H_0 R_{H_0} - H_1 R_{H_1}\|, \quad H_0, H_1 \in \mathbb{L}_{sa}(\mathcal{H}).$$

The identities

$$(H - \imath\mathbf{1})^{-1} = (H + \imath\mathbf{1})(H^2 + \mathbf{1})^{-1} = HR_H + \imath R_H,$$
$$(H + \imath\mathbf{1})^{-1} = (H - \imath\mathbf{1})(H^2 + \mathbf{1})^{-1} = HR_H - \imath R_H$$

imply

$$R_H = \frac{1}{2\imath}((H - \imath\mathbf{1})^{-1} - (H + \imath\mathbf{1})^{-1}),$$
$$HR_H = \frac{1}{2}((H - \imath\mathbf{1})^{-1} + (H + \imath\mathbf{1})^{-1}).$$

Therefore the metric d'_G is equivalent to the metric d''_G as, for $H_0, H_1 \in \mathbb{L}_{sa}(\mathcal{H})$,

$$d''_G(H_0, H_1) = \|(H_0 + \imath\mathbf{1})^{-1} - (H_1 + \imath\mathbf{1})^{-1}\| + \|(H_0 - \imath\mathbf{1})^{-1} - (H_1 - \imath\mathbf{1})^{-1}\|,$$

where it was used that $\|A\| = \|A^*\|$ for all $A \in \mathbb{B}(\mathcal{H})$. Now all claims follow from Corollary 6.3.7 and (6.19). □

Theorem 6.3.13. *With respect to the gap metric, the set $\mathbb{B}_{sa}(\mathcal{H})$ is dense in $\mathbb{L}_{sa}(\mathcal{H})$.*

Proof. For $H \in \mathbb{L}_{sa}(\mathcal{H})$, let the spectral resolution of H be denoted by $(E_\lambda)_{\lambda\in\mathbb{R}}$. For $n \in \mathbb{N}$, let us define the bounded self-adjoint operator

$$H_n = \int_{[-n,n]} \lambda dE_\lambda + \int_{|\lambda|>n} n\,\mathrm{sgn}(\lambda) dE_\lambda .$$

Then using the metric d''_G as in (6.19), one has

$$\begin{aligned} d''_G(H, H_n) &= 2\|(H + \imath\mathbf{1})^{-1} - (H_n + \imath\mathbf{1})^{-1}\| \\ &= 2\left\| \int_{|\lambda|>n} (\lambda + \imath)^{-1} - \big(n\,\mathrm{sgn}(\lambda) + \imath\big)^{-1} dE_\lambda \right\| \\ &\le \frac{4}{n} . \end{aligned}$$

Hence H_n converges to H with respect to the metric d''_G and, by Theorem 6.3.12, also with respect to the gap metric. □

Next let us focus on the set

$$\mathbb{F}_{sa}(\mathcal{H}) = \{H \in \mathbb{F}(\mathcal{H}) : H = H^*\}$$

of self-adjoint (unbounded) Fredholm operators on $\mathcal{H}$. By Corollary 6.3.8, the Cayley transform maps $\mathbb{F}_{sa}(\mathcal{H})$ bijectively onto $\mathbb{FU}^0(\mathcal{H}) = \mathbb{FU}(\mathcal{H}) \cap \mathbb{U}^0(\mathcal{H})$. Hence Theorem 6.3.12 also implies the second statement of Theorem 6.3.5, namely

Theorem 6.3.14. *The Cayley transform*

$$\mathcal{C} : (\mathbb{F}_{sa}(\mathcal{H}), d_G) \to (\mathbb{FU}^0(\mathcal{H}), d_N)$$

is a Lipshitz-continuous homeomorphism.

Theorem 6.3.14 directly implies the following because $\mathbb{FU}^0(\mathcal{H}) \subset \mathbb{U}^0(\mathcal{H})$ is open.

Corollary 6.3.15. *With respect to the gap metric, the set* $\mathbb{F}_{sa}(\mathcal{H})$ *is open in* $\mathbb{L}_{sa}(\mathcal{H})$.

In contrast to the set of bounded self-adjoint Fredholm operators consisting of three connected components as studied in Section 3.6, $\mathbb{F}_{sa}(\mathcal{H})$ is connected when equipped with the gap metric. Following [31], this is now proved directly by a spectral-theoretic argument. Let us note that an alternative proof, actually leading to a stronger statement, is given in Section 8.6.

Theorem 6.3.16. *With respect to the gap metric, the set* $\mathbb{F}_{sa}(\mathcal{H})$ *is connected.*

Proof. We show that $\mathbb{FU}^0(\mathcal{H})$ is connected with respect to $\mathcal{O}_N$, which by Theorem 6.3.14 implies the claim. For $U \in \mathbb{FU}^0(\mathcal{H})$, we show that there is a norm-continuous path within $\mathbb{FU}^0(\mathcal{H})$ connecting U to $\imath\mathbf{1}$. First, we decompose $\mathcal{H}$ into the spectral subspaces $\mathcal{H}_\pm$ of U

corresponding to $\{e^{\imath\varphi} : \varphi \in [0,\pi)\}$ and $\{e^{\imath\varphi} : \varphi \in [\pi, 2\pi]\}$. Respectively, we decompose $U = U_+ \oplus U_-$. There is no intersection of the spectral subspaces as if $-1 \in \operatorname{spec}(U)$ it is an isolated eigenvalue and hence belongs to $\operatorname{spec}(U_-)$. And if $1 \in \operatorname{spec}(U)$, it does not contribute to the decomposition of U as it is not an eigenvalue. Then by spectral deformation we contract U_+ to $\imath\mathbf{1}_+$ and U_- to $-\imath\mathbf{1}_-$ where $\mathbf{1}_\pm$ denotes the identity on $\mathcal{H}_\pm$. During this contraction, 1 does not become an eigenvalue and -1 does not become an element of the essential spectrum. Thus we have connected U to $\imath\mathbf{1}_+ \oplus -\imath\mathbf{1}_-$ within $\mathbb{FU}^0(\mathcal{H})$.

If $\mathcal{H}_-$ is finite dimensional, we rotate $-\imath\mathbf{1}_-$ through -1 into $\imath\mathbf{1}_-$. Otherwise, we identify $\mathcal{H}_-$ with $L^2([0,1])$. Now the multiplication operator by $-\imath$ on $L^2([0,1])$ can be connected to the multiplication by the function $f : [0,1] \to \mathbb{S}^1$, $f(t) = e^{\imath(\frac{3}{2}\pi + t - \frac{1}{2})}$ within the unitaries in such a way that one does not introduce spectrum at ± 1. Then $s \in [0,\pi] \mapsto e^{\imath s} f$ connects f to $g : [0,1] \to \mathbb{S}^1$, $g(t) = e^{\imath(\frac{1}{2}\pi + t - \frac{1}{2})}$ such that -1 is not in the spectrum and 1 does not become an eigenvalue. Finally, g can be contracted to the multiplication by $\imath$. Thus, in both cases U can be connected to $\imath\mathbf{1}$ within $\mathbb{FU}^0(\mathcal{H})$ completing the argument. □

The following result is due to Nicolaescu [139], see also [126] and Proposition 6.3.3.

Proposition 6.3.17. *The Riesz topology on $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ is strictly finer than the gap topology.*

Proof. By Proposition 6.1.9, the topology induced by the Riesz metric on $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ is finer than the topology induced by the gap metric. In the proof of Proposition 6.1.9, a sequence $(H_n)_{n\in\mathbb{N}}$ of operators in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ converging to $H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ with respect to the gap topology, but not converging with respect to the Riesz topology was constructed. This implies the claim. □

Note that Proposition 6.3.17 implies, in particular, that every path in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ which is continuous with respect to the Riesz metric is also continuous with respect to the gap metric. Next let us transfer the theorem of Cordes and Labrousse (see Theorem 6.1.10) to the subset of self-adjoint operators. One immediately deduces the following result (also discussed in [126]).

Corollary 6.3.18. *With respect to the gap metric, the set $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ is open in $\mathbb{F}_{\mathrm{sa}}(H)$. On $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ the topologies induced by d_N, d_R, and d_G coincide.*

Finally, the next result is a direct consequence of Theorem 6.3.13 and Corollary 6.3.15.

Corollary 6.3.19. *With respect to the gap metric, the set $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ is dense in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$.*

6.4 Self-adjoint Fredholm operators with compact resolvent

This section analyzes the set $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ of self-adjoint Fredholm operators with compact resolvent

$$\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H}) : (H - \imath\mathbf{1})^{-1} \in \mathbb{K}(\mathcal{H})\}.$$

By the resolvent identity, the compactness of the resolvent $(H-z\mathbf{1})^{-1}$ at some other point $z \in \mathbb{C} \setminus \mathrm{spec}(H)$ is equivalent to the compactness of $(H-\imath\mathbf{1})^{-1}$. Further recall that the compactness of the resolvent directly implies the Fredholm property:

Proposition 6.4.1. *Let $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ have a compact resolvent $(H-\imath\mathbf{1})^{-1} \in \mathbb{K}(\mathcal{H})$. Then $H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ is a Fredholm operator so that $H \in \mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$.*

Proof. If $H \in \mathbb{L}_{\mathrm{sa}}(\mathcal{H})$ has a compact resolvent, then $0 \notin \mathrm{spec}_{\mathrm{ess}}(H)$, which, by Theorem 6.2.14, directly implies that H is a Fredholm operator. □

Operators from $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ play a central role in index theory and noncommutative geometry [63] where they appear as unbounded Fredholm modules, which are also a special case of unbounded Kasparov modules (namely those representing elements from $KK(\mathbb{B}(\mathcal{H}),\mathbb{C})$ or $KK(\mathbb{C},\mathbb{B}(\mathcal{H}))$). While both Riesz and gap topologies can be used on $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$, the focus will here be on the gap topology. One of the main final results of this section is the following:

Theorem 6.4.2. *Space $(\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_G)$ is homotopy equivalent to $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_G)$.*

The proof of this result is surprisingly intricate and will make up a large part of the remainder of the section. While it will mainly pend on the use of the bounded transform of the set $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$, let us start the analysis of the Cayley transform of $(\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}),\mathcal{O}_G)$. Recall from Section 3.7 that $\mathbb{U}^{\mathrm{C}}(\mathcal{H})$ is the set of unitaries U with $U-\mathbf{1} \in \mathbb{K}(\mathcal{H})$ and furthermore from (6.13) that $\mathbb{U}^{0}(\mathcal{H})$ is the set of unitaries U with $\mathrm{Ker}(U-\mathbf{1}) = \{0\}$. Here the intersection of these sets will appear naturally

$$\mathbb{U}^{\mathrm{C},0}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U-\mathbf{1} \in \mathbb{K}(\mathcal{H}),\ \mathrm{Ker}(U-\mathbf{1}) = \{0\}\}.$$

Theorem 6.4.3. *The Cayley transform*

$$\mathcal{C} : (\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}), d_G) \to (\mathbb{U}^{\mathrm{C},0}(\mathcal{H}), d_N)$$

is a Lipshitz-continuous homeomorphism.

Proof. By (6.15), the compactness of $\mathcal{C}(H)-\mathbf{1}$ and that of the resolvent are equivalent. Therefore the claim directly follows from Theorem 6.3.12 (or equivalently, Theorem 6.3.5). □

Next let us consider the bounded transform of the set $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$. For this purpose, let us introduce the set

$$\mathbb{FB}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}) : \mathbf{1}-H^2 \in \mathbb{K}(\mathcal{H}),\ \mathrm{Ker}(\mathbf{1}-H^2) = \{0\}\}.$$

Note that this is a subset of $\mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})$ studied in Proposition 3.6.3, specified by the supplementary condition $\mathrm{Ker}(\mathbf{1}-H^2) = \{0\}$.

Proposition 6.4.4. *The bounded transform $\mathcal{F}$ provides a bi-Lipshitz-continuous homeomorphism*

$$\mathcal{F} : (\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}), d_G) \to (\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H}), d_E).$$

Proof. This follows from Corollary 6.3.4 by implementing the compactness condition $\mathbf{1} - H^2 \in \mathbb{K}(\mathcal{H})$. Indeed, the identity

$$H = \mathcal{F}(H)\big(\mathbf{1} - \mathcal{F}(H)^2\big)^{-\frac{1}{2}}$$

following from (6.6) implies

$$(H - \imath\mathbf{1})^{-1} = \big(\mathbf{1} - \mathcal{F}(H)^2\big)^{\frac{1}{2}}\big(\mathcal{F}(H) - \imath\big(\mathbf{1} - \mathcal{F}(H)^2\big)^{\frac{1}{2}}\big)^{-1},$$

which shows that the compactness of the resolvent of H is equivalent to the compactness of $\mathbf{1} - \mathcal{F}(H)^2$ because $(\mathcal{F}(H) - \imath(\mathbf{1} - \mathcal{F}(H)^2)^{\frac{1}{2}})^{-1}$ is unitary and hence bounded. □

For the following it is necessary to use yet another topology on $\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\mathrm{sa}}(\mathcal{H})$ and some of its supersets. The so-called strong extended gap topology on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ is defined by

$$\mathcal{O}_{SE} = \mathcal{O}(\mathcal{O}_E, \mathcal{O}_S),$$

where on the right-hand side $\mathcal{O}_E$ denotes the extended gap topology generated by d_E, $\mathcal{O}_S$ is the strong operator topology, and the remaining $\mathcal{O}$ denotes the generated topology. In other words, $\mathcal{O}_{SE}$ is the weakest (or smallest) topology on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ containing both $\mathcal{O}_E$ and $\mathcal{O}_S$. The topology $\mathcal{O}_{SE}$ was introduced in [108] under the name strict extended gap topology, but in the Hilbert space framework the strict and strong topologies coincide. The strong topology is not metrizable on the set of all bounded operators, but on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ it is metrizable. This leads to the following statement which, in particular, implies that sequential compactness and compactness are equivalent in $(\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Lemma 6.4.5. *The topology $\mathcal{O}_{SE}$ on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ is metrizable.*

Proof. It will first be shown that $\mathcal{O}_S$ is metrizable on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$. For an orthonormal basis $(\phi_n)_{n\geq 1}$ of $\mathcal{H}$, consider the metric

$$d_S(H_1, H_0) = \sum_{n=1}^{\infty} 2^{-n}\|(H_1 - H_0)\phi_n\|,$$

and let $B_\varepsilon(H_0)$ be a ball of radius $\varepsilon > 0$ in $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ with respect to d_S. Let $N \in \mathbb{N}$ be sufficiently large such that $\sum_{n=N}^{\infty} 2^{-n+1} < \frac{\varepsilon}{2}$. With $H \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$, $\psi \in \mathcal{H}$, and $\eta > 0$, the sets

$$\mathcal{U}_\eta(H, \psi) = \{H' \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) : \|H'\psi - H\psi\| < \eta\}$$

form a subbase of $\mathcal{O}_S$ and thus

$$\mathcal{V} = \bigcap_{n=1}^{N} \mathcal{U}_{\frac{\varepsilon}{2}}(H_0, \phi_n) \in \mathcal{O}_S.$$

It follows for $H_1 \in \mathcal{V}$ that

$$\begin{aligned} d_S(H_1, H_0) &= \sum_{n=1}^{N} 2^{-n} \|(H_1 - H_0)\phi_n\| + \sum_{n=N+1}^{\infty} 2^{-n} \|(H_1 - H_0)\phi_n\| \\ &\leq \frac{\varepsilon}{2} \sum_{n=1}^{N} 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n+1} < \varepsilon. \end{aligned}$$

Thus $\mathcal{V} \subset B_\varepsilon(H_0)$, and it is shown that every open neighborhood of H_0 in the metric topology contains an open neighborhood of H_0 in $\mathcal{O}_S$. This implies that $\mathcal{O}_S$ is finer than the metric topology.

For the converse inclusion, let us first note that $\mathcal{O}_S$ is already generated by the sets $\mathcal{U}_\eta(H, \psi)$ with ψ only taken from any dense subset of $\mathcal{H}$. Moreover, the set of all finite linear combinations of elements of $(\phi_n)_{n\geq 1}$ is dense in $\mathcal{H}$. Now let $\psi = \sum_{n=1}^{N} a_n \phi_n$. Then

$$\begin{aligned} \|(H_1 - H_0)\psi\| &\leq \sum_{n=1}^{N} |a_n| \|(H_1 - H_0)\phi_n\| \\ &\leq 2^N \max_{n=1,\dots,N} |a_n| \sum_{n=1}^{\infty} 2^{-n} \|(H_1 - H_0)\phi_n\|. \end{aligned}$$

Thus if $d_S(H_1, H_0) < \frac{\varepsilon}{2^N \max\{|a_n|\}}$, then $H_1 \in \mathcal{U}_\varepsilon(H_0, \psi)$. As these sets are a subbase of $\mathcal{O}_S$, it follows that the metric topology is finer than $\mathcal{O}_S$.

Finally, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$. □

Proposition 6.4.6. *The following pairs of topological spaces are identical:*
(i) $(\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *and* $(\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$*;*
(ii) $(\mathbb{F}\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *and* $(\mathbb{F}\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$*;*
(iii) $(\mathbb{F}\mathbb{B}^{C,0}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *and* $(\mathbb{F}\mathbb{B}^{C,0}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$.

Proof. Let $(H_j)_{j\geq 1}$ be a sequence in $\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$ converging to $H \in \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$ with respect to d_E, namely $\|H_j^2 - H^2\| \to 0$ and $\|H_j(\mathbf{1} - H_j^2)^{\frac{1}{2}} - H(\mathbf{1} - H^2)^{\frac{1}{2}}\| \to 0$. One needs to show that for any $\phi \in \mathcal{H}$, one has $\|(H_j - H)\phi\| \to 0$ so that the sequence also converges strongly. As $H \in \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$, one has $\mathrm{Ker}(\mathbf{1} - H^2) = \{0\}$ and therefore the range of $(\mathbf{1} - H^2)^{\frac{1}{2}}$ is dense in $\mathcal{H}$. Hence, for a given $\epsilon > 0$ there exists $\psi \in \mathcal{H}$ with $\|\phi - (\mathbf{1} - H^2)^{\frac{1}{2}}\psi\| \leq \epsilon$. Then there is a j_0 such that for $j \geq j_0$

$$\begin{aligned} \|(H_j - H)\phi\| &\leq \|(H_j(\mathbf{1} - H^2)^{\frac{1}{2}} - H(\mathbf{1} - H^2)^{\frac{1}{2}})\psi\| + 2\epsilon \\ &\leq \|(H_j(\mathbf{1} - H^2)^{\frac{1}{2}} - H_j(\mathbf{1} - H_j^2)^{\frac{1}{2}})\psi\| + 3\epsilon \end{aligned}$$

$$\leq \|((\mathbf{1}-H^2)^{\frac{1}{2}} - (\mathbf{1}-H_j^2)^{\frac{1}{2}})\psi\| + 3\epsilon$$
$$\leq \|H^2 - H_j^2\|^{\frac{1}{2}} \|\psi\| + 3\epsilon,$$

where the last step follows from Proposition A.2.2. Choosing j_0 possibly even larger, this shows that $\|(H_j - H)\phi\| \leq 4\epsilon$ for all $j \geq j_0$. As ϵ was arbitrary, this shows the first claim, which directly implies the second and third. □

Proposition 6.4.7. *One has the following deformation retracts:*
(i) $(\mathbb{B}^0_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *is a deformation retract of* $(\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$*;*
(ii) $(\mathbb{FB}^0_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *is a deformation retract of* $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$*;*
(iii) $(\mathbb{FB}^{C,0}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *is a deformation retract of* $(\mathbb{FB}^{C}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$*.*

Proof. (Inspired by Proposition 2.13 in [108].) Let us focus on the proof of (ii) and later on explain that the argument also covers the cases (i) and (iii). Let $K \in \mathbb{K}(\mathcal{H})$ be a nonnegative compact operator with norm less than or equal to $\frac{1}{2}$. To construct such an operator, recall that $\mathcal{H}$ is separable and thus has a countable orthonormal basis $(\phi_n)_{n\geq 1}$. Then $K = \sum_{n\geq 1} \frac{1}{n+1} |\phi_n\rangle\langle\phi_n|$ has all the desired properties. Then define

$$f : \mathbb{FB}_{1,\text{sa}}(\mathcal{H}) \to \mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \quad f(H) = (\mathbf{1}-K)H(\mathbf{1}-K).$$

Note that $f(H)$ is indeed self-adjoint and Fredholm by the compact stability of the Fredholm operators, and that it has norm less than or equal to 1 because $\|H\| \leq 1$ and $\|\mathbf{1}-K\| \leq 1$. Now let ϕ be a normalized vector. Then, using again $\|H\| \leq 1$ and $\|\mathbf{1}-K\| \leq 1$, the Cauchy–Schwarz inequality implies

$$\begin{aligned}\langle\phi|f(H)^2\phi\rangle^2 &= \langle(\mathbf{1}-K)^{\frac{1}{2}}\phi|(\mathbf{1}-K)^{\frac{1}{2}}H(\mathbf{1}-K)^2H(\mathbf{1}-K)\phi\rangle^2 \\ &\leq \langle\phi|(\mathbf{1}-K)\phi\rangle\langle\phi||(\mathbf{1}-K)^{\frac{1}{2}}H(\mathbf{1}-K)^2H(\mathbf{1}-K)|^2\phi\rangle \\ &\leq \langle\phi|(\mathbf{1}-K)\phi\rangle \\ &= 1 - \langle\phi|K\phi\rangle.\end{aligned}$$

Therefore

$$\langle\phi|(\mathbf{1}-f(H)^2)\phi\rangle = 1 - \langle\phi|f(H)^2\phi\rangle \geq 1 - \sqrt{1-\langle\phi|K\phi\rangle} > 0,$$

because K has a trivial kernel. Hence $\text{Ker}(\mathbf{1}-f(H)^2) = \{0\}$ so that $f(H)$ indeed lies in $\mathbb{FB}^0_{1,\text{sa}}(\mathcal{H})$.

Let us now show that f is continuous with respect to the topology $\mathcal{O}_{SE}$. Hence let $(H_j)_{j\geq 1}$ be a sequence converging to H in $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. It has to be shown that then also $(f(H_j))_{j\geq 1}$ converges to $f(H)$ in $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. Clearly, $(f(H_j))_{j\geq 1}$ converges strongly to $f(H)$. For the convergence with respect to d_E, let us begin by estimating

$$\|f(H_j)^2 - f(H)^2\|$$

$$\begin{aligned}
&= \big\|(\mathbf{1}-K)H_j(\mathbf{1}-K)^2H_j(\mathbf{1}-K) - (\mathbf{1}-K)H(\mathbf{1}-K)^2H(\mathbf{1}-K)\big\| \\
&\leq \big\|(\mathbf{1}-K)(H_j^2-H^2)(\mathbf{1}-K)\big\| \\
&\quad + \big\|(\mathbf{1}-K)H_jK(2\mathbf{1}-K)H_j(\mathbf{1}-K) - (\mathbf{1}-K)HK(2\mathbf{1}-K)H(\mathbf{1}-K)\big\| \\
&\leq \big\|H_j^2-H^2\big\| + \big\|H_jK(2\mathbf{1}-K)H_j - HK(2\mathbf{1}-K)H\big\| \\
&\leq \big\|H_j^2-H^2\big\| + \big\|(H_j-H)K(2\mathbf{1}-K)H_j\big\| + \big\|HK(2\mathbf{1}-K)(H-H_j)\big\| \\
&\leq \big\|H_j^2-H^2\big\| + 4\big\|(H_j-H)K\big\|.
\end{aligned}$$

Now K can be approximated in the operator norm by a finite-dimensional matrix, namely for all $\epsilon > 0$ one can find some finite-rank operator M with $\|K-M\| < \epsilon$ (this can readily be written out explicitly from K as given above). Due to the strong convergence $\text{s-}\lim_{j\to\infty} H_j = H$, one can then find a j_0 such that $\|(H_j-H)M\| \leq \epsilon$ for all $j \geq j_0$ (this is just the standard argument showing that a strongly converging sequence of compact operators is norm convergent). Choosing j_0 possibly even larger so that also $\|H_j^2 - H^2\| < \epsilon$ for $j \geq j_0$, one then finds

$$\big\|f(H_j)^2 - f(H)^2\big\| < \epsilon + 8\epsilon + 4\big\|(H_j-H)M\big\| < 13\epsilon$$

for all $j \geq j_0$. Hence $\lim_{j\to\infty} \|f(H_j)^2 - f(H)^2\| = 0$. By a similar argument, one also checks that the second norm difference in the definition of d_E vanishes in the limit so that $\lim_{j\to\infty} d_E(f(H_j), f(H)) = 0$. In conclusion, f is a continuous map on $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Next it will be shown that the map f is actually a homotopy inverse to the inclusion $i : \mathbb{FB}^0_{1,\text{sa}}(\mathcal{H}) \to \mathbb{FB}_{1,\text{sa}}(\mathcal{H})$, namely both of the maps $i \circ f : \mathbb{FB}_{1,\text{sa}}(\mathcal{H}) \to \mathbb{FB}_{1,\text{sa}}(\mathcal{H})$ and $f \circ i : \mathbb{FB}^0_{1,\text{sa}}(\mathcal{H}) \to \mathbb{FB}^0_{1,\text{sa}}(\mathcal{H})$ are homotopic to the identity on $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ and $(\mathbb{FB}^0_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$, respectively. One can use the homotopy $h_s(H) = (\mathbf{1}-sK)H(\mathbf{1}-sK)$ which is indeed continuous by similar arguments as above, and it also satisfies the inclusion $h_s(\mathbb{FB}^0_{1,\text{sa}}(\mathcal{H})) \subset \mathbb{FB}^0_{1,\text{sa}}(\mathcal{H})$ so that the case of $f \circ i$ is also dealt with.

The argument directly covers item (i) and also (iii), the latter because indeed one has $f(H) \in \mathbb{FB}^{\mathrm{C}}_{1,\text{sa}}(\mathcal{H})$ for $H \in \mathbb{FB}^{\mathrm{C}}_{1,\text{sa}}(\mathcal{H})$. □

Remark 6.4.8. The essence of the above proof is that the perturbation can be chosen such that it eliminates the point spectrum of H^2 at 1 for *all* H. The above proof also shows that $(\mathbb{FB}^0_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$, and also that $(\mathbb{B}^0_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$. ◇

Remark 6.4.9. In Proposition 6.4.7, on $\mathbb{FB}_{1,\text{sa}}(\mathcal{H})$ the strong essential gap topology $\mathcal{O}_{SE}$ appears. It is strictly weaker than the norm topology $\mathcal{O}_N$ on $\mathbb{FB}_{1,\text{sa}}(\mathcal{H})$. This can be seen by analyzing the bounded transform of the sequence $(H_n)_{n\geq 1}$ studied in the proof of Proposition 6.1.9 and realizing that $\mathcal{F}(H_n) \to \mathcal{F}(H)$ in the strong topology. Another manifestation is that $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$ has 3 components, while $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ has one component by Theorem 6.3.16 combined with Corollary 6.3.4. ◇

Example 6.4.10. Proposition 6.4.6 showed that the extended gap topology $\mathcal{O}_E$ and the strong extended gap topology $\mathcal{O}_{SE}$ coincide on $\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$. In view of Proposition 6.4.7, one might wonder whether the same holds true for the supersets $\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$ and $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$. In fact, this is *not* true as shows the following example. Consider the sequence $(H_j)_{j\geq 1}$ in $\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$ given by $H_j = -(1-\frac{1}{j})\mathbf{1}$. It converges to $H = \mathbf{1}$ with respect to d_E because

$$\|(H_j)^2 - H^2\| = \left(1-\frac{1}{j}\right)^2 - 1 \to 0,$$

$$\|H_j(\mathbf{1}-H_j^2)^{\frac{1}{2}} - H(\mathbf{1}-H^2)^{\frac{1}{2}}\| = \left(1-\frac{1}{j}\right)\left(1-\left(1-\frac{1}{j}\right)^2\right)^{\frac{1}{2}} \to 0.$$

However, the sequence $(H_j)_{j\geq 1}$ does not converge strongly to H as $H_j\phi \to -\phi \neq \phi = H\phi$ for all $\phi \in \mathcal{H}\setminus\{0\}$. Hence $\mathcal{O}_{SE}$ is strictly stronger than $\mathcal{O}_E$ on $\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$.

Working with the same sequence, one can show that the map f defined in the proof of Proposition 6.4.7 is *not* continuous with respect to $\mathcal{O}_E$ on $\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H})$. Indeed,

$$f(H_j) = -\left(1-\frac{1}{j}\right)(\mathbf{1}-K)^2, \quad f(H) = f(\mathbf{1}) = (\mathbf{1}-K)^2,$$

so that

$$\|f(H_j)(\mathbf{1}-f(H_j)^2)^{\frac{1}{2}} - f(\mathbf{1})(\mathbf{1}-f(\mathbf{1})^2)^{\frac{1}{2}}\| \to \|-2(\mathbf{1}-K)^2(\mathbf{1}-(\mathbf{1}-K)^4)^{\frac{1}{2}}\|.$$

Hence $d_E(f(H_j), f(\mathbf{1}))$ does not converge to zero. The problem is that f moves all the spectrum away from ± 1 to the inside which is a discontinuous procedure at $\mathbf{1}$ in the topology $\mathcal{O}_E$ (but the sequence $(H_j)_{j\geq 1}$ does not converge to $\mathbf{1}$ with respect to $\mathcal{O}_{SE}$ and hence does not disprove continuity with respect to $\mathcal{O}_{SE}$). ◇

Example 6.4.11. This example shows that the quotient topologies $\mathcal{O}_E^{\sim}$ and $\mathcal{O}_{SE}^{\sim}$ on $\mathbb{FB}^{C,\sim}_{1,\mathrm{sa}}(\mathcal{H}) = \mathbb{FB}^{C}_{1,\mathrm{sa}}(\mathcal{H})/\sim$ do not coincide. Let us consider $\mathcal{H} = \ell^2(\mathbb{N})$ and the following sequence of operators from $\mathbb{FB}^{C}_{1,\mathrm{sa}}(\mathcal{H})$:

$$H_n = \sum_{k\neq 1,n}\left(1-\frac{1}{kn}\right)|k\rangle\langle k| + \left(1-\frac{1}{n}\right)(|1\rangle, |n\rangle)^T \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}(\langle 1|, \langle n|).$$

Then $\|H_n^2 - \mathbf{1}\| \to 0$ and hence $d_E(H_n, \mathbf{1}) \to 0$. Thus $\{H_n : n\geq 2\}$ is not closed with respect to $\mathcal{O}_E$. As each class $[H_n]$ with respect to $\sim$ has only one representative, it follows that also $\{[H_n] : n\geq 2\}$ is not closed with respect to $\mathcal{O}_E^{\sim}$. On the other hand, it will be shown that the set $\{[H_n] : n\geq 2\}$ is closed with respect to $\mathcal{O}_{SE}^{\sim}$. Indeed, as

$$\|(H_n - H_m)|1\rangle\| = \frac{1}{\sqrt{2}}\left\|\left(-\frac{1}{n}+\frac{1}{m}\right)|1\rangle + \left(1-\frac{1}{n}\right)|n\rangle - \left(1-\frac{1}{m}\right)|m\rangle\right\| \geq \frac{1}{2\sqrt{2}},$$

the sequence $(H_n)_{n\geq 2}$ has no strongly convergent subsequence so that the subspace topology on $\{[H_n] : n \geq 2\}$ induced by $\mathcal{O}_{\widetilde{SE}}$ is the discrete topology. ◇

The next step will be to realize that $\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})$ is a deformation retract of $\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ by spectral analysis. This was already proved in Proposition 3.6.3, see also Remark 3.6.4, however, for the norm topology. It turns out that one can prove that this retraction is also continuous with respect to $\mathcal{O}_{SE}$. A generalization of this fact is proved in [108]. Here we provide an elementary proof.

Proposition 6.4.12. *The space* $(\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *is a deformation retract of the space* $(\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Proof. It will be checked that the maps in the proof of Proposition 3.6.3 are continuous with respect to $\mathcal{O}_{SE}$ so that they provide the desired retraction. For $H \in \mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$, let us define $\delta(H) = \min\{1, \min(\mathrm{spec}_{\mathrm{ess}}(H^2))^{\frac{1}{2}}\} > 0$. Then by the spectral radius theorem in the Calkin algebra, it follows that $H \mapsto \delta(H)$ is continuous with respect to $\mathcal{O}_{SE}$. For $\delta \in (0,1]$, let now $f_\delta : [-1,1] \to \mathbb{R}$ be the monotone continuous function defined by

$$f_\delta(x) = \chi_{[\delta,1]}(x) - \chi_{[-1,-\delta]}(x) + \frac{x}{\delta}\chi_{(-\delta,\delta)}(x).$$

Then set $\tilde{f} : \mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})$ defined by $\tilde{f}(H) = f_{\delta(H)}(H)$ and consider the linear homotopy

$$h : \mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}) \times [0,1] \to \mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}), \quad h(H,t) = (1-t)H + t\tilde{f}(H).$$

To show that this homotopy is continuous, let us first note that if a sequence $(H_n)_{n\geq 1}$ in $\mathbb{F}\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ converges to H with respect to $\mathcal{O}_{SE}$ and f is a continuous function, then also $(f(H_n))_{n\geq 1}$ converges strongly to $f(H)$. Indeed, for all even polynomials p, this follows from the convergence of $(H_n^2)_{n\geq 1}$ to H^2 in norm, while odd polynomials can be written as $H_n p(H_n)$ for an even polynomial p so that the strong convergence of H_n to H implies that $\text{s-}\lim_{n\to\infty} H_n p(H_n) = Hp(H)$. Then the strong continuity for any function follows from the Weierstrass approximation theorem which can be applied since $\|H_n^2 - H^2\| \to 0$ and therefore the sequence $(H_n)_{n\geq 1}$ is bounded.

To show that the homotopy h is continuous, it is shown that for any sequence $(H_n, t_n)_{n\geq 1}$ in $\mathbb{F}\mathbb{B}_{1,\mathrm{sa}} \times [0,1]$ converging to $(H,t) \in \mathbb{F}\mathbb{B}_{1,\mathrm{sa}} \times [0,1]$ with respect to $\mathcal{O}_{SE} \times |\cdot|$, the sequence $h(H_n, t_n)$ converges to $h(H,t)$ with respect to $\mathcal{O}_{SE}$. By Lemma 6.4.5, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ where

$$d_S(H_0', H_1') = \sum_{n=1}^{\infty} 2^{-n} \|(H_1' - H_0')\phi_n\|, \quad H_0', H_1' \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}),$$

for a fixed orthonormal basis $(\phi_n)_{n\geq 1}$ of $\mathcal{H}$ as in the proof of Lemma 6.4.5. Thus it is sufficient to show that

$$\lim_{n\to\infty}\big(d_E\big(h(H_n,t_n),h(H,t)\big)+d_S\big(h(H_n,t_n),h(H,t)\big)\big)=0. \tag{6.20}$$

The second summand is bounded by

$$\begin{aligned}d_S\big(h(H_n,t_n),h(H,t)\big)&=d_S\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H+t\tilde f(H)\big)\\&\le d_S\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n)\big)\\&\quad+d_S\big((1-t)H_n+t\tilde f(H_n),(1-t)H+t\tilde f(H)\big).\end{aligned}$$

The second summand in this expression converges to 0 because $f_{\delta(H_n)}$ converges to $f_{\delta(H)}$ with respect to $\|\cdot\|_{L^\infty}$ and therefore by the first part of the above argument $\text{s-}\lim_{n\to\infty}h(H_n,t)=h(H,t)$. The first summand is bounded by

$$\begin{aligned}&d_S\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n)\big)\\&\quad=\sum_{m=1}^{\infty}2^{-m}\big\|(t-t_n)H_n\phi_n+(t_n-t)\tilde f(H_n)\phi_m\big\|\\&\quad\le\sum_{m=1}^{\infty}2^{-m}|t_n-t|\big(\|H_n\|+\|\tilde f(H_n)\|\big)\\&\quad\le 2\sum_{m=1}^{\infty}2^{-m}|t_n-t|.\end{aligned}$$

Thus $d_S((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n))$ converges to 0 uniformly in H_n, and one concludes that $\lim_{n\to\infty}d_S(h(H_n,t_n),h(H,t))=0$. The first summand in (6.20) is bounded by

$$\begin{aligned}d_E\big(h(H_n,t_n),h(H,t)\big)&=d_E\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H+t\tilde f(H)\big)\\&\le d_E\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n)\big)\\&\quad+d_E\big((1-t)H_n+t\tilde f(H_n),(1-t)H+t\tilde f(H)\big).\end{aligned}$$

By Lemma 6.1.8,

$$\begin{aligned}&d_E\big((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n)\big)\\&\quad\le 2\sqrt2\big\|(1-t_n)H_n+t_n\tilde f(H_n)-(1-t)H_n+t\tilde f(H_n)\big\|^{\frac12}\\&\quad\le 2\sqrt2\big(|t-t_n|\big(\|H_n\|+\|\tilde f(H_n)\|\big)\big)^{\frac12}\\&\quad\le 2\sqrt2(2|t-t_n|)^{\frac12}.\end{aligned}$$

Thus $d_E((1-t_n)H_n+t_n\tilde f(H_n),(1-t)H_n+t\tilde f(H_n))$ converges to 0 uniformly in H_n. It remains to show $\lim_{n\to\infty}d_E((1-t)H_n+t\tilde f(H_n),(1-t)H+t\tilde f(H))=0$. It is therefore sufficient to show that $h_t:\mathbb{FB}_{1,\mathrm{sa}}(\mathcal H)\to\mathbb{FB}_{1,\mathrm{sa}}(\mathcal H)$ defined by $h_t(H)=h(t,H)$ is continuous with respect to $\mathcal O_E$. Because h_t is a class map with respect to $\sim$, this can be checked using the last claim of Lemma 4.6.6, namely it is sufficient to prove the continuity of the map $h_t^{\sim} : \mathbb{FB}_{1,\mathrm{sa}}^{\sim}(\mathcal{H}) \to \mathbb{FB}_{1,\mathrm{sa}}^{\sim}(\mathcal{H})$ with respect to $\mathcal{O}_E^{\sim}$. By Corollary 4.6.11, this is equivalent to the continuity of $\mathcal{G} \circ h_t^{\sim} \circ \mathcal{G}^{-1}$ on $(\mathbb{FU}(\mathcal{H}), \mathcal{O}_N)$. This is, in turn, guaranteed by the continuity of the map

$$(e^{\imath\varphi}, \delta) \in \mathbb{S}^1 \times (0,1] \mapsto \mathcal{G} \circ h_{t,\delta}^{\sim} \circ \mathcal{G}^{-1}(e^{\imath\varphi}), \tag{6.21}$$

where $\varphi \in (0, 2\pi]$ and $h_{t,\delta} : [-1,1] \to [-1,1]$ is defined by

$$h_{t,\delta}(x) = (1-t)x + tf_\delta(x).$$

As

$$\mathcal{G} \circ h_{t,\delta}^{\sim} \circ \mathcal{G}^{-1}(e^{\imath\varphi}) = \mathcal{G}\Big(-(1-t)\cos\Big(\frac{\varphi}{2}\Big) + tf_\delta\Big(-\cos\Big(\frac{\varphi}{2}\Big)\Big)\Big) \in \mathbb{S}^1,$$

for $(e^{\imath\varphi}, \delta) \in \mathbb{S}^1 \times (0,1]$ the continuity of (6.21) can readily be checked. □

Corollary 6.4.13. *The space* $(\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C},0}(\mathcal{H}), \mathcal{O}_{SE})$ *is homotopy equivalent to the space* $(\mathbb{FB}_{1,\mathrm{sa}}^{0}(\mathcal{H}), \mathcal{O}_{SE})$.

Proof. Proposition 6.4.7(iii) implies that $(\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C},0}(\mathcal{H}), \mathcal{O}_{SE})$ is homotopy equivalent to $(\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_{SE})$, which, by Proposition 6.4.12, is homotopy equivalent to the space $(\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. But Proposition 6.4.7(ii) shows that the latter is homotopy equivalent to $(\mathbb{FB}_{1,\mathrm{sa}}^{0}(\mathcal{H}), \mathcal{O}_{SE})$. □

Proof of Theorem 6.4.2. By Proposition 6.4.6, the topologies $\mathcal{O}_E$ and $\mathcal{O}_{SE}$ coincide on both $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C},0}(\mathcal{H})$ and $\mathbb{FB}_{1,\mathrm{sa}}^{0}(\mathcal{H})$. Due to Corollary 6.4.13, one concludes that $(\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C},0}(\mathcal{H}), \mathcal{O}_E)$ and $(\mathbb{FB}_{1,\mathrm{sa}}^{0}(\mathcal{H}), \mathcal{O}_E)$ are homotopy equivalent. The claim now follows from Proposition 6.4.4 and Corollary 6.3.4. □

In order to further complete the analysis of the strong extended gap topology $\mathcal{O}_{SE}$ on $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H})$, let us prove that it is equivalent to the Kasparov topology as introduced by Bunke, Joachim, and Stolz [44].

Definition 6.4.14. The Kasparov topology $\mathcal{O}_K$ on $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H})$ is the weakest topology containing the strong topology $\mathcal{O}_S$ and such that the map

$$H \in \big(\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_K\big) \mapsto \mathbf{1} - H^2 \in \big(\mathbb{K}(\mathcal{H}), \mathcal{O}_N\big)$$

is continuous.

Proposition 6.4.15. *The strong extended gap topology* $\mathcal{O}_{SE}$ *on* $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H})$ *is identical to the Kasparov topology* $\mathcal{O}_K$.

Proof. (Following Proposition 3.3 in [108].) The extended gap topology on $\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H})$ is the weakest topology such that $H \in (\mathbb{FB}_{1,\mathrm{sa}}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H^2 \in (\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ and

$H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H(\mathbf{1}-H^2)^{\frac{1}{2}} \in (\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ are continuous. Clearly, the continuity of the two maps $H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE}) \mapsto \mathbf{1}-H^2 \in (\mathbb{FB}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ and $H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H^2 \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ are equivalent. As both $\mathcal{O}_{SE}$ and $\mathcal{O}_K$ contain $\mathcal{O}_S$, it follows that $\mathcal{O}_{SE}$ is finer than $\mathcal{O}_K$.

Next let us come to the the converse. It will be used that the continuity of the map $H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_K) \mapsto \mathbf{1}-H^2 \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ implies, by Proposition A.2.2, also the continuity of $H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_K) \mapsto (\mathbf{1}-H^2)^{\frac{1}{2}} \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$. Because a strongly continuous map of compact operators is norm-continuous, it follows that also the map $H \in (\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_K) \mapsto H(\mathbf{1}-H^2)^{\frac{1}{2}} \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ is continuous, showing that $\mathcal{O}_K$ is also finer than $\mathcal{O}_{SE}$. □

Next let us provide an application of the Kasparov topology. In the set $\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H})$, there are two subsets with opposite properties: one is $\mathbb{FB}^{\mathbb{C},0}_{1,\mathrm{sa}}(\mathcal{H})$ in which neither -1 nor 1 is an eigenvalue, the other has both as eigenvalues with infinite multiplicity,

$$\mathbb{FB}^{\mathbb{C},\infty}_{1,\mathrm{sa}}(\mathcal{H}) = \{H \in \mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}) : \dim(\mathrm{Ker}(H \pm \mathbf{1})) = \infty\}.$$

The analogue of Proposition 6.4.7 is the following result (that is not used for the proof of Theorem 6.4.2):

Proposition 6.4.16. *The space* $(\mathbb{FB}^{\mathbb{C},\infty}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ *is homotopy equivalent to the space* $(\mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Proof. (Inspired by Lemma 2.5 of [44].) Let us denote $L^2 = L^2([0,1]) \otimes \mathbb{C}^2$ and choose a unitary

$$U : \mathcal{H} \to L^2.$$

Further let $Q_0 = \mathbf{1} \otimes \mathrm{diag}(1,-1)$ be a proper symmetry on L^2. Next let us introduce the unitary $W = (W_0, W_1) : L^2 \to L^2 \oplus L^2$ by

$$(W_0\psi)(x) = 2^{-\frac{1}{2}}\psi\left(\frac{x}{2}\right), \quad (W_1\psi)(x) = 2^{-\frac{1}{2}}\psi\left(\frac{x+1}{2}\right),$$

where $x \in [0,1]$ and the 2×2 matrix component is the identity and suppressed in the notation. Then set

$$\widetilde{H} = U^* W^* (UHU^* \oplus Q_0) WU$$

for $H \in \mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H})$. By construction, one has $\widetilde{H} \in \mathbb{FB}^{\mathbb{C},\infty}_{1,\mathrm{sa}}(\mathcal{H})$. It remains to construct a homotopy $h : \mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H}) \times [\frac{1}{2}, 1] \to \mathbb{FB}^{\mathbb{C}}_{1,\mathrm{sa}}(\mathcal{H})$ from $h_1(H) = H$ to $h_{\frac{1}{2}}(H) = \widetilde{H}$, continuous with respect to $\mathcal{O}_{SE}$. For this purpose, one can now proceed using a family of partial isometries $V_t : L^2 \to L^2$ first introduced by Dixmier and Douady [73]. Set

$$(V_t\psi)(x) = \begin{cases} t^{-\frac{1}{2}}\psi(\frac{x}{t}), & x \in [0,t], \\ 0, & x \in (t,1]. \end{cases}$$

Note that $V_t^* V_t = \mathbf{1}$ and $V_t V_t^* = \chi_{[0,t]}$ is the projection onto $L^2([0,t])$ (again tensorized with the identity on $\mathbb{C}^2$), so that, in particular, V_1 is unitary. Also $[V_t, Q_0] = 0$. Moreover, by a standard approximation argument with smooth functions, one can check that both $t \in [\frac{1}{2}, 1] \mapsto V_t$ and $t \in [\frac{1}{2}, 1] \mapsto V_t^*$ are strongly continuous. Then set

$$h_t(H) = U^*(V_t U H U^* V_t^* + (\mathbf{1} - V_t V_t^*) Q_0) U.$$

Due to $V_t^*(\mathbf{1} - V_t V_t^*) = 0$, $(\mathbf{1} - V_t V_t^*)^2 = \mathbf{1} - V_t V_t^*$ and $Q_0^2 = \mathbf{1}$, one has

$$\begin{aligned}\mathbf{1} - h_t(H)^2 &= U^*(\mathbf{1} - V_t U H U^* V_t^* V_t U H U^* V_t^* - (\mathbf{1} - V_t V_t^*)^2 Q_0^2) U \\ &= U^*(V_t V_t^* - V_t U H^2 U^* V_t^*) U \\ &= U^* V_t U (\mathbf{1} - H^2) U^* V_t^* U,\end{aligned}$$

which is compact so that indeed $h_t(H) \in \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})$. Next let us verify that h is continuous and therefore a homotopy on $(\mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. For this purpose, it is shown that, for any sequence $(H_n, t_n)_{n\geq 1}$ in $\mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}} \times [\frac{1}{2}, 1]$ converging to $(H, t) \in \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}} \times [\frac{1}{2}, 1]$ with respect to $\mathcal{O}_{SE} \times |\cdot|$, the sequence $h_{t_n}(H_n)$ converges to $h_t(H)$ with respect to $\mathcal{O}_{SE}$. By Lemma 6.4.5, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ where

$$d_S(H_0', H_1') = \sum_{n=1}^{\infty} 2^{-n} \|(H_1' - H_0')\phi_n\|, \quad H_0', H_1' \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H}),$$

for a fixed orthonormal basis $(\phi_n)_{n\geq 1}$ of $\mathcal{H}$ as in the proof of Lemma 6.4.5. Thus it is sufficient to show that

$$\lim_{n\to\infty} \big(d_E(h_{t_n}(H_n), h_t(H)) + d_S(h_{t_n}(H_n), h_t(H))\big) = 0. \tag{6.22}$$

The second summand is bounded by

$$d_S(h_{t_n}(H_n), h_t(H)) \leq d_S(h_{t_n}(H_n), h_t(H_n)) + d_S(h_t(H_n), h_t(H)). \tag{6.23}$$

Then

$$\lim_{n\to\infty} d_S(h_t(H_n), h_t(H)) = \lim_{n\to\infty} \sum_{m=1}^{\infty} 2^{-m} \|U^* V_t U (H_n - H) U^* V_t^* U \phi_m\| = 0$$

because $\text{s-}\lim_{n\to\infty} H_n = H$ by assumption and $\|U^* V_t U (H_n - H) U^* V_t^* U\| \leq 2$ for all $(n, t) \in \mathbb{N} \times [\frac{1}{2}, 1]$. The first summand in (6.23) is bounded by

$$\begin{aligned}&d_S(h_{t_n}(H_n), h_t(H_n)) \\ &\quad = \sum_{m=1}^{\infty} 2^{-m} \|U^*(V_{t_n} U H_n U^* V_{t_n}^* - V_t U H_n U^* V_t^* + Q_0(V_t V_t^* - V_{t_n} V_{t_n}^*)) U \phi_m\|\end{aligned}$$

$$\leq \sum_{m=1}^{\infty} 2^{-m} \|U^*(V_{t_n} U H_n U^* V_{t_n}^* - V_t U H_n U^* V_t^*) U\phi_m\|$$
$$+ \sum_{m=1}^{\infty} 2^{-m} \|(V_t V_t^* - V_{t_n} V_{t_n}^*) U\phi_m\|.$$

The second summand converges to 0 and the first summand is bounded by

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{-m} \|U^*(V_{t_n} U H_n U^* V_{t_n}^* - V_t U H_n U^* V_t^*) U\phi_m\| \\
&\quad \leq \sum_{m=1}^{\infty} 2^{-m} (\|V_{t_n} U H_n U^* (V_{t_n}^* - V_t^*) U\phi_m\| \\
&\quad\quad + \|(V_{t_n} - V_t) U H_n U^* V_t^* U\phi_m\|) \\
&\quad \leq \sum_{m=1}^{\infty} 2^{-m} (\|V_{t_n} U H_n U^* (V_{t_n}^* - V_t^*) U\phi_m\| \\
&\quad\quad + \|(V_{t_n} - V_t) U (H_n - H) U^* V_t^* U\phi_m\| \\
&\quad\quad + \|(V_{t_n} - V_t) U H U^* V_t^* U\phi_m\|) \\
&\quad \leq \sum_{m=1}^{\infty} 2^{-m} (2\|(V_{t_n}^* - V_t^*) U\phi_m\| + 2\|(H_n - H) U^* V_t^* U\phi_m\| \\
&\quad\quad + \|(V_{t_n} - V_t) U H U^* V_t^* U\phi_m\|)
\end{aligned}$$

and all three summands converge to 0 by the same argument as above using that $\text{s-}\lim_{n\to\infty} V_{t_n}^* = V_t^*$ and $\text{s-}\lim_{n\to\infty} H_n = H$. Finally, it remains to verify that $\lim_{n\to\infty} d_E(h_{t_n}(H_n), h_t(H)) = 0$. As $\mathcal{O}_{SE} = \mathcal{O}_K$ on $\mathbb{FB}^c_{1,\mathrm{sa}}(\mathcal{H})$ by Proposition 6.4.15 and $\text{s-}\lim_{n\to\infty} h_{t_n}(H_n) = h_t(H)$ by the above, it is sufficient to show

$$\lim_{n\to\infty} \|h_{t_n}(H_n)^2 - h_t(H)^2\| = 0.$$

This follows from

$$\begin{aligned}
\|h_{t_n}(H_n)^2 - h_t(H)^2\| &= \|V_{t_n} U(\mathbf{1} - H_n^2) U^* V_{t_n}^* - V_t U(\mathbf{1} - H^2) U^* V_t^*\| \\
&\leq \|(V_{t_n} - V_t) U(\mathbf{1} - H_n^2) U^* V_{t_n}^*\| \\
&\quad + \|V_t U((\mathbf{1} - H_n^2) - (\mathbf{1} - H^2)) U^* V_{t_n}^*\| \\
&\quad + \|V_t U(\mathbf{1} - H^2) U^* (V_{t_n}^* - V_t^*)\| \\
&\leq \|(V_{t_n} - V_t) U(\mathbf{1} - H_n^2)\| \\
&\quad + \|H_n^2 - H^2\| \\
&\quad + \|(\mathbf{1} - H^2) U^* (V_{t_n}^* - V_t^*)\| \\
&\leq \|(V_{t_n} - V_t) U(\mathbf{1} - H^2)\| \\
&\quad + \|(V_{t_n} - V_t) U(H^2 - H_n^2)\|
\end{aligned}$$

$$+ \|H_n^2 - H^2\|$$
$$+ \|(\mathbf{1} - H^2)U^*(V_{t_n}^* - V_t^*)\|.$$

The first summand converges to 0 because $\text{s-}\lim_{n\to\infty} V_{t_n} = V_t$ and $\mathbf{1} - H^2 \in \mathbb{K}(\mathcal{H})$ is compact. Because $((V_{t_n} - V_t)U(\mathbf{1}-H^2))^* = (\mathbf{1}-H^2)U^*(V_{t_n}^* - V_t^*)$, this implies that also the last summand converges to 0. By assumption $\lim_{n\to\infty} d_E(H_n, H) = 0$ and by definition of the extended gap metric, this implies that the third summand converges to 0 and therefore also the second summand converges to 0. One concludes that (6.22) holds and therefore h is continuous.

It only remains to note that indeed $h_1(H) = H$ and $h_{\frac{1}{2}}(H) = \widetilde{H}$, and furthermore $h_t(H) \in \mathbb{FB}^{\mathrm{C},\infty}_{1,\mathrm{sa}}(\mathcal{H})$ for all $H \in \mathbb{FB}^{\mathrm{C},\infty}_{1,\mathrm{sa}}(\mathcal{H})$ and all $t \in [\frac{1}{2}, 1]$. Therefore also the map $f : \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{FB}^{\mathrm{C},\infty}_{1,\mathrm{sa}}(\mathcal{H})$ defined by $f(H) = \widetilde{H}$ is a homotopy equivalence with respect to $\mathcal{O}_{SE}$ with homotopy inverse given by the embedding $i : \mathbb{FB}^{\mathrm{C},\infty}_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{FB}^{\mathrm{C}}_{1,\mathrm{sa}}(\mathcal{H})$. □

The following is a direct consequence of Theorems 6.4.2 and 6.3.16.

Theorem 6.4.17. *With respect to the gap metric, the set* $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ *is connected.*

Because this result may seem surprising at first sight, a direct proof is provided.

Proof. It is shown that $\mathbb{U}^{\mathrm{C},0}(\mathcal{H})$ is connected, which, by Theorem 6.4.3, implies the claim. For $U \in \mathbb{U}^{\mathrm{C},0}(\mathcal{H})$, a norm-continuous path within $\mathbb{U}^{\mathrm{C},0}(\mathcal{H})$ connecting U to

$$U_{\mathrm{ref}} = \sum_{n\geq 1} e^{\frac{\imath}{n+1}} |\phi_n\rangle\langle\phi_n|,$$

where $(\phi_n)_{n\geq 1}$ is an orthonormal basis of $\mathcal{H}$, is constructed. Note that $U_{\mathrm{ref}} = e^{\imath K_{\mathrm{ref}}}$ for the self-adjoint and compact operator $K_{\mathrm{ref}} = \sum_{n\geq 1} \frac{1}{n+1}|\phi_n\rangle\langle\phi_n|$.

First, let us decompose $\mathcal{H}$ into the spectral subspaces $\mathcal{H}_\pm$ of U corresponding to $\{e^{\imath\varphi} : \varphi \in [0, \pi]\}$ and $\{e^{\imath\varphi} : \varphi \in (\pi, 2\pi]\}$. Respectively, we decompose $U = U_+ \oplus U_-$. There is no intersection of the spectral subspaces as, if $-1 \in \mathrm{spec}(U)$, it is an isolated eigenvalue and hence belongs to $\mathrm{spec}(U_+)$. And if $1 \in \mathrm{spec}(U)$, it does not contribute to the decomposition of U as it is not an eigenvalue.

If $\mathcal{H}_-$ is finite dimensional, we rotate U_- through -1 into $U_-' = -U_-$. More precisely, the path $t \in [0,1] \mapsto e^{-\imath\pi t}U_-$ lies entirely in $\mathbb{U}^{\mathrm{C},0}(\mathcal{H}_-)$ and connects U_- to U_-' where $\mathrm{spec}(U_-') \subset \{e^{\imath\varphi} : \varphi \in (0,\pi)\}$. Otherwise, we identify $\mathcal{H}_-$ with $L^2([0,1])$. Then U_- is of the form $U_- = e^{\imath K_-}$ for some self-adjoint injective compact operator $K_- \in \mathbb{K}(L^2([0,1]))$ with $\mathrm{spec}(K_-) \subset (-\pi, 0]$. For $t \in [0,2]$, let $M_t \in \mathbb{B}(L^2([0,1]))$ denote the multiplication operator given by multiplication with the function $f_t : [0,1] \to [0,1]$ defined by

$$f_t(x) = (-1 + 2tx)\chi_{[0,1]}(t) + \big(2(1-x)(t-2) + 1\big)\chi_{(1,2]}(t)$$

for $t \in [0,2]$ and $x \in [0,1]$. Then

$$t \in [0,2] \mapsto K_t = -|K_-|^{\frac{1}{2}} M_t |K_-|^{\frac{1}{2}}$$

is a continuous path of injective compact operators connecting K_- to $-K_-$ such that $\|K_t\| \leq \|K_-\| < \pi$ for all $t \in [0,2]$. Therefore the path $t \in [0,2] \mapsto e^{\imath K_t}$ lies in $\mathbb{U}^{C,0}(\mathcal{H})$ and connects U_- to $U'_- = e^{-\imath K_-}$.

In both cases taking the pointwise direct sum of the constructed path and the constant path $t \mapsto U_+$ gives a path in $\mathbb{U}^{C,0}(\mathcal{H})$ connecting U to $U_+ \oplus U'_-$ with spectrum satisfying $\mathrm{spec}(U_+ \oplus U'_-) \subset \{e^{\imath\varphi} : \varphi \in [0,\pi]\}$. Then there is an injective compact operator $K \in \mathbb{K}(\mathcal{H})$ with $\mathrm{spec}(K) \subset [0,\pi]$ such that $U_+ \oplus U'_- = e^{\imath K}$. The linear path $t \in [0,1] \mapsto K'_t = (1-t)K + tK_{\mathrm{ref}}$ connecting K to K_{ref} is within the injective compact operators with spectrum $\mathrm{spec}(K_t) \subset [0,\pi]$. Therefore the path $t \in [0,1] \mapsto e^{\imath K'_t}$ is within $\mathbb{U}^{C,0}(\mathcal{H})$ and connects $U_+ \oplus U'_-$ to U_{ref}. Thus U can be connected to U_{ref} within $\mathbb{U}^{C,0}(\mathcal{H})$, which implies the claim. □

7 Spectral flow for unbounded self-adjoint Fredholm operators

In this chapter, the spectral flow of paths of self-adjoint unbounded Fredholm operators is introduced like in [31] as the spectral flow of unitary operators obtained from the unbounded operators via the Cayley transform. As such, the spectral flow of unbounded self-adjoint Fredholm operators inherits many natural properties which are also listed in Section 7.1. In the next Section 7.2, the paths are in the subset of Fredholm operators with compact resolvent and satisfy certain summability conditions which then allow connecting them to so-called η-invariants. Section 7.3 is an application of spectral flow of unbounded self-adjoint Fredholm operators to certain paths arising from Hamiltonian systems. The chapter is concluded by Section 7.4 which shows that for certain paths of unbounded self-adjoint Fredholm operators the spectral flow is still given as the index of an (unbounded) Fredholm operator.

7.1 Definition of spectral flow and its basic properties

In this section the notion of spectral flow is generalized to gap continuous paths of possibly unbounded Fredholm operators. As in [31], this will be achieved by taking the Cayley transform of the path to use the spectral flow of the resulting path of unitaries. More precisely, let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{sa}(\mathcal{H})$ be continuous with respect to the gap metric. Then by Theorem 6.3.14, the path $t \in [0,1] \mapsto \mathcal{C}(H_t) \in \mathbb{FU}^0(\mathcal{H})$ is norm-continuous. Therefore its spectral flow is well defined in the sense of Section 4.5, and one can define the spectral flow of $t \in [0,1] \mapsto H_t$ in a similar manner as in Proposition 4.6.2.

Definition 7.1.1. Let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{sa}(\mathcal{H})$ be continuous with respect to the gap metric. Then the spectral flow of this path is defined by

$$\mathrm{Sf}([0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto \mathcal{C}(H_t)).$$

The first result shows that for a path of bounded self-adjoint Fredholm operators this definition coincides with Definition 4.1.2. This implies that all examples of paths with nontrivial spectral flow from Chapter 4 also provide examples of nontrivial spectral flow in the sense of Definition 7.1.1. A path of truly unbounded operators with nonvanishing spectral flow will be given in Example 7.1.4 below.

Proposition 7.1.2. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ be a norm-continuous path of self-adjoint Fredholm operators. Its spectral flow defined by* (4.4) *fulfils*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto \mathcal{C}(H_t)).$$

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Proof. To show this, let us choose a partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ and $a_m \geq 0$, $m = 1, \ldots, M$ as in Definition 4.1.2. By item (i) of Corollary 6.3.8,

$$\operatorname{Ran}\big(\chi_{\mathcal{C}([-a,a])}(\mathcal{C}(H_t))\big) = \operatorname{Ran}\big(\chi_{[-a,a]}(H_t)\big)$$

and therefore

$$t \in [t_{m-1}, t_m] \mapsto \chi_{\mathcal{C}([-a_m,a_m])}(\mathcal{C}(H_t))$$

is norm-continuous. For $a \geq 0$, let us set

$$P^{>}_{a,t} = \chi_{(0,a]}(H_t), \quad P^{<}_{a,t} = \chi_{[-a,0)}(H_t)$$

and similarly

$$P^{>}_{\mathcal{C}(a),t} = \chi_{\mathcal{C}((0,a])}(\mathcal{C}(H_t)), \quad P^{<}_{\mathcal{C}(a),t} = \chi_{\mathcal{C}([-a,0))}(\mathcal{C}(H_t)).$$

Again by item (i) of Corollary 6.3.8, one has

$$\operatorname{Ran}(P^{>}_{\mathcal{C}(a_m),t}) = \operatorname{Ran}(P^{>}_{a_m,t})$$

and

$$\operatorname{Ran}(P^{<}_{\mathcal{C}(a_m),t}) = \operatorname{Ran}(P^{<}_{a_m,t})$$

for $t \in [t_{m-1}, t_m]$. Therefore

$$\operatorname{Tr}(P^{>}_{\mathcal{C}(a_m),t}) = \operatorname{Tr}(P^{>}_{a_m,t})$$

and

$$\operatorname{Tr}(P^{<}_{\mathcal{C}(a_m),t}) = \operatorname{Tr}(P^{<}_{a_m,t})$$

again for $t \in [t_{m-1}, t_m]$. One can conclude that

$$\begin{aligned}
&\operatorname{Sf}(t \in [0,1] \mapsto H_t) \\
&= \frac{1}{2} \sum_{m=1}^{M} \operatorname{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m}) - \operatorname{Tr}(P^{>}_{a_m,t_{m-1}} - P^{<}_{a_m,t_{m-1}}) \\
&= \frac{1}{2} \sum_{m=1}^{M} \operatorname{Tr}(P^{>}_{\mathcal{C}(a_m),t_m} - P^{<}_{\mathcal{C}(a_m),t_m}) - \operatorname{Tr}(P^{>}_{\mathcal{C}(a_m),t_{m-1}} - P^{<}_{\mathcal{C}(a_m),t_{m-1}}) \\
&= \operatorname{Sf}(t \in [0,1] \mapsto \mathcal{C}(H_t)),
\end{aligned}$$

as claimed. □

Remark 7.1.3. It is also possible to define the spectral flow of a gap-continuous path $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ in a similar way as for paths of self-adjoint bounded Fredholm operators in Section 4.1. More precisely, using Corollary 6.3.8, one can check that for $H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ there is a number $a \geq 0$ and a neighborhood $\mathcal{N}$ of H in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous, finite-rank projection-valued function on $\mathcal{N}$. Thus, by compactness it is possible to choose a finite partition $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = 1$ of $[0,1]$ and $a_m \geq 0$, $m = 1,\dots,M$, such that

$$t \in [t_{m-1}, t_m] \mapsto P_{a_m,t} = \chi_{[-a,a]}(H_t)$$

is norm-continuous with constant finite rank. Furthermore, as in Section 4.1, let us introduce the spectral projections

$$P^{>}_{a,t} = \chi_{(0,a]}(H_t), \quad P^{<}_{a,t} = \chi_{[-a,0)}(H_t).$$

Then the spectral flow of the path $t \in [0,1] \mapsto H_t$ can be defined by

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2} \sum_{m=1}^{M} \mathrm{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}).$$

As in Section 4.1, one can show that this does not depend on the partition of $[0,1]$ or on the values a_m, but only on the path $t \in [0,1] \mapsto H_t$. An argument similar to the one leading to Proposition 7.1.2 shows that both definitions coincide, namely

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto \mathcal{C}(H_t)).$$

Further details on the equivalence of this approach and Definition 7.1.1 are given in [31]. ◇

Next let us give an example of a path in $\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ with a nonvanishing spectral flow. It actually lies in the set $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ of self-adjoint operators with compact resolvent that was extensively studied in Section 6.4.

Example 7.1.4. Let us consider $\mathcal{H} = \ell^2(\mathbb{Z})$ and the operator path

$$t \in [0,1] \mapsto H_t = \sum_{n\in\mathbb{Z}} \left(n + \frac{1}{2} + t\right)|n\rangle\langle n|.$$

Clearly, $H_t \in \mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ and

$$\mathcal{C}(H_t) = \sum_{n\in\mathbb{Z}} \frac{n + \frac{1}{2} + t - \imath}{n + \frac{1}{2} + t + \imath}|n\rangle\langle n|$$

is norm-continuous in t. One readily checks that $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = 1$. Now one readily checks that $H_1 = S^* H_0 S$ where $S|n\rangle = |n+1\rangle$ is the right shift on $\ell^2(\mathbb{Z})$. This unitary S is connected to the identity $\mathbf{1}$ by a continuous path of unitaries (either by Kuiper's theorem or by using a logarithm of S, or by writing out an explicit path as in Example 8.3.4 below). Hence $t \in [0,1] \mapsto H_t$ can be closed to a loop by a path of invertibles. This shows that there are loops in $(\mathbb{U}^{C,0}(\mathcal{H}), d_N)$ that cannot be retracted, which may seem surprising at first sight because 1 is never an eigenvalue along this loop. ◇

In this example, the path is actually also Riesz-continuous (and thus, in particular, gap-continuous). This follows from the following result that allows checking the Riesz-continuity in numerous applications.

Proposition 7.1.5. *Let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ be such that $t \in [0,1] \mapsto H_t - H_0$ extends to a path of bounded operators which is norm-continuous. Then $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ is also Riesz-continuous.*

Proof. First of all, it follows from the Kato–Rellich theorem that the domains $\mathcal{D}(H_t)$ all coincide. By definition of the Riesz topology, one has to show that $t \in [0,1] \mapsto \mathcal{F}(H_t)$ is norm-continuous. For this purpose, let us use the following functional calculus:

$$\mathcal{F}(H_t) = H_t(\mathbf{1} + H_t^2)^{-\frac{1}{2}} = H_t \int_0^\infty \frac{d\lambda}{\pi\lambda^{\frac{1}{2}}} ((\lambda+1)\mathbf{1} + H_t^2)^{-1}.$$

As $\|((\lambda+1)\mathbf{1} + H_t^2)^{-1}\| \leq \frac{1}{\lambda+1}$, the integral is norm-convergent and its image lies in the domain of H_t. Therefore

$$\begin{aligned}
&\mathcal{F}(H_t) - \mathcal{F}(H_s) - (H_s - H_t)(\mathbf{1} + H_t^2)^{-\frac{1}{2}} \\
&\quad = H_s \int_0^\infty \frac{d\lambda}{\pi\lambda^{\frac{1}{2}}} \big[((\lambda+1)\mathbf{1} + H_t^2)^{-1} - ((\lambda+1)\mathbf{1} + H_s^2)^{-1}\big] \\
&\quad = \int_0^\infty \frac{d\lambda}{\pi\lambda^{\frac{1}{2}}} ((\lambda+1)\mathbf{1} + H_s^2)^{-1} H_s\big(H_s(H_t - H_s) + (H_t - H_s)H_t\big)((\lambda+1)\mathbf{1} + H_t^2)^{-1}.
\end{aligned}$$

By assumption, one clearly has $\lim_{s\to t} \|(H_t - H_s)(\mathbf{1} + H_t^2)^{-\frac{1}{2}}\| = 0$. To show that the integral also vanishes in the limit, let us note that the spectral theorem implies

$$\begin{aligned}
\big\|H_t((\lambda+1)\mathbf{1} + H_t^2)^{-1}\big\| &\leq \sup_{\mu\in\mathbb{R}} \frac{|\mu|}{\lambda + 1 + \mu^2} \leq \lambda^{-\frac{1}{2}}, \\
\big\|H_t^2((\lambda+1)\mathbf{1} + H_t^2)^{-1}\big\| &\leq 1.
\end{aligned}$$

This indeed allows checking $\lim_{s\to t} \|\mathcal{F}(H_t) - \mathcal{F}(H_s)\| = 0$. □

Remark 7.1.6. For a path $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ that is continuous with respect to the Riesz metric, one can also directly define its spectral flow using the bounded transform $\mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{BF}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ via

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto \mathcal{F}(H_t)). \tag{7.1}$$

Indeed, by the very definition of the Riesz metric, one then has the norm-continuity of $t \in [0,1] \mapsto \mathcal{F}(H_t)$. Because the Riesz topology is stronger than the gap topology by Proposition 6.1.9, the definition (7.1) directly coincides with Definition 7.1.1 due to the results in Section 4.6. However, there are paths which are gap-continuous, but not Riesz-continuous. If one has a merely gap-continuous path $t \in [0,1] \mapsto H_t$, then, by Corollary 6.3.4, one knows that $t \in [0,1] \mapsto \mathcal{F}(H_t)$ is continuous with respect to the extended gap metric d_E. In this situation, one can still use (7.1) to compute the spectral flow because the continuity of the low lying spectrum is the same for the two metrics d_E and d_N, see the proof of Proposition 4.6.16. ◇

Let us conclude this section by collecting some basic properties of the spectral flow of paths of (unbounded) Fredholm operators.

Theorem 7.1.7. *Let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ be a path that is continuous with respect to the gap metric.*

(i) *If $t \in [0,1] \mapsto \dim(\mathrm{Ker}(H_t))$ is constant, then* $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = 0$.

(ii) *The spectral flow has a concatenation property, namely if $t \in [1,2] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H})$ is a second gap-continuous path, composable to the first one in the sense that the endpoint of the first path is the initial point of the second path, then*

$$\mathrm{Sf}(t \in [0,2] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [1,2] \mapsto H_t).$$

(iii) *Changing the orientation of the path leads to a change of the sign of the spectral flow*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto H_{1-t}).$$

(iv) *The spectral flow has a reflection property, namely*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = -\,\mathrm{Sf}(t \in [0,1] \mapsto -H_t).$$

(v) *The spectral flow has an additivity property under direct sums, namely given a second gap-continuous path $t \in [0,1] \mapsto S_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H}')$*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t \oplus S_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [0,1] \mapsto S_t).$$

(vi) *The spectral flow is invariant under conjugation by a path $t \in [0,1] \mapsto U_t \in \mathbb{U}(\mathcal{H})$ of unitaries*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto U_t^* H_t U_t).$$

Proof. All items are directly inherited from the basic properties of the spectral flow of a path of unitaries, see Theorem 4.5.5. □

Theorem 7.1.8. *Let $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t$ be two paths in $\mathbb{F}_{sa}(\mathcal{H})$ with $H_0 = H'_0$ and $H_1 = H'_1$ and such that there exists a gap-continuous homotopy between the two paths leaving the endpoints fixed. Then* $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(s \in [0,1] \mapsto H'_t)$.

Proof. Let $h : [0,1] \times [0,1] \to \mathbb{F}_{sa}(\mathcal{H})$ be a gap-continuous homotopy between the paths $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t$. Then Theorem 6.3.14 implies that the composition $\mathcal{C} \circ h : [0,1] \times [0,1] \to \mathbb{F}\mathbb{U}^0(\mathcal{H})$ is a norm-continuous homotopy between the paths $t \in [0,1] \mapsto \mathcal{C}(H_t)$ and $t \in [0,1] \mapsto \mathcal{C}(H'_t)$. Therefore, by Theorem 4.5.6,

$$\mathrm{Sf}\big([0,1] \mapsto \mathcal{C}(H_t)\big) = \mathrm{Sf}\big([0,1] \mapsto \mathcal{C}(H'_t)\big).$$

The claim follows from Definition 7.1.1. □

7.2 The η-invariant and spectral flow

As already briefly mentioned in Section 1.3, Atiyah, Patodi, and Singer [14] introduced the η-invariant as a measure of the spectral asymmetry of an invertible self-adjoint operator $H = H^*$ under the condition that H has compact resolvent with eigenvalues decaying sufficiently fast. Let us first start with a formal definition of the η-function by

$$\eta(H,s) = \mathrm{Tr}\big(H|H|^{-s-1}\big) = \sum_j \mathrm{sgn}(\lambda_j)|\lambda_j|^{-s}, \tag{7.2}$$

where $s > 0$ and λ_j are the eigenvalues of H (this clearly makes sense if $|H|^{-s}$ is trace class). It is then often possible (e. g., for certain classes of pseudo-differential operators [177, 14, 28, 125]) to show that the η-function has a meromorphic extension given by

$$\eta(H,s) = \mathrm{Tr}\big(H(H^2)^{-\frac{s+1}{2}}\big) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt\, t^{\frac{s-1}{2}}\, \mathrm{Tr}\big(He^{-tH^2}\big). \tag{7.3}$$

Whenever $\eta(H,s)$ is regular at $s = 0$, one says that the η-invariant of H is well defined and given by $\eta(H) = \eta(H,0)$. Here the analyticity in s and its possible poles will not be further analyzed. We rather proceed with the following definition.

Definition 7.2.1. Let H be a self-adjoint operator with compact resolvent and such that He^{-tH^2} is trace class for all $t > 0$. Then the η-invariant of H is defined by

$$\eta(H) = \frac{1}{\sqrt{\pi}} \int_0^\infty dt\, t^{-\frac{1}{2}}\, \mathrm{Tr}\big(He^{-tH^2}\big), \tag{7.4}$$

provided that the integral is finite, in which case we will say that the η-invariant exists.

Let us first note that $\eta(H) = \operatorname{Sig}(H)$ if H is a matrix, see (1.10). Furthermore, one clearly has $\eta(H) \in \mathbb{R}$ as $H = H^*$ so that $\operatorname{Tr}(He^{-tH^2}) \in \mathbb{R}$. In some situations such as Proposition 10.5.10 in Section 10.5, one can even show $\eta(H) \in \mathbb{Z}$. Also let us note that Lemma 7.2.3 below shows that the trace class property of e^{-tH^2} also implies that He^{-tH^2} is trace class. What is required in Definition 7.2.1 is, moreover, the integrability condition in (7.4). Proving the existence of the η-invariant is, in general, a delicate issue. It is known to exist for Dirac operators on compact closed manifolds [14, 28]. Later on in Section 10.5, it will be shown that the η-invariant of the spectral localizer associated to a low-dimensional index pairing exists. It is the aim of this section to prove a connection between the η-variant and the spectral flow of a suitable path, first stated in the work of Getzler [96] and further analyzed in [93]. While we essentially follow the intuitive line of proof in [96], some essential modifications are necessary. In particular, the DuHamel formula stated and used in [96] does not hold (as pointed out via a counterexample in [56]). Another idea to prove Theorem 7.2.2 below was put forward by Carey and Phillips [55, 56]. It is based on a variation of the formula given in Theorem 5.7.9 which is briefly discussed in Remark 5.7.10. This alternative approach, moreover, extends to the semifinite setting of Chapter 11, see [56]. Section 10.5 will present an application of the theorem below to a particular situation in which the corrective term given by the limit actually vanishes.

Theorem 7.2.2. *Let $t \in [0,1] \mapsto H_t = H_0 + V_t \in \mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ be such that the endpoints H_0 and H_1 are invertible with existing η-invariants and $t \mapsto V_t \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ is differentiable. Then*

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}(\eta(H_1) - \eta(H_0)) + \lim_{\epsilon\to 0} \frac{\epsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^1 dt \operatorname{Tr}(\partial_t H_t e^{-\epsilon H_t^2}).$$

As a first preparation for the proof, let us state a stability result for the trace $\operatorname{Tr}(He^{-tH^2})$ entering into (7.4). For later use, it will be stated a bit more generally than needed.

Lemma 7.2.3. *Let $H_0 \in \mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ such that $\operatorname{Tr}(e^{-tH_0^2}) < \infty$ for some $t > 0$. Further let $V \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ and set $H = H_0 + V$. Then one has*

$$\operatorname{Tr}(e^{-tH^2}) \le e^{2t\|V\|^2} \operatorname{Tr}(e^{-\frac{t}{2}H_0^2}),$$

and for all $\alpha > 0$,

$$\operatorname{Tr}(|H^\alpha e^{-tH^2}|) \le \left(\frac{\alpha}{et}\right)^{\frac{\alpha}{2}} e^{t\|V\|^2} \operatorname{Tr}(e^{-\frac{t}{4}H_0^2}).$$

Proof. First of all, for all $\delta \in (0,1)$,

$$(H_0 + V)^2 = (1 - \delta^2)H_0^2 + (\delta H_0 + \delta^{-1}V)^2 + (1 - \delta^{-2})V^2$$

$$\geq (1-\delta^2)H_0^2 - \delta^{-2}V^2$$
$$\geq (1-\delta^2)H_0^2 - \delta^{-2}\|V\|^2\mathbf{1}.$$

Now, as $\mathrm{Tr}(e^{-A}) < \mathrm{Tr}(e^{-B})$ for $0 \leq B \leq A$,

$$\mathrm{Tr}\big(e^{-t(H_0+V)^2}\big) \leq e^{t\delta^{-2}\|V\|^2}\,\mathrm{Tr}\big(e^{-t(1-\delta^2)H_0^2}\big).$$

Choosing $\delta^2 = \frac{1}{2}$ implies the first claim. For the second, simply bound

$$\mathrm{Tr}\big(\big|(H_0+V)^\alpha e^{-t(H_0+V)^2}\big|\big) \leq \big\||H_0+V|^\alpha e^{-\frac{t}{2}(H_0+V)^2}\big\|\,\mathrm{Tr}\big(e^{-\frac{t}{2}(H_0+V)^2}\big)$$
$$\leq \left(\frac{\alpha}{et}\right)^{\frac{\alpha}{2}}\mathrm{Tr}\big(e^{-\frac{t}{2}(H_0+V)^2}\big),$$

because $\lambda^\alpha e^{-\frac{t}{2}\lambda^2} \leq (\frac{\alpha}{et})^{\frac{\alpha}{2}}$ for $\lambda \geq 0$. Now the first bound implies the claim. □

Proof of Theorem 7.2.2. First of all, let us state a fact that will be used, but not proved in detail: by an arbitrarily small bounded perturbation, the path $t \in [0,1] \mapsto H_t$ can be moved into a generic position in which all eigenvalue crossings are simple and transversal, namely if $\mathrm{Ker}(H_t) \neq \{0\}$ then $\dim(\mathrm{Ker}(H_t)) = 1$ and the eigenvalue λ_t satisfies $\partial_t\lambda_t \neq 0$. This can be achieved by adapting the proof of Proposition 4.3.2. Clearly, moving the path into a generic path does not change the spectral flow and for the generic path one can then use the sum of eigenvalue crossings to compute the spectral flow, just as in Proposition 4.3.3.

Next let us show that also the right-hand side in the claim of Theorem 7.2.2 does not change when the path is moved into a generic position. For that purpose, it is sufficient to show that this is true for the integral on the right-hand side. Indeed, this integral can be understood as the integral over a 1-form α_ϵ on the linear space $\mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ defined by

$$\alpha_{\epsilon,H}(X) = \mathrm{Tr}\big(Xe^{-\epsilon H^2}\big),$$

namely one has

$$\int_0^1 dt\,\mathrm{Tr}\big(\partial_t H_t e^{-\epsilon H_t^2}\big) = \int_{[t\in[0,1]\mapsto H_t]} \alpha_\epsilon.$$

The independence of the integral under deformations follows by standard arguments (see the proof of Proposition 5.7.6) once it is shown that the 1-form α_ϵ is closed, namely for all $X, Y \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ (which is the tangent space to $\mathbb{B}_{\mathrm{sa}}(\mathcal{H})$) one has

$$\partial_s|_{s=0}\alpha_{\epsilon,H+sY}(X) = \partial_s|_{s=0}\alpha_{\epsilon,H+sX}(Y).$$

To check this, let us begin by computing

$$\begin{aligned}\partial_s|_{s=0}\alpha_{\epsilon,H+sY}(X) &= \partial_s|_{s=0}\operatorname{Tr}\big(Xe^{-\frac{\epsilon}{2}(H+sY)^2}e^{-\frac{\epsilon}{2}(H+sY)^2}\big)\\ &= \partial_s|_{s=0}\operatorname{Tr}\big(Xe^{-\frac{\epsilon}{2}(H+sY)^2}e^{-\frac{\epsilon}{2}H^2}\big)\\ &\quad+\partial_s|_{s=0}\operatorname{Tr}\big(Xe^{-\frac{\epsilon}{2}H^2}e^{-\frac{\epsilon}{2}(H+sY)^2}\big).\end{aligned}$$

Note that both summands contain a trace class factor $e^{-\frac{\epsilon}{2}H^2}$. Now the other factor will be rewritten as the absolutely convergent integral

$$e^{-\frac{\epsilon}{2}(H+sY)^2} = \frac{1}{\sqrt{\epsilon}}\int_{\mathbb{R}}\frac{d\lambda}{\sqrt{2\pi}}e^{-\frac{\lambda^2}{2\epsilon}}e^{\imath\lambda(H+sY)}.$$

Now the derivative can be computed using DuHamel's formula [163, p. 69]:

$$\begin{aligned}\partial_s|_{s=0}\alpha_{\epsilon,H+sY}(X) &= \frac{1}{\sqrt{\epsilon}}\int_{\mathbb{R}}\frac{d\lambda}{\sqrt{2\pi}}e^{-\frac{\lambda^2}{2\epsilon}}\int_0^1 dr\big[\operatorname{Tr}\big(Xe^{\imath(1-r)\lambda H}\imath\lambda Ye^{\imath r\lambda H}e^{-\frac{\epsilon}{2}H^2}\big)\\ &\quad+\operatorname{Tr}\big(Xe^{-\frac{\epsilon}{2}H^2}e^{\imath(1-r)\lambda H}\imath\lambda Ye^{\imath r\lambda H}\big)\big]\\ &= \frac{1}{\sqrt{\epsilon}}\int_{\mathbb{R}}\frac{d\lambda}{\sqrt{2\pi}}e^{-\frac{\lambda^2}{2\epsilon}}\int_0^1 dr\big[\operatorname{Tr}\big(Ye^{-\frac{\epsilon}{2}H^2}e^{\imath r\lambda H}\imath\lambda Xe^{\imath(1-r)\lambda H}\big)\\ &\quad+\operatorname{Tr}\big(Ye^{\imath r\lambda H}\imath\lambda Xe^{\imath(1-r)\lambda H}e^{-\frac{\epsilon}{2}H^2}\big)\big]\\ &= \partial_s|_{s=0}\alpha_{\epsilon,H+sX}(Y).\end{aligned}$$

In conclusion, for the remainder of the proof one can assume that the path is in a generic position.

To continue the argument, let us introduce the regularized η-invariant by

$$\eta_\epsilon(H) = \frac{1}{\sqrt{\pi}}\int_\epsilon^\infty ds\, s^{-\frac{1}{2}}\operatorname{Tr}\big(He^{-sH^2}\big), \tag{7.5}$$

where $\epsilon > 0$. If the η-invariant of H exists, then clearly $\lim_{\epsilon\to 0}\eta_\epsilon(H) = \eta(H)$. This is known for H_0 by hypothesis, and it will be shown next that $\eta_\epsilon(H)$ exists for all $H = H_0+V$ for $V \in \mathbb{B}_{sa}(\mathcal{H})$ so that it holds, in particular, along the path H_t. First of all, let us note that if H has a kernel with an associated orthogonal projection P, then $\operatorname{Tr}(PHe^{-\epsilon H^2}) = 0$. Now, as H has compact resolvent, one can assume that there is a constant $g > 0$ such that $H^2 \geq g^2$. Then $\eta_\epsilon(H_t)$ can be bounded as follows:

$$\begin{aligned}\sqrt{\pi}\eta_\epsilon(H) &\leq \int_\epsilon^\infty ds\, s^{-\frac{1}{2}}\operatorname{Tr}\big(He^{-sH^2}\big)\\ &\leq \int_\epsilon^\infty ds\, s^{-\frac{1}{2}}\operatorname{Tr}\big(|H|e^{(-\epsilon+\epsilon-s)H^2}\big)\end{aligned}$$

$$\leq \int_{\epsilon}^{\infty} ds\, s^{-\frac{1}{2}}\, \mathrm{Tr}\big(|H|e^{-\epsilon H^2}\big)\|e^{-(s-\epsilon)H^2}\|$$
$$\leq \epsilon^{-\frac{1}{2}}\, \mathrm{Tr}\big(|H|e^{-\epsilon H^2}\big) \int_{\epsilon}^{\infty} ds\, e^{-(s-\epsilon)g^2}$$
$$= \epsilon^{-\frac{1}{2}}\, \mathrm{Tr}\big(|H|e^{-\epsilon H^2}\big)\frac{1}{g^2} < \infty.$$

Next let us show that the map $t \mapsto \eta_\epsilon(H_t)$ is continuously differentiable in every point $t \in [0,1]$ where $\mathrm{Ker}(H_t)$ is trivial (in the other points, it will be shown below that it is not even continuous and has a jump). In these points, there exists a $g_t > 0$ such that $H_t^2 \geq g_t^2$ and, consequently, as above

$$\mathrm{Tr}(e^{-\frac{s}{2}H_t^2}) \leq \mathrm{Tr}(e^{-\frac{s}{4}H_t^2})e^{-\frac{s}{4}g_t^2},$$

which is integrable at $s = \infty$. Moreover, $\|H_t e^{-\frac{s}{2}H_t^2}\| \leq (es)^{-\frac{1}{2}}$ as in the proof of Lemma 7.2.3. Hence in the following computation based on the Leibniz rule, all terms are absolutely convergent:

$$\sqrt{\pi}\partial_t\eta_\epsilon(H_t) = \int_{\epsilon}^{\infty} ds\, s^{-\frac{1}{2}}\big[\mathrm{Tr}(\partial_t H_t e^{-sH_t^2}) + \mathrm{Tr}(e^{-\frac{s}{2}H_t^2}H_t(\partial_t e^{-\frac{s}{2}H_t^2})) + \mathrm{Tr}(H_t e^{-\frac{s}{2}H_t^2}(\partial_t e^{-\frac{s}{2}H_t^2}))\big].$$

For the evaluation of the latter two summands, let us appeal as above to the Fourier transform and DuHamel formula:

$$\partial_t e^{-\frac{s}{2}H_t^2} = \frac{1}{\sqrt{s}}\int_{\mathbb{R}} d\lambda\, e^{-\frac{1}{2}\frac{\lambda^2}{s}} \int_0^1 dr\, e^{\imath(1-r)\lambda H_t}\imath\lambda\partial_t H_t e^{\imath r\lambda H_t}.$$

Replacing and using the cyclicity of the trace, allowed due to the trace class factor $e^{-\frac{s}{2}H_t^2}$, one finds

$$\mathrm{Tr}(H_t e^{-\frac{s}{2}H_t^2}(\partial_t e^{-\frac{s}{2}H_t^2})) = \frac{1}{\sqrt{s}}\int_{\mathbb{R}} d\lambda\, e^{-\frac{1}{2}\frac{\lambda^2}{s}}\, \mathrm{Tr}(\imath\lambda H_t e^{\imath\lambda H_t} e^{-\frac{s}{2}H_t^2}\partial_t H_t)$$
$$= -\mathrm{Tr}(sH_t^2 e^{-\frac{s}{2}H_t^2} e^{-\frac{s}{2}H_t^2}\partial_t H_t).$$

The other summand has exactly the same value. Therefore, using integration by parts,

$$\sqrt{\pi}\partial_t\eta_\epsilon(H_t) = \int_{\epsilon}^{\infty} ds\, s^{-\frac{1}{2}}\, \mathrm{Tr}(\partial_t H_t e^{-sH_t^2}) - 2\int_{\epsilon}^{\infty} ds\, s^{\frac{1}{2}}\, \mathrm{Tr}(\partial_t H_t H_t^2 e^{-sH_t^2})$$
$$= \int_{\epsilon}^{\infty} ds\, s^{-\frac{1}{2}}\, \mathrm{Tr}(\partial_t H_t e^{-sH_t^2}) + 2\int_{\epsilon}^{\infty} ds\, s^{\frac{1}{2}}\partial_s\, \mathrm{Tr}(\partial_t H_t e^{-sH_t^2})$$

$$= -2\epsilon^{\frac{1}{2}} \operatorname{Tr}(\partial_t H_t e^{-\epsilon H_t^2}).$$

Let us also note that it is possible to show that $t \mapsto \partial_t \eta_\epsilon(H_t)$ is continuous at all points where $\operatorname{Ker}(H_t)$ is trivial. This readily follows by invoking once again Lemma 7.2.3, but we do not spell out the details.

Next let us focus on a point t where $\operatorname{Ker}(H_t)$ is nontrivial. Because the path is in a generic position, one then has $\dim(\operatorname{Ker}(H_t)) = 1$ and the crossing eigenvalue λ_t satisfies $\partial_t \lambda_t \neq 0$. Let $\delta \mapsto P_{t+\delta}$ denote the associated kernel projection which for δ sufficiently small is one-dimensional. Inserting $\mathbf{1} = P_t + (\mathbf{1} - P_t)$ in the trace in $\eta_\epsilon(H_t)$ and using that the contribution of $\mathbf{1} - P_t$ is continuous in t, one finds

$$\begin{aligned}
\eta_\epsilon(H_{t+0}) - \eta_\epsilon(H_{t-0}) &= \lim_{\delta \downarrow 0} \big(\eta_\epsilon(H_{t+\delta}) - \eta_\epsilon(H_{t-\delta})\big) \\
&= \lim_{\delta \downarrow 0} \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty ds\, s^{-\frac{1}{2}} \operatorname{Tr}\big(H_{t+\delta} e^{-sH_{t+\delta}^2} - H_{t-\delta} e^{-sH_{t-\delta}^2}\big) \\
&= \lim_{\delta \downarrow 0} \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty ds\, s^{-\frac{1}{2}} \operatorname{Tr}\big(P_{t+\delta} H_{t+\delta} e^{-sH_{t+\delta}^2} - P_{t-\delta} H_{t-\delta} e^{-sH_{t-\delta}^2}\big).
\end{aligned}$$

Next expanding the eigenvalue shows $P_{t+\delta} H_{t+\delta} = \lambda_{t+\delta} P_{t+\delta} = \partial_t \lambda_t P_{t+\delta} \delta + \mathcal{O}(\delta^2)$. Now let us set

$$\operatorname{sgn}_\epsilon(\lambda) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty ds\, s^{-\frac{1}{2}} \lambda e^{-s\lambda^2}, \quad \lambda \in \mathbb{R}.$$

Then the following integral identities (similar to (1.12) in the introductory chapter) can be used:

$$\lim_{\lambda \uparrow 0} \operatorname{sgn}_\epsilon(\lambda) = -1, \quad \lim_{\lambda \downarrow 0} \operatorname{sgn}_\epsilon(\lambda) = 1.$$

This implies

$$\begin{aligned}
&\eta_\epsilon(H_{t+0}) - \eta_\epsilon(H_{t-0}) \\
&\quad = \lim_{\delta \downarrow 0} \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty ds\, s^{-\frac{1}{2}} (2\partial_t \lambda_t \delta + \mathcal{O}(\delta^2)) e^{-s(\delta^2 (\partial_t \lambda_t)^2 + \mathcal{O}(\delta^3))} \\
&\quad = 2 \operatorname{sgn}(\partial_t \lambda_t).
\end{aligned}$$

Finally, one can compute the spectral flow of $t \in [0,1] \mapsto H_t$ as the sum of contributions over all $t \in [0,1]$ such that $\operatorname{Ker}(H_t) \neq \{0\}$:

$$\operatorname{Sf}\big(t \in [0,1] \mapsto H_t\big) = \sum_{\operatorname{Ker}(H_t) \neq \{0\}} \operatorname{sgn}(\partial_t \lambda_t)$$

$$= \frac{1}{2} \sum_{\mathrm{Ker}(H_t) \neq \{0\}} (\eta_\epsilon(H_{t+0}) - \eta_\epsilon(H_{t-0}))$$

$$= \frac{1}{2}(\eta_\epsilon(H_1) - \eta_\epsilon(H_0)) - \frac{1}{2} \int_0^1 dt\, \partial_t \eta_\epsilon(H_t)$$

$$= \frac{1}{2}(\eta_\epsilon(H_1) - \eta_\epsilon(H_0)) + \frac{\epsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^1 dt\, \mathrm{Tr}(\partial_t H_t e^{-\epsilon H_t^2}),$$

due to the fundamental theorem and the above formula for $\partial_t \eta_\epsilon(H_t)$. This holds for all $\epsilon > 0$, but as the limit $\epsilon \to 0$ of the first two terms exists, also the limit of the last one exists. This proves the claimed formula for the spectral flow. □

7.3 Conley–Zehnder index as spectral flow

As an application of the last sections, it is here shown that the Conley–Zehnder index associated to a path of monodromy matrices of a one-parameter family of periodic linear Hamiltonian systems is equal to the spectral flow of the Fredholm operator associated to the Hamiltonian systems. Such a connection goes back to the work of Robbin and Salamon, see Theorem 7.42 in [160]. The setup is, moreover, the same as in Section 12.3 where the bifurcations of the family are analyzed.

The family of periodic linear Hamiltonian systems are of the form

$$\begin{cases} I\partial_s u(s) + A_t(s)u(s) = 0, & s \in \mathbb{S}^1 \cong [0, 2\pi], \\ u(0) = u(2\pi), \end{cases} \tag{7.6}$$

where $(t, s) \in [0, 1] \times \mathbb{R} \mapsto A_t(s)$ is a continuous family of self-adjoint $2N \times 2N$ matrices that is 2π-periodic in s, and I is the standard symplectic form given in (2.1). The real variable t is an external parameter. Note that if A_t is real, then also u can be chosen real, and actually all objects in the following are real. Let us also note that after the basis change given by the Cayley transform (2.2), one can also use the standard indefinite Krein form J multiplied by $\imath$. In this representation the reality condition takes a different form. Actually, the algebraic manipulations in this section become a bit more transparent in the standard representation because then no Cayley transform is needed. We rather stick with the standard from (7.6), also to illustrate the implementation of the basis change.

The crucial remark is that the solutions of (7.6) for each fixed value of t are the kernel of the self-adjoint Fredholm operator

$$H_t : W^{1,2}(\mathbb{S}^1, \mathbb{C}^{2N}) \subset L^2(\mathbb{S}^1, \mathbb{C}^{2N}) \to L^2(\mathbb{S}^1, \mathbb{C}^{2N})$$

given by

$$H_t u = I\partial_s u + A_t u, \quad u \in W^{1,2}(\mathbb{S}^1, \mathbb{C}^{2N}). \tag{7.7}$$

These operators have a constant domain and are self-adjoint Fredholm operators which depend continuously on t when considered as bounded operators between $W^{1,2}(\mathbb{S}^1, \mathbb{C}^{2N})$ and $L^2(\mathbb{S}^1, \mathbb{C}^{2N})$ (see [160]). Its use in context with (7.6) is based on

$$\dim(\operatorname{Ker}(H_t)) = \#\{\text{linear independent solutions of (7.6) for fixed } t\}. \tag{7.8}$$

Next let $\Psi_t(s)$ be the fundamental solution of (7.6), namely the unique time-dependent $2N \times 2N$ matrix solution of the initial value problem

$$I\partial_s \Psi_t(s) + A_t(s)\Psi_t(s) = 0, \quad \Psi_t(0) = \mathbf{1}. \tag{7.9}$$

The solution $\Psi_t(s)$ is I-unitary (and actually symplectic if A_t is real) for all (t, s) as $\partial_s(\Psi_t^* I \Psi_t) = 0$ and $\Psi_t^*(0) I \Psi_t(0) = I$. In particular, the same holds for the monodromy matrix

$$M_t = \Psi_t(2\pi).$$

This monodromy matrix provides another way to approach the solutions of (7.6), namely one has

$$\dim(\operatorname{Ker}(M_t - \mathbf{1})) = \#\{\text{linear independent solutions of (7.6) for fixed } t\}. \tag{7.10}$$

Now the kernel $\operatorname{Ker}(M_t \mp \mathbf{1})$ is precisely the object that can be accessed via intersection theory of Lagrangian planes in the Krein space $(\mathbb{C}^{4N}, (-I) \oplus I)$. As in (2.23), one has

$$\dim(\operatorname{Ker}(M_t \mp \mathbf{1})) = \dim\left(\operatorname{Ran}\left(\begin{pmatrix} \mathbf{1} \\ M_t \end{pmatrix}\right) \cap \mathcal{F}_\pm\right),$$

where $\mathcal{F}_\pm = \operatorname{Ran}(\binom{\mathbf{1}}{\pm\mathbf{1}})$ is the $2N$-dimensional reference plane in $\mathbb{C}^{4N}$. This is explained in detail in Section 2.3 for the Krein space $(\mathbb{C}^{4N}, (-J) \oplus J)$, but, as already stressed, the whole Section 2.3 directly transposes to $(\mathbb{C}^{4N}, (-I) \oplus I)$ after the basis change (2.2) is carried out. In particular, the Conley–Zehnder index of the path $t \in [0, 1] \mapsto M_t$ of I-unitaries is defined by

$$\operatorname{CZ}(t \in [0, 1] \mapsto M_t) = \operatorname{CZ}(t \in [0, 1] \mapsto \mathcal{C} M_t \mathcal{C}^*),$$

where on the right-hand side there is a path of J-unitaries. Next recall that the Conley–Zehnder index is related to the eigenvalue passages through -1. As here the focus is rather on the periodic solution and therefore the eigenvalue passages through 1, one rather looks at the Conley–Zehnder index of the path $t \in [0, 1] \mapsto -M_t$ which is

$$\operatorname{CZ}(t \in [0, 1] \mapsto -M_t) = \operatorname{BM}_{\mathcal{F}_+}(t \in [0, 1] \mapsto \operatorname{Ran}(\mathbf{1} \oplus M_t \mathcal{F}_+)).$$

Based on (7.8) and (7.10), it is now reasonable to expect that there is a tight-connection between the spectral flow of $t \mapsto H_t$ and the Conley–Zehnder index of $t \mapsto -M_t$. Indeed, the dimensions at the intersection points are the same. The following result states that also their orientations are the same.

Proposition 7.3.1. *If* (7.6) *only has the trivial solution for* $t = 0, 1$, *then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = -\mathrm{CZ}\big(t \in [0,1] \mapsto -M_t\big). \tag{7.11}$$

Proof. Let us first note without a detailed proof that Propositions 4.3.5 and 4.3.6 can be proved verbatim for the unbounded self-adjoint Fredholm operators H_t with constant domain (see [200]). Thus, after perturbing by $\delta\mathbf{1}$ for a sufficiently small δ, it can be assumed that all crossings of $t \mapsto H_t$ are regular. Moreover, this perturbation does not affect the Conley–Zehnder index because it shifts $A_t \mapsto A_t + \delta\mathbf{1}$ and M_t depends continuously on this shift so that, due to the fact that there are no nontrivial solutions at the boundary points, the Conley–Zehnder index is not changed. Hence it is sufficient to show (7.11) under the additional assumption that $t \mapsto H_t$ only has regular crossings.

Let now t_0 be a regular crossing of $t \mapsto H_t$. The crossing form at t_0 is

$$\begin{aligned}\Gamma_{t_0}(\phi) &= \langle \phi | (\partial_t H_t)_{t_0} \phi \rangle \\ &= \int_0^{2\pi} \langle \phi(s) | \big(\partial_t A_t(s)\big)_{t_0} \phi(s) \rangle ds \\ &= \int_0^{2\pi} \langle u | \Psi_{t_0}^*(s) \big(\partial_t A_t(s)\big)_{t_0} \Psi_{t_0}(s) u \rangle ds,\end{aligned} \tag{7.12}$$

where $\phi \in \mathrm{Ker}(H_{t_0}) \subset L^2(\mathbb{S}^1, \mathbb{C}^{2N})$ and $u = \phi(0) \in \mathbb{C}^{2N}$. In particular, $M_{t_0} u = u$. Now rewriting (7.9) leads to that

$$\partial_t A_t(s) \Psi_t(s) = -I \partial_t \partial_s \Psi_t(s) - A_t(s) \partial_t \Psi_t(s) \tag{7.13}$$

and

$$\Psi_t^*(s) A_t(s) = \partial_s\big(\Psi_t^*(s) I\big). \tag{7.14}$$

Plugging (7.13) and (7.14) into (7.12) yields, using $\Psi_t(0) = \mathbf{1}$,

$$\begin{aligned}\Gamma_{t_0}(\phi) &= -\int_0^{2\pi} \langle u | \big(\Psi_{t_0}^*(s) I \big(\partial_t \partial_s \Psi_t(s)\big)_{t_0} + \Psi_{t_0}^*(s) A_{t_0}(s) \big(\partial_t \Psi_t(s)\big)_{t_0}\big) u \rangle ds \\ &= -\int_0^{2\pi} \langle u | \big(\Psi_{t_0}^*(s) I \partial_s \big(\partial_t \Psi_t(s)\big)_{t_0} + \partial_s\big(\Psi_{t_0}^*(s) I\big) \big(\partial_t \Psi_t(s)\big)_{t_0}\big) u \rangle ds\end{aligned}$$

$$= -\int_0^{2\pi} \langle u | \partial_s (\Psi_{t_0}^*(s) I(\partial_t \Psi_t(s))_{t_0}) u \rangle ds$$
$$= -\langle u | \Psi_{t_0}^*(2\pi) I(\partial_t \Psi_t(2\pi))_{t_0} u \rangle$$
$$= -\langle u | M_{t_0}^* I(\partial_t M_t)_{t_0} u \rangle,$$

where still $\phi \in \mathrm{Ker}(H_{t_0}) \subset L^2(\mathbb{S}^1, \mathbb{C}^{2N})$ and $u = \phi(0) \in \mathbb{C}^{2N}$ with $M_{t_0} u = u$.

On the other hand,

$$\mathrm{CZ}(t \in [0,1] \mapsto -M_t) = \mathrm{Sf}(t \in [0,1] \mapsto S(-\mathcal{C} M_t \mathcal{C}^*)).$$

Now setting $v = \mathcal{C}u$ and using Theorem 2.3.3,

$$\mathcal{C} M_{t_0} \mathcal{C}^* v = v \quad \Longleftrightarrow \quad S(\mathcal{C} M_{t_0} \mathcal{C}^*) v = v \quad \Longleftrightarrow \quad S(-\mathcal{C} M_{t_0} \mathcal{C}^*) J v = -J v.$$

Then by Lemma 2.3.9,

$$\begin{aligned}\langle Jv | S(-\mathcal{C} M_{t_0} \mathcal{C}^*)^* \partial_t S(-\mathcal{C} M_t \mathcal{C}^*)_{t_0} | Jv \rangle &= \langle v | (\mathcal{C} M_{t_0} \mathcal{C}^*)^* J \partial_t (\mathcal{C} M_t \mathcal{C}^*)_{t_0} | v \rangle \\ &= \langle u | M_{t_0}^* (\mathcal{C}^* J \mathcal{C}) (\partial_t M_t)_{t_0} | u \rangle \\ &= \imath \langle u | M_{t_0}^* I (\partial_t M_t)_{t_0} | u \rangle.\end{aligned}$$

Therefore the crossing from for the spectral flow of unitaries, see Definition 4.5.7, is given by

$$-\imath \langle Jv | S(-\mathcal{C} M_{t_0} \mathcal{C}^*)^* \partial_t S(-\mathcal{C} M_t \mathcal{C}^*)_{t_0} | Jv \rangle = -\Gamma_{t_0}(\phi).$$

This implies the claim. □

Remark 7.3.2. The invertibility of the endpoints of $t \mapsto H_t$ in Proposition 7.3.1 is actually not necessary. It is possible to work out the necessary amendments, taking into account the boundary terms in Propositions 4.3.6 and 1.5.11. ◇

Now let us spell out the implications of a nontrivial spectral flow of $t \mapsto H_t$, or, due to Proposition 7.3.1, equivalently a nontrivial Conley–Zehnder index of the monodromy matrices $t \mapsto M_t$. The setup described above directly implies the following statement:

Proposition 7.3.3. *Let* (7.6) *be such that it only has the trivial solution for* $t = 0, 1$. *For each* $t \in [0,1]$, *let* $m_t = \dim(\mathrm{Ker}(M_t - \mathbf{1}))$ *denote the dimension of the solution space of* (7.6). *Then*

$$\sum_{t \in [0,1]} m_t \geq \left| \mathrm{CZ}(t \in [0,1] \mapsto -M_t) \right|. \tag{7.15}$$

The estimate (7.15) gives a lower bound on the number of linearly independent solutions of (7.6) in terms of a topological quantity. In the present situation, the path

$t \mapsto A_t$ is fairly arbitrary and, consequently, there is also no monotonicity, say of the unitaries $t \mapsto S(-\mathcal{C}M_t\mathcal{C}^*)$. This lack of monotonicity implies that there is only an inequality in (7.15). For special paths, a monotonicity may hold, and then one can boost (7.15) to an equality. An example for this is oscillation theory in the energy variable. More precisely, if A_t stems from a regular matrix-valued Sturm–Liouville operator on the interval $\mathbb{S}^1 \cong [0, 1)$ and t is the spectral parameter of the associated self-adjoint operator, then one can prove a monotonicity statement exactly as in Section 2.5, see [35, 173], and then conclude that an analogous result to Theorem 2.5.1 holds. For this reason, the claim of Proposition 7.3.3 is also referred to as a result of relative oscillation theory.

7.4 Spectral flow as index via semiclassics

This section presents a generalization of Section 3.5 to a setting with self-adjoint Fredholm operators acting on infinite dimensional fibers, namely the spectral flow of a path of self-adjoint Fredholm operators is shown to be equal to the index of a Fredholm operator. In contrast to Section 3.5, this connection does not hold for all paths of Fredholm operators, but only for some types of paths. Therefore let us consider a path of self-adjoint Fredholm operators of the form

$$t \in \mathbb{R} \mapsto H_t = H + K_t \in \mathbb{F}_{\text{sa}}(\mathcal{H}), \tag{7.16}$$

where $H : \mathcal{W} \to \mathcal{H}$ is a possibly unbounded self-adjoint Fredholm operator with domain $\mathcal{W}$ and $t \in \mathbb{R} \mapsto K_t \in \mathbb{B}_{\text{sa}}(\mathcal{H})$ is a continuous path of bounded self-adjoints. Furthermore, one of the following two assumptions is supposed to hold:

Case A: $\mathcal{W}$ equipped with the H-norm is compactly embedded in $\mathcal{H}$ and $K_\pm = \lim_{t\to\pm\infty} K_t$ exist and are such that $H_\pm = H + K_\pm$ are invertible.

Case B: H is bounded, K_t is compact and $K_\pm = \lim_{t\to\pm\infty} K_t$ exist and are such that $H_\pm = H + K_\pm$ are invertible.

Note that in both cases the spectral flow of the path $t \in \mathbb{R} \mapsto H_t$ is well defined as there is a compact interval $I \subset \mathbb{R}$ such that H_t is invertible for all $t \in \mathbb{R} \setminus I$. Then one defines the spectral flow as

$$\text{Sf}(t \in \mathbb{R} \mapsto H_t) = \text{Sf}(t \in I \mapsto H_t).$$

Now the following result is similar to Theorem 3.5.1.

Theorem 7.4.1. *Let $t \in \mathbb{R} \mapsto H_t$ be a path as above which is either in Case A or in Case B. Similar as in* (3.3), *we define the operator $D_H : W^{1,2}(\mathbb{R}, \mathcal{W}) \to L^2(\mathbb{R}, \mathcal{H})$ by*

$$D_H = \partial_t - H_t.$$

Then D_H is a Fredholm operator with index given by

$$\text{Ind}(D_H) = -\,\text{Sf}(t \in \mathbb{R} \mapsto H_t). \tag{7.17}$$

Case A of this theorem was proved by Robbin and Salamon [160], while Case B is the one-dimensional case of Callias index theorem [45], see also Abbondandolo and Majer [1], Pushnitski [155], as well as [94, 95]. Before going into the proof, let us stress that $\mathrm{Ind}(D_H) = -\mathrm{Sf}(t \in \mathbb{R} \mapsto H_t)$ does not hold for all continuous paths $t \in \mathbb{R} \mapsto H_t$ of self-adjoint Fredholm operators. The following example is essentially taken from [1].

Example 7.4.2. Let $P_0, P_1 \in \mathbb{B}(\mathcal{H})$ be projections with infinite-dimensional kernels and ranges such that (P_0, P_1) is a Fredholm pair of index $k \in \mathbb{Z} \setminus \{0\}$. Let $U \in \mathbb{U}(\mathcal{H})$ be a unitary such that $U^* P_0 U = P_1$. Choose a smooth path $t \in \mathbb{R} \mapsto U_t$ such that $U_t = \mathbf{1}$ for $t \leq 0$ and $U_t = U$ for $t \geq 1$. For every $\epsilon > 0$, consider the smooth path

$$t \in \mathbb{R} \mapsto H_{\epsilon,t} = U^*_{\frac{t}{\epsilon}} (\mathbf{1} - 2P_0) U_{\frac{t}{\epsilon}}$$

of self-adjoint and invertible operators. Clearly, $\mathrm{Sf}(t \in \mathbb{R} \mapsto H_{\epsilon,t}) = 0$ for all $\epsilon > 0$. When ϵ converges to zero, $H_{\epsilon,t}$ converges in $L^1(\mathbb{R}, \mathbb{B}(\mathcal{H}))$ to the piecewise-continuous path

$$t \in \mathbb{R} \mapsto H_{0,t} = (\mathbf{1} - 2P_0)\chi_{(-\infty,0]} + (\mathbf{1} - 2P_1)\chi_{(0,\infty)}.$$

The stable and unstable directions of H_0 (notably $\mathcal{E}^s_{H_0}$ and $\mathcal{E}^u_{H_0}$, as defined and denoted in [1]) are given by $\mathrm{Ran}(P_1)$ and $\mathrm{Ran}(\mathbf{1} - P_0)$, respectively. Therefore by Theorem 5.1 in [1], D_{H_0} is a Fredholm operator of index k. As $\lim_{\epsilon \to 0} \|D_{H_\epsilon} - D_{H_0}\| = 0$, this implies that D_{H_ϵ} is a Fredholm operator and $\mathrm{Ind}(D_{H_\epsilon}) = k \neq \mathrm{Sf}(t \in \mathbb{R} \mapsto H_{t,\epsilon})$ for ϵ sufficiently small. ◇

Here we provide a new proof based on a semiclassical argument similar to Witten's proof of the Morse inequalities (e. g., [67]). The argument will heavily use the operators $D_{H,\kappa} : W^{1,2}(\mathbb{R}, \mathcal{W}) \to L^2(\mathbb{R}, \mathcal{H})$ defined by

$$D_{H,\kappa} = \kappa \partial_t - H_t$$

for $\kappa > 0$ and $L_\kappa : W^{1,2}(\mathbb{R}, \mathcal{W}) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}, \mathcal{H}) \otimes \mathbb{C}^2$ given by

$$L_\kappa = \begin{pmatrix} 0 & D^*_{H,\kappa} \\ D_{H,\kappa} & 0 \end{pmatrix}.$$

The latter satisfies the so-called supersymmetry relation

$$J L_\kappa J = -L_\kappa, \quad J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

It will be shown that L_κ is a self-adjoint Fredholm operator. Hence its index vanishes, but actually the kernel allows computing the index of $D_{H,\kappa}$ via

$$\mathrm{Sig}(J|_{\mathrm{Ker}(L_\kappa)}) = \mathrm{Ind}(D_{H,\kappa}). \tag{7.18}$$

Note that the left-hand side makes sense because $\mathrm{Ker}(L_\kappa)$ is a J-invariant subspace. To compute the kernel, one may assume that the path $t \in \mathbb{R} \mapsto K_t$ is differentiable, see Lemma 7.4.5 below. Denote the derivatives by $H'_t = K'_t$. Next let us use $\mathrm{Ker}(L_\kappa) = \mathrm{Ker}(L_\kappa^2)$ and compute

$$(L_\kappa)^2 = -\kappa^2\partial_t^2 + \begin{pmatrix} (H_t)^2 + \kappa H'_t & 0 \\ 0 & (H_t)^2 - \kappa H'_t \end{pmatrix}. \tag{7.19}$$

Now let us note that $(L_\kappa)^2$ is an operator-valued one-dimensional Schrödinger operator with semiclassical constant κ. It has a compact resolvent. It will be shown that all eigenvalues close to 0 can, at least for κ sufficiently small, be computed as a sum of contributions that are *localized* in the variable t. By Krein type stability arguments, one can then easily access the desired signature in (7.18).

Lemma 7.4.3. *The operator*

$$L_\kappa : W^{1,2}(\mathbb{R}, \mathcal{W}) \otimes \mathbb{C}^2 \subset L^2(\mathbb{R}, \mathcal{W}) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}, \mathcal{W}) \otimes \mathbb{C}^2$$

is self-adjoint for all $\kappa > 0$.

Proof. Let us first decompose L_κ as follows:

$$L_\kappa = \kappa \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} - \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} - \begin{pmatrix} 0 & K_t \\ K_t & 0 \end{pmatrix}. \tag{7.20}$$

The first two summands are commuting self-adjoint operators acting on the space $W^{1,2}(\mathbb{R}) \otimes \mathcal{H} \otimes \mathbb{C}^2$ and $L^2(\mathbb{R}) \otimes \mathcal{W} \otimes \mathbb{C}^2$, respectively. Their sum is then self-adjoint on the intersection $W^{1,2}(\mathbb{R}, \mathcal{W})^2 \cong W^{1,2}(\mathbb{R}) \otimes \mathcal{W} \otimes \mathbb{C}^2$. The last summand is by hypothesis a bounded self-adjoint and therefore the Kato–Rellich theorem (e. g., [112, Theorem V.4.4]) implies that L_κ is self-adjoint on the same domain. □

Lemma 7.4.4. *The operator L_κ, as well as the operator $D_{H,\kappa}$, is Fredholm for all $\kappa > 0$.*

Proof. Let us note that $\sigma_1 L_\kappa = D_{H,\kappa} \oplus D_{H,\kappa}^*$ where σ_1 denotes the first Pauli matrix. Therefore L_κ is Fredholm if and only if $D_{H,\kappa}$ is Fredholm by Corollary 3.3.2.

Let us first deal with Case A by adapting an argument from [193]. Set

$$L_\kappa^\pm = \begin{pmatrix} 0 & -\kappa\partial_t - H_\pm \\ \kappa\partial_t - H_\pm & 0 \end{pmatrix}.$$

Because $(L_\kappa^\pm)^2 = -\kappa^2\partial_t^2 + H_\pm^2 > 0$ by the invertibility of $H_\pm$, it follows that the operator $L_\kappa^\pm : W^{1,2}(\mathbb{R}, \mathcal{W}) \to L^2(\mathbb{R}, \mathcal{H})$ is invertible. Thus there is a constant $c > 0$ such that $L_\kappa^\pm + A_t$ is invertible for all paths $t \in \mathbb{R} \mapsto A_t$ of bounded operators $A_t \in \mathbb{B}(\mathcal{H})$ such that $\sup_{t\in\mathbb{R}} \|A_t\| < c$. Let $t_\pm$ be such that $\|K_t - K_-\| < c$ on $(-\infty, t_-]$ and $\|K_t - K_+\| < c$ on $[t_+, \infty)$. Further let $\chi_\pm : \mathbb{R} \to [0, 1]$ be two smooth functions supported on these sets and equal to

1 on $(-\infty, t_- - 1]$ and $[t_+ + 1, \infty)$, respectively. Introduce $\chi : \mathbb{R} \to [0,1]$ by $\chi^2 = 1 - \chi_-^2 - \chi_+^2$. Then χ is supported on $I = [t_- - 1, t_+ + 1]$. As L_κ is self-adjoint by Lemma 7.4.3, $L_\kappa + \imath$ is invertible. Moreover, by the above, L_κ restricted to the support of χ_+ and χ_- is invertible. Hence one can set

$$Q = \chi_-(L_\kappa)^{-1}\chi_- + \chi(L_\kappa + \imath)^{-1}\chi + \chi_+(L_\kappa)^{-1}\chi_+.$$

The Fredholm property of L_κ will follow once it is shown that Q is a pseudoinverse to L_κ, namely an inverse up to a compact operator. One has

$$\begin{aligned} L_\kappa Q - \mathbf{1} &= L_\kappa\chi(L_\kappa + \imath)^{-1}\chi + \sum_{\sigma=\pm} L_\kappa\chi_\sigma(L_\kappa)^{-1}\chi_\sigma - \mathbf{1} \\ &= \chi L_\kappa(L_\kappa + \imath)^{-1}\chi + [L_\kappa, \chi](L_\kappa + \imath)^{-1}\chi + \sum_{\sigma=\pm} [L_\kappa, \chi_\sigma](L_\kappa)^{-1}\chi_\sigma - \chi^2 \\ &= -\imath\chi(L_\kappa + \imath)^{-1}\chi + [L_\kappa, \chi](L_\kappa + \imath)^{-1}\chi + \sum_{\sigma=\pm} [L_\kappa, \chi_\sigma](L_\kappa)^{-1}\chi_\sigma. \end{aligned}$$

It remains to show that each summand is compact. Let us recall from Lemma 7.4.3 that $L_\kappa : W^{1,2}(\mathbb{R}, \mathcal{W}) \to L^2(\mathbb{R}, \mathcal{H})$ is self-adjoint, hence $(L_\kappa + \imath)^{-1} : L^2(\mathbb{R}, \mathcal{H}) \to W^{1,2}(\mathbb{R}, \mathcal{W})$ is bounded. Because χ is compactly supported on $I = [t_- - 1, t_+ + 1]$, one concludes that also $\chi(L_\kappa + \imath)^{-1} : L^2(\mathbb{R}, \mathcal{H}) \to W^{1,2}(I, \mathcal{W})$ is bounded. Moreover, note that the inclusion $W^{1,2}(I, \mathcal{W}) \hookrightarrow L^2(\mathbb{R}, \mathcal{H})$ is compact by the Rellich embedding theorem combined with the hypothesis that $\mathcal{W} \hookrightarrow \mathcal{H}$ is compact. Therefore $\chi(L_\kappa + \imath)^{-1}$ is a compact operator on $L^2(\mathbb{R}, \mathcal{H})$. For the other summands, one can argue in the same manner, using that $(L_\kappa)^{-1}\chi_\sigma : L^2(\mathbb{R}, \mathcal{H}) \to W^{1,2}(\mathbb{R}, \mathcal{W})$ is bounded and $[L_\kappa, \chi_\sigma]$ is compactly supported. As L_κ is self-adjoint, $QL_\kappa - \mathbf{1} = (L_\kappa Q - \mathbf{1})^*$ is also compact and therefore Q is a pseudoinverse to L_κ.

Let us next come to Case B (this is covered by the results of [1], but we provide a different more direct proof). We show that $D_{H,\kappa}$ is Fredholm, which is equivalent to the Fredholm property of L_κ. The proof is similar to the argument leading to Theorem 3.5.1 dealing with the finite-dimensional situation. We first show that for all $\epsilon > 0$ there is a finite-dimensional projection $P_\epsilon \in \mathbb{B}(\mathcal{H})$ such that

$$\|P_\epsilon H(\mathbf{1} - P_\epsilon)\| < \epsilon \quad \text{and} \quad \|K_t(\mathbf{1} - P_\epsilon)\| < \epsilon \quad \text{for all } t \in \mathbb{R}.$$

Because $K_t \in \mathbb{B}(\mathcal{H})$ is compact for all $t \in \mathbb{R}$, there is a projection $P_t \in \mathbb{B}(\mathcal{H})$ such that $\|K_t(\mathbf{1} - P_t)\| < \frac{\epsilon}{3}$. Moreover, there are projections $P_\pm$ such that $\|K_\pm(\mathbf{1} - P_\pm)\| < \frac{\epsilon}{3}$. There is $t_* > 0$ such that $\|K_- - K_t\| < \frac{\epsilon}{3}$ for $t \in (-\infty, -t_*)$ and $\|K_+ - K_t\| < \frac{\epsilon}{3}$ for $t \in (t_*, \infty)$. As $t \in [-t_*, t_*] \mapsto K_t$ is uniformly continuous, there is $\delta > 0$ such that $\|K_{t'} - K_{t''}\| < \frac{\epsilon}{3}$ for all $t', t'' \in [-t_*, t_*]$ such that $|t' - t''| < \delta$. Then we choose a finite set of points $t_1 = -t_* < t_2 < \cdots < t_M = t_*$ such that $|t_{m-1} - t_m| < \delta$ for all $m = 1, \dots, M$. Then for $t < -t_*$, one has

$$\|K_t(\mathbf{1} - P_-)\| = \|K_t P_- - K_t\|$$

$$= \|K_tP_- - K_-P_- + K_-P_- - K_- + K_- - K_t\|$$
$$\leq \|K_tP_- - K_-P_-\| + \|K_-P_- - K_-\| + \|K_- - K_t\| < \epsilon$$

and analogously for $t > t_*$. For $t \in [-t_*, t_*]$, there is t_m such that $|t - t_m| < \delta$. Then

$$\begin{aligned}\|K_t(\mathbf{1} - P_{t_m})\| &= \|K_tP_{t_m} - K_t\| \\ &= \|K_tP_{t_m} - K_{t_m}P_{t_m} + K_{t_m}P_{t_m} - K_{t_m} + K_{t_m} - K_t\| \\ &\leq \|K_tP_{t_m} - K_{t_m}P_{t_m}\| + \|K_{t_m}P_{t_m} - K_{t_m}\| + \|K_{t_m} - K_t\| < \epsilon.\end{aligned}$$

Next choose a finite-dimensional projection $\widetilde{P}$ such that $\widetilde{P} \geq P_{t_m}$ for all m. By the above, $\|K_t(\mathbf{1}-\widetilde{P})\| < \epsilon$ for all $t \in \mathbb{R}$. By the spectral theorem and because H is self-adjoint, there is $\widetilde{H} \in \mathbb{B}(\mathcal{H})$ such that $\|H - \widetilde{H}\| < \epsilon$ and such that $\widetilde{H}$ is of the form $\widetilde{H} = \sum_{n=1}^{N} a_nP_n$ for $a_n \in \mathbb{R}$ and projections P_n such that $\sum_{n=1}^{N} P_n = \mathbf{1}$. Let $\widetilde{P}_n$ be the projection onto the range of $P_n\widetilde{P}$. Then $\widetilde{P}_n$ is finite dimensional and $\widetilde{P}_n \leq P_n$ holds for all n. Therefore $P_\epsilon = \sum_{n=1}^{N} \widetilde{P_n}$ is a finite-dimensional projection commuting with $\widetilde{H}$ and fulfilling $P_\epsilon \geq \widetilde{P}$. By construction, $\|K_t(\mathbf{1} - P_\epsilon)\| < \epsilon$ for all $t \in \mathbb{R}$ and

$$\begin{aligned}\|P_\epsilon H(\mathbf{1} - P_\epsilon)\| &= \|P_\epsilon(H - \widetilde{H} + \widetilde{H})(\mathbf{1} - P_\epsilon)\| \\ &\leq \|P_\epsilon(H - \widetilde{H})(\mathbf{1} - P_\epsilon)\| + \|P_\epsilon\widetilde{H}(\mathbf{1} - P_\epsilon)\| \\ &= \|P_\epsilon(H - \widetilde{H})(\mathbf{1} - P_\epsilon)\| < \epsilon.\end{aligned}$$

Next let us show that there are constants $a, c > 0$ and $\epsilon > 0$ such that

$$\|\varphi\|_{W^{1,2}} \leq c\big(\|P_{[-a,a]}\varphi\|_{L^2([-a,a])} + \|D_{H,\kappa}\varphi\|_{L^2}\big) \tag{7.21}$$

for all $\varphi \in W^{1,2}(\mathbb{R}, \mathcal{H})$, where $P_{[-a,a]} = \chi_{[-a,a]} \otimes P_\epsilon$. This then allows finishing the proof as follows. As the restriction $\varphi \mapsto P_{[-a,a]}\varphi$ is known to be a compact operator from $W^{1,2}(\mathbb{R}, \mathcal{H})$ into $L^2([-a, a], \mathcal{H})$ by the Rellich embedding theorem, D_H has a closed range and a finite-dimensional kernel by Proposition 3.2.6. As the same is true for $D_H^* = -D_{-H}$, the cokernel of D_H is finite dimensional and D_H is Fredholm.

For the proof of (7.21), let us first note that

$$\begin{aligned}\|\varphi\|_{W^{1,2}} &= \|\varphi\|_{L^2} + \|\varphi'\|_{L^2} \\ &= \|\varphi\|_{L^2} + \frac{1}{\kappa}\|(D_{H,\kappa} + H)\varphi\|_{L^2} \\ &\leq c_1\big(\|\varphi\|_{L^2} + \|D_{H,\kappa}\varphi\|_{L^2}\big),\end{aligned} \tag{7.22}$$

for some constant $c_1 > 0$. Second, assume that $H_t = \hat{H}$ is constant, where $\hat{H} \in \mathbb{B}(\mathcal{H})$ is self-adjoint and invertible. As $D_{\hat{H},\kappa}^* D_{\hat{H},\kappa} = D_{\hat{H},\kappa} D_{\hat{H},\kappa}^* = -\kappa^2\partial_t^2 + \hat{H}^2 > \tilde{c} > 0$ and therefore $D_{\hat{H},\kappa}$ is invertible, there is a constant $c_2(\hat{H})$ such that

$$\|\varphi\|_{W^{1,2}} \leq c_2(\hat{H})\|D_{\hat{H},\kappa}\varphi\|_{L^2}. \tag{7.23}$$

Let us now decompose $\varphi \in W^{1,2}(\mathbb{R}, \mathcal{H})$ into $\varphi_1 = (\mathbf{1} \otimes P_\epsilon)\varphi$ and $\varphi_2 = (\mathbf{1} - (\mathbf{1} \otimes P_\epsilon))\varphi$. For the considered (in general nonconstant) path $t \in \mathbb{R} \mapsto H_t$, there is a constant $\tilde{a}$ such that

$$\|H_t - H_\pm\| \le \frac{1}{2c_2} \quad \text{for } \pm t \ge \tilde{a},$$

where $c_2 = \max\{c_2(H_+), c_2(H_-)\}$. For φ_1 such that $\varphi_1(t) = 0$ for $t \in [-\tilde{a}, \tilde{a}]$ let us set $\varphi_{1,+}(t) = \varphi_1(t)\chi(t > 0)$ and $\varphi_{1,-}(t) = \varphi_1(t)\chi(t < 0)$. Then by (7.23),

$$\begin{aligned}
\|\varphi_{1,+}\|_{W^{1,2}} &\le c_2\|D_{H_+,\kappa}\varphi_{1,+}\|_{L^2} \\
&\le c_2\big(\big\|(D_{H_+,\kappa} - D_{H,\kappa})\varphi_{1,+}\big\|_{L^2} + \|D_{H,\kappa}\varphi_{1,+}\|_{L^2}\big) \\
&= c_2\big(\big\|(H_+ - H)\varphi_{1,+}\big\|_{L^2} + \|D_{H,\kappa}\varphi_{1,+}\|_{L^2}\big) \\
&\le \frac{1}{2}\|\varphi_{1,+}\|_{L^2} + c_2\|D_{H,\kappa}\varphi_{1,+}\|_{L^2} \\
&\le \frac{1}{2}\|\varphi_{1,+}\|_{W^{1,2}} + c_2\|D_{H,\kappa}\varphi_{1,+}\|_{L^2}.
\end{aligned}$$

Therefore

$$\|\varphi_+\|_{W^{1,2}} \le 2c_2\|D_{H,\kappa}\varphi_+\|_{L^2}$$

and similarly

$$\|\varphi_-\|_{W^{1,2}} \le 2c_2\|D_{H,\kappa}\varphi_-\|_{L^2}.$$

In conclusion,

$$\|\varphi\|_{W^{1,2}} \le 4c_2\|D_{H,\kappa}\varphi\|_{L^2} \tag{7.24}$$

for all φ_1 such that $\varphi_1(t) = 0$ for $t \in [-\tilde{a}, \tilde{a}]$. For general $\varphi_1 \in \operatorname{Ran}(\mathbf{1}\otimes P_\epsilon)$, choose a smooth cutoff function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(t) = 0$ for $|t| > \tilde{a} + 1$ and $\chi(t) = 1$ for $t \in [-\tilde{a}, \tilde{a}]$. Using (7.22) for $\chi\varphi_1$ and (7.24) for $(1 - \chi)\varphi_1$, one obtains

$$\begin{aligned}
\|\varphi_1\|_{W^{1,2}} &\le \|\chi\varphi_1\|_{W^{1,2}} + \big\|(1-\chi)\varphi_1\big\|_{W^{1,2}} \\
&\le c_1\big(\|\chi\varphi_1\|_{L^2} + \big\|D_{H,\kappa}(\chi\varphi_1)\big\|_{L^2}\big) + 4c_2\big\|D_{H,\kappa}\big((1-\chi)\varphi_1\big)\big\|_{L^2} \\
&\le c_1\big(\big\|P_{[-a,a]}\varphi_1\big\|_{L^2([-a,a])} + \big\|D_{H,\kappa}(\chi\varphi_1)\big\|_{L^2}\big) + 4c_2\big\|D_{H,\kappa}\big((1-\chi)\varphi_1\big)\big\|_{L^2} \\
&\le c_1\big(\|P_{[-a,a]}\varphi_1\|_{L^2([-a,a])} + \kappa\|\chi'\varphi_1\|_{L^2} + \|D_{H,\kappa}\varphi_1\|_{L^2}\big) \\
&\quad + 4c_2\big\|D_{H,\kappa}(\chi\varphi_1)\big\|_{L^2} + 4c_2\|D_{H,\kappa}\varphi_1\|_{L^2} \\
&\le c_1\Big(\|P_{[-a,a]}\varphi_1\|_{L^2([-a,a])} + \kappa\max_{t\in\mathbb{R}}\big|\chi'(t)\big|\|P_{[-a,a]}\varphi_1\|_{L^2} + \|D_{H,\kappa}\varphi_1\|_{L^2}\Big) \\
&\quad + 4c_2\kappa\max_{t\in\mathbb{R}}\big|\chi'(t)\big|\|P_{[-a,a]}\varphi_1\|_{L^2} + 8c_2\|D_{H,\kappa}\varphi_1\|_{L^2}
\end{aligned}$$

$$\leq c_3\big(\|P_{[-a,a]}\varphi_1\|_{L^2([-a,a])} + \|D_{H,\kappa}\varphi_1\|_{L^2}\big),$$

where $a = \tilde{a} + 1$ and $c_3 = \max\{c_1 + (c_1 + 4c_2)\kappa \max_{t\in\mathbb{R}} |\chi'(t)|, c_1 + 8c_2\}$.

To bound $\|\varphi_2\|_{W^{1,2}}$, without loss of generality, we assume that H is invertible which can be achieved by replacing H by H_{t_0} for some suitable $t_0 \in \mathbb{R}$. Then for $\epsilon \leq \frac{1}{2c_2(H)}$, using (7.23), one gets

$$\begin{aligned}
\|\varphi_2\|_{W^{1,2}} &\leq c_2(H)\big\|(\kappa\partial_t - H)\varphi_2\big\|_{L^2}\\
&= c_2(H)\big\|(\kappa\partial_t - H - K_t + K_t)\varphi_2\big\|_{L^2}\\
&\leq c_2(H)\big(\|D_{H,\kappa}\varphi_2\|_{L^2} + \|K_t\varphi_2\|_{L^2}\big)\\
&= c_2(H)\big(\|D_{H,\kappa}\varphi_2\|_{L^2} + \big\|K_t(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon))\varphi_2\big\|_{L^2}\big)\\
&\leq c_2(H)\|D_{H,\kappa}\varphi_2\|_{L^2} + \frac{1}{2}\|\varphi_2\|_{L^2}\\
&\leq c_2(H)\|D_{H,\kappa}\varphi_2\|_{L^2} + \frac{1}{2}\|\varphi_2\|_{W^{1,2}}.
\end{aligned}$$

Therefore

$$\|\varphi_2\|_{W^{1,2}} \leq 2c_2(H)\|D_{H,\kappa}\varphi_2\|_{L^2}.$$

One can conclude that

$$\begin{aligned}
\|\varphi\|_{W^{1,2}} &\leq \|\varphi_1\|_{W^{1,2}} + \|\varphi_2\|_{W^{1,2}}\\
&\leq c_3\big(\|P_{[-a,a]}\varphi_1\|_{L^2([-a,a])} + \|D_{H,\kappa}\varphi_1\|_{L^2}\big) + 2c_2(H)\|D_{H,\kappa}\varphi_2\|_{L^2}\\
&\leq c_3\|P_{[-a,a]}\varphi\|_{L^2([-a,a])} + \max\{c_3, 2c_2(H)\}\big(\|D_{H,\kappa}\varphi_1\|_{L^2} + \|D_{H,\kappa}\varphi_2\|_{L^2}\big).
\end{aligned} \tag{7.25}$$

Moreover,

$$\begin{aligned}
&\|D_{H,\kappa}\varphi_1\|_{L^2} + \|D_{H,\kappa}\varphi_2\|_{L^2}\\
&\quad\leq \big\|\big((\mathbf{1}\otimes P_\epsilon) + \big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)\big)D_{H,\kappa}\varphi_1\big\|_{L^2}\\
&\qquad+ \big\|\big((\mathbf{1}\otimes P_\epsilon) + \big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)\big)D_{H,\kappa}\varphi_2\big\|_{L^2}\\
&\quad\leq \big\|(\mathbf{1}\otimes P_\epsilon)D_{H,\kappa}\varphi_1\big\|_{L^2} + \big\|\big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)(H + K_t)\varphi_1\big\|_{L^2}\\
&\qquad+ \big\|\big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)D_{H,\kappa}\varphi_2\big\|_{L^2} + \big\|(\mathbf{1}\otimes P_\epsilon)(H + K_t)\varphi_2\big\|_{L^2}\\
&\quad\leq 2\big\|(\mathbf{1}\otimes P_\epsilon)D_{H,\kappa}\varphi_1 + \big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)D_{H,\kappa}\varphi_2\big\|_{L^2} + 4\epsilon\|\varphi\|_{L^2}\\
&\quad\leq 2\big\|(\mathbf{1}\otimes P_\epsilon)D_{H,\kappa}\varphi_1 + \big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)D_{H,\kappa}\varphi_1 + \big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)D_{H,\kappa}\varphi_2\\
&\qquad+ (\mathbf{1}\otimes P_\epsilon)D_{H,\kappa}\varphi_2\big\|_{L^2} + 4\epsilon\|\varphi\|_{L^2} + 2\big\|\big(\mathbf{1} - (\mathbf{1}\otimes P_\epsilon)\big)D_{H,\kappa}\varphi_1\big\|_{L^2}\\
&\qquad+ 2\big\|(\mathbf{1}\otimes P_\epsilon)D_{H,\kappa}\varphi_2\big\|_{L^2}\\
&\quad\leq 2\|D_{H,\kappa}\varphi\|_{L^2} + 12\epsilon\|\varphi\|_{L^2}\\
&\quad\leq 2\|D_{H,\kappa}\varphi\|_{L^2} + 12\epsilon\|\varphi\|_{W^{1,2}}.
\end{aligned}$$

Inserting this into (7.25) leads to

$$\|\varphi\|_{W^{1,2}} \le c_3\|P_{[-a,a]}\varphi\|_{L^2([-a,a])} + 2\max\{c_3, 2c_2(H)\}\|D_{H,\kappa}\varphi\|_{L^2} + 12\epsilon\max\{c_3, 2c_2(H)\}\|\varphi\|_{W^{1,2}}.$$

For ϵ sufficiently small,

$$12\epsilon\max\{c_3, 2c_2(H)\} < 1,$$

therefore (7.21) holds for some constant c. □

Note that, by Proposition 6.2.13 for Case A and Theorem 3.3.4 for Case B, the index of $D_{H,\kappa}$ is independent of $\kappa > 0$ as $\kappa \in (0,\infty) \mapsto \mathrm{Ind}(D_{H,\kappa})$ is constant. Therefore it is sufficient to prove (7.17) for one $\kappa > 0$. Next we show that it is also sufficient to consider smooth paths $t \in \mathbb{R} \mapsto K_t$.

Lemma 7.4.5. *Let $t \in \mathbb{R} \mapsto H_t = H + K_t \in \mathbb{SF}(\mathcal{H})$ be of the form described in Case A or Case B. For any $\epsilon > 0$, there exists a smooth path $t \in \mathbb{R} \mapsto \widetilde{K}_t \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ of bounded and in Case B of even compact operators with $\|K_t - \widetilde{K}_t\| < \epsilon$ uniformly in t such that eigenvalue crossings of the path $t \in \mathbb{R} \mapsto H + \widetilde{K}_t$ are simple and transversal, namely* $\dim(\mathrm{Ker}(H + \widetilde{K}_t)) \le 1$ *for all $t \in [0,1]$ and* $\mathrm{Ker}(H + \widetilde{K}_t) = \{0\}$ *except for a discrete set of crossings. For any crossing t_0, there is $\delta > 0$ such that $t \in (t_0 - \delta, t_0 + \delta) \mapsto \widetilde{K}_t$ is real analytic and $\widetilde{K}'_t|_{\mathrm{Ker}(H+\widetilde{K}_t)} \neq 0$. Moreover, the path $t \in \mathbb{R} \mapsto \tilde{K}_t$ can be chosen such that* $\lim_{t\to\pm\infty}(\widetilde{K})'_t = 0$.

Proof. As $t \in \mathbb{R} \mapsto K_t$ is uniformly continuous, there is $\delta' > 0$ such that $\|K_{t'} - K_{t''}\| < \frac{\epsilon}{16}$ for all $t', t'' \in \mathbb{R}$ such that $|t' - t''| < \delta'$. Let $b > 0$ be such that $\|K_t - K_{-\infty}\| < \frac{\epsilon}{16}$ for $t < -b$ and $\|K_t - K_{+\infty}\| < \frac{\epsilon}{16}$ for $t > b$ and such that $H + K_t$ is invertible for $t \notin [-b, b]$. For a partition $-b = t_1, \dots, t_M = b$ such that $|t_m - t_{m-1}| < \delta'$ for all $m = 1, \dots, M$, one can replace K_t on $[t_{m-1}, t_m]$ by the linear path $\widehat{K}_t = \frac{t-t_m}{t_{m-1}-t_m}K_{t_{m-1}} + \frac{t-t_{m-1}}{t_m-t_{m-1}}K_{t_m}$. Then the path $t \in [-b, b] \mapsto \widehat{K}_t$ is continuous and piecewise real analytic and $\|\widehat{K}_t - K_t\| < \frac{\epsilon}{8}$ uniformly in t. If $[-a, a] \cap \mathrm{spec}_{\mathrm{ess}}(H + \widehat{K}_t) = \emptyset$ for $a > 0$ and $t \in (t_{m-1}, t_m)$, by Theorem VII.1.8 in [112] for Case B and Section VII.3.1 in [112] for Case A, one can cover the set $\{(t, \lambda) \in [t_{m-1}, t_m] \times [-a, a] : \lambda \in \mathrm{spec}(H + \widehat{K}_t)\}$ by finitely many graphs of real-analytic functions λ_j, each possibly defined on some subinterval of $[t_{m-1}, t_m]$ if the eigenvalue leaves $[-a, a]$. In particular, $\mathrm{Ker}(H + \widehat{K}_t) = \{0\}$ except for finitely many crossings $t \in [t_{m-1}, t_m]$ or $\mathrm{Ker}(H + \widehat{K}_t) \neq \{0\}$ for all $t \in [t_{m-1}, t_m]$. In the latter case, we choose a partition $t_{m-1} = t_{m,0}, \dots, t_{m,l} = t_m$ and $c_{m,k}$ such that $\pm c_{m,k} \notin \mathrm{spec}(H + \widehat{K}_t)$ for all $t \in [t_{m,l-1}, t_{m,l}]$ and replace $H + \widehat{K}_t$ by $H + \widetilde{\mathcal{K}}_t = (H + \widehat{K}_t) + \tilde{\epsilon}(t - t_{m,l-1})(t - t_{m,l})(H + \widehat{K}_t)\chi_{[-c_{m,k}, c_{m,k}]}(H + \widehat{K}_t)$ where $0 < \tilde{\epsilon} < \frac{\epsilon}{8}$ is chosen such that $\mathrm{Ker}(H + \widetilde{\mathcal{K}}_t) = \{0\}$ except for finitely many crossings $t \in [t_{m-1}, t_m]$. Note that this path restricted to $[-b, b]$ is piecewise real analytic as $t \in [t_{m,l-1}, t_{m,l}] \mapsto \chi_{[-c_{m,k}, c_{m,k}]}(H + \widehat{K}_t)$ is real analytic, for the same reasons as above. Therefore there is a path $t \in \mathbb{R} \mapsto \widehat{\mathcal{K}}_t$ such that $H + \widehat{\mathcal{K}}_t$ is invertible except for a discrete set of points and such that $\|\widehat{\mathcal{K}}_t - K_t\| < \frac{\epsilon}{4}$ uniformly in t. For Case B, $\widehat{\mathcal{K}}_t$ is compact for all t.

If $t_0 \in [0,1]$ is such that $t \mapsto \widehat{\mathcal{K}}_t$ is not analytic in t_0 and such that $\mathrm{Ker}(H + \widehat{\mathcal{K}}_{t_0}) \neq 0$, there is an $\epsilon_0 > 0$ such that $H+\widehat{\mathcal{K}}_{t_0 \pm \epsilon_0}$ is invertible. We then replace $\widehat{\mathcal{K}}_t$ on $[t_0-\epsilon_0, t_0+\epsilon_0]$ by $\frac{t_0+\epsilon_0-t}{2\epsilon_0}\widehat{\mathcal{K}}_{t_0-\epsilon_0} + \frac{t-t_0+\epsilon_0}{2\epsilon_0}\widehat{\mathcal{K}}_{t_0+\epsilon_0}$. Therefore, one can assume that $\mathrm{Ker}(H+\widehat{\mathcal{K}}_t) = \{0\}$ except for a discrete set of crossings, and for each crossing t_0, there is ϵ_0 such that the map $t \in (t_0-\epsilon_0, t_0+\epsilon_0) \mapsto H + \widehat{\mathcal{K}}_t$ is real analytic. There are values $a > 0$ and $0 < \delta < \epsilon_0$ such that $\pm a \notin \mathrm{spec}(H + \widehat{\mathcal{K}}_t)$ and $[-a,a] \cap \mathrm{spec}(H+\widehat{\mathcal{K}}_t)$ for $t \in (t_0-\delta, t_0+\delta)$ consists of finitely many eigenvalues of finite multiplicity. Then, again by Theorem VII.1.8 in [112], there is a real-analytic path $t \in (t_0-\delta, t_0+\delta) \mapsto U_t \in \mathbb{U}(\mathcal{H})$ of unitaries such that $U_t \,\mathrm{Ran}(\chi_{[-a,a]}(H+\widehat{\mathcal{K}}_t)) = \mathrm{Ran}(\chi_{[-a,a]}(H+\widehat{\mathcal{K}}_{t_0}))$. Then $t \mapsto U_t(H+\widehat{\mathcal{K}}_t)U_t^*|_{\mathrm{Ran}(\chi_{[-a,a]}(H+\widehat{\mathcal{K}}_{t_0}))}$ is a real-analytic path of finite-dimensional operators and, by Theorem II.1.10 in [112], there is a real-analytic path of unitaries $t \in [0,1] \mapsto V_t \in \mathbb{B}(\mathrm{Ran}(\chi_{[-a,a]}(H+\widehat{\mathcal{K}}_{t_0})), \mathbb{C}^M)$ such that $V_tU_t(H+\widehat{\mathcal{K}}_t)U_t^*V_t^* = \mathrm{diag}(\lambda_1(t),\dots,\lambda_M(t))$ where $t \mapsto \lambda_k(t)$ are real-analytic functions representing the eigenvalues of $H+\widehat{\mathcal{K}}_t$. By Sard's theorem, the complement of the set of regular values of the eigenvalues λ_k, $k = 1,\dots,M$ in $(t_0-\delta, t_0+\delta)$ has measure zero. Therefore there are $\delta_1,\dots,\delta_M \in (-\frac{\epsilon}{8}, \frac{\epsilon}{8})$ such that 0 is a common regular value of the functions $t \mapsto \lambda_k(t)+\delta_k$ for $k=1,\dots,M$ and such that $\dim(\mathrm{Ker}(\mathrm{diag}(\lambda_1(t)+\delta_1,\dots,\lambda_M(t)+\delta_M))) \leq 1$ for all $t \in (t_0-\delta, t_0+\delta)$. Then setting $H+\breve{K}_t = U_t^*(V_t^*\,\mathrm{diag}(\lambda_1(t)+\delta_1,\dots,\lambda_N(t)+\delta_N)V_t + (H+\widehat{\mathcal{K}}_t)(\mathbf{1}-\chi_{[-a,a]}(H+\widehat{\mathcal{K}}_t)))U_t$ the path $t \in (t_0-\delta, t_0+\delta) \mapsto H+\breve{K}_t$ is a real-analytic and has only simple and transversal eigenvalue crossings. Moreover, there is $\widehat{\epsilon} > 0$ such that $\|\breve{K}_t - \widehat{\mathcal{K}}_{t_0-\delta}\| < \frac{\epsilon}{16}$ for all $t \in (t_0-\delta-\widehat{\epsilon}, t_0-\delta)$ and such that $\|\breve{K}_t - \widehat{\mathcal{K}}_{t_0+\delta}\| < \frac{\epsilon}{16}$ for all $t \in (t_0+\delta+\widehat{\epsilon}, t_0+\delta)$. We then replace $t \in (t_0-\delta-\widehat{\epsilon}, t_0-\delta) \mapsto \widehat{\mathcal{K}}_t$ by the linear path $t \mapsto \breve{K}_t$ connecting $\widehat{\mathcal{K}}_{t_0-\delta-\widehat{\epsilon}}$ to $\breve{K}_{t_0-\delta}$ and similar for $t \in (t_0+\delta, t_0+\delta+\widehat{\epsilon})$. Then, by construction, the path $t \in [0,1] \mapsto \breve{K}_t$, where $\breve{K}_t = \widehat{\mathcal{K}}_t$ for $t \notin (t_0-\delta-\widehat{\epsilon},, t_0+\delta+\widehat{\epsilon})$ for all crossings t_0, is continuous and all eigenvalue crossings are simple and transversal. Its restriction to $[-b,b]$ is piecewise real analytic. Let $t_{*,1},\dots,t_{*,L} \in [-b,b]$ be the points at which $\breve{K}$ is not analytic. For $\delta_* > 0$, let $t \in \mathbb{R} \mapsto \chi_{\delta_*}(t) \in [0,1]$ be a smooth function such that $\chi_{\delta_*}(t) = 0$ if $|t-t_{*,l}| > 2\delta_*$ for all l and $|t| < b-\delta_*$ and $\chi_{\delta_*}(t) = 1$ if there is an l such that $|t-t_{*,l}| < \delta_*$ or if $|t| > b$. Then for δ_* sufficiently small, the path

$$
\begin{aligned}
t \in \mathbb{R} \mapsto \widetilde{K}_t = \breve{K}_t(1-\chi_{\delta_*}(t)) &+ \sum_{l=1}^{L} \breve{K}_{t_{*,l}}\chi_{\delta_*}(t)\chi_{[t_{*,l}-2\delta_*, t_{*,l}+2\delta_*]}(t) \\
&+ K_b\chi_{\delta_*}(t)\chi(t > b-\delta_*) + K_{-b}\chi_{\delta_*}(t)\chi(t < -b+\delta_*)
\end{aligned}
$$

has the desired properties. □

Due to the homotopy invariance of the spectral flow and the constancy of the index of a Fredholm operator, it is sufficient to show (7.17) for generic paths $t \in \mathbb{R} \mapsto \widetilde{K}_t$ described in Lemma 7.4.5. Therefore, from now on, we will assume that $t \in \mathbb{R} \mapsto K_t$ is a generic path.

Proof of Theorem 7.4.1. The detailed proof will be separated in several steps.

Step 1 (IMS localization). Let $t_1^*, \dots, t_N^* \in \mathbb{R}$ be the finite number N of points for which $\dim(\mathrm{Ker}(H_{t_i^*})) = 1$. By diagonalizing the smooth path $t \mapsto H_t$ on each $B_i^\delta = (t_i^* - \delta, t_i^* + \delta)$, one obtains a differentiable unitary basis change $t \in B_i^\delta \mapsto W_t$ such that

$$W_t H_t W_t^* = \begin{pmatrix} H_{i,t}^0 & 0 \\ 0 & d_i(t - t_i^*) + H_{i,t}^R \end{pmatrix}, \tag{7.26}$$

where $d_i \in \mathbb{R} \setminus \{0\}$, $H_{i,t}^0$ is invertible for $t \in B_i^\delta$ and $\|H_{i,t}^R\| \le c|t - t_i^*|^2$.

Now a variation on the IMS localization procedure [181, 67] will be used. For each t_i^*, let $\chi_i^\delta : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function, supported on $[-\delta, \delta]$ and such that $\chi_i^\delta(t) = 1$ for $t \in [-\frac{1}{2}\delta, \frac{1}{2}\delta]$. We naturally extend χ_i^δ to a multiplication operator on $W^{1,2}(\mathbb{R}, \mathcal{H})$. One then has the following properties:

- $[\chi_{i,t}^\delta, H_t] = 0$;
- The support of χ_i^δ lies in the ball B_i^δ;
- $\|\partial_t \chi_i^\delta\| \le c' \delta^{-1}$ for some constant c' uniform in i.

Furthermore, let us set

$$(\chi_0^\delta)^2 = \mathbf{1} - \sum_{i=1}^N (\chi_i^\delta)^2.$$

Note that $\sum_{i=0}^N (\chi_i^\delta)^2 = \mathbf{1}$. Now computing the double commutator $[\chi_i^\delta, [\chi_i^\delta, (L_\kappa)^2]]$ twice shows

$$(\chi_i^\delta)^2 (L_\kappa)^2 + (L_\kappa)^2 (\chi_i^\delta)^2 - 2\chi_i^\delta (L_\kappa)^2 \chi_i^\delta = -2\kappa^2 (\partial_t \chi_i^\delta)^2.$$

Summing over i shows the ILS localization formula

$$(L_\kappa)^2 = \chi_0^\delta (L_\kappa)^2 \chi_0^\delta + \sum_{i=1}^N \chi_i^\delta (L_\kappa)^2 \chi_i^\delta - \kappa^2 \sum_{i=0}^N (\partial_t \chi_i^\delta)^2. \tag{7.27}$$

Step 2 (Local toy model). Let us now focus on one summand $\chi_i^\delta (L_\kappa)^2 \chi_i^\delta$ in order to obtain a local toy model for $(L_\kappa)^2$ in a neighborhood of t_i^*, by replacing (7.26). Set $W = \int_{\mathbb{R}}^\oplus dt\, W_t$ and extend the unitaries W naturally to 2×2 matrices. Then

$$\begin{aligned}
\chi_i^\delta (L_\kappa)^2 \chi_i^\delta &= \chi_i^\delta W^* (W L_\kappa W^*)^2 W \chi_i^\delta \\
&= \chi_i^\delta W^* \left[\begin{pmatrix} 0 & -\kappa W \partial_t W^* \\ \kappa W \partial_t W^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -WHW^* \\ -WHW^* & 0 \end{pmatrix} \right]^2 W \chi_i^\delta \\
&= \chi_i^\delta W^* \left[\begin{pmatrix} 0 & -\kappa \partial_t - WHW^* \\ \kappa \partial_t - WHW^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\kappa W (W')^* \\ \kappa W (W')^* & 0 \end{pmatrix} \right]^2 W \chi_i^\delta \\
&= \chi_i^\delta W^* \left[L_{\kappa,i}^0 \oplus L_{\kappa,i}^T + \begin{pmatrix} 0 & -\kappa W (W')^* - 0 \oplus H_i^R \\ \kappa W (W')^* - 0 \oplus H_i^R & 0 \end{pmatrix} \right]^2 W \chi_i^\delta,
\end{aligned} \tag{7.28}$$

where

$$L^0_{\kappa,i} = \begin{pmatrix} 0 & -\kappa\partial_t - H^0_{i,t} \\ \kappa\partial_t - H^0_{i,t} & 0 \end{pmatrix}$$

and the main toy model is

$$L^T_{\kappa,i} = \begin{pmatrix} 0 & -\kappa\partial_t - d_i(t - t_i^*) \\ \kappa\partial_t - d_i(t - t_i^*) & 0 \end{pmatrix}.$$

Even though in the above only its restriction to the range of χ_i^δ is of relevance, it will now be analyzed as an operator on $L^2(\mathbb{R}, \mathbb{C}^2)$. Its square is

$$(L^T_{\kappa,i})^2 = \begin{pmatrix} -\kappa^2\partial_t^2 + (d_i)^2(t - t_i^*)^2 + \kappa d_i & 0 \\ 0 & -\kappa^2\partial_t^2 + (d_i)^2(t - t_i^*)^2 - \kappa d_i \end{pmatrix}.$$

Both operators are direct sums of two harmonic oscillators, shifted by $\pm\kappa d_i$. The spectra are given by $\{(2n + 1)\kappa|d_i| \pm \kappa|d_i| \operatorname{sgn}(d_i) \,:\, n \in \mathbb{N}_0\}$. Therefore, if $d_i > 0$, $(L^T_{\kappa,i})^2$ has a vector in the kernel in the second component, while for $d_i < 0$ it has a kernel vector in the first component. In both cases, it is given by a Gaussian state $\phi_{\kappa,i} \in L^2(\mathbb{R}, \mathbb{C}^2)$ with variance $\kappa^{\frac{1}{2}}$. It follows that

$$\operatorname{Sig}(J|_{\operatorname{Ker}(L^T_{\kappa,i})}) = -\operatorname{Sig}(d_i).$$

Moreover, the first excited state of $(L^T_{\kappa,i})^2$ is of order κ.

Step 3 (Bounds on error terms in the IMS localization formula). Let us now bound the terms in (7.27). The operator $\chi_0^\delta(L_\kappa)^2\chi_0^\delta$ is of the form

$$\chi_0^\delta(L_\kappa)^2\chi_0^\delta = \chi_0^\delta \begin{pmatrix} -\kappa^2\partial_t^2 + H_t^2 + \kappa K_t' & 0 \\ 0 & -\kappa^2\partial_t^2 + H_t^2 - \kappa K_t' \end{pmatrix} \chi_0^\delta.$$

As H_t^2 restricted to the support of χ_0^δ is strictly positive and bounded below by $c_1\delta^2$ for δ sufficiently small and $\|K_t'\|$ is bounded by c_2 for all $t \in \mathbb{R}$,

$$\chi_0^\delta(L_\kappa)^2\chi_0^\delta \geq c_1\delta^2(\chi_0^\delta)^2 - c_2\kappa(\chi_0^\delta)^2.$$

Combined with the bound on the derivative $\partial_t\chi_i^\delta$, one thus deduces

$$(L_\kappa)^2 \geq c_1\delta^2(\chi_0^\delta)^2 - c_2\kappa(\chi_0^\delta)^2 + \sum_{i=1}^{N} \chi_i^\delta(L_\kappa)^2\chi_i^\delta - c_3\kappa^2\delta^{-2}. \tag{7.29}$$

Next let us bound the localized terms by using (7.28). First recall that, for two self-adjoint operators A and B (here A is unbounded and B is bounded), one has the operator inequality $\{A, B\} \leq A^2 + B^2$ for the anticommutator, so that

$$(A+B)^2 = A^2 + B^2 + \{\delta^{\frac{\eta}{2}}A, \delta^{-\frac{\eta}{2}}B\} \geq (1-\delta^\eta)A^2 + (1-\delta^{-\eta})B^2,$$

for $\eta > 0$ to be chosen later (such that the second negative term becomes small due to a δ-dependence of B). Therefore

$$\begin{aligned}
&\chi_i^\delta (L_\kappa)^2 \chi_i^\delta \\
&\quad \geq (1-\delta^\eta)\chi_i^\delta W^*\big((L^0_{\kappa,i})^2 \oplus (L^T_{\kappa,i})^2\big)W\chi_i^\delta \\
&\qquad + (1-\delta^{-\eta})\chi_i^\delta W^* \begin{pmatrix} 0 & -\kappa W(W')^* - 0\oplus H_i^R \\ \kappa W(W')^* - 0 \oplus H_i^R & 0 \end{pmatrix}^2 W\chi_i^\delta .
\end{aligned}$$

The above analysis of $L^T_{\kappa,i}$ shows that there is a rank 1 operator F_i such that

$$\chi_i^\delta W^* (L^T_{\kappa,i})^2 W\chi_i^\delta \geq C_i^* \kappa (\chi_i^\delta)^2 + F_i,$$

for some constant C_i^*. Moreover, $(L^0_{\kappa,i})^2$ has a mass gap that is uniformly (in κ) bounded from below so that

$$\chi_i^\delta W^* (L^0_{\kappa,i})^2 W\chi_i^\delta \geq c_4(\chi_i^\delta)^2 - c_2\kappa(\chi_i^\delta)^2 \geq \frac{1}{2}c_4(\chi_i^\delta)^2,$$

for κ sufficiently small. As to the second summand, let us note that $W(W')^*$ is bounded and H_i^R is of order δ^2 on the support of χ_i^δ, thus one has

$$\chi_i^\delta W^* \begin{pmatrix} 0 & -\kappa W(W')^* - 0\oplus H_i^R \\ \kappa W(W')^* - 0 \oplus H_i^R & 0 \end{pmatrix}^2 W\chi_i^\delta = \mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4).$$

Collecting the estimates, one thus has

$$\chi_i^\delta (L_\kappa)^2 \chi_i^\delta \geq C_i^*\kappa(1-\delta^\eta)(\chi_i^\delta)^2 + (1-\delta^\eta)F_i + (1-\delta^{-\eta})\mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4),$$

for κ so small that $C_i^*\kappa \leq \frac{1}{2}c_4$. Finally, introduce $F = \sum_{i=1}^N F_i$ which is of rank N and set $C^* = \min\{C_1^*, \dots, C_N^*\}$. Substituting into (7.29), one concludes

$$\begin{aligned}
(L_\kappa)^2 &\geq c_1\delta^2(\chi_0^\delta)^2 - c_2\kappa(\chi_0^\delta)^2 - c_3\kappa^2\delta^{-2} \\
&\qquad + C^*\kappa(1-\delta^\eta)(\mathbf{1} - (\chi_0^\delta)^2) + (1-\delta^\eta)F + (1-\delta^{-\eta})\mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4).
\end{aligned}$$

Now the size of the balls is chosen to be $\delta = \kappa^\alpha$ with $2\alpha < 1$. Then for $\eta \in (0,1)$ such that $\delta^\eta < \frac{1}{2}$ and $\tilde{C}$ such that $c_1\delta^2 - c_2\kappa - c_3\kappa^2\delta^{-2} \geq \tilde{C}\kappa$ and $\tilde{C} \leq \frac{1}{2}C^* - c_3\kappa\delta^{-2}$ for κ sufficiently small,

$$(L_\kappa)^2 \geq \tilde{C}\kappa\mathbf{1} + \frac{1}{2}F - c_5\kappa^{(4-\eta)\alpha}.$$

Now choosing any $\alpha < \frac{1}{2}$ and $\eta < 1$ such that $(4-\eta)\alpha > 1$ (one possible choice is $\alpha = \frac{1}{3}$ and $\eta = \frac{1}{4}$) shows that

$$(L_\kappa)^2 \geq \frac{1}{4}\tilde{C}\kappa\mathbf{1} + \frac{1}{2}F. \tag{7.30}$$

Step 4 (Spectral bounds on L_κ). The bound (7.30) combined with the Rayleigh–Ritz principle implies that $(L_\kappa)^2$ has at most N eigenvalues in $[0, \frac{1}{4}\tilde{C}\kappa]$. By providing N test functions (again following closely [181]), it will next be shown that there are at least N eigenvalues of $(L_\kappa)^2$ in $[0, c^2\kappa^{\frac{4}{3}}]$. Combining these facts then gives a detailed information on the low-lying spectrum of L_κ, namely there are eigenvalues $\nu_{\kappa,1}, \dots, \nu_{\kappa,N} \in \operatorname{spec}(L_\kappa) \cap [-c\kappa^{\frac{2}{3}}, c\kappa^{\frac{2}{3}}]$ (listed with their multiplicity) such that

$$\operatorname{spec}(L_\kappa) \cap \left[-\frac{1}{2}\sqrt{\tilde{C}\kappa}, \frac{1}{2}\sqrt{\tilde{C}\kappa}\right] = \{\nu_{\kappa,1}, \dots, \nu_{\kappa,N}\}.$$

The trial functions are constructed from the Gaussian zero modes $\phi_{\kappa,i}$ of $L^T_{\kappa,i}$ with variance $\kappa^{\frac{1}{2}}$. These states thus differ (locally) little from the normalized vectors

$$\Phi_{\kappa,i} = a_i \chi_i^\delta W^*(0 \oplus \phi_{\kappa,i}) \in L^2(\mathbb{R}, \mathcal{H}), \tag{7.31}$$

as long as $\delta \gg \kappa^{\frac{1}{2}}$. As their supports are disjoint, the functions $\Phi_{\kappa,1}, \dots, \Phi_{\kappa,N}$ are orthogonal. Similar as above (with $\eta = 0$), the operator Cauchy–Schwarz inequality implies that

$$\chi_i^\delta (L_\kappa)^2 \chi_i^\delta \leq 2\chi_i^\delta W^*(L^0_{\kappa,i} \oplus L^T_{\kappa,i})^2 W \chi_i^\delta + 2\mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4).$$

Hence, using as in the proof of the ILS localization formula

$$2\chi_i^\delta (L^T_{\kappa,i})^2 \chi_i^\delta = (\chi_i^\delta)^2 (L^T_{\kappa,i})^2 + 2\kappa^2 (\partial\chi_i^\delta)^2 + (L^T_{\kappa,i})^2 (\chi_i^\delta)^2,$$

we obtain

$$\begin{aligned}
\langle \Phi_{\kappa,i} | (L_\kappa)^2 | \Phi_{\kappa,i} \rangle &\leq |a_i|^2 2 \langle (0 \oplus \phi_{\kappa,i}) | \chi_i^\delta (L^0_{\kappa,i} \oplus L^T_{\kappa,i})^2 \chi_i^\delta | (0 \oplus \phi_{\kappa,i}) \rangle + \mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4) \\
&= |a_i|^2 2 \langle \phi_{\kappa,i} | \chi_i^\delta (L^T_{\kappa,i})^2 \chi_i^\delta | \phi_{\kappa,i} \rangle + \mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4) \\
&= |a_i|^2 \langle \phi_{\kappa,i} | (\chi_i^\delta)^2 (L^T_{\kappa,i})^2 + 2\kappa^2 (\partial\chi_i^\delta)^2 + (L^T_{\kappa,i})^2 (\chi_i^\delta)^2 | \phi_{\kappa,i} \rangle + \mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4) \\
&= \mathcal{O}(\kappa^2 \delta^{-2}) + \mathcal{O}(\kappa^2, \kappa\delta^2, \delta^4).
\end{aligned}$$

Choosing again $\delta = \kappa^\alpha$ now with $\alpha = \frac{1}{3}$, it follows that $\langle \Phi_{\kappa,i} | (L_\kappa)^2 | \Phi_{\kappa,i} \rangle = \mathcal{O}(\kappa^{\frac{4}{3}})$ for all $i = 1, \dots, N$. By the Rayleigh–Ritz principle, there are thus at least N eigenvalues of $(L_\kappa)^2$ that are smaller than $\mathcal{O}(\kappa^{\frac{4}{3}})$. This implies the claim stated above.

Step 5 (Stability of signature). Let $\Psi_\kappa = (\psi_{\kappa,1}, \dots, \psi_{\kappa,N}) : \mathbb{C}^N \to L^2(\mathbb{R}, \mathcal{H})$ be a partial isometry onto the normalized eigenstates $\psi_{\kappa,1}, \dots, \psi_{\kappa,N}$ corresponding to the low-lying spectrum of L_κ as identified in Step 4, namely

$$\Psi_\kappa^* L_\kappa \Psi_\kappa = \mathrm{diag}(\nu_{\kappa,1}, \dots, \nu_{\kappa,N}).$$

As all low-lying eigenvalues are included (see Step 4), the identity $JL_\kappa J = -L_\kappa$ implies that $\{\nu_{\kappa,1}, \dots, \nu_{\kappa,N}\}$ is invariant under reflection and that the set $\psi_{\kappa,1}, \dots, \psi_{\kappa,N}$ of eigenvectors is J-invariant. By Theorem 2.3 in [181], the approximate eigenstates $\phi_{\kappa,i}$ constructed in Step 4 are close to $\psi_{\kappa,i}$, namely $\|\psi_{\kappa,i} - \phi_{\kappa,i}\| \to 0$ as $\kappa \to 0$. Hence $\Psi_{\kappa,i}^* J \Psi_{\kappa,i} - \Phi_{\kappa,i}^* J \Phi_{\kappa,i} \to 0$ and thus, for κ sufficiently small,

$$\Psi_\kappa^* J \Psi_\kappa = -\,\mathrm{diag}\big(\mathrm{sgn}(d_1), \dots, \mathrm{sgn}(d_N)\big).$$

Those eigenvalues ν of L_κ which are different from 0 lead to symmetric pairs $(\nu, -\nu)$ with eigenstates given by $(\psi, J\psi)$. The latter are two-dimensional J-invariant subspaces with vanishing J-signature because $J(\psi, J\psi) = \sigma_1(\psi, J\psi)$ where σ_1 is the first Pauli matrix. Therefore, even though the kernel of L_κ is not determined,

$$\mathrm{Sig}(J|_{\mathrm{Ker}(L_\kappa)}) = -\,\mathrm{Sig}\big(\Psi_\kappa^* J \Psi_\kappa\big) = -\sum_{i=1}^{N} \mathrm{sgn}(d_i) = -\,\mathrm{Sf}(t \in \mathbb{R} \mapsto H_t).$$

Combined with (7.18), this implies the claim first for $\kappa > 0$ sufficiently small, but then κ can be raised without harming the Fredholm property and thus changing the index. □

8 Homotopy theory of Fredholm operators

This chapter is about homotopy groups of the sets of Fredholm operators, unitary and self-adjoint Fredholm operators, Fredholm pairs, and other operator classes. Both bounded and unbounded Fredholm operators with various topologies are dealt with, and as an application a characterization of the spectral flow is proved in Section 8.4. Part of the presentation below follows closely the textbook by Booß-Bavnbek and Wojciechowski [32], as well as the excellent lecture notes by Schröder [165] which are unfortunately only available in German. When dealing with unbounded self-adjoint operators equipped with the gap metric, another crucial element of proof is taken from a paper of Joachim [108] that is apparently not particularly well known. Along the way, several homotopy equivalences are proved and this is summarized in Section 8.7. Several fundamental results are needed (in particular, the long exact sequence of homotopy groups of fiber bundles and the stable homotopy groups of the general linear groups as computed by Bott) and are recalled in Appendix A.3 for the convenience of the reader.

8.1 Homotopy groups of essentially gapped unitaries

For the stable general linear group $\mathrm{GL}(\infty, \mathbb{C})$, the homotopy groups are known by Bott's celebrated result, see (A.3) in Appendix A.3. It was then proved by Palais [141] and Shvarts [179] that one can enlarge $\mathrm{GL}(\infty, \mathbb{C})$ to the invertible operators $\mathbb{G}^c(\mathcal{H})$ in the unitization of the compact operators without changing the homotopy groups. More precisely, consider

$$\mathbb{K}(\mathcal{H})^{\sim} = \{T \in \mathbb{B}(\mathcal{H}) : T - \mathbf{1} \in \mathbb{K}(\mathcal{H})\} = \mathbf{1} + \mathbb{K}(\mathcal{H}),$$

and the subset of invertibles

$$\mathbb{G}^c(\mathcal{H}) = \mathbb{G}(\mathcal{H}) \cap \mathbb{K}(\mathcal{H}),$$

equipped with the norm topology. Then

$$\pi_k(\mathbb{G}^c(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \tag{8.1}$$

A proof of the latter fact can also be found in [165]. Based on this, one can directly state the homotopy groups of the essentially gapped unitary operators. Indeed, by Proposition 3.7.2, the set $\mathbb{FU}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : -1 \notin \mathrm{spec}_{\mathrm{ess}}(U)\}$ of essentially gapped unitaries can be retracted to the set $\mathbb{U}^c(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U - \mathbf{1} \in \mathbb{K}(\mathcal{H})\}$. Furthermore, the polar decomposition provides the following:

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Proposition 8.1.1. *With respect to the norm topology, $\mathbb{U}^c(\mathcal{H})$ is a deformation retract of $\mathbb{G}^c(\mathcal{H})$.*

Proof. For $T \in \mathbb{G}^c(\mathcal{H})$, clearly, also $|T|^2 = T^*T \in \mathbb{G}^c(\mathcal{H})$. Moreover, for $s \in \mathbb{R}$,

$$|T|^s = \oint_\Gamma \frac{dz}{2\pi\imath} z^{\frac{s}{2}} (z\mathbf{1} - |T|^2)^{-1},$$

for some contour surrounding the (positive) spectrum of $|T|^2$ once in the positive sense. Due to the resolvent identity, one has $|T|^s \in \mathbb{G}^c(\mathcal{H})$. Therefore $U = T|T|^{-1} \in \mathbb{U}^c(\mathcal{H})$ and the path $s \in [0,1] \mapsto U|T|^s$ lies in $\mathbb{G}^c(\mathcal{H})$. Thus the homotopy

$$h : \mathbb{G}^c(\mathcal{H}) \times [0,1] \to \mathbb{G}^c(\mathcal{H}), \quad h(T,t) = T|T|^{-t}$$

is well defined and clearly norm-continuous. Moreover, $h(T,0) = T$ for $T \in \mathbb{G}^c(\mathcal{H})$, $h(T,1) \in \mathbb{U}^c(\mathcal{H})$ for $T \in \mathbb{G}^c(\mathcal{H})$ by the above, and $h(U,t) = U$ for $U \in \mathbb{U}^c(\mathcal{H})$ and $t \in [0,1]$, and therefore h is a deformation retraction of $\mathbb{G}^c(\mathcal{H})$ onto $\mathbb{U}^c(\mathcal{H})$. □

Now the homotopy groups of $\mathbb{G}^c(\mathcal{H})$ are given by (8.1) by the results of Bott and Palais. Therefore we obtain

Corollary 8.1.2. *With respect to the norm topology, the homotopy groups of the essentially gapped unitary operators are*

$$\pi_k(\mathbb{F}\mathbb{U}(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Corollary 8.1.3. *The spectral flow on closed loops establishes an isomorphism*

$$\mathrm{Sf} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{H})) \to \mathbb{Z}.$$

Proof. Clearly, $\mathrm{Sf} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{H})) = \mathbb{Z} \to \mathbb{Z}$ is a homomorphism. Example 4.5.4 shows that this homomorphism is surjective. It is then a fact that every surjective homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is injective. □

8.2 Homotopy groups of Fredholm operators

It was proved in Theorem 3.3.5 and Corollary 3.3.6 that the connected components $\mathbb{F}_n\mathbb{B}(\mathcal{H})$ of the set of bounded Fredholm operators $\mathbb{F}\mathbb{B}(\mathcal{H})$ with respect to the norm topology are labeled by the index $n \in \mathbb{Z}$, so that

$$\pi_0(\mathbb{F}\mathbb{B}(\mathcal{H})) = \mathbb{Z}.$$

Moreover, one can restate Corollary 3.3.6 as

Corollary 8.2.1. *The index establishes a bijection* $\mathrm{Ind} : \pi_0(\mathbb{F}\mathbb{B}(\mathcal{H})) \to \mathbb{Z}$.

As all connected components of $\mathbb{F}\mathbb{B}(\mathcal{H})$ are homotopy equivalent by Theorem 3.3.5, the task left is to determine the homotopy groups of the identity component $\mathbb{F}_0\mathbb{B}(\mathcal{H})$. This is done by applying the tools described Appendix A.3.

Theorem 8.2.2. *The homotopy groups of the identity component of the bounded Fredholm operators on* $\mathcal{H}$ *are given by*

$$\pi_k\big(\mathbb{F}_0\mathbb{B}(\mathcal{H})\big) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}. \end{cases}$$

As all connected components of $\mathbb{F}\mathbb{B}(\mathcal{H})$ are homotopy equivalent, this directly implies

Corollary 8.2.3. *For all* $n \in \mathbb{Z}$*, the homotopy groups of the component* $\mathbb{F}_n\mathbb{B}(\mathcal{H})$ *of the bounded Fredholm operators are*

$$\pi_k\big(\mathbb{F}_n\mathbb{B}(\mathcal{H})\big) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}. \end{cases}$$

Strictly speaking, one has

$$\pi_k\big(\mathbb{F}_n\mathbb{B}(\mathcal{H}), T\big) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}, \end{cases}$$

for all basepoints $T \in \mathbb{F}_n\mathbb{B}(\mathcal{H})$ where, for any topological space X and $b \in X$, the homotopy group $\pi_k(X, b)$ is made by the homotopy classes of continuous maps $f : \mathbb{S}^k \to X$ mapping some fixed point $a_k \in \mathbb{S}^k$ onto b. As the homotopy groups of a connected space X are independent of the basepoint, this is also written as

$$\pi_k\big(\mathbb{F}_n\mathbb{B}(\mathcal{H})\big) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k > 0 \text{ even}. \end{cases}$$

The proof of Theorem 8.2.2 will use the Bartle and Graves selection theorem as a crucial element for the construction of fiber bundles [34]. A version that is sufficient for the present purposes can be stated as follows: if $\mathcal{E}$ is a Banach space, $\mathcal{U} \subset \mathcal{E}$ a closed subspace, and $\pi : \mathcal{E} \to \mathcal{E}/\mathcal{U}$ is the quotient map, then there exists a continuous (homogeneous but not necessarily linear) right inverse $\rho : \mathcal{E}/\mathcal{U} \to \mathcal{E}$ of π, namely $\pi \circ \rho = \mathrm{id}_{\mathcal{E}/\mathcal{U}}$. A short proof of this version is given in [165].

Proof of Theorem 8.2.2. The proof is split into several steps:

Fact 1. $\mathbb{F}_0\mathbb{B}(\mathcal{H})$ is homotopy equivalent to the identity component $\mathbb{G}_0\mathbb{Q}(\mathcal{H})$ of the invertible elements in the Calkin algebra.

Indeed, let ρ be the right inverse of the Calkin projection $\pi : \mathbb{B}(\mathcal{H}) \to \mathbb{Q}(\mathcal{H})$ as given by the Bartle–Graves selection theorem. Because Fredholm operators with vanishing index are compact perturbations of an invertible operator $\mathbb{F}_0\mathbb{B}(\mathcal{H}) = \pi^{-1}(\mathbb{G}_0\mathbb{Q}(\mathcal{H}))$. Thus $T \in \mathbb{F}_0\mathbb{B}(\mathcal{H})$ can be uniquely decomposed into $T = \rho(\pi(T)) + K$ with some compact operator $K \in \mathbb{K}(\mathcal{H})$. Hence $\mathbb{F}_0\mathbb{B}(\mathcal{H})$ is homeomorphic to $\mathbb{G}_0\mathbb{Q}(\mathcal{H}) \times \mathbb{K}(\mathcal{H})$. The contractibility of the compact operators then implies the first fact.

Fact 2. The restriction of the Calkin projection $\hat{\pi} = \pi|_{\mathbb{G}(\mathcal{H})} : \mathbb{G}(\mathcal{H}) \to \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ is a fiber bundle with fiber $\mathbb{G}^c(\mathcal{H})$.

First note that π indeed maps the bounded invertibles into the identity component of the invertibles of the Calkin algebra, due to the connectedness of $\mathbb{G}(\mathcal{H})$. Moreover, $\hat{\pi} : \mathbb{G}(\mathcal{H}) \to \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ is surjective because for each $\widehat{T} \in \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ there is an operator $S = \rho(\widehat{T}) \in \mathbb{F}\mathbb{B}(\mathcal{H})$ with $\mathrm{Ind}(S) = 0$ so that there exists a compact operator $K \in \mathbb{K}(\mathcal{H})$ such that $T = S + K$ is invertible, and clearly $\hat{\pi}(T) = \widehat{T}$. Now fix an operator $\widehat{T}_0$ with associated invertible lift T_0, set $K_0 = T_0 - \rho(\widehat{T}_0)$ and next consider a neighborhood $\mathcal{U}$ of $\widehat{T}_0$. By choosing $\mathcal{U}$ sufficiently small, there is a continuous injective map $\ell : \mathcal{U} \to \mathbb{G}(\mathcal{H})$ defined by $\ell(\widehat{T}) = \rho(\widehat{T}) + K_0$. Note that the image of ℓ lies indeed in the set $\mathbb{G}(\mathcal{H})$ of invertibles because $\mathbb{G}(\mathcal{H})$ is open in $\mathbb{B}(\mathcal{H})$. Moreover,

$$\hat{\pi}^{-1}(\{\widehat{T}\}) = \{\rho(\widehat{T}) + K \text{ invertible} : K \in \mathbb{K}(\mathcal{H})\} = (\rho(\widehat{T}) + K_0)\mathbb{G}^c(\mathcal{H}),$$

the latter because

$$\rho(\widehat{T}) + K = (\rho(\widehat{T}) + K_0)\big(\mathbf{1} + (\rho(\widehat{T}) + K_0)^{-1}(K - K_0)\big).$$

Hence $\hat{\pi}^{-1}(\mathcal{U})$ is homeomorphic to $\mathcal{U} \times \mathbb{G}^c(\mathcal{H})$ as claimed. (Note that the fiber bundle is actually a principal bundle with fiber group $\mathbb{G}^c(\mathcal{H})$.)

Fact 3. The homotopy groups $\pi_k(\mathbb{F}_0\mathbb{B}(\mathcal{H}))$ are as stated.

This now uses the long exact sequence of homotopy theory associated to the fiber bundle of Fact 2. It reduces to isomorphisms $\pi_k(\mathbb{G}_0\mathbb{Q}(\mathcal{H})) \cong \pi_{k-1}(\mathbb{G}^c(\mathcal{H}))$ because $\mathbb{G}(\mathcal{H})$ is contractible by Kuiper's theorem and hence has vanishing homotopy groups. Using Fact 1, one deduces $\pi_k(\mathbb{F}_0\mathbb{B}(\mathcal{H})) \cong \pi_{k-1}(\mathbb{G}^c(\mathcal{H}))$ and therefore (8.1) concludes the proof. □

The next aim is to consider the set of unbounded Fredholm operators $\mathbb{F}(\mathcal{H})$ as defined in Definition 6.2.1. They form a subset of the densely defined closed operators $\mathbb{L}(\mathcal{H})$ on which Section 6.1 studied two natural topologies, namely the Riesz and gap topologies. The definition of the Riesz topology is tightly linked to the bounded transform $\mathcal{F}(T) = T(\mathbf{1} + T^*T)^{-\frac{1}{2}}$ and this leads to Proposition 6.2.18 which states that the spaces $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}\mathbb{B}(\mathcal{H}), \mathcal{O}_N)$ are homotopy equivalent. This directly implies the following main result on the set of unbounded Fredholm operators.

Theorem 8.2.4. *The homotopy groups of* $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ *are the same as those of* $(\mathbb{FB}(\mathcal{H}), \mathcal{O}_N)$ *as given by Corollaries* 8.2.1 *and* 8.2.3.

Let us briefly comment on the space $(\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$. By the bounded transform, it is homeomorphic to $(\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_E)$, which in turn can be shown to be homeomorphic to $(\mathbb{FB}_1(\mathcal{H}), \mathcal{O}_{SE})$ by adapting the argument in the proof of Proposition 6.4.7 (note that d_E is, however, only a pseudometric on $\mathbb{FB}_1(\mathcal{H})$). In [154] it is shown that the identity provides a homotopy equivalence $I : (\mathbb{F}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$. Therefore the homotopy groups of $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}(\mathcal{H}), \mathcal{O}_G)$ coincide and are given by Theorem 8.2.4.

8.3 Homotopy groups of bounded self-adjoint Fredholm operators

Recall from Section 3.6 that the set $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ of bounded self-adjoint Fredholm operators equipped with the norm topology has three connected components $\mathbb{FB}_{\mathrm{sa}}^+(\mathcal{H})$, $\mathbb{FB}_{\mathrm{sa}}^-(\mathcal{H})$, and $\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})$, consisting respectively of those self-adjoint Fredholm operators having only positive essential spectrum, only negative essential spectrum, and having both positive and negative essential spectrum. The components $\mathbb{FB}_{\mathrm{sa}}^+(\mathcal{H})$ and $\mathbb{FB}_{\mathrm{sa}}^-(\mathcal{H})$ are contractible so that the main task here is to determine the homotopy groups of $\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})$.

Theorem 8.3.1. *With respect to the norm topology, the homotopy groups of* $\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})$ *are*

$$\pi_k\big(\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})\big) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Corollary 8.3.2. *The spectral flow on closed loops establishes an isomorphism*

$$\mathrm{Sf} : \pi_1\big(\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})\big) \to \mathbb{Z}.$$

Proof. Clearly, $\mathrm{Sf} : \pi_1(\mathbb{FB}_{\mathrm{sa}}^*(\mathcal{H})) = \mathbb{Z} \to \mathbb{Z}$ is a homomorphism. By Example 8.3.4 further down, this homomorphism is surjective. As every surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is injective, this implies the claim. □

The proof of Theorem 8.3.1 parallels that of Theorem 8.2.2, but there is an extra element stated first.

Lemma 8.3.3. *Let* $Q_0 \in \mathbb{U}_{\mathrm{sa}}^*(\mathcal{H})$ *be a proper symmetry with neighborhood*

$$\mathcal{U} = \big\{Q \in \mathbb{U}_{\mathrm{sa}}^*(\mathcal{H}) : \|Q - Q_0\| < 2\big\}.$$

Then there is a continuous map $Q \in \mathcal{U} \mapsto U \in \mathbb{U}(\mathcal{H})$ *such that*

$$Q = UQ_0U^*.$$

Proof. (Note that this is essentially the same argument as in the proof of Proposition 5.3.20.) The unitary will be explicitly constructed, using the orthogonal projections $P = \frac{1}{2}(\mathbf{1} - Q)$ and $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$. Consider the operator

$$M = \mathbf{1} + (P - P_0)(2P_0 - \mathbf{1}).$$

By assumption, $\|(P - P_0)(2P_0 - \mathbf{1})\| < 1$ so that M is invertible. One readily checks

$$PM = PP_0 = MP_0.$$

Hence also $M^*P = P_0M^*$. Therefore $P = MP_0M^{-1}$ and $M^*P(M^*)^{-1} = P_0$ so that upon replacing also

$$P = (MM^*)P(MM^*)^{-1}.$$

This implies $P = (MM^*)^{-\frac{1}{2}}P(MM^*)^{\frac{1}{2}}$. Now set

$$U = (MM^*)^{-\frac{1}{2}}M.$$

This is indeed unitary and satisfies the claim. □

Proof of Theorem 8.3.1.

Fact 1. $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ is homotopy equivalent to the set $\mathbb{G}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H})$ of self-adjoint invertible elements in the Calkin algebra having both positive and negative spectrum, which in turn can be retracted to the set $\mathbb{U}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H})$ of proper symmetries in the Calkin algebra

$$\mathbb{U}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H}) = \{\widehat{Q} = \widehat{Q}^* \in \mathbb{Q}(\mathcal{H}) : \mathrm{spec}(\widehat{Q}) = \{-1, 1\}\}.$$

For the proof of this fact, let $\pi_{\mathrm{sa}} : \mathbb{B}_{\mathrm{sa}}(\mathcal{H}) \to \mathbb{Q}_{\mathrm{sa}}(\mathcal{H})$ be the restriction of the Calkin projection π to the self-adjoint bounded operators. Then $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}) = \pi_{\mathrm{sa}}^{-1}(\mathbb{G}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H}))$. A continuous right inverse ρ_{sa} to π_{sa} is given in terms of the right inverse ρ of π by setting $\rho_{\mathrm{sa}}(\widehat{H}) = \frac{1}{2}(\rho(\widehat{H}) + \rho(\widehat{H})^*)$. Then $H \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ can be uniquely decomposed into $H = \rho_{\mathrm{sa}}(\pi_{\mathrm{sa}}(H)) + K$ with some $K \in \mathbb{K}_{\mathrm{sa}}(\mathcal{H})$, the set of self-adjoint compact operators. Hence $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ is homeomorphic to $\mathbb{G}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H}) \times \mathbb{K}_{\mathrm{sa}}(\mathcal{H})$. The contractibility of the self-adjoint compact operators then implies the first claim. The retraction to $\mathbb{U}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H})$ can then be done by spectral calculus.

Fact 2. Let $\mathbb{U}\mathbb{Q}(\mathcal{H})$ denote the unitary elements in the Calkin algebra and fix some proper symmetry $\widehat{Q}_0 \in \mathbb{U}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H})$. Then the map $\pi_0 : \mathbb{U}\mathbb{Q}(\mathcal{H}) \to \mathbb{U}\mathbb{Q}^*_{\mathrm{sa}}(\mathcal{H})$ defined by

$$\pi_0(\widehat{U}) = \widehat{U}\widehat{Q}_0\widehat{U}^*$$

is the base projection of a principal bundle with connected base space and fiber group given by the stabilizer group of $\widehat{Q}_0$,

$$G_0 = \{\widehat{U} \in \mathbb{UQ}(\mathcal{H}) : \widehat{U}\widehat{Q}_0\widehat{U}^* = \widehat{Q}_0\}.$$

For the justification, let us first note that for every $\widehat{Q} \in \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ there is a self-adjoint lift $T = \rho_{\mathrm{sa}}(\widehat{Q}) \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$, namely $\pi_{\mathrm{sa}}(T) = \pi(T) = \widehat{Q}$. Its essential spectrum is $\mathrm{spec}_{\mathrm{ess}}(T) = \{-1, 1\}$. Hence there is a gap $\Delta \subset (-1, 1)$ somewhere in the spectrum of T, and one can choose an increasing continuous function $f : \mathbb{R} \to [-1, 1]$ with $f(-1) = -1$, $f(1) = 1$, and $\mathrm{supp}(f') \subset \Delta$. Then $Q = f(T)$ is a symmetry on $\mathcal{H}$ and $\pi(Q) = \widehat{Q}$ because $Q - T \in \mathbb{K}_{\mathrm{sa}}(\mathcal{H})$. In particular, for $T_0 = \rho_{\mathrm{sa}}(\widehat{Q}_0)$ there is a symmetry $Q_0 = f(T_0)$ such that $\widehat{Q}_0 = \pi(Q_0)$. By continuity of the spectrum, there is a neighborhood $\mathcal{U}$ of $\widehat{Q}_0$ such that the function f can be chosen uniformly for all $\widehat{Q} \in \mathcal{U}$, and one obtains a continuous local map $\widehat{Q} \in \mathcal{U} \mapsto Q = f(\rho_{\mathrm{sa}}(\widehat{Q})) \in \mathbb{U}_{\mathrm{sa}}(\mathcal{H})$. As both Q and Q_0 are proper, there exists a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $Q = UQ_0U^*$ and the map $\widehat{Q} \in \mathcal{U} \mapsto U$ can be chosen continuously by Lemma 8.3.3. Then $\widehat{U} = \pi(U)$ is a unitary in the Calkin algebra and $\widehat{Q} = \widehat{U}\widehat{Q}_0\widehat{U}^*$. Thus $\widehat{\rho} : \mathcal{U} \to \mathbb{UQ}(\mathcal{H})$ defined by $\widehat{\rho}(\widehat{Q}) = \widehat{U}$ is a local section, namely $\widehat{\rho}$ is continuous and $\pi_0 \circ \widehat{\rho} = \mathrm{id}$. This fact combined with the bundle structure theorem (see the paragraph after the proof of Theorem A.3.7) implies that $\pi_0 : \mathbb{UQ}(\mathcal{H}) \to \mathbb{UQ}(\mathcal{H})/G_0 \approx \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ is a principal bundle. (Note that one can spell out a version of Lemma 8.3.3 directly for symmetries in the Calkin algebra and this shortens the proof a little.)

It remains to show that the base space is connected. Let $\widehat{Q}_0, \widehat{Q}_1 \in \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ have symmetry lifts Q_0 and Q_1 (constructed as above). As both Q_0 and Q_1 are proper, there exists a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $UQ_1U^* = Q_0$. Deforming U to $\mathbf{1}$ (e. g., taking roots of U) one obtains a path of symmetries connecting Q_0 to Q_1, and consequently also a path connecting $\widehat{Q}_0$ to $\widehat{Q}_1$. Hence $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ is indeed pathwise connected.

Fact 3. The homotopy groups $\pi_k(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$ are as stated.

The long exact sequence of homotopy theory for the principal bundle of Fact 2 combined with Fact 1 leads to

$$\cdots \to \pi_k(\mathbb{UQ}(\mathcal{H})) \to \pi_k(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})) \to \pi_{k-1}(G_0) \to \pi_{k-1}(\mathbb{UQ}(\mathcal{H})) \to \cdots .$$

The set $\mathbb{UQ}(\mathcal{H})$ of unitaries in the Calkin algebra is a retract (using the polar decomposition) of the set $\mathbb{GQ}(\mathcal{H})$ of the invertibles in the Calkin algebra. The connected components of $\mathbb{GQ}(\mathcal{H})$ are $\mathbb{G}_n\mathbb{Q}(\mathcal{H}) = \{\widehat{T} \in \mathbb{GQ}(\mathcal{H}) : \rho(\widehat{T}) \in \mathbb{F}_n\mathbb{B}(\mathcal{H})\}$ for $n \in \mathbb{Z}$. These components are homeomorphic and therefore have the same homotopy groups. The homotopy groups of $\mathbb{G}_0\mathbb{Q}(\mathcal{H})$ were determined in the proof of Theorem 8.2.2, so that

$$\pi_k(\mathbb{U}_0\mathbb{Q}(\mathcal{H})) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k \geq 2 \text{ even}, \end{cases}$$

where $\mathbb{U}_0\mathbb{Q}(\mathcal{H}) = \mathbb{UQ}(\mathcal{H}) \cap \mathbb{G}_0\mathbb{Q}(\mathcal{H})$ and therefore

$$\pi_k(\mathbb{UQ}(\mathcal{H})) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z}, & k \text{ even}. \end{cases} \tag{8.2}$$

Moreover, the stabilizer group G_0 consists of all $\widehat{U} \in \mathbb{UQ}(\mathcal{H})$ commuting with the projection $\widehat{P}_0 = \frac{1}{2}(\mathbf{1} - \widehat{Q}_0)$, hence all block diagonal unitaries with block form $\widehat{P}_0 \mathbb{Q}(\mathcal{H}) \widehat{P}_0$ and $(\mathbf{1} - \widehat{P}_0)\mathbb{Q}(\mathcal{H})(\mathbf{1} - \widehat{P}_0)$. The lift $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$ of P_0 with Q_0 as above is a projection, and one can identify $\widehat{P}_0 \mathbb{Q}(\mathcal{H}) \widehat{P}_0$ with $\mathbb{Q}(P_0\mathcal{H})$, and similarly with the other block. Hence

$$G_0 \cong \mathbb{UQ}(P_0\mathcal{H}) \oplus \mathbb{UQ}((\mathbf{1} - P_0)\mathcal{H}).$$

Consequently, the homotopy groups of G_0 can be read from (8.2) as

$$\pi_k(G_0) = \begin{cases} 0, & k \text{ odd}, \\ \mathbb{Z} \oplus \mathbb{Z}, & k \text{ even}. \end{cases}$$

Thus for k odd, the above exact sequence becomes

$$\cdots \to 0 \to \pi_k(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to \pi_{k-1}(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})) \to 0 \to \cdots,$$

where i_* is the induced map of the inclusion $i : G_0 \hookrightarrow \mathbb{UQ}(\mathcal{H})$. However, i_* is surjective (actually, it is just the addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ in the homotopy groups as one can check using the fact that $i : G_0 \cong \mathbb{UQ}(P_0\mathcal{H}) \oplus \mathbb{UQ}((\mathbf{1} - P_0)\mathcal{H}) \to \mathbb{UQ}(\mathcal{H})$ is the embedding as a block diagonal operator) and thus exactness implies that the homotopy groups of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ are as stated. □

Example 8.3.4. This example is a continuation of Example 5.7.4 in which $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $|n\rangle$, $n \in \mathbb{Z}$ and

$$Q_k = \sum_{n \geq k} |n\rangle\langle n| - \sum_{n<k} |n\rangle\langle n|.$$

Then $Q_k = U^* Q_0 U$ where U is the left-shift by k and $\mathrm{Sf}(Q_0, Q_k) = -k$. This path shall now be closed to a loop. For that purpose, let us first rotate the states in the subspace spanned by $|n\rangle$ and $|k - n\rangle$. For even k this is done by

$$V_s = \sum_{n > \frac{k}{2}} (|n-1\rangle |k-n\rangle) \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} \langle n-1| \\ \langle k-n| \end{pmatrix},$$

while for odd k

$$V_s = \sum_{n > \frac{k}{2}} (|n\rangle |k-n-1\rangle) \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} \langle n| \\ \langle k-n-1| \end{pmatrix} + |\tfrac{k-1}{2}\rangle\langle\tfrac{k-1}{2}|.$$

One readily checks that the rotation $s \in [0,1] \mapsto V_s^* Q_k V_s$ connects Q_k to $V_1^* Q_k V_1 = -Q_0$. Let us concatenate with a second path, given in terms of

$$\tilde{V}_s = \sum_{n\geq 0} (|n\rangle| - 1 - n\rangle) \begin{pmatrix} \cos(\frac{\pi}{2}s) & -\sin(\frac{\pi}{2}s) \\ \sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix} \begin{pmatrix} \langle n| \\ \langle -1-n| \end{pmatrix}.$$

Then $s \in [0,1] \mapsto \tilde{V}_s^*(-Q_0)\tilde{V}_s$ connects $-Q_0$ to $\tilde{V}_1^*(-Q_0)\tilde{V}_1 = Q_0$. These last two paths are isospectral, so there is no spectral flow. As a consequence, the spectral flow along the whole closed loop obtained after concatenation is equal to $-k$. ◇

In Example 8.3.4, a concrete path of self-adjoint Fredholm operators with nontrivial spectral flow was constructed from a pair of symmetries (Q, UQU^*) which satisfies $[U, Q]$ compact. Actually, there is a far more general statement that implies the existence of a such a path. For any fixed proper symmetry Q, let us set

$$\mathbb{U}_Q(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : [U, Q] \in \mathbb{K}(\mathcal{H})\}. \tag{8.3}$$

Note that $\mathbb{U}_Q(\mathcal{H})$ is a subgroup of $\mathbb{U}(\mathcal{H})$.

Proposition 8.3.5. *For any proper symmetry Q on $\mathcal{H}$ and any $k \in \mathbb{N}$, $\pi_k(\mathbb{U}_Q(\mathcal{H}))$ is isomorphic to $\pi_{k+1}(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$.*

Proof. First recall from the proof of Theorem 8.3.1 that $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is homotopy equivalent to the set $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ of proper symmetries in the Calkin algebra. Note that Q also provides a base point $\pi(Q)$ in $\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$. Now let us define a map $\beta_Q : \mathbb{U}(\mathcal{H}) \to \mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$ via $\beta_Q(U) = \pi(UQU^*)$. By the arguments in the proof of Theorem 8.3.1, one can check that this is a principal bundle with structure group

$$\begin{aligned} (\beta_Q)^{-1}(\pi(Q)) &= \{U \in \mathbb{U} : \pi(UQU^*) = \pi(Q)\} \\ &= \{U \in \mathbb{U} : \pi([U, Q]U^*) = 0\} \\ &= \mathbb{U}_Q(\mathcal{H}). \end{aligned}$$

Hence one can use the long exact sequence of homotopy groups, which, due to the triviality of the homotopy groups of $\mathbb{U}(\mathcal{H})$, proves the proposition. (Let us note that there is an equivalent statement to Proposition 8.3.5 for $\mathbb{Z}_2$-valued index pairings given in [48] and Theorem 7.1 of [57]; the argument in these works was based on the claim that the connecting maps in the long exact sequence of homotopy groups of a fibre bundle are induced by a homotopy equivalence, which was not proved there; this claim is not used here.) □

8.4 An application: characterization of spectral flow

The aim of this section is to present an axiomatic characterization of the spectral flow that is due to Lesch [126], with modifications taken from [184]. Let us denote by $\Omega^*(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$ the set of norm-continuous paths in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$. Let us stress that the paths are not necessarily closed, which is why the notation Ω^* instead of Ω is used. Let us consider maps $\mu : \Omega^*(\mathbb{FB}^*_{\rm sa}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ that are invariant under orientation-preserving reparametrizations of the paths. Such maps can have the following properties:

Homotopy invariance. If $(s,t) \in [0,1] \times [0,1] \mapsto H_{s,t} \in \mathbb{FB}^*_{\rm sa}(\mathcal{H})$ is norm-continuous and $H_{s,0}$ as well as $H_{s,1}$ are invertible for all $s \in [0,1]$, then

$$\mu(t \in [0,1] \mapsto H_{0,t}) = \mu(t \in [0,1] \mapsto H_{1,t}).$$

Concatenation. If $t \in [0,2] \mapsto H_t$ is an element of $\Omega^*(\mathbb{FB}^*_{\rm sa}(\mathcal{H}))$, then

$$\mu(t \in [0,2] \mapsto H_t) = \mu(t \in [0,1] \mapsto H_t) + \mu(t \in [1,2] \mapsto H_t).$$

Integrality. If $t \in [0,1] \mapsto H_t$ is a path in $\mathbb{FB}^*_{\rm sa}(\mathcal{H})$ with invertible endpoints, then

$$\mu(t \in [0,1] \mapsto H_t) \in \mathbb{Z}.$$

Normalization. For every $H \in \mathbb{FB}^*_{\rm sa}(\mathcal{H})$ and associated $H_t = H + t\mathbf{1}$, one has

$$\mu(t \in [-\delta_H, 0] \mapsto H_t) = \mu(t \in [0, \delta_H] \mapsto H_t) = \frac{1}{2}\dim(\mathrm{Ker}(H)),$$

where $\delta_H = \frac{1}{2}\min\{|\lambda| : 0 \neq \lambda \in \mathrm{spec}(H)\}$.

Note that the spectral flow is a map $\mathrm{Sf} : \Omega^*(\mathbb{FB}^*_{\rm sa}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ that satisfies concatenation by Theorem 4.2.1 and homotopy invariance by Theorem 4.2.4. Moreover, integrality and normalization are immediate consequences of the definition of the spectral flow.

Theorem 8.4.1. *If $\mu : \Omega^*(\mathbb{FB}^*_{\rm sa}(\mathcal{H})) \to \frac{1}{2}\mathbb{Z}$ satisfies homotopy invariance, concatenation, integrality, and normalization, then $\mu = \mathrm{Sf}$.*

Proof. Let $H_0 \in \mathbb{FB}^*_{\rm sa}(\mathcal{H})$ have a one-dimensional kernel and set $\delta_0 = \delta_{H_0}$. Then let $t \in [-\delta_0, \delta_0] \mapsto H_t = H_0 + t\mathbf{1}$ be the corresponding path. By the concatenation, homotopy invariance, and integrality properties, it is clear that μ and Sf induce homomorphisms

$$\mu, \mathrm{Sf} : \pi_1(\mathbb{FB}^*_{\rm sa}(\mathcal{H}), H_{-\delta_0}) \to \mathbb{Z}, \tag{8.4}$$

where $\pi_1(\mathbb{FB}^*_{\rm sa}(\mathcal{H}), H_{-\delta_0})$ is the fundamental group. As the set of invertible operators in $\mathbb{FB}^*_{\rm sa}(\mathcal{H})$ is connected, there is a path $t \in [0,1] \mapsto H'_t$ of invertible operators connecting H_{δ_0} to $H_{-\delta_0}$ in $\mathbb{FB}^*_{\rm sa}(\mathcal{H})$. Now the (time rescaled) concatenation $t \in [0,1] \mapsto (H * H')_t$ is an element in $\pi_1(\mathbb{FB}^*_{\rm sa}(\mathcal{H}), H_{-\delta_0})$ and thus

$$\begin{aligned}\mu(t \in [0,1] \mapsto (H * H')_t) &= \mu(t \in [0,1] \mapsto H_t) + \mu(t \in [0,1] \mapsto H'_t)\\ &= \mu(t \in [0,1] \mapsto H_t)\\ &= \dim(\mathrm{Ker}(H_0))\\ &= \mathrm{Sf}(t \in [0,1] \mapsto H_t) + \mathrm{Sf}(t \in [0,1] \mapsto H'_t)\end{aligned}$$

$$= \mathrm{Sf}(t \in [0,1] \mapsto (H * H')_t),$$

where $\mu(t \in [0,1] \mapsto H'_t) = 0 = \mathrm{Sf}(t \in [0,1] \mapsto H'_t)$ was used which follows from the homotopy invariance and concatenation. As $\dim(\mathrm{Ker}(H_0)) = 1$, this firstly shows that $t \in [0,1] \mapsto (H * H')_t$ is a generator of the infinitely cyclic group $\pi_1(\mathbb{FB}^*_{\mathrm{sa}}, H_{-\delta_0})$ (see Theorem 8.3.1), and secondly that μ and Sf have the same value on it. Hence the maps in (8.4) coincide. Note that this also holds for any other invertible base point than $H_{-\delta_0}$, which follows by using once again that the set of invertible elements in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is connected.

Let now $t \in [0,1] \mapsto H_t$ be an arbitrary norm-continuous path in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$. Let us first consider the endpoints H_0 and H_1 and set

$$t \in [-\delta_0, 0] \mapsto H^0_t = H_0 + t\mathbf{1}, \quad t \in [0, \delta_1] \mapsto H^1_t = H_1 + t\mathbf{1},$$

where still $\delta_0 = \delta_{H_0}$ and $\delta_1 = \delta_{H_1}$. It follows from the normalization property that

$$\mathrm{Sf}(t \in [-\delta_0, 0] \mapsto H^0_t) = \mu(t \in [-\delta_0, 0] \mapsto H^0_t) = \frac{1}{2}\dim(\mathrm{Ker}(H_0))$$

and

$$\mathrm{Sf}(t \in [0, \delta_1] \mapsto H^1_t) = \mu(t \in [0, \delta_1] \mapsto H^1_t) = \frac{1}{2}\dim(\mathrm{Ker}(H_1)).$$

Let now $t \in [0,1] \mapsto \tilde{H}_t$ be a path of invertible operators in $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ connecting $H^1_{\delta_1}$ to $H^0_{-\delta_0}$. It follows from the first part of the proof and the concatenation property that

$$\begin{aligned}
&\mu(t \in [0,1] \mapsto H_t) \\
&\quad = \mu(t \in [0,1] \mapsto (H^0 * H * H^1)_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)) \\
&\quad = \mu(t \in [0,1] \mapsto (H^0 * H * H^1 * \tilde{H})_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)) \\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto (H^0 * H * H^1 * \tilde{H})_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)) \\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto (H^0 * H * H^1)_t) - \frac{1}{2}\dim(\mathrm{Ker}(H_0)) - \frac{1}{2}\dim(\mathrm{Ker}(H_1)) \\
&\quad = \mathrm{Sf}(t \in [0,1] \mapsto H_t),
\end{aligned}$$

and so the claim is shown. □

Let us note that Theorem 8.4.1 slightly differs from Lesch's work [126] as here it is not assumed that the endpoints of the paths in $\Omega^*(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$ are invertible. There are other axiomatic characterizations of the spectral flow. For example, Ciriza, Fitzpatrick, and Pejsachowicz showed in [60] that the spectral flow for paths in all three components of $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ is also uniquely determined by the homotopy invariance in Theorem 4.2.2, the two basic properties (i) and (v) in Theorem 4.2.1 and the fact that it is the difference of the Morse indices of the endpoints for paths in $\mathbb{FB}^+_{sa}(\mathcal{H})$ (see Proposition 4.3.1). Finally, let us note that Georgescu proved a similar characterization for paths of unbounded self-adjoint Fredholm operators [93].

8.5 Homotopy groups of Fredholm pairs

Let us introduce a notation for the set of proper orthogonal projections which form a Fredholm pair with a fixed proper orthogonal projections P_{ref}:

$$\mathbb{FP}(\mathcal{H}) = \{P : P \text{ proper orthogonal projection with } (P_{\mathrm{ref}}, P) \text{ Fredholm pair}\}.$$

This set will be equipped with norm topology $\mathcal{O}_N$ on $\mathbb{B}(\mathcal{H})$. Results on the homotopy groups of $\mathbb{FP}(\mathcal{H})$ and various modifications of it go back to Wojciechowski [207] and Abbondandolo and Majer [2].

Theorem 8.5.1. *The homotopy groups of* $\mathbb{FP}(\mathcal{H})$ *are given by*

$$\pi_k(\mathbb{FP}(\mathcal{H})) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

To show this, let us introduce the set

$$\mathbb{FP}^{\mathrm{C}}(\mathcal{H}) = \{P \in \mathbb{FP}(\mathcal{H}) : P_{\mathrm{ref}} - P \in \mathbb{K}(\mathcal{H})\}$$

of orthogonal projections P such that $P - P_{\mathrm{ref}}$ is compact. The proof of Theorem 8.5.1 is then based on the following fact.

Proposition 8.5.2. *The space* $(\mathbb{FP}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_N)$ *is a deformation retract of* $(\mathbb{FP}(\mathcal{H}), \mathcal{O}_N)$.

Proof. (Based on the proof of Proposition 5.3.19.) For $P \in \mathbb{FP}(\mathcal{H})$, let $Q = \mathbf{1} - 2P$ be the associated symmetry and let $Q_{\mathrm{ref}} = \mathbf{1} - 2P_{\mathrm{ref}}$ be the symmetry associated to P_{ref} and then set

$$R = QQ_{\mathrm{ref}} + Q_{\mathrm{ref}}Q = 2\mathbf{1} - 4(P - P_{\mathrm{ref}})^2.$$

Then one has $[R, Q] = 0 = [R, Q_{\mathrm{ref}}]$. Let us set $a = \sup \mathrm{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2)$. Because (P_{ref}, P) is a Fredholm pair, $1 \notin \mathrm{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2)$ and therefore $a \in [0, 1)$. Then also $b = \min\{\frac{a+1}{2}, 2a\} \in [0, 1)$ and the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \chi_{[0,a]}(x) + (x - b)(a - b)^{-1}\chi_{(a,b)}(x) \tag{8.5}$$

is continuous for $a > 0$. For

$$H_t = \mathbf{1} + R\cos\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big)\sin\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big), \quad t \in [0,1],$$

clearly, $[H_t,(P-P_{\text{ref}})^2] = 0$, and next it is shown that H_t is invertible for all t. Setting $\mathcal{H}' = \text{Ran}(\chi((P-P_{\text{ref}})^2 \geq b))$ the restriction of H_t to this space is $\mathbf{1}_{\mathcal{H}'}$. On the orthogonal complement $(\mathcal{H}')^\perp$ one has $(P-P_{\text{ref}})^2 \leq b$. Therefore $R\mathbf{1}_{(\mathcal{H}')^\perp} > (2-4b)\mathbf{1}_{(\mathcal{H}')^\perp}$ and, because $\|\cos(\frac{\pi}{2}tf((P-P_{\text{ref}})^2))\sin(\frac{\pi}{2}tf((P-P_{\text{ref}})^2))\| \leq \frac{1}{2}$, one gets $H_t\mathbf{1}_{(\mathcal{H}')^\perp} > (1+\frac{1}{2}(2-4b))\mathbf{1}_{(\mathcal{H}')^\perp} = (2-2b)\mathbf{1}_{(\mathcal{H}')^\perp} > 0$. Combined with the fact that $[H_t,(P-P_{\text{ref}})^2] = 0$ and therefore H_t is diagonal with respect to the grading $\mathcal{H} = \mathcal{H}' \oplus (\mathcal{H}')^\perp$, this implies that H_t is invertible for all t. Therefore one can set

$$Q_t = (H_t)^{-\frac{1}{2}}\Big(Q\cos\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big) + Q_{\text{ref}}\sin\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big)\Big), \quad t \in [0,1].$$

Clearly, $Q_t^* = Q_t$ and computing the square shows $Q_t^2 = \mathbf{1}$, so this is a path of symmetries. Moreover, $Q_0 = Q$. To show that (P_{ref}, P_t) is a Fredholm pair for all $t \in [0,1]$ where $P_t = \frac{1}{2}(\mathbf{1}-Q_t)$, let us compute

$$(P_t-P_{\text{ref}})^2 = \frac{1}{2}\mathbf{1} - \frac{1}{4}(H_t)^{-\frac{1}{2}}\Big(R\cos\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big) + 2\sin\Big(\frac{\pi}{2}tf((P-P_{\text{ref}})^2)\Big)\mathbf{1}\Big).$$

Suppose that $a > 0$. Then the right-hand side is a continuous function of the self-adjoint operator $(P-P_{\text{ref}})^2$. Namely,

$$(P_t-P_{\text{ref}})^2 = g_t((P-P_{\text{ref}})^2)$$

for the continuous function $g_t : [0,1] \to [0,1]$ defined by

$$\begin{aligned} g_t(x) = \frac{1}{2} - \frac{1}{4}\Big(1+(2-4x)\cos\Big(\frac{\pi}{2}tf(x)\Big)\sin\Big(\frac{\pi}{2}tf(x)\Big)\Big)^{-\frac{1}{2}} \\ \cdot\Big((2-4x)\cos\Big(\frac{\pi}{2}tf(x)\Big) + 2\sin\Big(\frac{\pi}{2}tf(x)\Big)\Big). \end{aligned} \tag{8.6}$$

By the spectral mapping theorem in the Calkin algebra, one gets $\text{spec}_{\text{ess}}((P_t-P_{\text{ref}})^2) = g_t(\text{spec}_{\text{ess}}((P-P_{\text{ref}})^2))$ and therefore

$$\sup \text{spec}_{\text{ess}}((P_t-P_{\text{ref}})^2) \leq \sup_{t\in[0,1],x\in[0,a]} g_t(x) \leq \sup_{t\in[0,1],x\in[0,a]} h_t(x)$$

for

$$h_t(x) = \frac{1}{2} - \frac{1}{4}\Big(1+(2-4x)\cos\Big(\frac{\pi}{2}t\Big)\sin\Big(\frac{\pi}{2}t\Big)\Big)^{-\frac{1}{2}}\Big((2-4x)\cos\Big(\frac{\pi}{2}t\Big) + 2\sin\Big(\frac{\pi}{2}t\Big)\Big). \tag{8.7}$$

The supremum is given by

$$\sup_{t\in[0,1],x\in[0,a]} h_t(x) = \sup_{x\in[0,a]} h_0(x) = \sup_{x\in[0,a]} \frac{1}{2} - \frac{1}{4}(2-4x) = a < 1.$$

Then $\sup \operatorname{spec}_{\mathrm{ess}}((P_t - P_{\mathrm{ref}})^2) < 1$ by Corollary 5.3.13 implies that (P_{ref}, P_t) is a Fredholm pair for all $t \in [0,1]$. Moreover, $\operatorname{spec}_{\mathrm{ess}}((P_1 - P_{\mathrm{ref}})^2) = g_1(\operatorname{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2)) = \{0\}$ and therefore $P_1 - P_{\mathrm{ref}}$ is compact.

Suppose that $P - P_{\mathrm{ref}}$ is compact, or equivalently that $a = 0$. Then it follows that $f((P_t - P_{\mathrm{ref}})^2) = \chi((P_t - P_{\mathrm{ref}})^2 = 0)$ is the projection onto $\operatorname{Ker}((P_t - P_{\mathrm{ref}})^2) = \operatorname{Ker}(P_t - P_{\mathrm{ref}})$. We show that in this case $t \in [0,1] \mapsto Q_t$ is constant. Clearly, Q_t commutes with $(P_t - P_{\mathrm{ref}})^2$ and thus Q_t is diagonal with respect to the grading $\mathcal{H} = \operatorname{Ker}(P_t - P_{\mathrm{ref}}) \oplus \operatorname{Ker}(P_t - P_{\mathrm{ref}})^{\perp}$. On $\operatorname{Ker}(P_t - P_{\mathrm{ref}})^{\perp}$ one has $f((P_t - P_{\mathrm{ref}})^2) = 0$ and thus $Q_t = Q$. On $\operatorname{Ker}(P_t - P_{\mathrm{ref}})$ one has $Q = Q_{\mathrm{ref}}$ and therefore

$$Q_t = \left(\mathbf{1} + 2\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}} \left(Q\cos\left(\frac{\pi}{2}t\right) + Q\sin\left(\frac{\pi}{2}t\right)\right) = Q.$$

Thus $Q_t = Q$ for all $t \in [0,1]$ on all of $\mathcal{H}$.

Next let us consider the homotopy

$$h : \mathbb{FP}(\mathcal{H}) \times [0,1] \to \mathbb{FP}(\mathcal{H}), \quad h(P,t) = \frac{1}{2}(\mathbf{1} - Q_t)$$

It is shown that h is continuous at any point $(P,t) \in \mathbb{FP}(\mathcal{H}) \times [0,1]$. This is verified by a rather lengthy argument in the remainder of the proof which an experienced reader may want to skip.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{FP}(\mathcal{H})$ converging to P. Associated to it is the sequence $(Q_n)_{n\in\mathbb{N}}$ of symmetries, where $Q_n = \mathbf{1} - 2P_n$. Moreover, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ converging to t. Let us first assume that $a = \sup \operatorname{spec}_{\mathrm{ess}}((P - P_{\mathrm{ref}})^2) > 0$. Then for n sufficiently large, $a_n = \sup \operatorname{spec}_{\mathrm{ess}}((P_n - P_{\mathrm{ref}})^2) > 0$ and $b_n = \min\{\frac{a_n+1}{2}, 2a_n\} \in (a_n, 1)$, and the function $f_n : [0,1] \to [0,1]$ defined by

$$f_n(x) = \chi_{[0,a_n]}(x) + (x - b_n)(a_n - b_n)^{-1}\chi_{(a_n,b_n)}(x)$$

is continuous. Moreover,

$$\sup_{x\in[0,1]} \left|f(x) - f_n(x)\right| \to 0 \tag{8.8}$$

for $f : [0,1] \to [0,1]$ as in (8.5). Furthermore, let us set

$$R_n = Q_n Q_{\mathrm{ref}} + Q_{\mathrm{ref}} Q_n = 2\mathbf{1} - 4(P_n - P_{\mathrm{ref}})^2$$

and

$$H_{n,t} = \mathbf{1} + R_n \cos\left(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\right) \sin\left(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\right), \quad t \in [0,1],$$

which by the same argument as above is invertible with inverse bounded by

$$\|H_{n,t}^{-1}\| \leq \min\{1, 2 - 2b_n\}^{-1}. \tag{8.9}$$

Clearly,

$$\|h(P_n, t_n) - h(P, t)\| \leq \|h(P_n, t_n) - h(P_n, t)\| + \|h(P_n, t) - h(P, t)\|.$$

For the first summand, one has

$$\begin{aligned}
&\|h(P_n, t_n) - h(P_n, t)\| \\
&\quad = \frac{1}{2}\Big\|(H_{n,t_n})^{-\frac{1}{2}}\Big(Q_n \cos\Big(\frac{\pi}{2}t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2}t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big) \\
&\quad\quad - (H_{n,t})^{-\frac{1}{2}}\Big(Q_n \cos\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big)\Big\| \\
&\quad \leq \frac{1}{2}\Big\|(H_{n,t})^{-\frac{1}{2}}\Big[Q_n\Big(\cos\Big(\frac{\pi}{2}t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big) \\
&\quad\quad + Q_{\mathrm{ref}}\Big(\sin\Big(\frac{\pi}{2}t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big)\Big]\Big\| \\
&\quad\quad + \frac{1}{2}\Big\|((H_{n,t_n})^{-\frac{1}{2}} - (H_{n,t})^{-\frac{1}{2}}) \\
&\quad\quad \cdot \Big(Q_n \cos\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2}t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big)\Big\| \\
&\quad \leq \frac{1}{2}\sup_{n,t}\|(H_{n,t})^{-\frac{1}{2}}\|\Big(\sup_{x\in[0,1]}\Big|\cos\Big(\frac{\pi}{2}t_n x\Big) - \cos\Big(\frac{\pi}{2}tx\Big)\Big| + \sup_{x\in[0,1]}\Big|\sin\Big(\frac{\pi}{2}t_n x\Big) - \sin\Big(\frac{\pi}{2}tx\Big)\Big|\Big) \\
&\quad\quad + \|(H_{n,t_n})^{-\frac{1}{2}} - (H_{n,t})^{-\frac{1}{2}}\|.
\end{aligned}$$

Clearly, the first summand converges to 0 for $t_n \to t$ (uniformly in P_n). To bound the second summand, let us first note that $\lim_{n\to\infty} b_n = b$ and therefore by (8.9) $\|H_{n,t}^{-1}\|$ is uniformly bounded in t and P_n, namely the is a constant $C \in \mathbb{R}_{>0}$ such that

$$\sup_{t\in[0,1], n\in\mathbb{N}} \|H_{n,t}^{-1}\| < C.$$

Then

$$\begin{aligned}
\|(H_{n,t_n})^{-\frac{1}{2}} - (H_{n,t})^{-\frac{1}{2}}\| &= \|(H_{n,t_n})^{-\frac{1}{2}}((H_{n,t})^{\frac{1}{2}} - (H_{n,t_n})^{\frac{1}{2}})(H_{n,t})^{-\frac{1}{2}}\| \\
&\leq \|(H_{n,t_n})^{-\frac{1}{2}}\|\|(H_{n,t})^{\frac{1}{2}} - (H_{n,t_n})^{\frac{1}{2}}\|\|(H_{n,t})^{-\frac{1}{2}}\| \\
&\leq C\|H_{n,t} - H_{n,t_n}\|^{\frac{1}{2}},
\end{aligned} \tag{8.10}$$

where the last step follows from Proposition A.2.2. Because

$$\begin{aligned}\|H_{n,t} - H_{n,t_n}\| = \Big\| R_n \Big[&\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) \sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) \\ &- \cos\Big(\frac{\pi}{2} t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big) \sin\Big(\frac{\pi}{2} t_n f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le 2 \sup_{x\in[0,1]} \Big|\cos\Big(\frac{\pi}{2} tx\Big)\sin\Big(\frac{\pi}{2} tx\Big) - \cos\Big(\frac{\pi}{2} t_n x\Big)\sin\Big(\frac{\pi}{2} t_n x\Big)\Big|,\end{aligned}$$

one has

$$\lim_{m\to 0} \|H_{n,t_m} - H_{n,t}\| = 0$$

uniformly in n. Thus $\lim_{t_n\to t} \|h(P_n, t_n) - h(P_n, t)\| = 0$ uniformly in P_n. It remains to show $\lim_{n\to\infty} \|h(P_n, t) - h(P, t)\| = 0$. One has

$$\begin{aligned}&\|h(P_n, t) - h(P, t)\| \\ &= \Big\|(H_{n,t})^{-\frac{1}{2}} \Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big] \\ &\quad - (H_t)^{-\frac{1}{2}} \Big[Q \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le \Big\|(H_t)^{-\frac{1}{2}} \Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - Q \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big) \\ &\quad + Q_{\mathrm{ref}}\Big(\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big)\Big]\Big\| \\ &\quad + \Big\|((H_{n,t})^{-\frac{1}{2}} - (H_t)^{-\frac{1}{2}}) \\ &\quad \cdot \Big[Q_n \cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) + Q_{\mathrm{ref}} \sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big]\Big\| \\ &\le \|(H_t)^{-\frac{1}{2}}\| \Big(\Big\|(Q_n - Q)\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \Big\|Q\Big(\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big)\Big\| \\ &\quad + \Big\|\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big\|\Big) \\ &\quad + 2\|(H_{n,t})^{-\frac{1}{2}} - (H_t)^{-\frac{1}{2}}\| \\ &\le \|(H_t)^{-\frac{1}{2}}\|\|Q_n - Q\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\|\Big\|\cos\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\|\Big\|\cos\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big) - \cos\Big(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\Big)\Big\| \\ &\quad + \|(H_t)^{-\frac{1}{2}}\|\Big\|\sin\Big(\frac{\pi}{2} t f_n((P_n - P_{\mathrm{ref}})^2)\Big) - \sin\Big(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\Big)\Big\|\end{aligned}$$

$$+ \|(H_t)^{-\frac{1}{2}}\| \left\| \sin\left(\frac{\pi}{2} t f((P_n - P_{\mathrm{ref}})^2)\right) - \sin\left(\frac{\pi}{2} t f((P - P_{\mathrm{ref}})^2)\right) \right\|$$
$$+ C\|H_{n,t} - H_t\|^{\frac{1}{2}},$$

where the last step follows from a similar argument as that leading to (8.10). Clearly, the first and last summands converge to 0 for $n \to \infty$. Because f is a continuous function on $[0,1] \supset \bigcup_{n\in\mathbb{N}} \operatorname{spec}((P_n - P_{\mathrm{ref}})^2) \cup \operatorname{spec}((P - P_{\mathrm{ref}})^2))$ the third and fifth summands converge to 0. Finally the second and fourth summands converge to 0 by the same argument using (8.8). This shows the claim for $a > 0$. Finally, let us consider the case $a = 0$. Then, by the above, $P_t = P$ for all $t \in [0,1]$. One has to show

$$\|h(P_n, t_n) - P\| \to 0,$$

or equivalently

$$\|\tilde{Q}_n - Q\| \to 0,$$

for $\tilde{Q}_n = \mathbf{1} - 2h(P_n, t_n)$. Note that by the spectral radius theorem in the Calkin algebra $a_n = \sup \operatorname{spec}_{\mathrm{ess}}((P_n - P_{\mathrm{ref}})^2) \to 0$. For $\frac{1}{2} > \epsilon > 0$, there is $\lambda_\epsilon \in (0,\epsilon) \setminus \operatorname{spec}(P_{\mathrm{ref}} - P)^2$. Note that

$$\|\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| \to 0.$$

Then for n sufficiently large, $a_n < \frac{1}{2}\lambda_\epsilon$ and therefore

$$\tilde{Q}_n(\mathbf{1} - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)) = Q_n(\mathbf{1} - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)).$$

One then has

$$\|\tilde{Q}_n - Q\| \le \|(\tilde{Q}_n - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|(\tilde{Q}_n - Q)(\mathbf{1} - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\|$$

and the second term is bounded by

$$\begin{aligned}
&\|(\tilde{Q}_n - Q)(\mathbf{1} - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad \le \|\tilde{Q}_n(\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\qquad + \|(Q_n - Q)(\mathbf{1} - \chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\qquad + \|Q(\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon))\| \\
&\quad \le 2\|\chi((P_n - P_{\mathrm{ref}})^2 \le \lambda_\epsilon) - \chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\| + \|Q_n - Q\| \\
&\quad \to 0.
\end{aligned}$$

For the first summand, one has

$$\|(\tilde{Q}_n - Q)\chi((P - P_{\mathrm{ref}})^2 \le \lambda_\epsilon)\|$$

$$
\begin{aligned}
&= \big\|(Q_{\text{ref}} - \tilde{Q}_n)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big) - (Q_{\text{ref}} - Q)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| \\
&\le \big\|(Q_{\text{ref}} - \tilde{Q}_n)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\| + \|(Q_{\text{ref}} - Q)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| \\
&\le \big\|(Q_{\text{ref}} - \tilde{Q}_n)\big(\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big) - \chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big)\big\| \\
&\quad + \big\|(Q_{\text{ref}} - \tilde{Q}_n)\chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| + \big\|(Q_{\text{ref}} - Q)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\|.
\end{aligned}
$$

The first summand is bounded by

$$
\begin{aligned}
&\big\|(Q_{\text{ref}} - \tilde{Q}_n)\big(\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big) - \chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big)\big\| \\
&\quad \le 2\big\|\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big) - \chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| \\
&\quad \to 0.
\end{aligned}
$$

For the third summand, one has

$$
\begin{aligned}
\big\|(Q_{\text{ref}} - Q)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| &= \big\|(2P_{\text{ref}} - 2P)\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| \\
&= 2\big\|(P_{\text{ref}} - P)^2\chi\big((P - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\|^{\frac{1}{2}} \\
&\le 2\lambda_\epsilon^{\frac{1}{2}} \\
&\le 2\epsilon^{\frac{1}{2}}.
\end{aligned}
$$

And finally, the second summand is bounded by

$$
\begin{aligned}
\big\|(Q_{\text{ref}} - \tilde{Q}_n)\chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| &= \big\|(2P_{\text{ref}} - 2h(P_n, t_n))\chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\| \\
&= 2\big\|\big(P_{\text{ref}} - h(P_n, t_n)\big)^2\chi\big((P_n - P_{\text{ref}})^2 \le \lambda_\epsilon\big)\big\|^{\frac{1}{2}} \\
&\le 2\Big(\sup_{t\in[0,1],x\in[0,\lambda_\epsilon]} h_t(x)\Big)^{\frac{1}{2}} \\
&= 2\lambda_\epsilon^{\frac{1}{2}} \\
&\le 2\epsilon^{\frac{1}{2}},
\end{aligned}
$$

where $h_t(x)$ is defined in (8.7). Because $\epsilon > 0$ was arbitrary $\|\tilde{Q}_n - Q\| \to 0$ follows and therefore the considered homotopy is continuous. Thus, one can conclude that $(\mathbb{FP}^{\mathrm{C}}(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{FP}(\mathcal{H}), \mathcal{O}_N)$. □

For $n \in \mathbb{Z}$ let us introduce the sets

$$
\mathbb{F}_n\mathbb{P}(\mathcal{H}) = \{P \in \mathbb{FP}(\mathcal{H}) : \operatorname{Ind}(P_{\text{ref}}, P) = n\}
$$

and

$$
\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H}) = \{P \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H}) : \operatorname{Ind}(P_{\text{ref}}, P) = n\}.
$$

The next result shows that these are the connected components of $\mathbb{FP}(\mathcal{H})$ and $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$, respectively.

Proposition 8.5.3. *The sets $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ and $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ are connected with respect to the operator norm. Moreover, the space $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ is homeomorphic to $\mathbb{F}_0\mathbb{P}(\mathcal{H})$ and $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ is homeomorphic to $\mathbb{F}_0\mathbb{P}^{\mathrm{C}}(\mathcal{H})$.*

Proof. The argument leading to Proposition 5.3.23 shows that both $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ and $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ are connected.

To show that $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ is homeomorphic to $\mathbb{F}_0\mathbb{P}(\mathcal{H})$, let $P_{\mathrm{ref},n} \in \mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ be a fixed projection. Then by Corollary 5.3.13, for any projection $P \in \mathbb{P}(\mathcal{H})$, (P_{ref}, P) is a Fredholm pair if and only if the pair $(P_{\mathrm{ref},n}, P)$ is Fredholm. By Proposition 5.3.15,

$$\mathbb{F}_n\mathbb{P}(\mathcal{H}) = \{P \in \mathbb{P}(\mathcal{H}) : (P_{\mathrm{ref},n}, P) \text{ Fredholm}, \mathrm{Ind}(P_{\mathrm{ref},n}, P) = 0\}.$$

Moreover, by Proposition 5.1.7, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $U^* P_{\mathrm{ref},n} U = P_{\mathrm{ref}}$. Then, by the above, one has $P \in \mathbb{F}_n\mathbb{P}(\mathcal{H})$ if and only if $U^*PU \in \mathbb{F}_0\mathbb{P}(\mathcal{H})$. Therefore

$$f : \mathbb{F}_n\mathbb{P}(\mathcal{H}) \to \mathbb{F}_0\mathbb{P}(\mathcal{H}), \quad P \mapsto U^*PU$$

is a homeomorphism. Thus the claim on $\mathbb{F}_n\mathbb{P}(\mathcal{H})$ is shown. Restricting f to $\mathbb{F}_n\mathbb{P}^{\mathrm{C}}(\mathcal{H})$ implies the last claim. □

Proof of Theorem 8.5.1. By Proposition 8.5.2, the homotopy groups of $\mathbb{FP}(\mathcal{H})$ and $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ coincide, namely $\pi_k(\mathbb{FP}(\mathcal{H})) = \pi_k(\mathbb{FP}^{\mathrm{C}}(\mathcal{H}))$ for all $k \in \mathbb{N}_0$.

Recall from (8.3) that $\mathbb{U}_{Q_{\mathrm{ref}}}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : [U, Q_{\mathrm{ref}}] \in \mathbb{K}(\mathcal{H})\}$ denotes the set of all unitaries that have a compact commutator with $Q_{\mathrm{ref}} = \mathbf{1} - 2P_{\mathrm{ref}}$. Then the map $\pi_0 : \mathbb{U}_{Q_{\mathrm{ref}}}(\mathcal{H}) \to \mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ defined by

$$\pi_0(U) = UP_{\mathrm{ref}}U^*$$

is the base projection of a fiber bundle with fiber given by

$$\mathbb{U}_{Q_{\mathrm{ref}},0}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : [U, Q_{\mathrm{ref}}] = 0\}. \tag{8.11}$$

For the justification, let us first note that π_0 is surjective. For $P_0 \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\mathrm{ref}}U^* = P_0\}.$$

For the neighborhood

$$\mathcal{U} = \{P \in \mathbb{FP}^{\mathrm{C}}(\mathcal{H}) : \|P - P_0\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\text{ref}}U^* \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \pi_0^{-1}(\mathcal{U})$, $P \mapsto V_P$ such that $P_{\text{ref}} = V_P^* P V_P$. For $U \in \pi_0^{-1}(P)$, one gets $UP_{\text{ref}}U^* = P = V_P P_{\text{ref}} V_P^*$ and therefore $[P_{\text{ref}}, U^* V_P] = 0$. Thus

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$$

defined by

$$\phi(U) = \left(UP_{\text{ref}}U^*, U^* V_{UP_{\text{ref}}U^*}\right)$$

is a homeomorphism.

Even though the base space $\mathbb{FP}^{\mathrm{C}}(\mathcal{H})$ is not connected, Proposition 8.5.3 implies that the homotopy groups of all connected components are the same. Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains isomorphisms $\pi_k(\mathbb{FP}^{\mathrm{C}}(\mathcal{H})) \cong \pi_k(\mathbb{U}_{Q_{\text{ref}}}(\mathcal{H}))$ because $\mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$ is contractible by Kuiper's theorem and hence has vanishing homotopy groups. Then Proposition 8.3.5 combined with Theorem 8.3.1 shows the claim. □

Recall from (5.19) that $\mathbb{FPP}(\mathcal{H})$ denotes the set of Fredholm pairs of proper orthogonal projections. This set is then equipped with the topology $\mathcal{O}_N \times \mathcal{O}_N$ where $\mathcal{O}_N$ denotes the norm topology on $\mathbb{B}(\mathcal{H})$. The following result goes back to Abbondandolo and Majer [2], but the proof below is different.

Theorem 8.5.4. *The homotopy groups of* $\mathbb{FPP}(\mathcal{H})$ *are given by*

$$\pi_k\big(\mathbb{FPP}(\mathcal{H})\big) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

The proof is based on the following fact on the homotopy groups of the set:

$$\mathbb{P}(\mathcal{H}) = \{P = P^* = P^2 \in \mathbb{B}(H) : \dim(\operatorname{Ran}(P)) = \dim(\operatorname{Ker}(P))\}$$

of all proper orthogonal projections on $\mathcal{H}$, which goes back to [15].

Proposition 8.5.5. *The space* $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ *is contractible.*

Proof. First note that, by Proposition 5.1.7, the space $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is connected. Let, as above, $P_{\text{ref}} \in \mathbb{P}(\mathcal{H})$ be one fixed proper orthogonal projection. Then $\pi_0 : \mathbb{U}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ defined by

$$\pi_0(U) = UP_{\text{ref}}U^*$$

is the base projection of a fiber bundle with connected base space and fiber given by $\mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$. For the justification, let us first note that π_0 is surjective by Proposition 5.1.7. For $P_0 \in \mathbb{P}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\text{ref}}U^* = P_0\}.$$

For the neighborhood

$$\mathcal{U} = \{P \in \mathbb{P}(\mathcal{H}) : \|P - P_0\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{U \in \mathbb{U}(\mathcal{H}) : UP_{\text{ref}}U^* \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \mathbb{U}(\mathcal{H})$, $P \mapsto V_P$, such that $P_0 = V_P^* P V_P$. Moreover, there is a unitary $U_0 \in \mathbb{U}(\mathcal{H})$ such that $U_0^* P_0 U_0 = P_{\text{ref}}$. Then for $P \in \mathcal{U}$ and $U \in \pi_0^{-1}(P)$, one has $UP_{\text{ref}}U^* = P = V_P P_0 V_P^*$ and therefore $V_P^* UP_{\text{ref}}U^* V_P = P_0$, or equivalently $U_0^* V_P^* UP_{\text{ref}}U^* V_P U_0 = P_{\text{ref}}$. Thus $U^* V_P U_0 \in \mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$ where as in (8.11) $\mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$ denotes the set of unitaries that commute with P_{ref} and

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$$

defined by

$$\phi(U) = \left(UP_{\text{ref}}U^*, U^* V_{UP_{\text{ref}}U^*} U_0\right)$$

is a homeomorphism.

Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains that $\pi_k(\mathbb{P}(\mathcal{H})) = 0$ for all $k \in \mathbb{N}_0$ because $\mathbb{U}(\mathcal{H})$ and $\mathbb{U}_{Q_{\text{ref}},0}(\mathcal{H})$ have vanishing homotopy groups by Kuiper's theorem. As $\mathbb{P}(\mathcal{H})$ is a metrizable Banach manifold (see [2]), $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is contractible by Theorem A.3.5 combined with Theorem A.3.6. □

Proof of Theorem 8.5.4. The map $\pi_0 : \mathbb{FPP}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ defined by

$$\pi_0\left((P_0, P_1)\right) = P_0$$

is the base projection of a fiber bundle with connected base space and fiber given by $\mathbb{FP}(\mathcal{H})$. For the justification, let us first note that π_0 is surjective. For $P_0' \in \mathbb{P}(\mathcal{H})$, one has

$$\pi_0^{-1}(P_0') = \{(P_0, P_1) \in \mathbb{FPP}(\mathcal{H}) : P_0 = P_0'\}.$$

For the neighborhood

$$\mathcal{U} = \{P_0 \in \mathbb{P}(\mathcal{H}) : \|P_0 - P_0'\| < 1\},$$

one gets

$$\pi_0^{-1}(\mathcal{U}) = \{(P_0, P_1) \in \mathbb{FPP}(\mathcal{H}) : P_0 \in \mathcal{U}\}.$$

By Lemma 8.3.3, there is a continuous map $\mathcal{U} \to \mathbb{U}(\mathcal{H}), P_0 \mapsto V_{P_0}$, such that $P_0' = V_{P_0}^* P V_{P_0}$. Moreover, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $U^* P_0' U = P_{\text{ref}}$. Then for $P_0 \in \mathcal{U}$ and $P_1 \in \mathbb{P}(\mathcal{H})$, the pair (P_0, P_1) is Fredholm if and only if $(P_0', V_{P_0}^* P_1 V_{P_0})$ is Fredholm, which is equivalent to the Fredholm property of $(P_{\text{ref}}, U^* V_{P_0}^* P_1 V_{P_0} U)$. Thus

$$\phi : \pi_0^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{FP}(\mathcal{H})$$

defined by

$$\phi((P_0, P_1)) = (P_0, U^* V_{P_0}^* P_1 V_{P_0} U)$$

is a homeomorphism.

Using the long exact sequence of homotopy theory associated to this fiber bundle, one obtains isomorphisms $\pi_k(\mathbb{FPP}(\mathcal{H})) \cong \pi_k(\mathbb{FP}(\mathcal{H}))$ because $\mathbb{P}(\mathcal{H})$ has vanishing homotopy groups by Proposition 8.5.5. Then Theorem 8.5.1 allows finishing the proof. □

8.6 Homotopy groups of unbounded self-adjoint Fredholm operators

On the set of unbounded self-adjoint Fredholm operators $\mathbb{F}_{\text{sa}}(\mathcal{H})$, there are two natural topologies, the Riesz and gap topologies, see Section 6.3. As to the Riesz topology, Proposition 6.3.3 already shows that $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ is homotopy equivalent to the set of bounded self-adjoint Fredholm operators $(\mathbb{FB}_{\text{sa}}(\mathcal{H}), \mathcal{O}_N)$. Hence their homotopy groups coincide, and one immediately deduces the next result.

Theorem 8.6.1. *The homotopy groups of* $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ *are the same as the homotopy groups of* $\mathbb{FB}_{\text{sa}}(\mathcal{H})$*, namely* $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ *has three connected components and the homotopy groups of the nontrivial component are as given by Theorem* 8.3.1.

Note that the proof of Theorem 8.6.1 merely implements self-adjointness in the proof of Theorem 8.2.4, because the same can be said already about Propositions 6.3.3 and 6.2.18.

The homotopy groups of the space $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$ are much more difficult to access. It was already proved in Theorem 6.3.16 that $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$ is connected, which is a striking difference to $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$. Of course, this reflects the fact that the Riesz topology is strictly finer than the gap topology. Moreover, item (ii) of Theorem 7.1.7 and Theorem 7.1.8 directly imply that the spectral flow on closed loops establishes a homomorphism $\text{Sf} : \pi_1(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G) \to \mathbb{Z}$. Whether this captures the whole fundamental group was an open question, as pointed out in [31, 126]. An affirmative answer was given in a paper by Joachim [108] which actually computed all the homotopy groups of $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$. This paper is placed in the more general framework of Hilbert modules and, unfortunately, this made parts of the paper difficult to understand for many (including ourselves). Very recently, Prokhorova provided a new and independent proof in a Hilbert space framework [154]. The arguments of [154] are in spirit close to the approach used in this book and are therefore followed closely in the remainder of this section. The outcome is the following:

Theorem 8.6.2. *Spaces* $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ *and* $(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ *are homotopy equivalent. In particular, their homotopy groups coincide and are as given by Theorem* 8.3.1.

Theorem 8.6.2 follows from the next result on the set-theoretic preimage

$$\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}) = \mathcal{F}^{-1}(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}))$$

of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ under the bounded transform.

Theorem 8.6.3. *The embedding* $I : (\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ *is a homotopy equivalence.*

Proof of Theorem 8.6.2. By Theorem 8.6.3, $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ is homotopy equivalent to $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$. Because $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ is homotopy equivalent to $(\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ by Proposition 6.3.3 this implies the claim. □

It now remains to prove Theorem 8.6.3, and this will make up the remainder of this section. Let us state at the very beginning that the main novel ingredient of [154] is to use a technique of tom Dieck to prove homotopy equivalence, stated in Theorem A.3.3 in Appendix A.3. This motivates many of the constructions that follow. Let us begin by analyzing the space $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ where for $a \geq 0$,

$$\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}) = \{H \in \mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap [-a, a] = \emptyset\}$$

denotes the set of operators in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ with spectral gap around 0 of size a.

Proposition 8.6.4. *The space* $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ *is contractible for every* $a \geq 0$.

Proof. Consider the map $f : (\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ defined by $f(H) = \chi(H > 0)$. Then f is well defined and continuous because for $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ one actually has $\chi(H > 0) = \chi(\mathcal{F}(H) > 0) \in \mathbb{P}(\mathcal{H})$ as $\mathcal{F}(H) \in \mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ has positive and negative essential spectrum. This map is a homotopy equivalence with homotopy inverse given by the map $g : (\mathbb{P}(\mathcal{H}), \mathcal{O}_N) \to (\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ defined by $g(P) = (2a + 1)(2P - \mathbf{1})$. Clearly, $f \circ g$ is the identity on $\mathbb{P}(\mathcal{H})$. And $g \circ f$ is homotopic to the identity on $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ via the homotopy $h : \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}) \times [0, 1] \to \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ defined by

$$h(H, t) = \mathcal{F}^{-1}\big(t\mathcal{F}(H) + (1 - t)\mathcal{F}\big((2a + 1)(2\chi(H > 0) - \mathbf{1})\big)\big),$$

where the argument in $\mathcal{F}^{-1}$ has no eigenvalues at ± 1 by spectral calculus. By the spectral mapping theorem,

$$\mathrm{spec}(\mathcal{F}(H)) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset$$

and

$$\mathrm{spec}(\mathcal{F}((2a+1)(2\chi(H>0)-\mathbf{1}))) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset.$$

Therefore and as $\chi(H>0) = \chi(\mathcal{F}(H)>0)$,

$$\mathrm{spec}\Big(t\mathcal{F}(H) + (1-t)\mathcal{F}((2a+1)(2\chi(H>0)-\mathbf{1}))\Big) \cap [-\mathcal{F}(a), \mathcal{F}(a)] = \emptyset$$

for all $t \in [0,1]$. Then, again by the spectral mapping theorem, $h(H,t)$ is indeed in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ for all $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ and $t \in [0,1]$. Thus h is well defined. By Proposition 6.2.17, h is continuous with respect to $\mathcal{O}_R$. Because $h(H,0) = (2a+1)(2\chi(H>0)-\mathbf{1}) = (g\circ f)(H)$ and $h(H,1) = H$ for all $H \in \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$, one can conclude that f is a homotopy equivalence with homotopy inverse g. By Proposition 8.5.5, $(\mathbb{P}(\mathcal{H}), \mathcal{O}_N)$ is contractible and combined with the above this concludes the proof. □

Next it is shown that the space $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ is contractible where for $a \geq 0$,

$$\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap [-a,a] = \emptyset\}$$

denotes the set of operators in $\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ with spectral gap around 0 of size a. The argument is based on the following fact.

Proposition 8.6.5. *The map $f : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}(\mathcal{H}), \mathcal{O}_N)$ given by $f(H) = H^{-1}$ is continuous. It provides a homeomorphism*

$$f_0 : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N),$$

where $\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) = \{H \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H}) : H \text{ injective}\}$ denotes the set of bounded self-adjoint injective operators. For $a > 0$, the restriction of f gives another homeomorphism

$$f_a : (\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N),$$

where $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}) = \{H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) : \|H\| < a^{-1}\}$.

Proof. For any $H \in \mathbb{F}_{\mathrm{sa},0}(\mathcal{H})$, the bounded linear map $g : \mathcal{H}\oplus\mathcal{H} \to \mathcal{H}\oplus\mathcal{H}$ defined by $g(\phi,\psi) = (\psi,\phi)$ maps the graph of H onto the graph of its inverse H^{-1}. This implies that the map $\tilde{f} : (\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{B}(\mathcal{H}), \mathcal{O}_G)$ defined by $\tilde{f}(H) = H^{-1}$ is continuous. However, $\mathcal{O}_G$ and $\mathcal{O}_N$ coincide on $\mathbb{B}(\mathcal{H})$ by Theorem 6.1.10 and therefore f is continuous. Because $H \in \mathbb{F}_{\mathrm{sa},0}(\mathcal{H})$ has a spectral gap containing 0, one has $H^{-1} \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. For $H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$ its inverse H^{-1} is, moreover, densely defined, closed, and symmetric. As $\mathrm{Ran}(H^{-1}) = \mathcal{H}$, this implies that H^{-1} is self-adjoint. Thus f_0 is a homeomorphism. By the spectral radius theorem, $H \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$ has norm less than a^{-1} if and only if $\mathrm{spec}(H) \subset (-a^{-1}, a^{-1})$. By the spectral mapping theorem, this is equivalent to the property $\mathrm{spec}(H^{-1}) \cap [-a,a] = \emptyset$. Thus also f_a is a homeomorphism. □

Proposition 8.6.6. *The space $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ is contractible for all $a \geq 0$.*

Proof. Let us first focus on the case $a = 0$. By Proposition 8.6.5, it is sufficient to show that $(\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N)$ is contractible. Let $K \in \mathbb{K}(\mathcal{H})$ be a positive semidefinite injective compact operator with $\|K\| < 1$, as already used in the proof of Proposition 6.4.7. Then

$$h_1 : \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) \times [0,1] \to \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \quad h_1(H,t) = \big((1-t)\mathbf{1} + tK\big)H\big((1-t)\mathbf{1} + tK\big),$$

is well defined because $(1-t)\mathbf{1} + tK$ is a convex combination of positive semidefinite injective operators and therefore injective for all $t \in [0,1]$. Thus $h_1(H,t) \in \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. Clearly, h_1 is a norm-continuous homotopy such that $h_1(H,0) = H$ and $h_1(H,1) = KHK$ is an injective compact operator lying in $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) = \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}) \cap \mathbb{K}(\mathcal{H}) \subset \mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H})$. It is thus sufficient to show that $(\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N)$ is contractible. To do so, let us identify $\mathcal{H}$ with $L^2([0,1])$, but suppress the unitary in the following. For $t \in (0,1]$, let us consider (inspired by [73] and as in the proof of Proposition 6.4.16) the partial isometry

$$V_t(\phi)(x) = \begin{cases} t^{-\frac{1}{2}}\phi(\frac{x}{t}), & \text{for } x \le t, \\ 0, & \text{for } x > t. \end{cases}$$

Similarly, for $t \in [0,1)$,

$$W_t(\phi)(x) = \begin{cases} 0, & \text{for } x \le t, \\ (1-t)^{-\frac{1}{2}}\phi(\frac{x-t}{1-t}), & \text{for } x > t, \end{cases}$$

is also a partial isometry that is complementary to V_t. Clearly, V_t, V_t^*, W_t, and W_t^* continuously depend on t in the strong operator topology. Moreover, the projections $P_t = V_tV_t^*$ and $Q_t = W_tW_t^*$ fulfill

$$s\text{-}\lim_{t\to 0} P_t = 0, \quad s\text{-}\lim_{t\to 1} Q_t = 0,$$

as well as

$$P_1 = V_1 = \mathbf{1} = W_0 = Q_0, \quad P_t + Q_t = \mathbf{1}. \tag{8.12}$$

For $H_0 \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$, we define the homotopy $h_2 : \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) \times [0,1] \to \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ by

$$h_2(H,t) = \begin{cases} H_0, & \text{for } t = 0, \\ tV_tHV_t^* + (1-t)W_tH_0W_t^*, & \text{for } t \in (0,1), \\ H, & \text{for } t = 1. \end{cases}$$

Clearly, $h_2(H,t)$ is self-adjoint and compact. Moreover, $V_tHV_t^*$ is an injective operator on $\mathrm{Ran}(P_t)$ and $W_tH_0W_t^*$ is an injective operator on $\mathrm{Ran}(Q_t)$ by (8.12) and therefore $h_2(H,t) \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ so that h_2 is well defined. Since H and H_0 are compact and the maps $t \in (0,1) \mapsto V_t$ and $t \in (0,1) \mapsto W_t$ are strongly continuous, h_2 is norm-continuous on $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) \times (0,1)$. Moreover, for every $H, \tilde{H} \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ and $t \in (0,1]$,

$$\begin{aligned}\|h_2(H,1) - h_2(\tilde{H},t)\| &\leq \|H - h_2(H,t)\| + \|h_2(H,t) - h_2(\tilde{H},t)\| \\ &\leq \|H - tV_tHV_t^*\| + (1-t)\|W_tH_0W_t^*\| + \|H - \tilde{H}\|.\end{aligned}$$

As $\lim_{t\to 1}\|H - tV_tHV_t^*\| = 0$, this implies that h_2 is continuous at all points $(H,1)$ for $H \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$. Similarly, one shows that h_2 is continuous at all points $(H,0)$ for $H \in \mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$. Thus h_2 is continuous on the whole domain $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H}) \times [0,1]$ and therefore $\mathbb{K}_{\mathrm{sa,inj}}(\mathcal{H})$ is contractible. Thus $(\mathbb{B}_{\mathrm{sa,inj}}(\mathcal{H}), \mathcal{O}_N)$ is contractible and the claim on $(\mathbb{F}_{\mathrm{sa},0}(\mathcal{H}), \mathcal{O}_G)$ follows from Proposition 8.6.5.

For $a > 0$, the homotopy h_1 defined as above maps $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}) \times [0,1]$ to the set $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H})$. Furthermore $\mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}) \times \{1\}$ is mapped to $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H}) = \mathbb{B}_{a,\mathrm{sa,inj}}(\mathcal{H}) \cap \mathbb{K}(\mathcal{H})$. For $H_0 \in \mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})$ the homotopy h_2 maps $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H}) \times [0,1]$ to $\mathbb{K}_{a,\mathrm{sa,inj}}(\mathcal{H})$ because $tV_tHV_t^* \in \mathbb{K}_{a,\mathrm{sa,inj}}(\mathrm{Ran}(P_t))$ and $(1-t)W_tH_0W_t^* \in \mathbb{K}_{a,\mathrm{sa,inj}}(\mathrm{Ran}(Q_t))$. Therefore the same argument as for the case $a = 0$ shows that $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ is contractible for all $a \geq 0$. □

Now all is prepared to complete the

Proof of Theorem 8.6.3. The proof is based on Theorem A.3.3. Note that both spaces $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$ are metric and therefore paracompact. Thus every open covering of these spaces is numerable. Let $\mathcal{T}$ denote the set of all finite symmetric (with respect to 0) nonempty subsets of $\mathbb{R}$, such as $\tau = \{-a, a\}$ and $\tau = \{-a, -b, b, a\}$ for $a > b > 0$. For $\tau \in \mathcal{T}$, let $\overline{\tau}$ denote the convex hull of τ which is a closed symmetric interval in $\mathbb{R}$. Then

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \tau = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \overline{\tau}\}$$

is open in the gap topology $\mathcal{O}_G$ because

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) = \mathcal{F}^{-1}\big(\{H \in \mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \mathcal{F}(\tau) = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \mathcal{F}(\overline{\tau})\}\big),$$

$\mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_E)$ is continuous by Corollary 6.3.4 and, moreover, $\{H \in \mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \mathcal{F}(\tau) = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \mathcal{F}(\overline{\tau})\}$ is an open subset of $\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$ with respect to the extended gap topology by the spectral mapping theorem and because τ is symmetric. Thus $(\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}))_{\tau\in\mathcal{T}}$ is an open and, by the above, numerable covering of $(\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G)$. On the other hand,

$$\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) = \{H \in \mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \tau = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \overline{\tau}\}$$

is open in the Riesz topology $\mathcal{O}_R$ because

$$\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) = \mathcal{F}^{-1}\big(\{H \in \mathbb{FB}^{*,0}_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \mathcal{F}(\tau) = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \mathcal{F}(\overline{\tau})\}\big),$$

the bounded transform $\mathcal{F} : (\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ is continuous by Corollary 6.3.2 and $\{H \in \mathbb{FB}^{*,0}_{1,\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}(H) \cap \mathcal{F}(\tau) = \emptyset = \mathrm{spec}_{\mathrm{ess}}(H) \cap \mathcal{F}(\overline{\tau})\}$ is an open subset of $\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$ with respect to the norm topology. Thus $(\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}))_{\tau\in\mathcal{T}}$ is an open and, by the above, numerable covering of $(\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R)$. For $\tau, \tau' \in \mathcal{T}$ one clearly has $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \cap \mathbb{F}^*_{\mathrm{sa},\tau'}(\mathcal{H}) = \mathbb{F}^*_{\mathrm{sa},\tau\cup\tau'}(\mathcal{H})$ and $\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) \cap \mathbb{F}_{\mathrm{sa},\tau'}(\mathcal{H}) = \mathbb{F}_{\mathrm{sa},\tau\cup\tau'}(\mathcal{H})$ with $\tau \cup \tau' \in \mathcal{T}$. Moreover, $I(\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H})) \subset \mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})$. Thus, by Theorem A.3.3, it is sufficient to show that the embedding $I_\tau : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G)$ is a homotopy equivalence for every fixed $\tau \in \mathcal{T}$.

Let $\mathbb{P}_{\mathrm{fin}}(\mathcal{H}) = \{P = P^* = P^2 \in \mathbb{B}(\mathcal{H}) : \dim(\mathrm{Ran}(P)) < \infty\}$ denote the set of finite-dimensional orthogonal projections on $\mathcal{H}$. Then $\pi_R : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{P}_{\mathrm{fin}}(\mathcal{H}), \mathcal{O}_N)$ defined by

$$\pi_R(H) = \chi_{\overline{\tau}}(H)$$

is the base projection of a fiber bundle with fiber over $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ given by

$$\mathbb{F}^*_{\mathrm{sa},\tau,P}(\mathcal{H}) = \pi_R^{-1}(P) = \{H \in \mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) : \chi_{\overline{\tau}}(H) = P\}.$$

Similarly, $\pi_G : (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G) \to (\mathbb{P}_{\mathrm{fin}}(\mathcal{H}), \mathcal{O}_N)$ defined by

$$\pi_G(H) = \chi_{\overline{\tau}}(H)$$

is the base projection of a fiber bundle with fiber over $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ given by

$$\mathbb{F}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \pi_G^{-1}(P) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) : \chi_{\overline{\tau}}(H) = P\}.$$

Both fiber bundles are locally trivial in the sense that, by Lemma 8.3.3, for $P_0 \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ there is a continuous map $\{P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) : \|P - P_0\| < 1\} \to \mathbb{U}(\mathcal{H})$, $P \mapsto V_P$, such that $P_0 = V_P^* P V_P$. Let

$$\mathcal{H} \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) = \mathcal{H}' \oplus \mathcal{H}''$$

be the canonical decomposition of the trivial Hilbert bundle over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ into the direct sum of two vector bundles, whose fibers are $\mathcal{H}'_P = \mathrm{Ran}(P)$ and $\mathcal{H}''_P = \mathrm{Ker}(P)$. Let $\mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ be the fiber bundles over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ associated with $\mathcal{H}'$, respectively $\mathcal{H}''$, with fibers given by

$$\mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathrm{Ran}(P)) : \mathrm{spec}(H) \subset \overline{\tau} \setminus \tau\}$$

and

$$\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}^*_{\mathrm{sa},\tau}(\mathrm{Ker}(P)) : \mathrm{spec}(H) \cap \overline{\tau} = \emptyset\},$$

where both fibers are equipped with the Riesz topology. By Lemma 8.3.3, these fiber bundles are again locally trivial. Then taking fiberwise the direct sums, $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H})$ can be seen as fiber product bundle over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ of the form

$$\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) = \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \times_{\mathbb{P}_{\mathrm{fin}}(\mathcal{H})} \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$$

with bundle maps $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ given by the restriction $H \mapsto H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))}$ and $\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ given by $H \mapsto H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))}$. In exactly the same way, one can view $\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H})$ as a fiber product

$$\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}) = \mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H}) \times_{\mathbb{P}_{\mathrm{fin}}(\mathcal{H})} \mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H}),$$

where $\mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H})$ are the fiber bundles over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ with fibers given by $\mathbb{F}'_{\mathrm{sa},\tau,P}(\mathcal{H}) = \mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H})$ and

$$\mathbb{F}''_{\mathrm{sa},\tau,P}(\mathcal{H}) = \{H \in \mathbb{F}_{\mathrm{sa},\tau}(\mathrm{Ker}(P)) : \mathrm{spec}(H) \cap \overline{\tau} = \emptyset\},$$

where both fibers are equipped with the gap topology. Now the embedding $I_\tau : (\mathbb{F}^*_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_R) \to (\mathbb{F}_{\mathrm{sa},\tau}(\mathcal{H}), \mathcal{O}_G)$ is a product of the maps

$$I'_\tau : \mathbb{F}^{*\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}'_{\mathrm{sa},\tau}(\mathcal{H}), \quad H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))} \mapsto H|_{\mathrm{Ran}(\chi_{\overline{\tau}}(H))}$$

and

$$I''_\tau : \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H}) \to \mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H}), \quad H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))} \mapsto H|_{\mathrm{Ker}(\chi_{\overline{\tau}}(H))}.$$

On $\mathbb{F}^{*\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) = \mathbb{F}'_{\mathrm{sa},\tau,P}(\mathcal{H})$, the Riesz topology and the gap topology coincide with the norm topology by Theorem 6.1.10, therefore I'_τ is a homeomorphism.

To show that I''_τ is a homotopy equivalence note that the Hilbert bundle $\mathcal{H}''$ over the (metric and thus) paracompact space $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ has infinite-dimensional separable fibers. Thus by Theorem A.3.12, it is a trivial Hilbert bundle. The trivialization map can be chosen to be unitary, namely there is a norm-continuous map $P \in \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \mapsto W_P \in \mathbb{B}(\mathcal{H})$ where W_P is a partial isometry with $\mathrm{Ker}(W_P) = \mathrm{Ran}(P)$ and $\mathrm{Ran}(W_P) = \mathcal{H}$. Then the map $(\phi, P) \in \mathcal{H}'' \mapsto (W_p\phi, P) \in \mathcal{H} \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ is a trivialization of $\mathcal{H}''$. Therefore the fiber bundles $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau}(\mathcal{H})$ over $\mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ are also trivial, namely isomorphic to the trivial bundles $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$, respectively $\mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$, via the trivalization maps $A \in \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,P}(\mathcal{H}) \mapsto (W_P A W_P^*, P) \in \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ and $A \in \mathbb{F}''_{\mathrm{sa},\tau,P}(\mathcal{H}) \mapsto (W_P A W_P^*, P) \in \mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$. After this, isomorphism I''_τ transposes to $I''_{\tau,0} : \mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H}) \to \mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) \times \mathbb{P}_{\mathrm{fin}}(\mathcal{H})$ simply given by $(H, P) \mapsto (H, P)$. As $\overline{\tau} = [-a, a]$ for some $a \geq 0$, one gets $\mathbb{F}^{*\prime\prime}_{\mathrm{sa},\tau,0}(\mathcal{H}) = \mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H})$ and $\mathbb{F}''_{\mathrm{sa},\tau,0}(\mathcal{H}) = \mathbb{F}_{\mathrm{sa},a}(\mathcal{H})$. By Propositions 8.6.4 and 8.6.6, the spaces $(\mathbb{F}^*_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa},a}(\mathcal{H}), \mathcal{O}_G)$ are contractible. This implies that I''_τ is a homotopy equivalence and therefore I_τ is a homotopy equivalence as it is the product of two homotopy equivalences. This concludes the argument. □

Remark 8.6.7. Let us point out that the proof of Theorem 8.4.1 merely uses that the fundamental group of $(\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_N)$ is infinitely cyclic. Thus, by the results of this section, Theorem 8.4.1 also holds for $(\mathbb{F}^*_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_R)$ and $(\mathbb{F}_{\mathrm{sa}}(\mathcal{H}), \mathcal{O}_G)$. ◇

8.7 Resumé: homotopy equivalences of operator classes

For the convenience of the reader, this section summarizes various of the homotopy equivalences of sets of Fredholm operators proved in this and earlier chapters. Let us begin with a diagram for self-adjoint Fredholm operators. It is quite extended, even though not all results proved in this book are included. For sake of compactness of the presentation, we drop the specification of the Hilbert space $\mathcal{H}$.

$$
\begin{array}{ccccccc}
(\mathbb{F}\mathbb{B}_{\text{sa}}, \mathcal{O}_N) & \xleftrightarrow[3.6.2]{\text{retr.}} & (\mathbb{F}\mathbb{B}_{1,\text{sa}}, \mathcal{O}_N) & \xleftrightarrow[6.3.2]{\mathcal{F}} & (\mathbb{F}_{\text{sa}}, \mathcal{O}_R) & \xleftrightarrow[6.3.3]{i} & (\mathbb{F}\mathbb{B}_{\text{sa}}, \mathcal{O}_N) \\
\uparrow{\scriptstyle \text{incl.}} & & & & & & \\
(\mathbb{F}\mathbb{B}^*_{\text{sa}}, \mathcal{O}_N) & & & & & & \\
{\scriptstyle 6.3.3}\updownarrow{\scriptstyle i} & & & & & & \\
(\mathbb{F}^*_{\text{sa}}, \mathcal{O}_R) & \xleftrightarrow[6.3.2]{\mathcal{F}} & (\mathbb{F}\mathbb{B}^{*,0}_{1,\text{sa}}, \mathcal{O}_N) & & & & \\
{\scriptstyle I}\updownarrow{\scriptstyle 8.6.3} & & & & & & \\
(\mathbb{F}_{\text{sa}}, \mathcal{O}_G) & \xleftrightarrow[6.3.4]{\mathcal{F}} & (\mathbb{F}\mathbb{B}^0_{1,\text{sa}}, \mathcal{O}_E) & \xleftrightarrow[6.4.6]{\text{id}} & (\mathbb{F}\mathbb{B}^0_{1,\text{sa}}, \mathcal{O}_{SE}) & \xleftrightarrow[4.6.12]{\mathcal{G}} & (\mathbb{F}\mathbb{U}^0, \mathcal{O}_N) \\
{\scriptstyle i}\updownarrow{\scriptstyle 6.4.2} & & & & & & \\
(\mathbb{F}^{\mathrm{C}}_{\text{sa}}, \mathcal{O}_G) & \xleftrightarrow[6.4.4]{\mathcal{F}} & (\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\text{sa}}, \mathcal{O}_E) & \xleftrightarrow[6.4.6]{\text{id}} & (\mathbb{F}\mathbb{B}^{\mathrm{C},0}_{1,\text{sa}}, \mathcal{O}_{SE}) & \xleftrightarrow[4.6.15]{\mathcal{G}} & (\mathbb{U}^{\mathrm{C},0}, \mathcal{O}_N) \\
& & & & {\scriptstyle \text{id}}\updownarrow{\scriptstyle 6.4.7} & & \\
& & (\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\text{sa}}, \mathcal{O}_K) & \xleftrightarrow[6.4.15]{\text{id}} & (\mathbb{F}\mathbb{B}^{\mathrm{C}}_{1,\text{sa}}, \mathcal{O}_{SE}) & \xleftrightarrow[6.4.12]{\text{retr.}} & (\mathbb{F}\mathbb{B}_{1,\text{sa}}, \mathcal{O}_{SE}) \\
& & & & {\scriptstyle \text{retr.}}\updownarrow{\scriptstyle 6.4.16} & & \\
& & & & (\mathbb{F}\mathbb{B}^{\mathrm{C},\infty}_{1,\text{sa}}, \mathcal{O}_{SE}) & &
\end{array}
$$

The diagram splits in the top row and the rest. However, they are tightly connected as it was shown in Proposition 3.6.1 that $\mathbb{F}\mathbb{B}_{\text{sa}} = \mathbb{F}\mathbb{B}^*_{\text{sa}} \cup \mathbb{F}\mathbb{B}^+_{\text{sa}} \cup \mathbb{F}\mathbb{B}^-_{\text{sa}}$ is a disjoint union in which the two components $\mathbb{F}\mathbb{B}^{\pm}_{\text{sa}}$ are contractible. Hence the nontrivial part of the higher homotopy groups (of degree greater or equal to $k = 1$) of the upper row stems from the component $\mathbb{F}\mathbb{B}^*_{\text{sa}}$, namely the lower part of the diagram. These homotopy groups have been computed in Section 8.3. Let us also note that many of the homotopy equivalences in the diagram also hold for the operator sets without Fredholm properties. The corresponding statements can always be found near by those on the Fredholm operators.

Next let us come to the set of (not necessarily self-adjoint) Fredholm operators. The results are summarize as follows:

$$\begin{array}{ccccccc}
(\mathbb{FB}(\mathcal{H}), \mathcal{O}_N) & \overset{i}{\underset{6.2.18}{\longleftrightarrow}} & (\mathbb{F}(\mathcal{H}), \mathcal{O}_R) & \overset{\mathcal{F}}{\underset{6.2.17}{\longleftrightarrow}} & (\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_N) & \overset{f}{\underset{6.4.7}{\longleftrightarrow}} & (\mathbb{FB}_1(\mathcal{H}), \mathcal{O}_N) \\
 & & {\scriptstyle \mathrm{id}} \big\downarrow & & & & \\
 & & (\mathbb{F}(\mathcal{H}), \mathcal{O}_G) & \overset{\mathcal{F}}{\underset{6.2.17}{\longleftrightarrow}} & (\mathbb{FB}_1^0(\mathcal{H}), \mathcal{O}_E). & &
\end{array}$$

Here the two-sided errors designate homotopy equivalences by the maps on top of them and the corresponding statement below them, while the hook error is a homotopy equivalence by [154]. Hence the homotopy groups of all spaces are given in Section 8.2.

9 Bott–Maslov index via spectral flow

This chapter develops the theory of the Bott–Maslov and Conley–Zehnder indices in the framework of a complex infinite-dimensional Krein space. It hence generalizes many of the results of Chapter 2 by imposing suitable Fredholm conditions. Standard monographs on Krein spaces are [29, 20]. The infinite-dimensional Bott–Maslov index was introduced and studied by Swanson [187], Nicolaescu [138], Booß-Bavnbek and Furutani [30], Kirk and Lesch [113], and Furutani [90], see also [137, 33, 168, 203]. Apart from these fundamental references, other literature is cited in the text below. Works on the finite-dimensional case are already mentioned in Chapter 2. As an application of the infinite-dimensional theory, Section 9.7 develops oscillation theory for the bound states of a high-dimensional scattering setup. Let us note that numerous other applications can be found in the literature, in particular, most notably in Morse theory [138, 113].

9.1 Krein spaces and operators thereon

In this chapter, the separable complex Hilbert space $\mathcal{K}$ is supposed to be equipped with a proper symmetry $J = J^* = J^{-1} \in \mathbb{B}(\mathcal{K})$, namely one which has infinite-dimensional eigenspaces for the eigenvalues 1 and −1. We will always assume to be in the spectral representation of J so that

$$J = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Thus J introduces a grading of $\mathcal{K}$, namely $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Because J is proper, both $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite dimensional and therefore they can be naturally identified with a separable Hilbert space $\mathcal{H}$, namely $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. One then calls the couple $(\mathcal{K}, J)$ a complex Krein space with fundamental symmetry J. Let us note that this excludes the class of infinite-dimensional Pontryagin spaces [29, 20] where one of the fibers $\mathcal{H}_+$ or $\mathcal{H}_-$ is finite-dimensional. However, later on (in particular, in Section 9.2) Pontryagin subspaces of a Krein space and their Krein signature will be relevant.

Definition 9.1.1. Let P be an orthogonal projection on a Krein space $(\mathcal{K}, J)$ and let Φ be a normalized frame for P, namely $P = \Phi\Phi^*$.
(i) P is called J-invariant if and only if $PJ = JP$.
(ii) P is called nondegenerate if $0 \notin \mathrm{spec}(\Phi^* J \Phi)$.
(iii) A nondegenerate orthogonal projection P is called a Pontryagin projection if and only if $\Phi^* J \Phi$ has only a finite number of positive eigenvalues or a finite number of negative eigenvalues. Then the Krein signature of a Pontryagin projection P is

$$\mathrm{KSig}(P) = \mathrm{Sig}(\Phi^* J \Phi) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

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(iv) A Pontryagin projection is called Krein-definite if $\Phi^* J\Phi$ is either positive or negative definite, or equivalently if the restricted quadratic form $J|_{\mathrm{Ran}(P)}$ is either positive or negative definite.

Identifying orthogonal projections with their range, all these notions directly transpose to closed subspaces $\mathcal{E}$ of $\mathcal{K}$.

Note that every J-invariant projection is nondegenerate. In this section, Pontryagin spaces and their Krein signature will not play any role, yet. The focus here is rather on the analysis of linear operators on the Krein space that preserve J as a quadratic form.

Definition 9.1.2. A bounded invertible operator $T \in \mathbb{B}(\mathcal{K})$ on a Krein space $(\mathcal{K}, J)$ is called J-unitary if

$$T^* J T = J. \tag{9.1}$$

The set of J-unitary operators on $\mathcal{K}$ is denoted by $\mathbb{U}(\mathcal{K}, J)$ and it will be equipped with the norm metric d_N and the associated norm topology $\mathcal{O}_N$.

Let us stress that $\mathbb{U}(\mathcal{K}, J)$ does *not* denote the unitary operators on $\mathcal{K}$ viewed as a Hilbert space. These latter operators are simply denoted by $\mathbb{U}(\mathcal{K})$. Also note that the relation (9.1) alone does not imply that T is invertible. For example, set $\mathcal{K} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ in the grading of J and define $T = S \oplus S$ where S denotes the right-shift on $\ell^2(\mathbb{N})$. Then (9.1) holds but T is not invertible and hence not in $\mathbb{U}(\mathbb{K}, J)$. Many of the basic algebraic properties of J-unitaries transfer from the finite-dimensional case. In particular, the spectrum satisfies (2.16) and the Riesz projection of $T \in \mathbb{U}(\mathcal{K}, J)$ are those given in Proposition 2.2.2. Furthermore, $\mathbb{U}(\mathcal{K}, J)$ is clearly a subgroup of the set $\mathbb{G}(\mathcal{K})$ of invertible operators on $\mathcal{K}$. One can also rewrite the definition of $\mathbb{U}(\mathcal{K}, J)$ as follows.

Proposition 9.1.3. *The group $\mathbb{U}(\mathcal{K}, J)$ is invariant under taking adjoints. In the grading of J, one has*

$$\mathbb{U}(\mathcal{K}, J) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{G}(\mathcal{K}) : A^*A - C^*C = \mathbf{1},\ D^*D - B^*B = \mathbf{1},\ A^*B = C^*D \right\}$$
$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{G}(\mathcal{K}) : AA^* - BB^* = \mathbf{1},\ DD^* - CC^* = \mathbf{1},\ AC^* = BD^* \right\},$$

and in this representation A and D are invertible and satisfy $\|A^{-1}\| \le 1$, $\|D^{-1}\| \le 1$. Also $\|A^{-1}B\| < 1$, $\|D^{-1}C\| < 1$, $\|BD^{-1}\| < 1$, and $\|CA^{-1}\| < 1$.

Proof. Inverting $T^*JT = J$ shows $T^{-1}J(T^*)^{-1} = J$ so that $J = TJT^*$. The fact that A is invertible follows from $AA^* \ge \mathbf{1}$ and $A^*A \ge \mathbf{1}$. Analogously, one shows that D is invertible. Furthermore, $AA^* - BB^* = \mathbf{1}$ implies that $A^{-1}B(A^{-1}B)^* = \mathbf{1} - A^{-1}(A^{-1})^* < \mathbf{1}$, so that $\|A^{-1}B\| < 1$. The same argument leads to the other inequalities. □

As in finite dimension (Proposition 2.2.5), the polar decomposition of J-unitary operators only involves J-unitary operators. In fact, the proof in finite dimension is based on Lemma 2.2.6 which directly generalizes to the Krein space framework.

Proposition 9.1.4. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ have the polar decomposition $T = W|T|$, namely where $|T| = (T^*T)^{\frac{1}{2}}$ and W is unitary. Then $|T| \in \mathbb{U}(\mathcal{K}, J)$ and $W \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$.*

Now let us turn to study the topology of $(\mathbb{U}(\mathcal{K}, J), \mathcal{O}_N)$ and some of its subspaces. As in the finite-dimensional case, one has the following.

Corollary 9.1.5. *The group $(\mathbb{U}(\mathcal{K}, J), \mathcal{O}_N)$ is path connected.*

Proposition 9.1.6. *The group $(\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K}), \mathcal{O}_N)$ is contractible and given by*

$$\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K}) = \{\mathrm{diag}(V_+, V_-) \in \mathbb{U}(\mathcal{K}) : V_+, V_- \in \mathbb{U}(\mathcal{H})\}.$$

Proof. Let us fist note that $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ is the set of unitaries commuting with J. These are the, in the grading of J, diagonal unitaries, just as stated. Therefore the contractibility of $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ follows as $\mathbb{U}(\mathcal{H})$ is contractible by Kuiper's theorem. □

Let us next consider another subgroup of $\mathbb{U}(\mathcal{K}, J)$, namely the set of J-unitary operators that are compact perturbations of the identity,

$$\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) = \{\mathbf{1} + K \text{ invertible} : K \in \mathbb{K}(\mathcal{K}), (\mathbf{1} + K)^* J (\mathbf{1} + K) = J\}.$$

This is the norm-closure of the finite-dimensional J-unitaries, under suitable embedding of the latter in $\mathbb{U}(\mathcal{K}, J)$. Proposition 9.1.4 directly implies the following result.

Corollary 9.1.7. *Let $T \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$ have the polar decomposition $T = W|T|$, then one has $|T| \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$ and $W \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$.*

The next result follows from Corollary 9.1.7 combined with (8.1).

Proposition 9.1.8. *The space $(\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K}), \mathcal{O}_N)$ is a deformation retract of the space $(\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J), \mathcal{O}_N)$. The homotopy groups of $(\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J), \mathcal{O}_N)$ are*

$$\pi_k(\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Proof. Using the polar decomposition in $\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$ as given in Corollary 9.1.7 and deforming the radial part shows that $\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$ can be retracted to $\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Moreover, Proposition 9.1.6 shows that $\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K}) = \mathbb{U}^{\mathrm{C}}(\mathcal{H}) \times \mathbb{U}^{\mathrm{C}}(\mathcal{H})$ where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Therefore, the claim follows from (8.1). □

Next let us come to the Lie algebra of $\mathbb{U}(\mathcal{K}, J)$.

Definition 9.1.9. A bounded operator $H \in \mathbb{B}(\mathcal{K})$ on a Krein space $(\mathcal{K}, J)$ is called J-self-adjoint if

$$JH^*J = H. \tag{9.2}$$

The set of J-self-adjoint bounded operators on $\mathcal{K}$ is denoted by $\mathbb{B}_{\rm sa}(\mathcal{K}, J)$ and it is equipped with the norm metric d_N and the associated norm topology $\mathcal{O}_N$.

There is a close connection between J-self-adjoint operators and self-adjoint operators on $\mathcal{K}$. More precisely, H is J-self-adjoint if and only if $S = JH$ is self-adjoint. Let us note that $\mathbb{B}_{\rm sa}(\mathcal{K}, J)$ is an $\mathbb{R}$-vector space. Moreover, $\mathbb{B}_{\rm sa}(\mathcal{K}, J)$ is the Lie algebra of $\mathbb{U}(\mathcal{K}, J)$ in the sense that

$$H \in \mathbb{B}_{\rm sa}(\mathcal{K}, J) \quad \Longrightarrow \quad e^{\imath H} \in \mathbb{U}(\mathcal{K}, J).$$

If $H \in \mathbb{B}_{\rm sa}(\mathcal{K}, J)$ is such that $H + \imath\mathbf{1} \in \mathbb{B}(\mathcal{K})$ is invertible, also the Cayley transform $\mathcal{C}(H) = (H - \imath\mathbf{1})(H + \imath\mathbf{1})^{-1}$ lies in $\mathbb{U}(\mathcal{K}, J)$. Finally, the set $\mathbb{B}_{\rm sa}(\mathcal{K}) \cap \mathbb{B}_{\rm sa}(\mathcal{K}, J)$ is the real vector space which is the Lie algebra of $\mathbb{U}(\mathcal{K}) \cap \mathbb{U}(\mathcal{K}, J)$, namely $H \in \mathbb{B}_{\rm sa}(\mathcal{K}) \cap \mathbb{B}_{\rm sa}(\mathcal{K}, J)$ implies $e^{\imath H} \in \mathbb{U}(\mathcal{K}) \cap \mathbb{U}(\mathcal{K}, J)$. Statements and formulas similar to those in Proposition 9.1.3 also hold for operators in the Lie algebra $\mathbb{B}_{\rm sa}(\mathcal{K}, J)$.

Proposition 9.1.10. *The $\mathbb{R}$-vector space $\mathbb{B}_{\rm sa}(\mathcal{K}, J)$ is invariant under taking adjoints. In the grading of J, one has*

$$\mathbb{B}_{\rm sa}(\mathcal{K}, J) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A = A^*,\ D = D^*,\ B = -C^* \right\}.$$

Proof. The claim follows directly by writing out (9.2). □

There is another natural class of bounded operators on the Krein space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, namely for a given operator $B \in \mathbb{B}(\mathcal{H})$ one can set

$$H = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

By construction, $H = H^*$ is self-adjoint and satisfies $JHJ = -H$, namely $\imath H$ is J-self-adjoint. In a quantum-mechanical setting, the operator H is then called a Hamiltonian and the relation $JHJ = -H$ either a supersymmetry [67] or a chiral symmetry [152]. This motivates the following definition.

Definition 9.1.11. A self-adjoint operator $H \in \mathbb{B}_{\rm sa}(\mathcal{K})$ satisfying $JHJ = -H$ is called chiral.

Chiral operators clearly have a spectral symmetry $\mathrm{spec}(H) = -\,\mathrm{spec}(H) \subset \mathbb{R}$. A particular example of a chiral operator is a chiral symmetry. For every chiral symmetry Q, there is a unitary operator U on $\mathcal{H}$ such that

$$Q = \begin{pmatrix} 0 & -U \\ -U^* & 0 \end{pmatrix}. \tag{9.3}$$

As will be discussed below, there is a tight connection between invertible chiral operators and J-Lagrangian projections, and this also explains why we choose to add the minus sign in (9.3).

9.2 *J*-isotropic subspaces

Definition 9.2.1. A closed subspace $\mathcal{E}$ of a Krein space $(\mathcal{K}, J)$ is called J-isotropic if J viewed as a hermitian sesquilinear form vanishes when restricted to $\mathcal{E}$. More explicitly, for all $\phi, \psi \in \mathcal{E}$, one has $\phi^* J \psi = 0$. Two closed subspaces $\mathcal{E}$ and $\mathcal{E}'$ are called J-orthogonal if and only if $\phi^* J \psi = 0$ for all $\phi \in \mathcal{E}$ and $\psi \in \mathcal{E}'$.

Let us note that any subspace on which J vanishes can be closed and its closure will then be J-isotropic so that it is natural to require closedness in Definition 9.2.1. Furthermore, every J-isotropic subspace is J-orthogonal to itself. As there is a bijection between closed subspaces of a Hilbert space and orthogonal projections (self-adjoint idempotents), the following definition is hence in line with the above.

Definition 9.2.2. An orthogonal projection P is called J-isotropic if $PJP = 0$. The set of J-isotropic projections will be denoted by $\mathbb{I}(\mathcal{K}, J)$. Two orthogonal projections P and P' are called J-orthogonal if and only if $PJP' = 0$.

One has the following characterization of J-isotropic projections.

Lemma 9.2.3. *An orthogonal projection P is J-isotropic if and only if*

$$P \leq J(\mathbf{1} - P)J.$$

Proof. First note that $J(\mathbf{1} - P)J$ is an orthogonal projection. Moreover, $P \leq J(\mathbf{1} - P)J$ is equivalent to $\mathrm{Ran}(P) \subset \mathrm{Ran}(J(\mathbf{1} - P)J)$ because if $P \leq J(\mathbf{1} - P)J$ holds and $\phi_1 \in \mathrm{Ran}(P)$ is a vector in the range of P then one has

$$\|\phi_1\|^2 = \langle \phi_1 | P\phi_1 \rangle \leq \langle \phi_1 | J(\mathbf{1} - P)J\phi_1 \rangle \leq \|\phi_1\|^2 \|J(\mathbf{1} - P)J\| = \|\phi_1\|^2.$$

Therefore, as the Cauchy–Schwarz inequality is an equality in this case, $J(\mathbf{1} - P)J\phi_1 = \phi_1$ and ϕ_1 is in the range of $J(\mathbf{1} - P)J$. Conversely, assume that $\mathrm{Ran}(P) \subset \mathrm{Ran}(J(\mathbf{1} - P)J)$ holds. Then for $\phi = \phi_0 + \phi_1 \in \mathcal{H}$ with $\phi_0 \in \mathrm{Ker}(P)$ and $\phi_1 \in \mathrm{Ran}(P)$, one has $J(\mathbf{1} - P)J\phi_1 = \phi_1$ and therefore

$$\begin{aligned}
&\langle \phi | J(\mathbf{1} - P)J\phi \rangle \\
&\quad = \langle \phi_1 | J(\mathbf{1} - P)J\phi_1 \rangle + \langle \phi_0 | J(\mathbf{1} - P)J\phi_1 \rangle + \langle \phi_1 | J(\mathbf{1} - P)J\phi_0 \rangle + \langle \phi_0 | J(\mathbf{1} - P)J\phi_0 \rangle \\
&\quad = \langle \phi_1 | \phi_1 \rangle + \langle \phi_0 | \phi_1 \rangle + \langle \phi_1 | \phi_0 \rangle + \langle \phi_0 | J(\mathbf{1} - P)J\phi_0 \rangle
\end{aligned}$$

$$= \langle \phi_1 | \phi_1 \rangle + \langle \phi_0 | J(\mathbf{1} - P) J \phi_0 \rangle \geq \langle \phi_1 | \phi_1 \rangle$$
$$= \langle \phi | P \phi \rangle,$$

thus $P \leq J(\mathbf{1} - P)J$ follows. We show

$$P \leq J(\mathbf{1} - P)J \quad \Longleftrightarrow \quad P = PJ(\mathbf{1} - P)JP = P - PJPJP. \tag{9.4}$$

Let us first suppose that $P = PJ(\mathbf{1} - P)JP$ holds. Then for $\phi_1 \in \text{Ran}(P)$, one obtains the equalities $\phi_1 = P\phi_1 = PJ(\mathbf{1} - P)J\phi_1 = J(\mathbf{1} - P)J\phi_1$ where the last step follows as $\|J(\mathbf{1} - P)J\phi_1\| \leq \|\phi_1\|$. This implies $\text{Ran}(P) \subset \text{Ran}(J(\mathbf{1} - P)J)$ and therefore $P \leq J(\mathbf{1} - P)J$. Conversely, $P \leq J(\mathbf{1}-P)J$ implies $\text{Ran}(P) \subset \text{Ran}(J(\mathbf{1}-P)J)$ and therefore $J(\mathbf{1}-P)J\phi_1 = \phi_1$ for $\phi_1 \in \text{Ran}(P)$. Thus $P\phi_1 = PJ(\mathbf{1}-P)JP\phi_1$ follows. As $P\phi_0 = 0 = PJ(\mathbf{1}-P)JP\phi_0$ for $\phi_0 \in \text{Ker}(P)$ is obvious, $P = PJ(\mathbf{1} - P)JP$ follows. This concludes the proof of (9.4). If P is J-isotropic, the right-hand side of (9.4) is obviously correct and therefore $P \leq J(\mathbf{1} - P)J$ holds. Conversely, $P \leq J(\mathbf{1} - P)J$ implies by (9.4) that $0 = PJPJP = (PJP)^2$, as PJP is self-adjoint, and this implies that P is J-isotropic. □

Associated to a given J-unitary operator $T \in \mathbb{U}(\mathcal{K}, J)$ there are numerous J-isotropic subspaces. Recall that a subset $\Delta \subset \text{spec}(T)$ is called separated spectral subset if it is a closed subset and has trivial intersection with the closure of $\text{spec}(T) \setminus \Delta$.

Proposition 9.2.4. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ and $\Delta, \Delta' \subset \text{spec}(T)$ be separated spectral subsets. Set $\overline{\Delta}^{-1} = \{z \in \mathbb{C} : \overline{z}^{-1} \in \Delta\}$.*

(i) *If $\Delta' \cap \overline{\Delta}^{-1} = \emptyset$, then the associated Riesz projections of T satisfy*

$$(R_\Delta)^* J R_{\Delta'} = 0.$$

(ii) *If $\Delta \cap \overline{\Delta}^{-1} = \emptyset$, then the range of the Riesz projection R_Δ is J-isotropic.*
(iii) *If $\Delta \cap \overline{\Delta}^{-1} = \emptyset$, then the projection on the cokernel of R_Δ is J-isotropic.*
(iv) *Suppose that* $\text{spec}(T) = \Delta \cup \Delta'$ *and*

$$\Delta = \overline{\Delta}^{-1}, \quad \Delta' = \overline{\Delta'}^{-1}, \quad \Delta \cap \Delta' = \emptyset.$$

Furthermore, let R_Δ only have finite-dimensional range. Then both $\text{Ran}(R_\Delta)$ *and* $\text{Ran}(R_{\Delta'})$ *are nondegenerate.*

Proof. First of all, let us note that Proposition 2.2.2 remains valid for infinite-dimensional Krein spaces by the same proof, namely the Riesz projections satisfy

$$(R_\Delta)^* = J R_{\overline{\Delta}^{-1}} J.$$

Therefore

$$(R_\Delta)^* J R_{\Delta'} = J R_{\overline{\Delta}^{-1}} J J R_{\Delta'} = J R_{\overline{\Delta}^{-1}} R_{\Delta'} = 0,$$

the latter by the assumption and the properties of the Riesz projections, see Appendix A.1. The claim (ii) is now a direct consequence and (iii) follows from the identity $R_\Delta J(R_\Delta)^* = 0$ obtained in a similar manner.

(iv) By item (i), $\mathrm{Ran}(R_\Delta)$ and $\mathrm{Ran}(R_{\Delta'})$ are J-orthogonal. Moreover, these two subspaces span all $\mathcal{K}$ due to $R_\Delta + R_{\Delta'} = \mathbf{1}$ which follows from Proposition A.1.1(iii). Suppose that $\mathrm{Ran}(R_\Delta)$ is not nondegenerate. Then there exists a nonvanishing vector $\phi \in \mathrm{Ran}(R_\Delta)$ such that $(\phi')^* J\phi = 0$ for all $\phi' \in \mathrm{Ran}(R_\Delta)$ and hence ϕ is J-orthogonal to all vectors in $\mathcal{K}$. This is a contradiction to the fact that J is invertible. Now let Φ be a normalized frame with $\mathrm{Ran}(\phi) = \mathrm{Ran}(R_{\Delta'})$. Then $\Phi\Phi^* J\Phi\Phi^* - J$ is finite dimensional. Hence the essential spectrum of $\Phi\Phi^* J\Phi\Phi^*$ is $\{-1, 1\}$. But $\mathrm{spec}(\Phi^* J\Phi) \cup \{0\} = \mathrm{spec}(\Phi\Phi^* J\Phi\Phi^*)$, unless $\Delta = \emptyset$. However, by the same argument as above, $\mathrm{Ker}(\Phi^* J\Phi) = \{0\}$. Taking these facts together, one deduces that also $R_{\Delta'}$ is nondegenerate. □

Remark 9.2.5. Results similar to Proposition 9.2.4 also hold for a J-self-adjoint operator H. One merely has to replace the spectral reflection on the unit circle $\mathbb{S}^1$ by a reflection on the real axis, namely by complex conjugation. For example, let $\Delta \subset \mathrm{spec}(H)$ be a spectral subset such that $\Delta \cap \overline{\Delta} = \emptyset$ where the complex conjugate is $\overline{\Delta} = \{z \in \mathbb{C} : \overline{z} \in \Delta\}$. Then the range and cokernel of the Riesz projection R_Δ of H are J-isotropic subspaces. Several of the results below transfer in the same way, even though this will not be spelled out. The reader may consult [29, 175]. ◇

Next let us note that for a J-isotropic orthogonal projection P, also $P + JPJ$ is an orthogonal projection. Its range is a J-invariant subspace, and so is therefore its orthogonal complement which will be denoted

$$\mathcal{F}_P = \mathrm{Ran}(P + JPJ)^\perp = \mathrm{Ker}(P) \cap \mathrm{Ker}(JPJ).$$

Definition 9.2.6. A J-isotropic projection P is called semi-Fredholm if $\mathcal{F}_P$ is a Pontryagin space, and it is called Fredholm if $\mathcal{F}_P$ is finite dimensional.

Let us establish an elementary link between the Fredholm property of J-isotropic projections and Fredholm pairs of projections.

Proposition 9.2.7. *A J-isotropic projection P is Fredholm if and only if $(\mathbf{1} - P, JPJ)$ is a Fredholm pair. Its index is given by*

$$\mathrm{Ind}(\mathbf{1} - P, JPJ) = \dim(\mathcal{F}_P).$$

Proof. The characterization of the Fredholm property given in Proposition 5.3.2 can be readily checked and

$$\begin{aligned}\mathrm{Ind}(\mathbf{1} - P, JPJ) &= \dim\big(\mathrm{Ran}(\mathbf{1} - P) \cap \mathrm{Ker}(JPJ)\big) - \dim\big(\mathrm{Ran}(JPJ) \cap \mathrm{Ker}(\mathbf{1} - P)\big) \\ &= \dim\big(\mathrm{Ker}(P) \cap \mathrm{Ker}(JPJ)\big) - \dim\big(\mathrm{Ran}(JPJ) \cap \mathrm{Ran}(P)\big),\end{aligned}$$

which is indeed equal to $\dim(\mathcal{F}_P)$ because JPJ and P are orthogonal. □

The invariant $\mathrm{Ind}(\mathbf{1}-P, JPJ)$ is, however, not the only interesting integer that can be associated to a Fredholm J-isotropic projection. Even for any semi-Fredholm J-isotropic projection P, one can furthermore consider the Krein signature

$$\mathrm{KSig}(\mathbf{1}-P-JPJ) = \mathrm{Sig}(J|_{\mathcal{F}_P}) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

It turns out that these two quantities are related for the following class of J-isotropic subspaces.

Definition 9.2.8. A J-isotropic subspace $\mathcal{E}$ is called maximally J-isotropic if there is no J-isotropic subspace $\mathcal{F}$ with $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{E} \neq \mathcal{F}$. A projection P is called maximally J-isotropic if its range is maximally J-isotropic. The set of all Fredholm maximally J-isotropic projections will be denoted by

$$\mathbb{FI}(\mathcal{K}, J) = \{P \in \mathbb{P}(\mathcal{K}) : P \text{ maximally } J\text{-isotropic and Fredholm}\}.$$

It is equipped with the norm topology $\mathcal{O}_N$.

Proposition 9.2.9. *A J-isotropic projection P is maximal if and only if $\mathbf{1} - (P + JPJ)$ is a Krein-definite Pontryagin projection. Moreover, for every maximally J-isotropic projection P, one has*

$$\mathrm{Ind}(\mathbf{1}-P, JPJ) = |\mathrm{KSig}(\mathbf{1}-P-JPJ)|.$$

Proof. Note that $\mathbf{1} - (P + JPJ)$ is the projection onto $\mathcal{F}_P$. The J-isotropic projection P is not maximal if and only if there exists a nontrivial subspace of $\mathcal{F}_P$ that is J-orthogonal to itself. For any unit vector ϕ in this subspace, $P + \phi\phi^*$ is a J-isotropic projection. Then $J\phi$ and ϕ are linearly independent vectors from $\mathcal{F}_P$, and J restricted to the two-dimensional subspace spanned by ϕ and $J\phi$ has eigenvalues 1 and -1, so that $\mathcal{F}_P$ is not Krein-definite. Conversely, if $\mathcal{F}_P$ is not Krein-definite, there is a unit vector $\phi \in \mathcal{F}_P$ that is J-orthogonal to itself. Then $P + \phi\phi^*$ is a J-isotropic projection and P is not maximal. The claim about $\mathrm{Ind}(\mathbf{1}-P, JPJ)$ directly follows from Proposition 9.2.7. □

For a finite-dimensional $\mathcal{H}$, every maximally J-isotropic subspace is of dimension $\dim(\mathcal{H})$ and is hence J-Lagrangian in the sense that $JPJ = \mathbf{1} - P$ holds for its range projection P, see Definition 9.3.1 below. However, in infinite dimension there are more maximally J-isotropic subspaces, namely maximally J-isotropic subspaces that are not J-Lagrangian (other than incorrectly stated in Section 2 of [168]). This shows the following example.

Example 9.2.10. Let us fix an orthonormal basis $\{b_l : l \in \mathbb{N}\}$ of $\mathrm{Ker}(J - \mathbf{1})$ and an orthonormal basis $\{e_l : l \in \mathbb{N}\}$ of $\mathrm{Ker}(J + \mathbf{1})$. For $k \in \mathbb{N}$, let us then define P_k as the projection onto $\mathrm{Ran}(P_k) = \overline{\mathrm{span}\{b_l + e_{l-k} : l \in \mathbb{N}, l > k\}}$. As $(\mathbf{1}-P_k)J(\mathbf{1}-P_k)b_l = b_l$ for all $l = 1, \dots, k$, one has $(\mathbf{1}-P_k)J(\mathbf{1}-P_k) \neq 0$. (Thus P_k is not J-Lagrangian.) One directly checks that P_k is J-isotropic and, equivalently, $\mathrm{Ran}(P_k)$ is J-isotropic. Because of the orthogonal decomposition

$$\mathcal{K} = \mathrm{Ran}(P_k) \oplus \mathrm{Ran}(JP_kJ) \oplus \mathrm{span}\{b_1, \dots, b_k\}, \tag{9.5}$$

there is no extension of $\mathrm{Ran}(P_k)$ to a J-isotropic subspace, namely $\mathrm{Ran}(P_k)$ is maximally J-isotropic. Along the same lines, it is also possible to construct an example of a maximally J-isometric projection P_∞ such that $\mathrm{Ran}(P_\infty) \oplus \mathrm{Ran}(JP_\infty J)$ has infinite codimension, by setting, e. g., $\mathrm{Ran}(P_\infty) = \overline{\mathrm{span}\{b_l + e_{2l} : l \in \mathbb{N}\}}$. ◇

The maximally J-isotropic projections P_k constructed in Example 9.2.10 are Fredholm for $k < \infty$ and their Krein signature is $k > 0$. In the same manner, it is also possible to construct maximally J-isotropic projections with negative Krein signature. On the other hand, P_∞ is not Fredholm and has Krein signature $+\infty$.

Proposition 9.2.11. *The space* $(\mathbb{FI}(\mathcal{K}, J), \mathcal{O}_N)$ *has* $\mathbb{Z}$ *connected components labeled by the Krein signature, that is, the map* $K : \pi_0(\mathbb{FI}(\mathcal{K}, J)) \to \mathbb{Z}$ *given by*

$$K(P) = \mathrm{KSig}(\mathbf{1} - P - JPJ)$$

is an isomorphism.

Proof. It is shown that every $P \in \mathbb{FI}(\mathcal{K}, J)$ with $\mathrm{KSig}(\mathbf{1}-P-JPJ) = k$ is unitarily equivalent to a fixed maximally J-isotropic projection P_k with $\mathrm{KSig}(\mathbf{1} - P_k - JP_kJ) = k$ via a unitary that commutes with J (Example 9.2.10 constructs such projections P_k for $k \geq 0$). From this unitary equivalence, one readily constructs the desired connecting path by taking a root of the unitary. For the construction of the unitary, it is convenient to use normalized frames (see Definition 5.1.5). Hence let Φ_k be a normalized frame for P_k, namely one has $P_k = \Phi_k \Phi_k^*$. Then $J\Phi_k$ is a normalized frame for JP_kJ. Further set $\Psi_k = (\Phi_k, J\Phi_k)^\perp$ which is then a normalized frame for the orthogonal projection onto the finite-dimensional space $\mathcal{F}_{P_k}$. Note that $J\Psi_k = \mathrm{sgn}(k)\Psi_k$. Similarly, let $\Phi, J\Phi$ and Ψ be associated to P. Then one checks that $U = (\Phi, J\Phi, \Psi)(\Phi_k, J\Phi_k, \Psi_k)^*$ is well defined, unitary, commutes with J, and satisfies $UP_kU^* = P$. □

Given a maximally J-isotropic projection P, one can set

$$T_P = \frac{1}{2}P + 2JPJ + (\mathbf{1} - P - JPJ), \tag{9.6}$$

and then readily checks that $T_P \in \mathbb{U}(\mathcal{K}, J)$ is J-unitary. The same holds, e. g., for the operator $zP + \overline{z}^{-1}JPJ + e^{\imath\varphi}(\mathbf{1}-P-JPJ)$ where z is a complex number with $|z| \in (0, 1)$ and $e^{\imath\varphi} \in \mathbb{S}^1$ a phase. Even further, one can spread out the spectrum on the unit circle within the class of J-unitary operators. On the other hand, it is impossible for these unit eigenvalues to leave the unit circle under any perturbation within the set of J-unitaries. Indeed, each such eigenvalue λ would lead to another eigenvalue $\overline{\lambda}^{-1}$ by (2.16), and a more detailed elementary analysis shows that the J-inertia on the joint eigenspace has to be $(1, 0, 1)$ which is impossible because $\mathrm{Ran}(\mathbf{1} - P - JPJ)$ is a J-definite subspace. This is the essence of Krein stability which associates a signature to each unit eigenvalue of a J-unitary operator, for details see, e. g., [168]. In Definition 9.2.12 below, the Krein signature is not associated to a single eigenvalue on the unit circle, but rather jointly to all eigenvalues on $\mathbb{S}^1$, which corresponds to taking the sum of all Krein signatures of unit eigenvalues. The example of T_P suggests the following natural situation in which Fredholm J-isotropic projections appear. This is relevant for applications, such as in [168, 175].

Definition 9.2.12. A J-unitary operator $T \in \mathbb{U}(\mathcal{K}, J)$ is said to be essentially $\mathbb{S}^1$-gapped if it only has discrete spectrum (isolated eigenvalues of finite algebraic multiplicity) on $\mathbb{S}^1$. The total Krein signature of such an essentially $\mathbb{S}^1$-gapped J-unitary T is

$$\mathrm{KSig}(T) = \mathrm{KSig}(P^{=}),$$

where $P^{=}$ denotes the finite-dimensional range projection of all eigenvalues on the unit circle $\mathbb{S}^1$.

Let us note that Proposition 9.2.4(iv) applies directly, in particular, to an essentially $\mathbb{S}^1$-gapped operator $T \in \mathbb{U}(\mathcal{K}, J)$ if one chooses $\Delta = \mathbb{S}^1 \cap \mathrm{spec}(T)$ and $\Delta' = \mathrm{spec}(T) \setminus \Delta$. Thus $\mathrm{Ran}(P^{=}) = \mathrm{Ran}(R_\Delta)$ and $\mathrm{Ran}(R_{\Delta'})$ are J-orthogonal and nondegenerate. As $P^{=}$ is finite dimensional, it is hence a Pontryagin projection with a well-defined Krein signature. Therefore $\mathrm{KSig}(T)$ is well defined.

Proposition 9.2.13. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ be an essentially $\mathbb{S}^1$-gapped J-unitary. Then let $R^<$ and $R^>$ be the Riesz projections of T associated to the spectral subsets* $\mathrm{spec}(T) \cap B_1(0)$ *and* $\mathrm{spec}(T) \setminus \overline{B_1(0)}$, *respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto the subspaces $\mathcal{E}^< = \mathrm{Ran}(R^<)$ and $\mathcal{E}^> = \mathrm{Ran}(R^>)$. Further let $P^{=}$ be the finite-dimensional range projection of all eigenvalues on the unit circle $\mathbb{S}^1$. The (total) Krein signature* $\mathrm{KSig}(T)$ *of the essentially $\mathbb{S}^1$-gapped J-unitary T is continuous in T. The projections $P^<$ and $P^>$ are Fredholm J-isotropic projections, which are maximal (namely in $\mathbb{FI}(\mathcal{K}, J)$) if the restriction $J|_{\mathrm{Ran}(P^{=})}$ of the quadratic form J to* $\mathrm{Ran}(P^{=})$ *is definite.*

Proof. The fact that $P^<$ and $P^>$ are J-isotropic follows from Proposition 9.2.4 applied to $\Delta = \mathrm{spec}(T) \cap B_1(0)$. The Fredholm property follows directly from the hypothesis because $JP^<J$ is the orthogonal projection onto $\mathrm{Ker}(R^>)^\perp$ and thus $\mathcal{F}_{P^<} = \mathrm{Ker}(R^>) \cap \mathrm{Ran}(R^<)^\perp$ is finite dimensional as $P^{=}$ is finite dimensional. The same argument shows that also $\mathcal{F}_{P^>} = \mathrm{Ker}(R^<) \cap \mathrm{Ran}(R^>)^\perp$ is finite dimensional. Now an eigenvalue λ of T can leave $\mathbb{S}^1$ only together with its reflected $\overline{\lambda}^{-1}$ (Krein collision). But on the span of the two corresponding eigenvectors, J has vanishing signature (this requires an addendum to the argument leading to Proposition 9.2.4, see [168]). Even though this process changes the projection $P^{=}$, it does therefore not change the Krein signature $\mathrm{KSig}(T)$. Once J is definite on the range of $P^{=}$, no eigenvalue can leave the unit circle and neither $P^<$ nor $P^>$ can be enlarged, and are thus maximal. □

Admittedly, the above proof of Proposition 9.2.13 is only a sketch of what is the heart of the celebrated Krein stability result [118]. The reader interested in further details is referred to [168, 175].

Remark 9.2.14. Once again, one can also introduce essentially $\mathbb{R}$-gapped bounded J-self-adjoint operators, namely those bounded J-self-adjoints that only have discrete spectrum of finite multiplicity on $\mathbb{R}$. Then one can consider their global Krein signature. Using Riesz projections for the upper and lower half-plane, as well as a perturbative argument for the real eigenvalues of indefinite signature, one can show that the set of all essentially $\mathbb{R}$-gapped bounded J-self-adjoint operators can be retracted to the set $\mathbb{FI}(\mathcal{K}, J)$, if the latter is identified with the J-self-adjoint operators $H = \imath P - \imath JPJ$ (this is similar to (9.6)). Moreover, it is possible to show by analytic Fredholm theory that the set of all essentially $\mathbb{R}$-gapped bounded J-self-adjoint operators is equal to the set $\{H \in \mathbb{B}_{sa}(\mathcal{K}, J) : H - \lambda\mathbf{1} \in \mathbb{FB}(\mathcal{K}) \text{ for all } \lambda \in \mathbb{R}\}$. Detailed proofs can be found in [175]. Such a characterization with a Fredholm property is not possible for the essentially $\mathbb{S}^1$-gapped J-unitaries, see [168] for a counterexample. It is likely also not true that the essentially $\mathbb{S}^1$-gapped J-unitaries can be retracted to $\mathbb{FI}(\mathcal{K}, J)$. ◇

9.3 *J*-Lagrangian subspaces

Definition 9.3.1. Projection $P = P^* = P^2 \in \mathbb{P}(\mathcal{K})$ is called J-Lagrangian if and only if $JPJ = \mathbf{1} - P$. A closed subspace is called J-Lagrangian if its range projection is J-Lagrangian. The J-Lagrangian Grassmannian is defined as

$$\mathbb{P}(\mathcal{K}, J) = \{P = P^* = P^2 \in \mathbb{B}(\mathcal{K}) : JPJ = \mathbf{1} - P\}.$$

It is equipped with the metric d_N and thus the norm topology $\mathcal{O}_N$.

A Fredholm maximally J-isotropic projection P is J-Lagrangian if and only if one has $\mathrm{KSig}(\mathbf{1} - P - JPJ) = 0$. Clearly, one can reformulate Definition 9.3.1 as

$$P \ J\text{-Lagrangian} \quad \Longleftrightarrow \quad P + JPJ = \mathbf{1}.$$

The definition implies that $\mathbf{1} - P$ is J-Lagrangian if and only if P is J-Lagrangian. Furthermore, every J-Lagrangian projection P provides a chiral symmetry $Q = \mathbf{1} - 2P$, and vice versa. More generally, the negative spectral projection $P = \chi(H < 0)$ of an invertible chiral operator H is J-Lagrangian. Definition 9.3.1 can further be reformulated algebraically. In view of (9.3), every J-Lagrangian projection is of the form

$$P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix}, \tag{9.7}$$

where U is a unitary on $\mathcal{H}$, see also Proposition 9.3.4 below. Let us next give another characterization of J-Lagrangian projections.

Lemma 9.3.2. *An orthogonal projection P is J-Lagrangian if and only if*

$$PJP = 0 \quad \text{and} \quad (\mathbf{1} - P)J(\mathbf{1} - P) = 0,$$

or alternatively if and only if the restrictions $J|_{\mathrm{Ran}(P)}$ and $J|_{\mathrm{Ran}(\mathbf{1}-P)}$ of the quadratic form J vanish.

Proof. Multiplying $JPJ = \mathbf{1} - P$ by P from the left and J from the right shows $PJP = 0$. Proceeding similarly with $P = J(\mathbf{1} - P)J$ shows $(\mathbf{1} - P)J(\mathbf{1} - P) = 0$. Conversely,

$$\begin{aligned} JPJ &= JPJP + JPJ(\mathbf{1} - P) \\ &= JPJ(\mathbf{1} - P) \\ &= J(P + \mathbf{1} - P)J(\mathbf{1} - P) \\ &= (\mathbf{1} - P), \end{aligned}$$

showing the claimed equivalence. □

The following result describes a natural situation in which J-Lagrangian subspaces arise. It is the infinite-dimensional analogue of Proposition 2.2.3.

Proposition 9.3.3. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ satisfy $\mathrm{spec}(T) \cap \mathbb{S}^1 = \emptyset$. Then let $R^<$ and $R^>$ be the Riesz projections of T associated to the separated spectral subsets $\mathrm{spec}(T) \cap B_1(0)$ and $\mathrm{spec}(T) \setminus B_1(0)$, respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto the subspaces $\mathcal{E}^< = \mathrm{Ran}(R^<)$ and $\mathcal{E}^> = \mathrm{Ran}(R^>)$. Then $P^<$ and $P^>$ are J-Lagrangian.*

Proof. This follows directly from Proposition 9.2.13. □

Next let us consider the set of all J-Lagrangian subspaces. Due to (9.7), the J-Lagrangian Grassmannian $\mathbb{P}(\mathcal{K}, J)$ on $\mathcal{K}$ can naturally be identified with the unitary group on $\mathcal{H}$.

Proposition 9.3.4. *The stereographic projection $\Pi : \mathbb{P}(\mathcal{K}, J) \to \mathbb{U}(\mathcal{H})$ defined by*

$$\Pi(P) = U, \quad P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix},$$

is a bijective isometry.

Proof. The stereographic projection is surjective because, for $U \in \mathbb{U}(\mathcal{H})$,

$$P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix} \in \mathbb{P}(\mathcal{K}, J)$$

is a Lagrangian projection and $\Pi(P) = U$. Moreover, the stereographic projection is injective as, for $P, P' \in \mathbb{P}(\mathcal{K}, J)$ with $U = \Pi(P)$ and $U' = \Pi(P')$, one has

$$\|P - P'\| = \frac{1}{2}\|\Pi(P) - \Pi(P')\| = \frac{1}{2}\|U - U'\|.$$

Therefore Π is injective. In conclusion, the stereographic projection is a bijection. The above identity also shows that it is bi-Lipshitz-continuous. □

The spectral theory in $\mathbb{U}(\mathcal{H})$ is of importance for the intersection of two Lagrangian subspaces, as shows the following result which is at the heart of intersection theory of J-Lagrangian subspaces and hence of crucial relevance for the Bott–Maslov index introduced and analyzed in the next section.

Proposition 9.3.5. *Let P_0 and P_1 be J-Lagrangian projections with stereographic projections $U_0 = \Pi(P_0)$ and $U_1 = \Pi(P_1)$. One has*

$$\begin{aligned}\dim(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)) &= \dim(\mathrm{Ker}(U_1^* U_0 + \mathbf{1}))\\ &= \dim(\mathrm{Ker}(U_1 U_0^* + \mathbf{1})),\end{aligned}$$

or alternatively

$$\dim(\mathrm{Ran}(P_0) \cap \mathrm{Ran}(JP_1J)) = \dim(\mathrm{Ker}(U_1^* U_0 + \mathbf{1})).$$

Proof. A vector $\phi = \binom{\phi_1}{\phi_2} \in \mathcal{K}$ with $\phi_1, \phi_2 \in \mathcal{H}$ is in the range of P_0 if and only if

$$P_0\phi = \frac{1}{2}\begin{pmatrix}\phi_1 + U_0\phi_2\\ U_0^*\phi_1 + \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

which is equivalent to $U_0\phi_2 = \phi_1$. Then

$$P_1\phi = \frac{1}{2}\begin{pmatrix}\phi_1 + U_1\phi_2\\ U_1^*\phi_1 + \phi_2\end{pmatrix} = 0$$

if and only if $-U_1^*\phi_1 = \phi_2$. In conclusion, $\phi \in \mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$ implies $\phi_2 = -U_1^* U_0\phi_2$ and $\dim(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)) \le \dim(\mathrm{Ker}(U_1^* U_0 + \mathbf{1}))$. Conversely, for $\phi_2 \in \mathrm{Ker}(U_1^* U_0 + \mathbf{1})$, one has $U_0\phi_2 = -U_1\phi_2$ and therefore $\phi = \binom{U_0\phi_2}{\phi_2} \in \mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$. This implies that $\dim(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)) \ge \dim(\mathrm{Ker}(U_1^* U_0 + \mathbf{1}))$ and thus the claim follows. □

A J-unitary operator $T \in \mathbb{U}(\mathcal{K}, J)$ sends a J-Lagrangian subspace $\mathcal{E}$ to a J-Lagrangian subspace $T\mathcal{E}$. Indeed, for all vectors $\psi_0 = T\phi_0 \in T\mathcal{E}$ and $\psi_1 = T\phi_1 \in T\mathcal{E}$, one deduces $\psi_0^* J\psi_1 = \phi_0^* T^* JT\phi_1 = \phi_0^* J\phi_1 = 0$. Analogously, for $\widetilde{\psi_0} = (T^*)^{-1}\widetilde{\phi_0} \in (T\mathcal{E})^\perp = (T^*)^{-1}\mathcal{E}^\perp$ and $\widetilde{\psi_1} = (T^*)^{-1}\widetilde{\phi_1} \in (T\mathcal{E})^\perp$, one has $(\widetilde{\psi_0})^* J\widetilde{\psi_0}$, so that Lemma 9.3.2 implies that $T\mathcal{E}$ is J-Lagrangian. (Note that this also shows that the image of J-isotropic subspaces under a J-unitary is J-isotropic.) If $P \in \mathbb{P}(\mathcal{K}, J)$ is the range projection of $\mathcal{E}$, then the range projection of $T\mathcal{E}$ is denoted by $T \cdot P$, namely $\cdot : \mathbb{U}(\mathcal{K},J) \times \mathbb{P}(\mathcal{K},J) \to \mathbb{P}(\mathcal{K},J)$ is a group action. This action is transitive. Actually, already the subgroup $\mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{K})$ does so as shows the following result.

Proposition 9.3.6. *The group $\mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{K})$ acts continuously and transitively on $\mathbb{P}(\mathcal{K},J)$.*

Proof. The action of $\mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{K})$ on $\mathbb{P}(\mathcal{K},J)$ is simply given by $V \cdot P = VPV^*$ for $V \in \mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{K})$ and $P \in \mathbb{P}(\mathcal{K},J)$. One directly checks that VPV^* is in $\mathbb{P}(\mathcal{K},J)$ and therefore the action is well defined. To show that the action is transitive, consider two J-Lagrangian projections

$$P_0 = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U_0 \\ U_0^* & \mathbf{1} \end{pmatrix} \quad \text{and} \quad P_1 = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U_1 \\ U_1^* & \mathbf{1} \end{pmatrix},$$

where $U_0, U_1 \in \mathbb{U}(\mathcal{H})$ are unitaries. One directly checks that $VP_0V^* = P_1$ for

$$V = \begin{pmatrix} U_1 & 0 \\ 0 & U_0 \end{pmatrix} \in \mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{K}),$$

finishing the proof. □

Now the action (5.7) of invertibles on projections becomes an action of $\mathbb{U}(\mathcal{K},J)$ on $\mathbb{P}(\mathcal{K},J)$. Recall that for $T \in \mathbb{U}(\mathcal{K},J)$ on $P \in \mathbb{P}(\mathcal{K},J)$, it is given by

$$T \cdot P = (TPT^*)\left(TPT^*\right)^{-2}(TPT^*).$$

The following elementary fact will be used later on.

Proposition 9.3.7. *For $T \in \mathbb{U}(\mathcal{K},J)$ and $P \in \mathbb{P}(\mathcal{K},J)$, one has*

$$T \cdot P = J((T^{-1})^* \cdot (\mathbf{1} - P))J.$$

Proof. The computation

$$\begin{aligned} J(T \cdot P)J &= (JTPT^*J)(JTPT^*J)^{-2}(JTPT^*J) \\ &= ((T^{-1})^*JPJT^{-1})((T^{-1})^*JPJT^{-1})^{-2}((T^{-1})^*JPJT^{-1}), \end{aligned}$$

combined with $JPJ = \mathbf{1} - P$, shows the claim. □

Under the stereographic projection, the action takes a simpler form.

Proposition 9.3.8. *The group $\mathbb{U}(\mathcal{K},J)$ acts continuously on the Siegel disc*

$$\mathbb{D}(\mathcal{H}) = \{U \in \mathbb{B}(\mathcal{H}) : \|U\| < 1\}$$

and also on the unitary group $\mathbb{U}(\mathcal{H})$ by Möbius transformation denoted by a dot and defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot U = (AU + B)(CU + D)^{-1}, \quad U \in \mathbb{D}(\mathcal{H}).$$

The Möbius action on $\mathbb{U}(\mathcal{H})$ *implements the action* $\cdot : \mathbb{U}(\mathcal{K}, J) \times \mathbb{P}(\mathcal{K}, J) \to \mathbb{P}(\mathcal{K}, J)$*, namely*

$$T \cdot \Pi(P) = \Pi(T \cdot P).$$

Proof. One first has to show that for $U \in \mathbb{B}(\mathcal{H})$ with $\|U\| \leq 1$ and $T \in \mathbb{U}(\mathcal{K}, J)$ the inverse in the Möbius transformation $T \cdot U$ is well defined. By Proposition 9.1.3, one concludes that $(CU + D) = D(D^{-1}CU + \mathbf{1})$ is indeed invertible. Then the identities of Proposition 9.1.3 imply

$$(CU + D)^*(CU + D) - (AU + B)^*(AU + B) = \mathbf{1} - U^*U. \tag{9.8}$$

Now multiplying (9.8) from the left by $((CU + D)^*)^{-1}$ and from the right by $(CU + D)^{-1}$ and using $\mathbf{1} - U^*U > 0$ for $U \in \mathbb{D}(\mathcal{H})$ shows $(T \cdot U)^*(T \cdot U) < \mathbf{1}$ so that $T \cdot U \in \mathbb{D}(\mathcal{H})$. By the same argument, if $U \in \mathbb{U}(\mathcal{H})$, then $T \cdot U \in \mathbb{U}(\mathcal{H})$. A short algebraic calculation also shows that $(TT') \cdot U = T \cdot (T' \cdot U)$.

To prove the last formula, let us note that the range of $P = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix}$ is

$$\mathrm{Ran}(P) = \left\{ \begin{pmatrix} U\phi \\ \phi \end{pmatrix} : \phi \in \mathcal{H} \right\},$$

therefore

$$\begin{aligned} \mathrm{Ran}(T \cdot P) &= \left\{ \begin{pmatrix} (AU + B)\phi \\ (CU + D)\phi \end{pmatrix} : \phi \in \mathcal{H} \right\} \\ &= \left\{ \begin{pmatrix} (AU + B)(CU + D)^{-1}\widetilde{\phi} \\ \widetilde{\phi} \end{pmatrix} : \widetilde{\phi} \in \mathcal{H} \right\} \\ &= \mathrm{Ran}\left(\frac{1}{2} \begin{pmatrix} \mathbf{1} & (AU + B)(CU + D)^{-1} \\ ((AU + B)(CU + D)^{-1})^* & \mathbf{1} \end{pmatrix} \right). \end{aligned}$$

Proposition 9.3.4 implies the claim. □

9.4 Fredholm pairs of *J*-Lagrangian projections

Recall from Section 5.2 the notion of Fredholm pairs (P_0, P_1) of orthogonal projections and their index given by the difference of the finite dimensions of the subspaces $\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1) = \mathrm{Ran}(P_0) \cap \mathrm{Ran}(P_1)^\perp$ and $\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0) = \mathrm{Ran}(P_1) \cap \mathrm{Ran}(P_0)^\perp$. It is now natural to consider Fredholm pairs of J-Lagrangian projections and introduce the following notation:

$$\mathbb{FPP}(\mathcal{K}, J) = \{(P_0, P_1) : P_0, P_1 \in \mathbb{P}(\mathcal{K}, J) \text{ and } (P_0, P_1) \text{ Fredholm pair}\}. \tag{9.9}$$

Again $\mathcal{O}_N$ (or more precisely, $\mathcal{O}_N \times \mathcal{O}_N$) is the natural topology on this set. Noting that J-Lagrangian Fredholm pairs are always proper and recalling the notation for proper Fredholm pairs from (5.19), this can be rewritten as

$$\mathbb{FPP}(\mathcal{K}, J) = \big(\mathbb{P}(\mathcal{K}, J), \mathbb{P}(\mathcal{K}, J)\big) \cap \mathbb{FPP}(\mathcal{K}).$$

Let us provide a simple way to produce Fredholm pairs of J-Lagrangian projections.

Proposition 9.4.1. *Let $P \in \mathbb{P}(\mathcal{K}, J)$ and $T \in \mathbb{U}^{\mathbb{C}}(\mathcal{K}, J)$. Then $(P, T \cdot P) \in \mathbb{FPP}(\mathcal{K}, J)$.*

Proof. The hypothesis implies that $T \cdot P - P \in \mathbb{K}(\mathcal{K})$, e. g., by using the formula in Proposition 9.3.8. Therefore the Fredholm property of the pair $(P, T \cdot P)$ follows from Proposition 5.2.4. □

The first aim will be to characterize the Fredholm property of pairs of J-Lagrangian projections (P_0, P_1) in terms of the associated stereographic projections. That this should be possible is plausible due to Proposition 9.3.5 which shows that the above finite-dimensional intersections can precisely be determined from the spectral theory of the stereographic projections. For the formulation of the result, which goes back at least to [113], let us recall the relevant spectral notions from Section 3.4. The discrete spectrum $\mathrm{spec}_{\mathrm{dis}}(A)$ of a normal operator $A \in \mathbb{B}(\mathcal{H})$ consists of all isolated eigenvalues of finite multiplicity, and then the essential spectrum is defined by $\mathrm{spec}_{\mathrm{ess}}(A) = \mathrm{spec}(A) \setminus \mathrm{spec}_{\mathrm{dis}}(A)$. Also recall form Section 3.7 that the set of unitaries $U \in \mathbb{U}(\mathcal{H})$ such that $-1 \notin \mathrm{spec}_{\mathrm{ess}}(U)$ is denoted by $\mathbb{FU}(\mathcal{H})$.

Theorem 9.4.2. *Let P_0 and P_1 be two J-Lagrangian projections with stereographic projections $U_0 = \Pi(P_0)$ and $U_1 = \Pi(P_1)$. Then*

$$\begin{aligned}(P_0, P_1) \in \mathbb{FPP}(\mathcal{K}, J) \textit{ Fredholm pair} \quad &\Longleftrightarrow \quad -1 \notin \mathrm{spec}_{\mathrm{ess}}(U_1^* U_0) \\ &\Longleftrightarrow \quad U_1^* U_0 \in \mathbb{FU}(\mathcal{H}) \\ &\Longleftrightarrow \quad U_1 U_0^* \in \mathbb{FU}(\mathcal{H}).\end{aligned}$$

Proof. As above, $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ are chiral symmetries. If (P_0, P_1) is a Fredholm pair,

$$(Q_0 + Q_1)^2 = \begin{pmatrix} (U_0 + U_1)(U_0 + U_1)^* & 0 \\ 0 & (U_0 + U_1)^*(U_0 + U_1) \end{pmatrix}$$

is Fredholm by Proposition 5.4.2 and therefore $0 \notin \mathrm{spec}_{\mathrm{ess}}((Q_0 + Q_1)^2)$. Multiplying out shows that

$$(U_0 + U_1)^*(U_0 + U_1) = 2\mathbf{1} + U_1^* U_0 + U_0^* U_1$$

and

$$(U_0 + U_1)(U_0 + U_1)^* = 2\mathbf{1} + U_1U_0^* + U_0U_1^*$$

are Fredholm. Let us define $U_1^*U_0 = \hat{U}$. Then

$$2\mathbf{1} + \hat{U} + \hat{U}^* = (\mathbf{1} + \hat{U})(\mathbf{1} + \hat{U})^* = (\mathbf{1} + \hat{U})^*(\mathbf{1} + \hat{U})$$

is Fredholm. Thus, by Corollary 3.4.4, $0 \notin \mathrm{spec}_{\mathrm{ess}}(2\mathbf{1} + \hat{U} + \hat{U}^*)$ and $\mathbf{1} + \hat{U}$ is Fredholm by Theorem 3.4.1. Again by Corollary 3.4.4, one has $-1 \notin \mathrm{spec}_{\mathrm{ess}}(\hat{U}) = \mathrm{spec}_{\mathrm{ess}}(U_0U_1^*)$.

Conversely, if $-1 \notin \mathrm{spec}_{\mathrm{ess}}(U_1^*U_0)$, then $\mathbf{1} + U_1^*U_0$ is Fredholm by Corollary 3.4.4. Therefore $U_0 + U_1$ and $(U_0 + U_1)^*$ are Fredholm. Thus $Q_0 + Q_1$ is Fredholm and, by Proposition 5.4.2, (P_0, P_1) is a Fredholm pair. □

Corollary 9.4.3. *Let (P_0, P_1) be a pair of J-Lagrangian projections and let furthermore $V \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Then*

$$(P_0, P_1) \in \mathbb{FPP}(\mathcal{K}, J) \quad \Longleftrightarrow \quad (V \cdot P_0, V \cdot P_1) \in \mathbb{FPP}(\mathcal{K}, J).$$

Proof. Recall from Proposition 9.1.6 that $V = \mathrm{diag}(V_+, V_-)$ with $V_\pm \in \mathbb{U}(\mathcal{H})$. By Proposition 9.3.8, one hence has $\Pi(V \cdot P_j) = V_+\Pi(P_j)V_-^*$ so that

$$\Pi(V \cdot P_1)^*\Pi(V \cdot P_0) = V_-\Pi(P_1)^*\Pi(P_0)V_-^*. \tag{9.10}$$

Hence the claim follows from Theorem 9.4.2. □

Proposition 9.4.4. *Let $P \in \mathbb{P}(\mathcal{K}, J)$ with $U = \Pi(P)$ and $T \in \mathbb{U}(\mathcal{K}, J)$. Then*

$$(P, T \cdot P) \in \mathbb{FPP}(\mathcal{K}, J) \quad \Longleftrightarrow \quad \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} \in \mathbb{FB}(\mathcal{H}).$$

Proof. By Theorem 9.4.2, the Fredholm property of $(P, T \cdot P) \in \mathbb{FPP}(\mathcal{K}, J)$ is equivalent to -1 not being in the essential spectrum of $U^*T \cdot U$, which is equivalent to 0 not being in the essential spectrum of the self-adjoint operator $\mathbb{R}e(U^*T \cdot U) + \mathbf{1}$. Now let A, B, C, D be the entries of T, e. g., as in Proposition 9.3.8. This proposition also shows that $(CU + D)^{-1}$ is invertible. Then

$$\begin{aligned} \mathbb{R}e(U^*T \cdot U) + \mathbf{1} &= \frac{1}{2}(U^*T \cdot U + (T \cdot U)^*U) + \mathbf{1} \\ &= \frac{1}{2}(U^*T \cdot U + \mathbf{1})^*(U^*T \cdot U + \mathbf{1}) \\ &= \frac{1}{2}((CU + D)^{-1})^*\left[\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}\right]^* \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}(CU + D)^{-1} \\ &= \frac{1}{2}\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}|(CU + D)^*|^{-2}\left[\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}\right]^*. \end{aligned}$$

Due to Theorem 3.4.1, the stated Fredholm property implies a lower bound on the essential spectrum of $\mathbb{R}e(U^* T \cdot U)+\mathbf{1}$, and, conversely, the Fredholm property is a consequence of the lower bound on the essential spectrum. □

Example 9.4.5. There are $P \in \mathbb{P}(\mathcal{K}, J)$ and $T \in \mathbb{U}(\mathcal{K}, J)$ such that $(P, T \cdot P)$ is *not* a Fredholm pair. For example, take $U = \mathbf{1}$ (which corresponds to P being the reference projection P_{ref} given in (9.11) below) and $T = J$. ◇

The following result provides another natural situation in which Fredholm pairs of J-Lagrangian projections arise. It merely extends Proposition 9.3.3.

Proposition 9.4.6. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ satisfy* $\text{spec}(T) \cap \mathbb{S}^1 = \emptyset$. *Let $R^<$ and $R^>$ be the Riesz projections of T associated to the spectral subsets* $\text{spec}(T) \cap B_1(0)$ *and* $\text{spec}(T) \setminus B_1(0)$, *respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto their ranges* $\mathcal{E}^< = \text{Ran}(R^<)$ *and* $\mathcal{E}^> = \text{Ran}(R^>)$. *Then $(P^<, \mathbf{1} - P^>)$ forms a Fredholm pair.*

Proof. In Proposition 9.3.3 it was already shown that $P^<$ and $P^>$ are J-Lagrangian so that also $\mathbf{1} - P^>$ is J-Lagrangian. It remains to check the conditions in Definition 5.3.2 for $P_0 = P^<$ and $P_1 = \mathbf{1} - P^>$. First of all, $\text{Ran}(P_0) + \text{Ran}(\mathbf{1} - P_1) = \mathcal{E}^< + \mathcal{E}^> = \mathcal{K}$ is closed. Secondly, $\text{Ran}(P_0) \cap \text{Ker}(P_1) = \mathcal{E}^< \cap \mathcal{E}^> = \{0\}$ is finite dimensional, and finally, $\text{Ker}(P_0)^\perp + \text{Ran}(P_1)^\perp = \text{Ran}(P_0) + \text{Ker}(P_1) = \mathcal{E}^< + \mathcal{E}^> = \mathcal{K}$ so that $\text{Ker}(P_0) \cap \text{Ran}(P_1) = \{0\}$ is also finite dimensional. □

The next results states that for a Fredholm pair of J-Lagrangian projections the index as defined in Section 5.2 is of little interest (for the finite-dimensional case, see already Remark 5.2.3).

Proposition 9.4.7. *For all* $(P_0, P_1) \in \mathbb{FPP}(\mathcal{K}, J)$, *one has*

$$\text{Ind}(P_0, P_1) = 0.$$

Moreover, $(\mathbb{FPP}(\mathcal{K}, J), \mathcal{O}_N)$ *is connected.*

Proof. Let (P_0, P_1) be a Fredholm pair of J-Lagrangian projections. Then, by Theorem 9.4.2, -1 is not in the essential spectrum of $\Pi(P_0)\Pi(P_1)^*$. By spectral calculus with a root for which the branch cut is chosen to be on the negative real axis, the paths $s \in [0,1] \mapsto (\Pi(P_0)\Pi(P_1)^*)^{1-s}$ lies entirely in $\mathbb{FU}(\mathcal{H})$. For $U(s) = (\Pi(P_0)\Pi(P_1)^*)^{1-s}\Pi(P_1)$, let us define a path of J-Lagrangian projections by

$$s \in [0,1] \mapsto P(s) = \frac{1}{2}\begin{pmatrix} \mathbf{1} & U(s) \\ U(s)^* & \mathbf{1} \end{pmatrix}.$$

Again by Theorem 9.4.2, one checks that $s \in [0,1] \mapsto (P(s), P_1)$ is a path of Fredholm pairs of J-Lagrangian projections. It connects (P_0, P_1) to (P_1, P_1). Therefore by Proposition 5.2.7,

$$\text{Ind}(P_0, P_1) = \text{Ind}(P_1, P_1) = 0.$$

The second claim follows because the set $\mathbb{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ is connected and therefore there is a path $s \in [0,1] \mapsto \hat{U}(s)$ of unitaries connecting $\Pi(P_1)$ to $\mathbf{1}$. Then $s \in [0,1] \mapsto \Pi^{-1}(\hat{U}(s))$ is a path of J-Lagrangian projections connecting P_1 to the reference J-Lagrangian projection $P_{\text{ref}} = \frac{1}{2}\left(\begin{smallmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{smallmatrix}\right)$. Thus $s \in [0,1] \mapsto (\Pi^{-1}(\hat{U}(s)), \Pi^{-1}(\hat{U}(s)))$ is a path of Fredholm pairs of J-Lagrangian projections connecting (P_1, P_1) to $(P_{\text{ref}}, P_{\text{ref}})$. In conclusion, there is a path of Fredholm pairs of J-Lagrangian projections connecting (P_0, P_1) to $(P_{\text{ref}}, P_{\text{ref}})$ and therefore the set of Fredholm pairs of J-Lagrangian projections is connected. □

In many applications of the Fredholm pairs of J-Lagrangian projections, one of the projections, say P_0, is fixed and given by a reference J-Lagrangian projection which we choose to be

$$P_{\text{ref}} = \frac{1}{2}\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{pmatrix}. \tag{9.11}$$

Thus let us introduce the Fredholm J-Lagrangian Grassmannian (with respect to P_{ref}) by

$$\mathbb{FP}(\mathcal{K}, J) = \{P \in \mathbb{P}(\mathcal{K}, J) : (P_{\text{ref}}, P) \in \mathbb{FPP}(\mathcal{K}, J)\}.$$

As $\Pi(P_{\text{ref}}) = \mathbf{1}$, Theorem 9.4.2 implies the following

Corollary 9.4.8. *The map* $\Pi : (\mathbb{FP}(\mathcal{K}, J), \mathcal{O}_N) \to (\mathbb{FU}(\mathcal{H}), \mathcal{O}_N)$ *is a bijective isometry.*

Due to Corollary 8.1.2, this directly implies the next statement:

Corollary 9.4.9. *The homotopy groups of* $(\mathbb{FP}(\mathcal{K}, J), \mathcal{O}_N)$ *are*

$$\pi_k(\mathbb{FP}(\mathcal{K}, J)) = \begin{cases}\mathbb{Z}, & k \text{ odd},\\ 0, & k \text{ even}.\end{cases}$$

The next result also allows accessing the homotopy groups of $\mathbb{FPP}(\mathcal{K}, J)$.

Proposition 9.4.10. *The space* $(\mathbb{FP}(\mathcal{K}, J), \mathcal{O}_N)$ *is homotopy equivalent to the space* $(\mathbb{FPP}(\mathcal{K}, J), \mathcal{O}_N)$.

Proof. Let $(P_0, P_1) \in \mathbb{FPP}(\mathcal{K}, J)$ be a pair. Recall that there is a unitary $U_0 \in \mathbb{U}(\mathcal{H})$ (where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ in the grading of J) such that

$$P_0 = \frac{1}{2}\begin{pmatrix}\mathbf{1} & U_0\\ U_0^* & \mathbf{1}\end{pmatrix}.$$

Set $V = \operatorname{diag}(\mathbf{1}, U_0)$, which is an element in $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Then

$$(P_0, P_1) = V^*(P_{\text{ref}}, VP_1V^*)V.$$

Due to the natural identification

$$\mathbb{FP}(\mathcal{K},J) \cong \{(P_{\text{ref}},P) : P \in \mathbb{P}(\mathcal{K},J) \text{ and } (P_{\text{ref}},P) \in \mathbb{FPP}(\mathcal{K},J)\},$$

one hence has $\mathbb{FPP}(\mathcal{K},J) = \mathbb{FP}(\mathcal{K},J) \times \mathbb{U}(\mathcal{H})$. Therefore the claim follows from the contractibility of $\mathbb{U}(\mathcal{H})$. □

Combining Corollary 9.4.9 with Proposition 9.4.10, one deduces

Theorem 9.4.11. *The homotopy groups of* $(\mathbb{FPP}(\mathcal{K},J), \mathcal{O}_N)$ *are*

$$\pi_k\big(\mathbb{FPP}(\mathcal{K},J)\big) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

9.5 Paths of Fredholm pairs of *J*-Lagrangian projections

Proposition 9.4.7 shows that the index of a Fredholm pair $(P_0,P_1) \in \mathbb{FPP}(\mathcal{K},J)$ of J-Lagrangian projections always vanishes. As already stated in Theorem 9.4.11, there is interesting topological information contained in paths in $\mathbb{FPP}(\mathcal{K},J)$. As shown in Corollary 9.5.7 at the end of this section, this is captured by the Bott–Maslov index which will be introduced and studied in this section. For the definition, recall the characterization of Fredholm pairs of J-Lagrangian projections as given in Theorem 9.4.2.

Definition 9.5.1. Let $t \in [0,1] \mapsto (P_0(t),P_1(t)) \in \mathbb{FPP}(\mathcal{K},J)$ be a path of Fredholm pairs of J-Lagrangian projections and set $U(t) = \Pi(P_0(t))^*\Pi(P_1(t)) \in \mathbb{FU}(\mathcal{H})$. Then the Bott–Maslov index of the path is defined by

$$\text{BM}\big(t \in [0,1] \mapsto (P_0(t),P_1(t))\big) = \text{Sf}\big(t \in [0,1] \mapsto U(t)\big).$$

By Proposition 9.3.5, the Bott–Maslov index counts the number of finite-dimensional intersections of $\text{Ker}(P_0(t))$ with $\text{Ran}(P_1(t))$ along the path, with the orientation of the passage through the intersection as a weight. This two-sidedness will be further discussed below, and we will also provide a crossing form formulation for the Bott–Maslov index. Let us first note a few obvious properties that the Bott–Maslov index directly inherits from the spectral flow. More precisely, the next Proposition 9.5.2 is a direct consequence of Theorem 4.5.6, and Proposition 9.5.3 further down follows from item (ii) of Theorem 4.5.5.

Proposition 9.5.2. *Let* $t \in [0,1] \mapsto (P_0(t),P_1(t)) \in \mathbb{FPP}(\mathcal{K},J)$ *be a path of Fredholm pairs of J-Lagrangian projections. Then its Bott–Maslov index is a homotopy invariant under homotopies within the set of paths of Fredholm pairs of J-Lagrangian projections keeping the endpoints* $(P_0(0),P_1(0))$ *and* $(P_0(1),P_1(1))$ *fixed.*

In particular, the Bott–Maslov index associates to every closed path of Fredholm pairs of J-Lagrangian projections an integer invariant. As will be shown in Corollary 9.5.7 below, this characterizes the fundamental group of $\mathbb{FPP}(\mathcal{K},J)$.

Proposition 9.5.3. *Let $t \in [0,1] \mapsto (P_0(t), P_1(t))$ and $t \in [0,1] \mapsto (P_0'(t), P_1'(t))$ be two paths in $\mathbb{FPP}(\mathcal{K}, J)$ such that $P_0(1) = P_0'(0)$ and $P_1(1) = P_1'(0)$. Then their concatenation $(P_0 * P_0', P_1 * P_1')$, defined by*

$$P_j * P_j'(t) = \begin{cases} P_j(2t), & t \in [0, \frac{1}{2}], \\ P_j'(2t-1), & t \in [\frac{1}{2}, 1], \end{cases}$$

has a Bott–Maslov index given by

$$\begin{aligned} &\mathrm{BM}\big(t \in [0,1] \mapsto (P_0 * P_0'(t), P_1 * P_1'(t))\big) \\ &\quad = \mathrm{BM}\big(t \in [0,1] \mapsto (P_0(t), P_1(t))\big) + \mathrm{BM}\big(t \in [0,1] \mapsto (P_0'(t), P_1'(t))\big). \end{aligned}$$

The next result also follows directly from the definition and the identity (9.10).

Proposition 9.5.4. *Let $t \in [0,1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a path of Fredholm pairs of J-Lagrangian projections and $V \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Then*

$$\mathrm{BM}\big(t \in [0,1] \mapsto (V \cdot P_0(t), V \cdot P_1(t))\big) = \mathrm{BM}\big(t \in [0,1] \mapsto (P_0(t), P_1(t))\big).$$

Next crossing forms for differentiable paths $t \in [0,1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ are introduced. Let us set $U(t) = \Pi(P_0(t))^* \Pi(P_1(t)) \in \mathbb{FU}(\mathcal{H})$ as in Definition 9.5.1. Then the crossing form at t as in Definition 4.5.7 is given by

$$\Gamma_t : \mathrm{Ker}\big(U(t) + \mathbf{1}\big) \to \mathbb{R}, \quad \Gamma_t(\phi) = -\imath \langle \phi | U(t)^* \partial_t U(t) \phi \rangle.$$

A crossing is called regular if Γ_t is nondegenerate. Now Proposition 4.5.9 immediately implies the following result.

Proposition 9.5.5. *Let $t \in [0,1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a continuously differentiable path having only regular crossings. Then*

$$\mathrm{BM}\big(t \in [0,1] \mapsto (P_0(t), P_1(t))\big) = \frac{1}{2}\,\mathrm{Sig}(\Gamma_0) + \sum_{t \in (0,1)} \mathrm{Sig}(\Gamma_t) + \frac{1}{2}\,\mathrm{Sig}(\Gamma_1). \tag{9.12}$$

As in the finite-dimensional case (Lemma 2.1.9), it is useful to have an explicit formula for the crossing form in terms of the projections. This can be deduced from the first part of the next statement.

Lemma 9.5.6. *Let $t \in [0,1] \mapsto (P_0(t), P_1(t))$ be a differentiable path of pairs of J-Lagrangian projections with associated $U_0(t) = \Pi(P_0(t))$ and $U_1(t) = \Pi(P_1(t))$. Then for $U(t) = U_0(t)^* U_1(t)$ one has*

$$U(t)^* \partial_t U(t) = 4 \begin{pmatrix} U_1(t) \\ 0 \end{pmatrix}^* P_0(t) \partial_t P_0(t) \begin{pmatrix} U_1(t) \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}^* P_1(t) \partial_t P_1(t) \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}.$$

If, moreover, $\partial_t P_0(t)$ and $\partial_t P_1(t)$ are trace class, then $U(t)^*\partial_t U(t)$ is trace class and given by

$$\mathrm{Tr}\big(U(t)^*\partial_t U(t)\big) = 2\,\mathrm{Tr}\big(JP_0(t)\partial_t P_0(t)^*\big) - 2\,\mathrm{Tr}\big(JP_1(t)\partial_t P_1(t)^*\big).$$

Proof. By the formulas in Proposition 9.3.4, one has for $j = 0, 1$,

$$P_j(t)\partial_t P_j(t) = \frac{1}{4}\begin{pmatrix} U_j(t)\partial_t U_j(t)^* & \partial_t U_j(t) \\ \partial_t U_j(t)^* & U_j(t)^*\partial_t U_j(t) \end{pmatrix}.$$

As

$$U(t)^*\partial_t U(t) = U_1(t)^*\big(U_0(t)\partial_t U_0(t)^*\big)U_1(t) + U_1(t)^*\partial_t U_1(t),$$

this implies the first formula. The summability in the second claim is now clear, and

$$\begin{aligned}\mathrm{Tr}\big(U(t)^*\partial_t U(t)\big) &= -\,\mathrm{Tr}\big(U(t)\partial_t U(t)^*\big) \\ &= 2\,\mathrm{Tr}\big(JP_0(t)\partial_t P_0(t)^*\big) - 2\,\mathrm{Tr}\big(JP_1(t)\partial_t P_1(t)^*\big),\end{aligned}$$

by taking the trace of the above formula for $P_j(t)\partial_t P_j(t)$ times J. □

Next let us state that the Bott–Maslov index restricted to closed paths identifies the fundamental group of $\mathbb{FPP}(\mathcal{K}, J)$ given in Theorem 9.4.11.

Corollary 9.5.7. *The Bott–Maslov index defined in Definition* 9.5.1 *establishes an isomorphism*

$$\mathrm{BM} : \pi_1\big(\mathbb{FPP}(\mathcal{K}, J)\big) \to \mathbb{Z}.$$

For differentiable closed paths and under a trace class condition on $\partial_t U(t)$, it is now possible to plug in the formula for $\mathrm{Tr}(U(t)^*\partial_t U(t))$ given in Lemma 9.5.6 into Proposition 4.5.11.

Corollary 9.5.8. *Let $t \in [0,1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a continuously differentiable closed path. Suppose that $\partial_t P_0(t)$ and $\partial_t P_1(t)$ are trace class. Then*

$$\begin{aligned}&\mathrm{BM}\big(t \in [0,1] \mapsto \big(P_0(t), P_1(t)\big)\big) \\ &\quad = \frac{1}{\pi\imath}\int_0^1 dt\big(\mathrm{Tr}\big(JP_0(t)\partial_t P_0(t)^*\big) - \mathrm{Tr}\big(JP_1(t)\partial_t P_1(t)^*\big)\big).\end{aligned}$$

Next let us note that one has an infinite-dimensional analogue of Proposition 2.2.14. Further down in Proposition 9.6.17 a link to the Conley–Zehnder index will be given.

Proposition 9.5.9. *Let $t \in [0,1] \mapsto T_t \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$ be a closed path. Then the Bott–Maslov index* $\mathrm{BM}(t \in [0,1] \mapsto (P, T_t \cdot P))$ *is well defined and independent of $P \in \mathbb{P}(\mathcal{K}, J)$.*

Proof. First of all, Proposition 9.4.1 indeed shows that $(P, T_t \cdot P) \in \mathbb{FPP}(\mathcal{K}, J)$ so that the Bott–Maslov index of the path is well defined. As the path is closed, it is given by a winding number in the sense of Proposition 4.5.10 and thus is homotopy invariant. Now given $P_0, P_1 \in \mathbb{P}(\mathcal{K}, J)$, there exists a unitary $U \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ such that $P_1 = U \cdot P_0$ by Proposition 9.3.6. Then, because $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ is connected, one can choose a path $s \in [0,1] \mapsto U_s \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ such that $P_s = U_s \cdot P_0$ connects P_0 to P_1 in $\mathbb{P}(\mathcal{K}, J)$. Then $s \in [0,1] \mapsto (P_s, T_t \cdot P_s)$ is a homotopy of closed loops in $\mathbb{FP}(\mathcal{K}, J)$, showing that the Bott–Maslov index of the stated path is independent of P. □

Based on Proposition 9.4.4, one can also deal with other situations than that in Proposition 9.5.9 in which $t \mapsto (P, T_t \cdot P)$ has a well-defined Bott–Maslov index that has stability properties in P. For example, suppose $T_t = T_0(\mathbf{1} + K_t)$ for some fixed T_0 and loop $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{K})$ in the Lie algebra such that $(P, T_0 \cdot P)$ is a Fredholm pair. Then indeed $(P, T_t \cdot P)$ is a Fredholm pair (by the same argument as in Proposition 9.4.1) and, furthermore, the Fredholm property is stable along this path under small perturbations of P due to Proposition 9.4.4. By homotopy invariance of the Bott–Maslov index, one then also deduces its stability as in Proposition 9.5.9. As to explicit formulas, of course, Corollary 9.5.8 applies to the case of differentiable closed paths $(P, T_t \cdot P)$ and actually only one of the summands remains. Further formulas (such as an infinite-dimensional analogue of Proposition 2.2.14) will be given below.

As already pointed out, often one of the two projections of a pair of J-Lagrangian projections is fixed. Also Proposition 9.5.9 considers such a situation. In the following, this reference projection is again chosen to be $P_0 = P_{\text{ref}}$, and then the Bott–Maslov index of the path $t \mapsto (P_{\text{ref}}, P(t))$ is considered for $P(t) \in \mathbb{FP}(\mathcal{K}, J)$. Moreover, it will be shown below (by essentially the same argument as in the proof of Proposition 9.4.10) that one can always arrange one of the J-Lagrangian projections to be moved into the reference J-Lagrangian projection P_{ref} (or any other one). In this situation, the following is just a special case of Definition 9.5.1, simply because $\Pi(P_{\text{ref}}) = \mathbf{1}$.

Definition 9.5.10. For a path $t \in [0,1] \mapsto P(t) \in \mathbb{FP}(\mathcal{K}, J)$ in the Fredholm J-Lagrangian Grassmannian, the Bott–Maslov index is defined by

$$\mathrm{BM}\big(t \in [0,1] \mapsto P(t)\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto U(t)\big),$$

where $U(t) = \Pi(P(t)) \in \mathbb{FU}(\mathcal{H})$.

Proposition 9.5.11. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a differentiable closed path in $\mathbb{U}(\mathcal{K}, J)$ and $P \in \mathbb{P}(\mathcal{K}, J)$. Suppose that $(P, T_t \cdot P) \in \mathbb{FPP}(\mathcal{K}, J)$ and that $\partial_t T_t J T_t^*$ is trace class. Then*

$$\mathrm{BM}\big(t \in [0,1] \mapsto T_t \cdot P\big) = \frac{1}{\imath\pi} \int_0^1 dt\, \mathrm{Tr}\big((\mathbf{1} - T_t \cdot P)(\partial_t T_t J T_t^*)\big).$$

Due to Corollary 4.5.10, the proof of Proposition 9.5.11 is completed by the following algebraic lemma which generalizes Lemma 2.2.13 dealing with the finite-dimensional case.

Lemma 9.5.12. *Let* $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ *be a differentiable path in* $\mathbb{U}(\mathcal{K}, J)$ *and* $P \in \mathbb{P}(\mathcal{K}, J)$*. Then* $U_t = \Pi(P_t)$ *associated to* $P_t = T_t \cdot P$ *satisfies*

$$U_t^* \partial_t U_t = \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}^* (\partial_t T_t J T_t^*) \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}.$$

Moreover, if $\partial_t T_t J T_t^*$ *is trace class, then also* $\partial_t U_t$ *is trace class and*

$$\operatorname{Tr}(U_t^* \partial_t U_t) = 2 \operatorname{Tr}((\mathbf{1} - P_t)(\partial_t T_t J T_t^*)).$$

Proof. For sake of notational simplicity, let us suppress the index $t$and set $W = \Pi(P)$. First note that

$$U^* \partial U = \Pi(T \cdot P)^* \partial \Pi(T \cdot P) = (T \cdot W)^* \partial (T \cdot W),$$

because $\Pi(T \cdot P) = T \cdot \Pi(P) = T \cdot W$. Using $(T \cdot W)^* = (T \cdot W)^{-1}$ and the laws of operator differentiation, one finds

$$\begin{aligned}
&(T \cdot W)^* \partial (T \cdot W) \\
&\quad = (T_t \cdot W)^* \big(\partial(AW + B)\big)(CW + D)^{-1} - \big(\partial(CW + D)\big)(CW + D)^{-1} \\
&\quad = \big((CW + D)^{-1}\big)^* \big[(AW + B)^* \partial (AW + B) - (CW + D)^* \partial (CW + D)\big](CW + D)^{-1} \\
&\quad = \big((CW + D)^{-1}\big)^* \begin{pmatrix} W \\ \mathbf{1} \end{pmatrix}^* T^* J \partial T \begin{pmatrix} W \\ \mathbf{1} \end{pmatrix} (CW + D)^{-1}.
\end{aligned}$$

But

$$\begin{pmatrix} W \\ \mathbf{1} \end{pmatrix} (CW + D)^{-1} = T^{-1} \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}.$$

Now

$$(T^{-1})^* (T^* J \partial T) T^{-1} = J (\partial T J T^*) J$$

concludes the proof of the first identity. Plugging it into the trace leads to the second one. □

It is always possible to recourse to the Bott–Maslov index with respect to a fixed reference plane as in Definition 9.5.10 by appealing to Proposition 9.3.6 to deform $P_0(t)$ into P_{ref}. More precisely, given a path $t \in [0, 1] \mapsto (P_0(t), P_1(t))$ of J-Lagrangian projections, set as in the proof of Proposition 9.4.10

$$V(t) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \Pi(P_0(t)) \end{pmatrix}.$$

Clearly, $t \in [0,1] \mapsto V(t)$ is a path in $\mathbb{U}(\mathcal{K},J) \cap \mathbb{U}(\mathcal{H})$, and one checks that

$$V(t)P_0(t)V(t)^* = P_{\mathrm{ref}}.$$

In this manner, one obtains the path $t \in [0,1] \mapsto (P_{\mathrm{ref}}, V(t)P_1(t)V(t)^*)$ which consists of Fredholm pairs if $(P_0(t), P_1(t))$ are Fredholm pairs. The basis change can be suppressed in the following by setting $P(t) = V(t)P_1(t)V(t)^*$. Then $U(t) = \Pi(P(t))$ lies in $\mathbb{F}\mathbb{U}(\mathcal{H})$ by Theorem 9.4.2.

Remark 9.5.13. Another alternative to attain a situation with a fixed reference frame is a doubling procedure, e. g., [90]. Suppose given $t \mapsto (P_0(t), P_1(t)) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K},J)$. Then one constructs a new Krein space $(\widehat{\mathcal{K}}, \widehat{J})$ by setting $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$ and $\widehat{J} = J \oplus (-J)$. Then $\widehat{P}(t) = P_0(t) \oplus (\mathbf{1} - P_1(t))$ is clearly $\widehat{J}$-Lagrangian by construction. Moreover, the doubled reference frame

$$\widetilde{P}_{\mathrm{ref}} = \frac{1}{2}\begin{pmatrix} \mathbf{1}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}$$

is also $\widehat{J}$-Lagrangian. One can then check that

$$\dim\big(\mathrm{Ran}(P_0(t)) \cap \mathrm{Ker}(P_1(t))\big) = \dim\big(\mathrm{Ran}(\widehat{P}(t)) \cap \mathrm{Ker}(\widetilde{P}_{\mathrm{ref}})\big)$$

and, with $U_0(t) = \Pi(P_0(t))$ and $U_1(t) = \Pi(P_1(t))$,

$$\begin{aligned} \mathrm{BM}\big(t \in [0,1] \mapsto (P_0(t), P_1(t))\big) &= \mathrm{Sf}\big(t \in [0,1] \mapsto U_0(t)^* U_1(t)\big) \\ &= \mathrm{Sf}\left(t \in [0,1] \mapsto \begin{pmatrix} 0 & -U_1(t) \\ U_0(t)^* & 0 \end{pmatrix}\right) \\ &= \mathrm{BM}\big(t \in [0,1] \mapsto (\widehat{P}(t), \widetilde{P}_{\mathrm{ref}})\big), \end{aligned}$$

i. e., the latter expression is a Bott–Maslov index in the sense of Definition 9.5.10. This approach may be of some theoretical use, but has the disadvantage of doubling dimension and consequently only producing a special type of $\widehat{J}$-Lagrangian subspaces, namely the diagonal ones $\widehat{P}(t)$. ◇

Combining Corollaries 9.4.9 and 9.4.8 with Corollary 8.1.3 now leads to

Corollary 9.5.14. *The Bott–Maslov index induces an isomorphism*

$$\mathrm{BM} : \pi_1\big(\mathbb{F}\mathbb{P}(\mathcal{K},J)\big) \to \mathbb{Z}.$$

Let us also note that both Corollary 9.5.8 and Proposition 9.5.11 cover the situation of a fixed reference P_{ref}. There is, however, an even more explicit formula extending Proposition 2.2.14 of the finite-dimensional case.

Proposition 9.5.15. *Let* $t \in [0,1] \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \in \mathbb{U}(\mathcal{K},J)$ *be a closed differentiable path such that all four entries of* $\partial_t T_t$ *are trace class. Then*

$$\begin{aligned}\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P_{\mathrm{ref}}) \\ = \frac{1}{2\pi\imath} \int_0^1 dt\, \mathrm{Tr}\big((A_t + B_t)^{-1}\partial_t(A_t + B_t) - (C_t + D_t)^{-1}\partial_t(C_t + D_t)\big).\end{aligned}$$

Proof. Set $U_t = \Pi(T_t \cdot P_{\mathrm{ref}})$. Then, by Lemma 9.5.12, the hypothesis implies that $U_t^* \partial_t U_t$ is trace class so that Corollary 4.5.10 can be applied to compute the Bott–Maslov index. As $U_t = T_t \cdot \Pi(P_{\mathrm{ref}}) = T_t \cdot \mathbf{1} = (A_t + B_t)(C_t + D_t)^{-1}$, the usual derivative rule and the cyclicity of the trace then immediately lead to the claimed identity. □

In the remainder of this section, let us next discuss a geometric interpretation of the Bott–Maslov index that has been put forward by Arnold. Let us now consider a path $t \in [0,1] \mapsto P(t) \in \mathbb{FP}(\mathcal{K},J)$. By Proposition 9.3.5, the Bott–Maslov index counts the number of intersections of $\mathrm{Ran}(P(t))$ with the fixed subspace $\mathrm{Ran}(JP_{\mathrm{ref}}J)$, with their multiplicity and with an orientation as a weight. The following definition, generalizing Arnold's definition [9], is hence natural.

Definition 9.5.16. The singular cycle of J-Lagrangian subspaces with nontrivial intersections with $JP_{\mathrm{ref}}J$ is

$$\mathbb{SP}(\mathcal{K},J) = \bigcup_{l \geq 1} \mathbb{SP}_l(\mathcal{K},J),$$

where

$$\mathbb{SP}_l(\mathcal{K},J) = \big\{P \in \mathbb{FP}(\mathcal{K},J) : \dim\big(\mathrm{Ran}(P) \cap \mathrm{Ker}(P_{\mathrm{ref}})\big) = l\big\}.$$

First of all, let us note that the Fredholm property assures that the intersection of $\mathrm{Ran}(P)$ with $\mathrm{Ran}(JP_{\mathrm{ref}}J) = \mathrm{Ker}(P_{\mathrm{ref}})$ is always finite dimensional. Note also that Proposition 9.3.5 implies

$$\Pi\big(\mathbb{SP}_l(\mathcal{K},J)\big) = \big\{U \in \mathbb{FU}(\mathcal{H}) : \dim\big(\mathrm{Ker}(U + \mathbf{1})\big) = l\big\}.$$

Hence the codimension of $\mathbb{SP}_l(\mathcal{K},J)$ increases with l and this makes $\mathbb{SP}(\mathcal{K},J)$ into a stratified space with strata $\mathbb{SP}_l(\mathcal{K},J)$. Finally, the singular cycle $\mathbb{SP}(\mathcal{K},J)$ is two-sided, namely a point close to $\mathbb{SP}(\mathcal{K},J)$ can either be on its right or its left, depending on whether the eigenvalue of its stereographic projection has a positive or negative imaginary part. Having in mind the image of the path under the stereographic projection, all these geometric properties become self-evident.

9.6 Conley–Zehnder index

Section 2.3 analyzed the Conley–Zehnder index in finite dimensions. It turned out that the Conley–Zehnder index is nothing but the Bott–Maslov index of the graphs of J-unitaries, considered as Lagrangian subspaces in a doubled Krein space. The same algebraic setup transposes to infinite-dimensional Krein spaces, provided that suitable Fredholm conditions are imposed. This is carried out in this section. Most of the algebraic expressions and identities are identical to those in Section 2.3, but several are repeated to facilitate readability.

Associated to a Krein space $(\mathcal{K}, J)$ and a given J-unitary T is another doubled Krein space $(\mathcal{K} \oplus \mathcal{K}, (-J) \oplus J)$ on which then acts $\mathbf{1} \oplus T$ as $((-J) \oplus J)$-unitary. The range of the operator $(\mathbf{1} \oplus T)\binom{\mathbf{1}}{\mathbf{1}}$ is the graph $\mathcal{G}_T$ of T. It is hence a $((-J) \oplus J)$-Lagrangian subspace. In order to use the stereographic projection in the form of Section 9.3, it is convenient to use the basis transformation $\widehat{F}$ given in (2.24). Note that it actually is a symmetry. It then leads to a standard form for the doubled Krein space,

$$(\widehat{\mathcal{K}}, \widehat{J}) = (\mathcal{K} \oplus \mathcal{K}, \widehat{F}((-J) \oplus J)\widehat{F}),$$

with $\widehat{J} = \operatorname{diag}(\mathbf{1}, -\mathbf{1})$. The group of $\widehat{J}$-unitary operators is again denoted by $\mathbb{U}(\widehat{\mathcal{K}}, \widehat{J})$. A particular operator therein is

$$\widehat{T} = \widehat{F}(\mathbf{1} \oplus T)\widehat{F} \in \mathbb{U}(\widehat{\mathcal{K}}, \widehat{J}),$$

and an example of a $\widehat{J}$-Lagrangian subspace is the $\widehat{F}$-transformed graph $\widehat{\mathcal{G}}_T = \widehat{F}\mathcal{G}_T$. The stereographic projection from the space $\mathbb{P}(\widehat{\mathcal{K}}, \widehat{J})$ of $\widehat{J}$-Lagrangian subspaces to $\mathbb{U}(\mathcal{K})$ defined as in Proposition 9.3.4 is denoted by $\widehat{\Pi}$. As a reference $\widehat{J}$-Lagrangian projection, we will use

$$\widehat{P}_{\mathrm{ref}} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}. \tag{9.13}$$

It satisfies $\widehat{\Pi}(\widehat{P}_{\mathrm{ref}}) = \mathbf{1}$ and $\widehat{F}\widehat{P}_{\mathrm{ref}}\widehat{F} = \widehat{P}_{\mathrm{ref}}$. Its range is denoted by $\widehat{\mathcal{E}}_{\mathrm{ref}} = \operatorname{Ran}(\widehat{P}_{\mathrm{ref}})$. Moreover, it allows writing the projection on $\widehat{\mathcal{G}}_T$ as $\widehat{T} \cdot \widehat{P}_{\mathrm{ref}}$. The algebraic proof of the following theorem is identical to that of Theorem 2.3.1 covering the finite-dimensional case.

Theorem 9.6.1. *To a given $T \in \mathbb{U}(\mathcal{K}, J)$ let us associate a unitary $S(T)$ by*

$$S(T) = \widehat{\Pi}(\widehat{\mathcal{G}}_T) = \widehat{\Pi}(\widehat{P}_{\mathrm{ref}})^* \,\widehat{\Pi}(\widehat{T} \cdot \widehat{P}_{\mathrm{ref}}) \in \mathbb{U}(\mathcal{K}). \tag{9.14}$$

If $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, *then*

$$S(T) = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix} = \begin{pmatrix} (A^*)^{-1} & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix}.$$

The map $T \in \mathbb{U}(\mathcal{K}, J) \mapsto S(T) \in \mathbb{U}(\mathcal{K})$ *is a continuous embedding with image*

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}(\mathcal{K}) : \alpha, \delta \in \text{invertible} \right\}. \tag{9.15}$$

Also the proof of the next structural result for $S(T)$ is as in the finite-dimensional case, see Proposition 2.3.2.

Proposition 9.6.2. *Given* $T \in \mathbb{U}(\mathcal{K}, J)$, *one has*

$$S(-T) = -JS(T)J,$$

and

$$S(T)^* = S(T)^{-1} = S(T^{-1}) = JS(T^*)J.$$

The following result justifies the above constructions. The algebraic proof is identical to the proof of Theorem 2.3.3.

Theorem 9.6.3. *Let* T *and* $S(T)$ *be as in Theorem* 9.6.1. *Then*

$$\operatorname{Ker}(T - \mathbf{1}) = \operatorname{Ker}(S(T) - \mathbf{1}), \quad \operatorname{Ker}(T + \mathbf{1}) = J \operatorname{Ker}(S(T) + \mathbf{1}).$$

Theorem 9.6.3, as well as the connection between eigenvectors, can easily be adapted to study other eigenvalues on the unit circle. Indeed, if $T\phi = z\phi$ for $z \in \mathbb{S}^1$, then also $(\overline{z}T)\phi = \phi$. But the operator $\overline{z}T$ is also J-unitary so that one can apply the above again to construct an associated unitary. This shows the following.

Proposition 9.6.4. *Let* $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a* J*-unitary and set, for* $z \in \mathbb{S}^1$,

$$S(\overline{z}T) = \begin{pmatrix} \overline{z}(A^*)^{-1} & BD^{-1} \\ -D^{-1}C & zD^{-1} \end{pmatrix}. \tag{9.16}$$

Then the geometric multiplicity of z *as eigenvalue of* T *is equal to the multiplicity of* 1 *as eigenvalue of* $S(\overline{z}T)$.

Therefore, the unitaries $S(\overline{z}T)$ are a tool to study eigenvalues of T which lie on the unit circle. Let us focus again on $z = \pm 1$. Theorem 9.6.3 concerns the kernel of $S(T) \pm \mathbf{1}$. It is natural to analyze how much more spectrum $S(T)$ has close to ± 1, or, what is equivalent, how much spectrum the self-adjoint operator $\Re e(S(T)) = \frac{1}{2}(S(T) + S(T)^*)$ has close to ± 1.

For this purpose, it is useful to have an explicit formula for $\mathbb{R}e(S(T))$. Again the algebraic proof is identical to that in the finite-dimensional cases stated in Proposition 2.3.7.

Proposition 9.6.5. *Let T be a J-unitary and $S(T)$ as above. Then*

$$\mathbb{R}e(S(T)) = (\mathbf{1} + T)(\mathbf{1} + T^*T)^{-1}(\mathbf{1} + T)^* - \mathbf{1}. \tag{9.17}$$

The most robust compactness property of J-unitaries implies the following:

Proposition 9.6.6. *For $T \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$, one has $S(T) \in \mathbb{U}^{\mathrm{C}}(\mathcal{K})$. Furthermore, the image of the map $S : T \in \mathbb{U}(\mathcal{K}, J) \mapsto S(T) \in \mathbb{U}(\mathcal{K})$ is*

$$S(\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}) : \alpha, \delta \in \text{invertible} \right\}.$$

Proof. As $T = \mathbf{1} + K \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$, the claim directly follows from Theorem 9.6.1. □

Remark 9.6.7. Proposition 9.6.6 holds irrespective of the choice of the reference projection. More precisely, if one uses some other reference plane $\widetilde{P}_{\mathrm{ref}} \in \mathbb{P}(\widehat{\mathcal{K}}, \widehat{J})$ to define $\widetilde{S}(T) = \widehat{\Pi}(\widetilde{P}_{\mathrm{ref}})^* \widehat{\Pi}(\widehat{T} \cdot \widetilde{P}_{\mathrm{ref}})$, then also $\widetilde{S}(T) \in \mathbb{U}^{\mathrm{C}}(\mathcal{K})$ for $T \in \mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$. ◇

Let us now come to a Fredholm condition for the J-unitaries.

Definition 9.6.8. A J-unitary $T \in \mathbb{U}(\mathcal{K}, J)$ is called Fredholm if $S(T) \in \mathbb{FU}(\mathcal{K})$. The set of all J-unitaries $T \in \mathbb{U}(\mathcal{K}, J)$ with this Fredholm property is denoted by $\mathbb{FU}(\mathcal{K}, J)$.

Clearly, one has $\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J) \subset \mathbb{FU}(\mathcal{K}, J)$. Let us now provide several characterizations of the Fredholm property of $T \in \mathbb{U}(\mathcal{K}, J)$, one of which shows that it is independent of the choice of $\widehat{P}_{\mathrm{ref}}$ (similar as in Remark 9.6.7). Another comment is that characterization (iv) below explains that operators in $\mathbb{FU}(\mathcal{K}, J)$ were called (−1)-Fredholm J-unitaries in [168] (and then the more restricted class of $\mathbb{S}^1$-Fredholm unitaries was considered there for which $T - z\mathbf{1}$ is Fredholm for all $z \in \mathbb{S}^1$, which is a strictly larger class than the essentially $\mathbb{S}^1$-gapped J-unitaries considered in Definition 9.2.12).

Proposition 9.6.9. *For $T \in \mathbb{U}(\mathcal{K}, J)$, the following are equivalent:*
(i) $T \in \mathbb{FU}(\mathcal{K}, J)$*;*
(ii) $(\widehat{P}_{\mathrm{ref}}, \widehat{T} \cdot \widehat{P}_{\mathrm{ref}}) \in \mathbb{FPP}(\widehat{\mathcal{K}}, \widehat{J})$*;*
(iii) $-1 \notin \mathrm{spec}_{\mathrm{ess}}(S(T))$*;*
(iv) $T + \mathbf{1} \in \mathbb{FB}(\mathcal{K})$.

Proof. (i) $\Longleftrightarrow$ (ii). This follows directly from Theorem 9.4.2 applied to the Krein space $(\widehat{\mathcal{K}}, \widehat{J})$ and the Fredholm pair of $\widehat{J}$-Lagrangian projections $(\widehat{P}_{\mathrm{ref}}, \widehat{T} \cdot \widehat{P}_{\mathrm{ref}})$.

(i) $\Longleftrightarrow$ (iii). This follows immediately from the definition because $S(T) \in \mathbb{FU}(\mathcal{K})$ is equivalent to $-1 \notin \mathrm{spec}_{\mathrm{ess}}(S(T))$.

(iii) $\Longleftrightarrow$ (iv). For any unitary S, $-1 \notin \mathrm{spec}_{\mathrm{ess}}(S)$ is equivalent to $\min \mathrm{spec}_{\mathrm{ess}}(\mathbb{R}e(S)+\mathbf{1}) > 0$. Now $\mathbb{R}e(S(T)) + \mathbf{1}$ is given by Proposition 9.6.5 which can also be rewritten as

$$\mathbb{R}e(S(T)) + \mathbf{1} = (\mathbf{1} + T)(\mathbf{1} + T^* T)^{-1}(\mathbf{1} + T)^*.$$

Because $(\mathbf{1} + T^* T)^{-1}$ is a bounded invertible operator and therefore Fredholm, (iv) implies by Corollary 3.3.2 that $\mathbb{R}e(S(T)) + \mathbf{1}$ is Fredholm which, by Corollary 3.4.4, is equivalent to (iii). Conversely, if $\mathbb{R}e(S(T)) + \mathbf{1}$ is Fredholm also

$$J(\mathbb{R}e(S(T)) + \mathbf{1})J = (\mathbf{1} + T^*)(\mathbf{1} + TT^*)^{-1}(\mathbf{1} + T)$$

is Fredholm. Therefore $\dim(\mathrm{Ker}(J(\mathbb{R}e(S(T))+\mathbf{1})J)) < \infty$ and, because one moreover has $\mathrm{Ker}(\mathbf{1} + T) \subset \mathrm{Ker}(J(\mathbb{R}e(S(T)) + \mathbf{1})J)$, this implies $\dim(\mathrm{Ker}(\mathbf{1} + T)) < \infty$. Furthermore, the range of $\mathbb{R}e(S(T)) + \mathbf{1}$ is closed. Thus

$$\mathrm{Ran}(\mathbf{1} + T) = \mathrm{Ran}(\mathbb{R}e(S(T)) + \mathbf{1}) \oplus \big(\mathrm{Ran}(\mathbf{1} + T) \ominus \mathrm{Ran}(\mathbb{R}e(S(T)) + \mathbf{1})\big)$$

is closed because $\mathrm{Ran}(\mathbf{1}+T)\ominus\mathrm{Ran}(\mathbb{R}e(S(T))+\mathbf{1}) \subset \mathrm{Ran}(\mathbb{R}e(S(T))+\mathbf{1})^{\perp}$ is finite dimensional and therefore closed. As $\mathrm{Ran}(\mathbf{1} + T)^{\perp} \subset \mathrm{Ran}(\mathbb{R}e(S(T)) + \mathbf{1})^{\perp}$ is finite dimensional, this implies that $\mathbf{1} + T$ is Fredholm. □

Combined with Theorem 9.6.1, more precisely (9.15), Proposition 9.6.9 implies the following:

Corollary 9.6.10. *The image of* $\mathbb{FU}(\mathcal{K},J)$ *under* $S : T \in \mathbb{U}(\mathcal{K},J) \mapsto S(T) \in \mathbb{U}(\mathcal{K})$ *is*

$$S(\mathbb{FU}(\mathcal{K},J)) = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathbb{FU}(\mathcal{K}) : \alpha, \delta \in \text{invertible}\right\}.$$

Corollary 9.6.10 suggests that $\mathbb{U}^{\mathrm{C}}(\mathcal{K},J)$ is a deformation retract of $\mathbb{FU}(\mathcal{K},J)$ because $\mathbb{U}^{\mathrm{C}}(\mathcal{K})$ is a deformation retract of $\mathbb{FU}(\mathcal{K})$ by Proposition 3.7.2. This is, however, not clear because the retract in the proof of Proposition 3.7.2 may not stay within the image $S(\mathbb{FU}(\mathcal{K},J))$ of the map S given in Corollary 9.6.10.

Now all preparations for the following definition are carried out.

Definition 9.6.11. The Conley–Zehnder index of a path $t \in [0,1] \mapsto T_t \in \mathbb{FU}(\mathcal{K},J)$ is defined as

$$\mathrm{CZ}(t \in [0,1] \mapsto T_t) = \mathrm{Sf}(t \in [0,1] \mapsto S(T_t)).$$

Note that, indeed, Proposition 9.6.9 implies that the spectral flow of unitaries on the right-hand side is well defined (as the spectral flow through -1 in the sense of Section 4.5). As such, the Conley–Zehnder index inherits several properties of the spectral flow which are not spelled out in detail: concatenation, homotopy invariance (with fixed endpoints), integrality, and additivity. From these properties, one directly deduces the following statement:

Proposition 9.6.12. *The Conley–Zehnder index applied to closed paths induces group homomorphisms* $\mathrm{CZ} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{K},J)) \to \mathbb{Z}$ *and* $\mathrm{CZ} : \pi_1(\mathbb{U}^{\mathrm{C}}(\mathcal{K},J)) \to \mathbb{Z}$.

Remark 9.6.13. Proposition 9.1.8 shows that $\pi_1(\mathbb{U}^{\mathrm{C}}(\mathcal{K},J)) = \mathbb{Z} \oplus \mathbb{Z}$. Hence the Conley–Zehnder index extracts one of these $\mathbb{Z}$. ◇

In many applications, one deals with differentiable paths $t \mapsto T_t$ of J-unitaries. Then it is useful to be able to compute the derivatives of the eigenvalues of $S(T_t)$ when they cross -1, namely those points which can contribute to the Conley–Zehnder index. The following proposition then leads to a crossing form formulation of the Conley–Zehnder index. This is not spelled out in detail as it is essentially the same as in Section 4.3. The formulas below also allow to analyze the transversality of the path.

Proposition 9.6.14. *Let* $t \mapsto T_t = \left(\begin{smallmatrix} A_t & B_t \\ C_t & D_t \end{smallmatrix}\right)$ *be a differentiable path in* $\mathbb{U}(\mathcal{K},J)$. *Then*

$$S(T_t)^*\partial_t S(T_t) = \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix}^* (T_t^* J \partial_t T_t) \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1}C_t & D_t^{-1} \end{pmatrix}.$$

For a vector $\phi_t \in \mathcal{K}$ *satisfying* $T_t\phi_t = -\phi_t$, *one has* $S(T_t)J\phi_t = -J\phi_t$ *and*

$$\phi_t^* J S(T_t)^* \partial_t S(T_t) J\phi_t = \phi_t^* T_t^* J \partial_t T_t \phi_t.$$

Proof. The proof is essentially the same as that of Lemma 2.3.9. □

Let us now provide an integral formula for the Conley–Zehnder index of differentiable closed paths. It is an infinite-dimensional version of Proposition 2.3.11 with an identical proof, provided supplementary trace class properties are imposed. In particular, the algebraic Lemma 2.3.10 transposes directly.

Proposition 9.6.15. *Let* $t \mapsto T_t = \left(\begin{smallmatrix} A_t & B_t \\ C_t & D_t \end{smallmatrix}\right)$ *be a continuously differentiable closed path in* $\mathbb{U}(\mathcal{K},J)$ *such that all four entries of* $\partial_t T_t$ *are trace class. Then*

$$\mathrm{CZ}\big(t \in [0,1] \mapsto T_t\big) = \frac{1}{2\pi\imath}\int_0^1 dt\, \mathrm{Tr}\big((A_t)^{-1}\partial_t A_t - (D_t)^{-1}\partial_t D_t\big).$$

Also the statement and proof of Corollary 2.3.12 transpose to the infinite-dimensional setting. It provides a connection between the Bott–Maslov and Conley–Zehnder indices.

Corollary 9.6.16. *Let* $t \mapsto T_t = \left(\begin{smallmatrix} A_t & B_t \\ C_t & D_t \end{smallmatrix}\right)$ *be a continuously differentiable closed path in* $\mathbb{U}^{\mathrm{C}}(\mathcal{K},J)$ *such that all four entries of* $\partial_t T_t$ *are trace class. Then for any* $P \in \mathbb{P}(\mathcal{K},J)$,

$$\mathrm{CZ}\big(t \in [0,1] \mapsto T_t\big) = \mathrm{BM}\big(t \in [0,1] \mapsto (P, T_t \cdot P)\big).$$

Based on Corollary 9.6.16, as well as Propositions 9.5.9 and 9.5.15, one can now prove an infinite-dimensional version of Corollary 2.3.13, simply by realizing that the finite-dimensional arguments transpose to a trace class situation. While it is certainly possible to weaken the hypothesis, this is not further studied here.

Proposition 9.6.17. *Let $t \in [0,1] \mapsto P_t \in \mathbb{FP}(\mathcal{K},J)$ and $t \in [0,1] \mapsto T_t \in \mathbb{U}^{\mathrm{C}}(\mathcal{K},J)$ be two continuously differentiable closed paths such that all four entries of $\partial_t T_t$ are trace class. Then*

$$\mathrm{BM}(t \in [0,1] \mapsto T_t \cdot P_t) = \mathrm{BM}(t \in [0,1] \mapsto P_t) + \mathrm{CZ}(t \in [0,1] \mapsto T_t).$$

9.7 Oscillation theory for bound states of scattering systems

This section provides an application of the Bott–Maslov index and spectral flow in the infinite-dimensional setting as described in this chapter. It is about oscillation theory for bound states of a higher-dimensional quantum scattering system (within a single-particle framework). This basically consists of transposing the setup and results of Section 2.6 to a situation where the fibers are infinite dimensional and the locality of the scattering perturbation directly leads to the required Fredholm property. Therefore it is possible to simply refer to Section 2.6 for most of the algebraic arguments, and merely add the required functional analytic elements to the proofs. Let us also note that we are not aware of other results on oscillation theory with infinite dimensional fibers except for [101] where, however, the Fredholm property rather holds in a Breuer–Fredholm sense and the spectral flow is with respect to a semifinite trace so that it determines the density of states.

Let us begin by describing the Hamiltonian. It acts on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ over a d-dimensional lattice with N internal degrees of freedom over every site and is of the next-neighbor form

$$(H\psi)_m = \sum_{|m-k|=1} a_{m,k}\psi_k + v_m\psi_m, \tag{9.18}$$

where $\psi = (\psi_m)_{m\in\mathbb{Z}^d}$ with $\psi_m \in \mathbb{C}^N$, the sum runs over all sites neighboring n (the distance $|n-m|$ is meant in the maximum norm on $\mathbb{Z}^d$), and $a_{m,k} = a_{k,m}^*$ and v_m are $N \times N$ matrices that are invertible and self-adjoint. As in Section 2.6, we will suppose to be in a scattering situation where the coefficient matrices $a_{m,k}$ and v_m are all equal to a and v except for a finite number of sites. Let $L > 0$ be such that all these sites lie in a strip $\mathbb{Z}^{d-1} \times \{1, \dots, L\}$. Hence H is a finite-rank perturbation of a periodic Hamiltonian

$$(H_{\mathrm{per}}\psi)_m = \sum_{|m-k|=1} a\psi_k + v\psi_m.$$

By discrete Fourier transform, one can show that H_{per} has purely absolutely continuous spectrum $\mathrm{spec}(H_{\mathrm{per}}) = \mathrm{spec}_{\mathrm{ac}}(H_{\mathrm{per}})$ consisting of at most N intervals. This section is about computing discrete eigenvalues of H not lying in $\mathrm{spec}(H_{\mathrm{per}})$, thus so-called bound states, by a formula similar as in Theorem 2.6.5. The dimension is throughout assumed to satisfy $d \geq 2$.

For this purpose, the Hamiltonian is rewritten as a (two-sided) infinite block Jacobi operator. The fiber Hilbert space will be $\mathcal{H} = \ell^2(\mathbb{Z}^{d-1}, \mathbb{C}^N)$. Then $\ell^2(\mathbb{Z}^d, \mathbb{C}^N) \cong \ell^2(\mathbb{Z}, \mathcal{H})$. Under this identification, the Hamiltonian (9.18) can be rewritten as

$$(H\psi)_n = A_{n+1}\psi_{n+1} + A_n\psi_{n-1} + V_n\psi_n,$$

where now $n \in \mathbb{Z}$ and $(A_n)_{n\in\mathbb{Z}}, (V_n)_{n\in\mathbb{Z}}$ are both sequences of invertible and self-adjoint operators on $\mathcal{H}$, respectively. We do not write out explicit formulas for A_n and V_n in terms of the $a_{m,k}$ and v_m, but stress that the coefficient operators are such that

$$A_n = A, \quad V_n = V, \quad n \notin \{1, \dots, L\}, \tag{9.19}$$

just as in Section 2.6. The Schrödinger equation $H\psi^E = E\psi^E$ will be considered for all sequences $\psi^E = (\psi^E_n)_{n\in\mathbb{Z}}$ of vectors $\psi^E_n \in \mathcal{H}$, and not only square-integrable states from $\ell^2(\mathbb{Z}, \mathcal{H})$. Explicitly written out, it becomes

$$A_{n+1}\psi^E_{n+1} + V_n\psi^E_n + A_n\psi^E_{n-1} = E\psi^E_n. \tag{9.20}$$

Regrouping two neighboring vectors into

$$\Psi^E_n = \begin{pmatrix} A_{n+1}\psi^E_{n+1} \\ \psi^E_n \end{pmatrix},$$

one can then rewrite (9.20) as

$$\Psi^E_n = M^E_n \Psi^E_{n-1}, \tag{9.21}$$

where the I-unitary transfer matrices M^E_n on the Krein space $(\mathcal{K}, I) = (\mathcal{H} \oplus \mathcal{H}, I)$ are defined by

$$M^E_n = \begin{pmatrix} (E\mathbf{1} - V_n)A_n^{-1} & -A_n \\ A_n^{-1} & \mathbf{0} \end{pmatrix}. \tag{9.22}$$

Let us stress that (9.21) looks exactly as the corresponding equation (2.50) in the setting with finite-dimensional fibers. Indeed, all structural algebraic facts transpose directly. In particular, we will use (9.21) also as an equation for frames $\Psi^E_n : \mathcal{H} \to \mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. If one of the Ψ^E_n spans an I-Lagrangian subspace, then all others do as well because all M^E_n are I-unitary. Let us note that due to (9.19) the M^E_n are for all $n \notin \{1, \dots, N\}$ equal to one fixed I-unitary

$$M^E = \begin{pmatrix} (E\mathbf{1} - V)A^{-1} & -A \\ A^{-1} & \mathbf{0} \end{pmatrix}.$$

The matrix entries of this transfer matrix specify H_{per}, and therefore M^E is also closely linked to the spectral properties of H_{per}. The following result extends Propositions 2.6.1 and 2.6.2.

Proposition 9.7.1. *The following statements hold:*
(i) $E \in \sigma(H_{\text{per}}) \iff \sigma(M^E) \cap \mathbb{S}^1 \neq \emptyset$.
(ii) *For real* $E \notin \sigma(H_{\text{per}})$, *the subspaces* $\mathcal{E}^{E,<}$ *and* $\mathcal{E}^{E,>}$ *given by the range of the Riesz projection of* M^E *on* $\text{spec}(M^E) \cap B_1(0)$ *and* $\text{spec}(M^E) \setminus B_1(0)$, *respectively are I-Lagrangian.*
(iii) *For real* $E \notin \sigma(H_{\text{per}})$, *the subspaces* $\mathcal{E}^{E,<}$ *and* $(\mathcal{E}^{E,>})^{\perp}$ *form a Fredholm pair of I-Lagrangian subspaces.*

Proof. The first claim follows by a Weyl sequence argument, just as in the proof of Proposition 2.6.1. The second and third claims follow from Proposition 9.4.6, after a Cayley transform. □

As in Section 2.6 now follows the analysis of the energy dependence of the unitaries

$$W^{E,<} = \Pi(\mathcal{C}\mathcal{E}^{E,<}), \quad W^{E,>} = \Pi(\mathcal{C}\mathcal{E}^{E,>}),$$

using the half-space restrictions of H_{per}. Let H^+_{per} and H^-_{per} be the (Dirichlet) restrictions of H_{per} to the subspaces $\ell^2(\mathbb{N}, \mathcal{H})$ and $\ell^2(\mathbb{N}^-, \mathcal{H})$, respectively, where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}^- = \{\ldots, -1, 0\}$. In the situation of Section 2.6, the fiber Hilbert space $\mathcal{H}$ is finite dimensional, and this implies that the new spectrum $\text{spec}(H^{\pm}_{\text{per}}) \setminus \text{spec}(H_{\text{per}})$ only consists of a finite number of eigenvalues (bound states) of finite multiplicity. In the present situation, it is possible that $\text{spec}(H^{\pm}_{\text{per}})$ acquires new essential spectrum resulting from surface states along the boundary. This spectrum is typically topologically protected. It can be studied via K-theoretic methods [152] or via transfer matrix methods along the boundary [17, 174]. We believe that the computation of the density of states of this boundary spectrum is possible by adapting Corollary 2.6.4 to a semifinite setting (either by using the Fourier decomposition along the boundary or, more generally, by transposing the techniques from [101]), but this is not carried out here. Irrespective of this, one can prove the following analogue of Proposition 2.6.3.

Proposition 9.7.2. *One has, for* $E \in \mathbb{R} \setminus \text{spec}(H_{\text{per}})$,

$$\frac{1}{\imath}(W^{E,<})^* \partial_E W^{E,<} < 0, \quad \frac{1}{\imath}(W^{E,>})^* \partial_E W^{E,>} > 0.$$

Proof. The whole setup is translation invariant with respect to shifts along the boundary. Hence it is possible to carry out a $(d-1)$-dimensional discrete Fourier decomposition of all objects involved. In particular,

$$H_{\rm per}^{\pm} \cong \int_{\mathbb{T}^{d-1}}^{\oplus} dk\, H_{\rm per}^{\pm}(k),$$

where $k \in \mathbb{T}^{d-1} \mapsto H_{\rm per}^{\pm}(k)$ is a real-analytic family of half-space block Jacobi matrices with a finite-dimensional fiber. Furthermore, also the transfer operators admit such a Fourier decomposition

$$M^E \cong \int_{\mathbb{T}^{d-1}}^{\oplus} dk\, M^E(k),$$

with finite-dimensional J-unitaries depending real analytically on $k \in \mathbb{T}^{d-1}$. Thus also $\mathcal{E}^{E,<}$ and $\mathcal{E}^{E,>}$, as well as $W^{E,<}$ and $W^{E,>}$, can be decomposed. For each $k \in \mathbb{T}^{d-1}$, one can now apply Proposition 2.6.3, and integrating over $\mathbb{T}^{d-1}$ concludes the proof. □

To continue the analysis of the scattering Hamiltonian H, let us now set

$$m^E = \text{multiplicity of } E \text{ as eigenvalue of } H.$$

Each eigenstate $\psi^E \in \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \cong \ell^2(\mathbb{Z}, \mathcal{H})$ decays both at $-\infty$ and $+\infty$. To construct such an eigenstate, one can again proceed as in Section 2.6. Outside of $[1, L] \cap \mathbb{Z}$, the decaying solution satisfies (9.21) with $M_n^E = M^E$. Hence neighboring sites must produce vectors lying in $\mathcal{E}^{E,>}$ on $(-\infty, 0] \cap \mathbb{Z}$ and lying in $\mathcal{E}^{E,<}$ on $[L+1, \infty) \cap \mathbb{Z}$. Matching of the solutions thus shows

$$m^E = \dim(M^E(L,1)\mathcal{E}^{E,>} \cap \mathcal{E}^{E,<}), \tag{9.23}$$

where $M^E(L,1) = M_L^E \cdots M_1^E$.

Proposition 9.7.3. *For $E \in \mathbb{R} \setminus \operatorname{spec}(H_{\rm per})$, the multiplicity m^E is finite and given by*

$$m^E = \dim(\operatorname{Ker}(\Pi(\mathcal{C}\mathcal{E}^{E,<})^* \Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>}) - \mathbf{1})). \tag{9.24}$$

Proof. By Proposition 9.7.1, the right-hand side of (9.23) is an intersection between the two I-Lagrangian subspaces $M^E(L,1)\mathcal{E}^{E,>}$ and $\mathcal{E}^{E,<}$. This intersection can thus be computed by (9.24) due to Proposition 9.3.5. It remains to show that this intersection is finite. For that purpose, let us first note that by Proposition 9.7.1 one has $\mathcal{E}^{E,<} \cap \mathcal{E}^{E,>} = \{0\}$, and therefore 1 is not in the spectrum of $\Pi(\mathcal{C}\mathcal{E}^{E,<})^*\Pi(\mathcal{C}\mathcal{E}^{E,>}) = (W^{E,<})^* W^{E,>}$, again by Proposition 9.3.5. Furthermore, let us note that $\mathcal{E}^{E,>}$ is M^E-invariant by construction. Therefore $(M^E)^L \mathcal{E}^{E,>} = \mathcal{E}^{E,>}$. Due to the assumption (9.19), $M_n^E - M^E$ is of finite rank and therefore $M_n^E (M^E)^{-1} = \mathbf{1} + F_n$ where F_n is of finite rank (and such that $\mathbf{1} + F_n$ is J-unitary). Iterating one concludes that $M^E(L,1)(M^E)^{-L} = \mathbf{1} + F$ where F is of finite rank. Finally, by Proposition 9.3.8,

$$\begin{aligned}\Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>}) &= \Pi(\mathcal{C}M^E(L,1)(M^E)^{-L}\mathcal{E}^{E,>}) \\ &= (\mathcal{C}(\mathbf{1}+F)\mathcal{C}^*)\cdot W^{E,>} \\ &= (\mathbf{1}+\mathcal{C}F\mathcal{C}^*)\cdot W^{E,>} \\ &= W^{E,>}+K,\end{aligned}$$

where K is some compact operator (such that $W^{E,>}+K$ is unitary). In conclusion, $\Pi(\mathcal{C}\mathcal{E}^{E,<})^*\Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>})$ is a compact perturbation of $(W^{E,<})^*W^{E,>}$ and therefore has no essential spectrum in a neighborhood of 1. □

Note that the above proof combined with Theorem 9.4.2 also shows that the subspaces $M^E(L,1)\mathcal{E}^{E,>}$ and $(\mathcal{E}^{E,<})^\perp$ form a Fredholm pair of I-Lagrangians. As in Section 2.6, let us now set

$$\begin{aligned}U^E &= -\Pi(\mathcal{C}\mathcal{E}^{E,<})^*\Pi(\mathcal{C}M^E(L,1)\mathcal{E}^{E,>}) \\ &= -(W^{E,<})^*(\mathcal{C}M^E(L,1)\mathcal{C}^*)\cdot W^{E,>}.\end{aligned}$$

Theorem 9.7.4. *One has*

$$\frac{1}{\imath}(U^E)^*\partial_E U^E > 0. \tag{9.25}$$

Suppose that $[E_0,E_1]\cap\operatorname{spec}(H_{\mathrm{per}})=\emptyset$ *and that* E_0 *and* E_1 *are not eigenvalues of* H. *Then the number of bound states of* H *in* $[E_0,E_1]$ *is given by*

$$\#\{\text{eigenvalues of } H \text{ in } [E_0,E_1]\} = \operatorname{Sf}(E\in[E_0,E_1]\mapsto U^E \textit{ through } -1).$$

Proof. Given the preparations in Propositions 9.7.2 and 9.7.3, the proof is identical to that of Theorem 2.6.5. □

10 Index pairings and spectral localizer

Index theory in the classical setting of the Atiyah–Singer index theorem pairs a Dirac operator on a compact spin manifold with a homotopy class of vector bundles, and constructs a Fredholm operator and its associated index [98, 32, 28]. The Dirac operator provides cohomological information via a certain de Rham differential form, while the vector bundle is thought of as homological data. Already Atiyah [12] interpreted this as a pairing of K-theory with what would become K-homology. The first and most elementary example of this type is Noether's index theorem [140] which expresses the winding number of an invertible complex function as the index of a Fredholm operator. Later on and based on Atiyah's ideas came far-fetching noncommutative generalizations, going back to the work of Connes and Kasparov [63, 111, 104, 92]. These noncommutative index theorems have numerous applications, in particular in the theory of disordered topological insulators [25, 19, 152]. For such solid state systems, it is of great interest to have local formulas for the index that can be implemented numerically. Such local expressions were provided in [128, 129] in terms of the so-called spectral localizer, which was motivated by earlier work of Kitaev [114, Appendix C] and Hastings and Loring [102]. This spectral localizer is a quite universal tool for the computation of index pairings and can be formulated in the purely functional-analytic framework of the earlier chapters. The main mathematical tool connecting the spectral localizer to index pairings is the spectral flow [130, 131, 170, 75], making this chapter a nice application of the theory developed above.

10.1 Fredholm modules and index pairings

Fredholm modules can be even or odd, reflecting whether the underlying (possibly noncommutative) manifold is even or odd dimensional. Let us start out with the even case.

Definition 10.1.1. An even unbounded Fredholm module for an invertible bounded operator $H = H^*$ on $\mathcal{H}$ consists of a self-adjoint, invertible operator D^{ev} on $\mathcal{H} \oplus \mathcal{H}$ satisfying the following conditions:
(i) D^{ev} has a compact resolvent;
(ii) D^{ev} is odd with respect to the symmetry

$$\Gamma = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \in \mathbb{U}(\mathcal{H} \oplus \mathcal{H}),$$

namely

$$\Gamma D^{\mathrm{ev}} \Gamma = -D^{\mathrm{ev}}$$

(iii) the domain $\mathcal{D}(D^{\mathrm{ev}})$ of D^{ev} is left invariant by $H \oplus H$;

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(iv) the commutator $[H \oplus H, D^{\mathrm{ev}}]$ extends to a bounded operator.

The operator D^{ev} is of the form

$$D^{\mathrm{ev}} = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}, \tag{10.1}$$

with an invertible unbounded operator D_0 on $\mathcal{H}$. One then extracts a unitary operator F on $\mathcal{H}$,

$$F = D_0 |D_0|^{-1}.$$

The operator D^{ev} is then called the Dirac operator and F the Dirac phase. The identity $\Gamma D^{\mathrm{ev}} \Gamma = -D^{\mathrm{ev}}$ is also referred to as the chirality of D^{ev}.

In the literature [63, 92], unbounded Fredholm modules are also called spectral triples or unbounded K-cycles. Furthermore, a Fredholm module usually involves representations of some C^*-algebra $\mathcal{A}$ and requires the bounded commutator property to hold for all elements of $\mathcal{A}$. Then H is supposed to be a representative of $\mathcal{A}$, or a matrix algebra over $\mathcal{A}$, and then specifies a class in the K_0-group $K_0(\mathcal{A})$ via the projection $P = \frac{1}{2}(\mathbf{1} - H|H|^{-1})$. Here in Definition 10.1.1 we rather work with a hands-on purely operator-theoretic approach in which $\mathcal{A}$ is simply the enveloping commutative algebra of H. Let us also stress, following Carey and Phillips [55], that the condition (iii) does *not* imply that (ii) holds, see Remark 10.1.5 below. An extension to Definition 10.1.1 is the so-called nonunital case in which $H = H^*$ is allowed to be unbounded (e. g., [54, 171]). Let us also note that it is not necessary to require that D^{ev} is invertible, as this can always be achieved by a standard doubling trick and adding a mass term.

Remark 10.1.2. If D^{ev} is not invertible, then replace $\mathcal{H}$ by $\mathcal{H} \oplus \mathcal{H}$ and

$$\tilde{D}^{\mathrm{ev}} = \begin{pmatrix} D^{\mathrm{ev}} & \mu\mathbf{1} \\ \mu\mathbf{1} & -D^{\mathrm{ev}} \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} H & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad \tilde{\Gamma} = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix},$$

for some $\mu > 0$. Then $\tilde{D}^{\mathrm{ev}}$ is invertible and an even Fredholm module for $\tilde{H}$. This leads to the same index pairings; see, e. g., [51] or [171]. ◇

Let us next provide a standard example of an even Fredholm module stemming from a flat manifold with a trivial spin bundle. It also indicates why the notation Γ is used for the symmetry in Definition 10.1.1, rather than J as in Chapter 9 on Krein spaces.

Example 10.1.3. Let d be even and $\gamma_1, \dots, \gamma_d$ be an irreducible self-adjoint representation of the Clifford algebra with d generators, namely one has

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{i,j}.$$

The representation space is $\mathbb{C}^{d'}$ with $d' = 2^{\frac{d}{2}}$. As d is even, there exists a symmetry Γ (often also denoted by γ_{d+1}) anticommuting with $\gamma_1, \dots, \gamma_d$. Now choose $\mathcal{H} \oplus \mathcal{H} = L^2(\mathbb{T}^d, \mathbb{C}^{d'})$ and set

$$D^{\mathrm{ev}} = \imath \sum_{j=1}^{d} \gamma_j \partial_j, \tag{10.2}$$

where ∂_j denotes the partial derivative on $L^2(\mathbb{T}^d)$ in the jth direction. The operator D^{ev} is self-adjoint and fulfills $\Gamma D^{\mathrm{ev}} \Gamma = -D^{\mathrm{ev}}$ as required. It has a d'-dimensional kernel spanned by the constant functions, but this is not of relevance as explained above. If H is a multiplication operator by a continuously differentiable function $x \in \mathbb{T}^d \mapsto H_x \in \mathbb{R}$, indeed D^{ev} is an unbounded Fredholm module for H. By replacing ∂_j by covariant derivatives and the constant γ_j by a varying representation, this example can readily be generalized to arbitrary Riemannian manifolds equipped with a spin structure, e. g., [122]. ◇

The following result is well known (e. g., [63, 55, 92]), and it is crucial for the later sections.

Theorem 10.1.4. *Let D^{ev} provide an even unbounded Fredholm module for a self-adjoint invertible H and let F be the associated Dirac phase. If $P = \chi(H < 0)$ is the spectral projection of H on the negative spectrum, then*

$$T = PFP + \mathbf{1} - P, \tag{10.3}$$

is a bounded Fredholm operator on $\mathcal{H}$.

Proof. (Following Section 2 of [55].) Set $H_2 = H \oplus H$ and drop the upper index on $D^{\mathrm{ev}} = D$ for the sake of notational simplicity. The proof consists in showing that both summands on the right-hand side of

$$[D|D|^{-1}, H_2] = [D, H_2]|D|^{-1} + D[|D|^{-1}, H_2]$$

are compact. For the first summand, this is immediately clear from assumptions (i) and (iv) of Definition 10.1.1. For the second summand, one uses the following spectral calculus for the square root:

$$|D|^{-1} = (D^2)^{-\frac{1}{2}} = \int_0^\infty \frac{d\lambda}{\pi \lambda^{\frac{1}{2}}} (\lambda \mathbf{1} + D^2)^{-1}.$$

Thus

$$D[|D|^{-1}, H_2] = \int_0^\infty \frac{d\lambda}{\pi \lambda^{\frac{1}{2}}} D[(\lambda \mathbf{1} + D^2)^{-1}, H_2].$$

The computation of the commutator has to be carried out with great care because it is *not* assumed that H_2 leaves the domain of D^2 invariant, but merely that $\mathcal{D}(D)$ is invariant. Let us start from

$$\begin{aligned}[(\lambda\mathbf{1}+D^2)^{-1},H_2] &= (\lambda\mathbf{1}+D^2)^{-1}H_2(\lambda\mathbf{1}+D^2)(\lambda\mathbf{1}+D^2)^{-1} - H_2(\lambda\mathbf{1}+D^2)^{-1}\\ &= ((\lambda\mathbf{1}+D^2)^{-1}H_2(\lambda\mathbf{1}+D^2) - H_2)(\lambda\mathbf{1}+D^2)^{-1}\\ &= ((\lambda\mathbf{1}+D^2)^{-1}H_2D^2 + (\lambda\mathbf{1}+D^2)^{-1}\lambda H_2 - H_2)(\lambda\mathbf{1}+D^2)^{-1}\\ &= ((\lambda\mathbf{1}+D^2)^{-1}H_2D^2 + (\mathbf{1}-D^2(\lambda\mathbf{1}+D^2)^{-1})H_2 - H_2)(\lambda\mathbf{1}+D^2)^{-1}\\ &= -(\lambda\mathbf{1}+D^2)^{-1}H_2D^2(\lambda\mathbf{1}+D^2)^{-1} + D^2(\lambda\mathbf{1}+D^2)^{-1}H_2(\lambda\mathbf{1}+D^2)^{-1}.\end{aligned}$$

Now $D^2(\lambda\mathbf{1}+D^2)^{-1} = D(\lambda\mathbf{1}+D^2)^{-1}D$ on $\mathcal{D}(D)$. Moreover, $\mathrm{Ran}((\lambda\mathbf{1}+D^2)^{-1}) = \mathcal{D}(D^2) \subset \mathcal{D}(D)$ so that, because H_2 leaves $\mathcal{D}(D)$ invariant by (ii) of Definition 10.1.1, one has on all of $\mathcal{H}$ the identity

$$D^2(\lambda\mathbf{1}+D^2)^{-1}H_2(\lambda\mathbf{1}+D^2)^{-1} = D(\lambda\mathbf{1}+D^2)^{-1}DH_2(\lambda\mathbf{1}+D^2)^{-1}.$$

In the same way, one shows that on all $\mathcal{H}$,

$$D(\lambda\mathbf{1}+D^2)^{-1}H_2D(\lambda\mathbf{1}+D^2)^{-1} = (\lambda\mathbf{1}+D^2)^{-1}DH_2D(\lambda\mathbf{1}+D^2)^{-1}.$$

Replacing in the above, one finds that on all $\mathcal{H}$,

$$\begin{aligned}&[(\lambda\mathbf{1}+D^2)^{-1},H_2]\\ &\quad= -(\lambda\mathbf{1}+D^2)^{-1}H_2D^2(\lambda\mathbf{1}+D^2)^{-1} + D(\lambda\mathbf{1}+D^2)^{-1}DH_2(\lambda\mathbf{1}+D^2)^{-1}\\ &\quad= -(\lambda\mathbf{1}+D^2)^{-1}H_2D^2(\lambda\mathbf{1}+D^2)^{-1} + (\lambda\mathbf{1}+D^2)^{-1}DH_2D(\lambda\mathbf{1}+D^2)^{-1}\\ &\qquad - D(\lambda\mathbf{1}+D^2)^{-1}H_2D(\lambda\mathbf{1}+D^2)^{-1} + D(\lambda\mathbf{1}+D^2)^{-1}DH_2(\lambda\mathbf{1}+D^2)^{-1}\\ &\quad= (\lambda\mathbf{1}+D^2)^{-1}[D,H_2]D(\lambda\mathbf{1}+D^2)^{-1} + D(\lambda\mathbf{1}+D^2)^{-1}[D,H_2](\lambda\mathbf{1}+D^2)^{-1}.\end{aligned}$$

Replacing shows

$$\begin{aligned}D[|D|^{-1},H_2] = \int_0^\infty \frac{d\lambda}{\pi\lambda^{\frac{1}{2}}}\,&[D(\lambda\mathbf{1}+D^2)^{-1}[D,H_2]D(\lambda\mathbf{1}+D^2)^{-1}\\ &+ D^2(\lambda\mathbf{1}+D^2)^{-1}[D,H_2](\lambda\mathbf{1}+D^2)^{-1}].\end{aligned}$$

Now $(\lambda+D^2)^{-\frac{1}{2}}$ is compact as the square root of a compact positive operator; because $D(\lambda+D^2)^{-1} = D(\lambda+D^2)^{-\frac{1}{2}}(\lambda+D^2)^{-\frac{1}{2}}$ is the product of a bounded with a compact operator, it is also compact. This shows that both summands under the integral are compact. All integrals are absolutely convergent in norm, so that one concludes that $D[|D|^{-1},H_2]$, and hence also $[D|D|^{-1},H_2]$, is compact. But

$$[D|D|^{-1}, H_2] = \left[\begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}, \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}\right] = \begin{pmatrix} 0 & [F^*, H] \\ [F, H] & 0 \end{pmatrix}.$$

Hence $[F, H]$ is compact. Writing P as a Riesz projection with a contour γ around the negative spectrum, one concludes that

$$[F, P] = \oint_\gamma \frac{dz}{2\pi\imath}[F, (z\mathbf{1} - H)^{-1}] = \oint_\gamma \frac{dz}{2\pi\imath}(z\mathbf{1} - H)^{-1}[F, H](z\mathbf{1} - H)^{-1}$$

is also compact. This implies that $PF^*PFP + \mathbf{1} - P$ and $PFPF^*P + \mathbf{1} - P$ are Fredholm operators, and this implies the claim by Theorem 3.4.1. □

Remark 10.1.5. It is not possible to remove the hypothesis (iii) from Definition 10.1.1 because otherwise the operator T defined in (10.3) may not be Fredholm. In [88] one finds several examples of Dirac operators (actually, odd ones in the sense of the Definition 10.1.7 below) satisfying (i) and (iv) of Definition 10.1.1 for which the operator T is not Fredholm. ◇

Definition 10.1.6. Given an even unbounded Fredholm module for an invertible $H = H^*$, the associated Fredholm operator T given in (10.3) and its index $\mathrm{Ind}(T)$ are referred to as the even index pairing.

Using the notion of the index of a pair of projection discussed at length in Chapter 5, the index pairing can be expressed as

$$\mathrm{Ind}(T) = \mathrm{Ind}(P, F^*PF),$$

see Proposition 5.5.3 for the explicit statement. In the literature on K-theory and K-homology (in particular, [63, 104, 92]), the index pairing is also denoted by $\langle [D^{\mathrm{ev}}]_0, [H]_0\rangle$ expressing that the pairing does not change when the unbounded Fredholm module and gapped self-adjoint are changed within their class in K-homology and K-theory (by suitable continuous homotopies).

Let us now turn to odd unbounded Fredholm modules for invertible bounded operators. In a K-theoretic formulation, these latter represent K_1-group elements of a suitable algebra, and the odd Fredholm modules specify odd K-homology classes. The definition of odd Fredholm modules is as that of even ones.

Definition 10.1.7. An odd unbounded Fredholm module for an invertible bounded operator A on $\mathcal{H}$ is a self-adjoint, invertible operator D_0 on $\mathcal{H}$ with compact resolvent such that A leaves the domain $\mathcal{D}(D_0)$ invariant and $[D_0, A]$ extends to a bounded operator. Then the spectral projection $E = \chi(D_0 > 0)$ is called the associated Hardy projection.

In slight deviation of standard terminology, the odd Dirac operator is not D_0, but rather

$$D^{\mathrm{od}} = \begin{pmatrix} D_0 & 0 \\ 0 & -D_0 \end{pmatrix} \tag{10.4}$$

acting on $\mathcal{H} \oplus \mathcal{H}$. This will allow treating the even and odd case in an analogous manner, if one associates an invertible self-adjoint operator H on $\mathcal{H} \oplus \mathcal{H}$ to A by

$$H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}. \tag{10.5}$$

The operator is odd with respect to $J = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$, namely satisfies $JHJ = -H$, which is also called a chiral symmetry. Let us provide a standard example of an odd Fredholm module.

Example 10.1.8. This example completely parallels Example 10.1.3 of even Fredholm modules on an even-dimensional torus. Let now d be odd and $\gamma_1, \dots, \gamma_d$ be an irreducible representation of the Clifford algebra with d generators. The representation space is $\mathbb{C}^{d'}$ with $d' = 2^{\frac{d-1}{2}}$. Then set

$$D_0 = \imath \sum_{j=1}^{d} \gamma_j \partial_j. \tag{10.6}$$

If A is now a multiplication operator by a differentiable function $x \in \mathbb{T}^d \mapsto A_x \in \mathbb{R}$, indeed D_0 specifies an unbounded odd Fredholm module for A. ◇

The next result is proved in a similar manner as Theorem 10.1.4.

Theorem 10.1.9. *Let D_0 specify an odd unbounded Fredholm module for an invertible bounded operator A and let $E = \chi(D_0 > 0)$ be the associated Hardy projection. If now $U = A|A|^{-1}$ denotes the unitary phase of A, then*

$$T = EUE + \mathbf{1} - E \tag{10.7}$$

is a bounded Fredholm operator on $\mathcal{H}$.

Definition 10.1.10. Given an odd unbounded Fredholm module for an invertible A, the associated Fredholm operator T given in (10.7) and its index $\mathrm{Ind}(T)$ are referred to as the odd index pairing.

Let us stress that both index pairings (10.3) and (10.7) result from a Fredholm pair of unitarily conjugate orthogonal projections as in Section 5.5, namely (P, F^*PF) and (E, U^*EU), respectively. However, in the even index pairing the projection P stems from H and is hence the cohomological (K-theoretic) input to the pairing, while in the odd index pairing the projection E rather stems from the homological input D^{od}.

10.2 Spectral flow formulas for index pairings

In the last section, it was shown that an odd unbounded Fredholm module for either a projection or an invertible operator leads to an index pairing. Section 5.5 shows that such index pairings are connected to a spectral flow. This leads directly to the following result.

Theorem 10.2.1. *Let D_0 specify an odd unbounded Fredholm module for a unitary operator U and let $E = \chi(D_0 > 0)$ be the associated Hardy projection. Then the linear path $t \in [0,1] \mapsto D_t = (1-t)D_0 + tU^* D_0 U$ is Riesz-continuous and lies entirely in $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$. Its spectral flow is equal to the index pairing,*

$$\mathrm{Ind}(EUE + \mathbf{1} - E) = -\,\mathrm{Sf}\big(t \in [0,1] \mapsto D_t\big). \tag{10.8}$$

Proof. First of all, note that $D_t = D_0 + tU^*[D_0, U]$ is a bounded perturbation of an operator D_0 with compact resolvent. Hence the domain $\mathcal{D}(D_t)$ is constant by the Kato–Rellich theorem. The path is continuous in the Riesz topology by Proposition 7.1.5 and indeed in $\mathbb{F}^{\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ so that its spectral flow is well defined as the spectral flow of $t \mapsto \mathcal{F}(F_t)$ where $\mathcal{F}$ is the bounded transform. Next let $\delta = \|D_0^{-1}\|^{-1}$ be the invertibility gap of D_0. Then D_t only has discrete spectrum in $(-\delta, \delta)$. Let $s \in [0,1] \mapsto g_s$ be the linear homotopy of nondecreasing smooth functions $g_s : \mathbb{R} \to \mathbb{R}$ between some nondecreasing continuous function g_0 satisfying $g_0(\lambda) = \mathrm{sgn}(\lambda)$ for $|\lambda| > \delta$ and $g_1(\lambda) = \mathcal{F}(\lambda)$. Then $(s,t) \mapsto g_s(D_t) \in \mathbb{F}\mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ is norm-continuous. By homotopy invariance of the spectral flow, one now has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto D_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto g_1(D_t)\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto g_0(D_t)\big).$$

But $g_0(D_0) = 2E - \mathbf{1}$ and $g_0(D_1) = U^*(2E - \mathbf{1})U$. Furthermore, the linear path between $g_0(D_0)$ and $g_0(D_1)$ is homotopic to $t \in [0,1] \mapsto g_0(D_t)$ within $\mathbb{F}\mathbb{B}_{\mathrm{sa}}(\mathcal{H})$. Therefore

$$\mathrm{Sf}\big(t \in [0,1] \mapsto D_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)(2E - \mathbf{1}) + tU^*(2E - \mathbf{1})U\big),$$

so that, by Corollary 5.6.2,

$$\begin{aligned}\mathrm{Sf}\big(t \in [0,1] \mapsto D_t\big) &= \mathrm{Ind}\big((\mathbf{1} - E)U(\mathbf{1} - E) + E\big)\\ &= -\,\mathrm{Ind}(EUE + \mathbf{1} - E),\end{aligned}$$

concluding the proof. □

Given that Theorem 10.2.1 connects the index pairing to the spectral flow of a path of self-adjoint Fredholm operators with compact resolvent, one can now use the results of Section 7.2 to provide an integral formula for the index pairing. For this purpose, the following supplementary property will be required.

Definition 10.2.2. Let D be the Dirac operator of a Fredholm module, hence $D = D^{\mathrm{ev}}$ as in (10.1) or $D = D^{\mathrm{od}}$ as in (10.4). Then D is said to be θ-summable if $\mathrm{Tr}(e^{-tD^2}) < \infty$ for any $t > 0$.

Theorem 10.2.3. *Let D_0 specify a θ-summable, odd, unbounded Fredholm module for a unitary operator U and let $E = \chi(D_0 > 0)$ be the associated Hardy projection. Then the spectral flow of $t \in [0,1] \mapsto D_t = (1-t)D_0 + tU^*D_0U$ satisfies, for any $\epsilon > 0$,*

$$\mathrm{Sf}(t \in [0,1] \mapsto D_t) = \frac{\epsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^1 dt\, \mathrm{Tr}(\partial_t D_t e^{-\epsilon D_t^2}) = -\,\mathrm{Ind}(EUE + \mathbf{1} - E).$$

It appears that this formula was first found by Wojciechowski (see the discussion in Section 8 of [26]) and then it is stated in Getzler's work [96]. It is dubbed the *easy adiabatic formula* in [26] and is the starting point for the proof of the Connes–Moscovici index formula [64]. The reference [26] also provides a semifinite version of this formula.

Proof of Theorem 10.2.3. In the final part of the proof of Theorem 7.2.2, it is shown that

$$\mathrm{Sf}(t \in [0,1] \mapsto D_t) = \frac{1}{2}\eta_\epsilon(D_1) - \frac{1}{2}\eta_\epsilon(D_0) + \frac{\epsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^1 dt\, \mathrm{Tr}(\partial_t D_t e^{-\epsilon D_t^2}),$$

where $\eta_\epsilon(D_1)$ and $\eta_\epsilon(D_0)$ are the regularized η-invariants defined in (7.5). Due to the unitary invariance of the trace, one directly deduces $\eta_\epsilon(D_1) = \eta_\epsilon(D_0)$ and therefore due to Theorem 10.2.1 the claim. □

Also even index pairings can be computed in terms of the heat semigroup of the Dirac operator. This is the celebrated McKean–Singer formula, see Section 3 in [54].

10.3 Spectral localizer for even index pairings

This section provides an alternative expression for an even index pairing as the signature of a suitable finite dimensional self-adjoint matrix called the finite volume even spectral localizer. This matrix will essentially be given by suitable matrix elements of D^{ev} and H, and it therefore provides a very efficient numerical algorithm for the computation of the index pairing. The corresponding result for odd index pairings will be given in Section 10.4 below.

Let us construct the even spectral localizer, directly following [129, 131, 170]. Therefore, let $H = H^* \in \mathbb{B}(\mathcal{H})$ be invertible and D^{ev} a self-adjoint, invertible Dirac operator specifying an unbounded even Fredholm module for H. The even spectral localizer is defined as the operator

$$L_\kappa^{\mathrm{ev}} = \begin{pmatrix} -H & \kappa D_0^* \\ \kappa D_0 & H \end{pmatrix}, \tag{10.9}$$

acting on $\mathcal{H} \oplus \mathcal{H}$, where $\kappa > 0$ is a tuning parameter. To construct finite volume restrictions of the spectral localizer, let us now set $(\mathcal{H} \oplus \mathcal{H})_\rho = \mathrm{Ran}(\chi(|D^{\mathrm{ev}}| \leq \rho))$, for a radius $\rho > 0$. Recall that D^{ev} has compact resolvent so that each $(\mathcal{H} \oplus \mathcal{H})_\rho$ is finite dimensional. Let $\pi_\rho : \mathcal{H} \oplus \mathcal{H} \to (\mathcal{H} \oplus \mathcal{H})_\rho$ denote the surjective partial isometry onto $(\mathcal{H} \oplus \mathcal{H})_\rho$ with $\mathrm{Ker}(\pi_\rho) = ((\mathcal{H} \oplus \mathcal{H})_\rho)^\perp$ and such that $\pi_\rho|_{(\mathcal{H}\oplus\mathcal{H})_\rho}$ is the identity on $(\mathcal{H} \oplus \mathcal{H})_\rho$. Let $\mathbf{1}_\rho = \pi_\rho \pi_\rho^*$ denote the identity on $(\mathcal{H} \oplus \mathcal{H})_\rho$. For any operator B on $\mathcal{H} \oplus \mathcal{H}$, we set $B_\rho = \pi_\rho B \pi_\rho^*$ which is an operator on $(\mathcal{H} \oplus \mathcal{H})_\rho$. With these notations, the finite volume spectral localizer on $(\mathcal{H} \oplus \mathcal{H})_\rho$ is

$$L^{\mathrm{ev}}_{\kappa,\rho} = \begin{pmatrix} -H & \kappa D_0^* \\ \kappa D_0 & H \end{pmatrix}_\rho .$$

The following connection of the index pairing to the half-signature of the spectral localizer was first shown in [129].

Theorem 10.3.1. *Let $g = \|H^{-1}\|^{-1}$ be the gap of the invertible self-adjoint operator H. Suppose that*

$$\kappa \leq \frac{g^3}{12\|H\|\|[D, H \oplus H]\|}, \qquad \frac{2g}{\kappa} < \rho. \tag{10.10}$$

Then $(L^{\mathrm{ev}}_{\kappa,\rho})^2 \geq \frac{g^2}{4}\mathbf{1}_\rho$. In particular, $L^{\mathrm{ev}}_{\kappa,\rho}$ is invertible and thus has a well-defined signature $\mathrm{Sig}(L^{\mathrm{ev}}_{\kappa,\rho})$ *which is independent of κ and ρ satisfying* (10.10)*, and*

$$\mathrm{Ind}(PFP + \mathbf{1} - P) = \frac{1}{2}\,\mathrm{Sig}(L^{\mathrm{ev}}_{\kappa,\rho}). \tag{10.11}$$

The proof of Theorem 10.3.1 given in [129] is of K-theoretic nature (and thus also provides a stronger K-theoretic result). Here, however, we rather provide a proof based on spectral flow and, more precisely, on Theorem 5.7.3 which gives a spectral flow formula for $\mathrm{Ind}(PFP + \mathbf{1} - P)$. Such a spectral flow proof was first put forward in the odd case in [130], and then for the even case in [131, 170]. The proof presented below is a further improvement requiring neither the normality of D_0 (as in [131]) nor the Lipshitz property (as in [170]).

Proof. For sake of notation simplicity, let us denote $D = D^{\mathrm{ev}}$. To show that the signature of $L^{\mathrm{ev}}_{\kappa,\rho}$ is independent of κ and ρ provided that (10.10) holds, we closely follow the argument in Section 3 of [129]. The proof will use an even and differentiable tapering function $G_\rho : \mathbb{R} \to [0, 1]$ with three properties:
(i) $G_\rho(x) = 1$ for $|x| \leq \frac{\rho}{2}$;
(ii) $G_\rho(x) = 0$ for $|x| \geq \rho$;
(iii) The Fourier transform $\widehat{G'_\rho} : \mathbb{R} \to \mathbb{R}$, $\widehat{G'_\rho}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath px} G'_\rho(x)dx$ of the derivative G'_ρ has an L^1-norm bounded by $\frac{8}{\rho}$.

Such a function can be constructed as follows, see also Lemma 4 in [128]. For $\rho = 0$, the function

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \begin{cases} 0, & x < 1, \\ \frac{1}{2}(1+x)^2, & x \in [-1,0], \\ 1 - \frac{1}{2}(1-x)^2, & x \in [0,1], \\ 1, & x > 1, \end{cases}$$

is defined. The Fourier transform of the derivative is $\widehat{g'}(p) = \frac{1-\cos(p)}{\pi p^2}$ with L^1-norm $\|\widehat{g'}\|_{L^1(\mathbb{R})} = 1$. Then one introduces $G_1 : \mathbb{R} \to \mathbb{R}$ by

$$G_1(x) = g(4x+3) - g(4x-3).$$

It satisfies $\|\widehat{G_1'}\|_{L^1(\mathbb{R})} \leq 8$. Finally, $G_\rho : \mathbb{R} \to \mathbb{R}$ defined by $G_\rho(x) = G_1(\frac{x}{\rho})$ has the desired properties. By Theorem 3.2.32 in [39], see also Lemma 10.15 in [92],

$$\big\|[G_\rho(D), H \oplus H]\big\| \leq \frac{8}{\rho}\big\|[D, H \oplus H]\big\|. \tag{10.12}$$

To connect the radii ρ and $\rho' \geq \rho$, let us consider the operator

$$L_{\kappa,\rho,\rho'}(\lambda) = \kappa\pi_{\rho'} D\pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho}((-H) \oplus H)G_{\lambda,\rho}\pi_{\rho'}^*,$$

acting on $(\mathcal{H} \oplus \mathcal{H})_{\rho'}$ where $0 \leq \lambda \leq 1$ and

$$G_{\lambda,\rho} = (1-\lambda)\pi_{\rho'}^*\pi_{\rho'} + \lambda G_\rho(D).$$

Also (10.10) is supposed to hold for the pair κ, ρ and thus also for the pair κ, ρ'. Notice that $L_{\kappa,\rho,\rho'}(0) = L^{\mathrm{ev}}_{\kappa,\rho'}$. The first goal is to show that $L_{\kappa,\rho,\rho'}(\lambda)$ is invertible for all $\lambda \in [0,1]$ and that its square is bounded from below by $\frac{g^2}{4}\mathbf{1}_{\rho'}$ when $\lambda = 0$. The square of $L_{\kappa,\rho,\rho'}(\lambda)$ is

$$\begin{aligned} L_{\kappa,\rho,\rho'}(\lambda)^2 = \kappa^2\pi_{\rho'} D^2\pi_{\rho'}^* + \left(\pi_{\rho'} G_{\lambda,\rho}((-H) \oplus H)G_{\lambda,\rho}\pi_{\rho'}^*\right)^2 \\ - \kappa\pi_{\rho'} G_{\lambda,\rho}[D, H \oplus H]\Gamma G_{\lambda,\rho}\pi_{\rho'}^*, \end{aligned}$$

where $\pi_{\rho'} D\pi_{\rho'}^*\pi_{\rho'} = \pi_{\rho'} D$ was used and $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. The second summand is bounded from below as follows:

$$\begin{aligned} &\left(\pi_{\rho'} G_{\lambda,\rho}(H \oplus (-H))G_{\lambda,\rho}\pi_{\rho'}^*\right)^2 \\ &\quad = \pi_{\rho'} G_{\lambda,\rho}(H \oplus H)G_{\lambda,\rho}^2(H \oplus H)G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad \geq \pi_{\rho'} G_{\lambda,\rho}(H \oplus H)G_\rho(D)^2(H \oplus H)G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad = \pi_{\rho'} G_{\lambda,\rho}G_\rho(D)(H \oplus H)^2 G_\rho(D)G_{\lambda,\rho}\pi_{\rho'}^* \end{aligned}$$

$$
\begin{aligned}
&+ \pi_{\rho'} G_{\lambda,\rho} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] G_{\lambda,\rho} \pi_{\rho'}^* \\
&\geq g^2 \pi_{\rho'} G_{\lambda,\rho}^2 G_\rho(D)^2 \pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] G_{\lambda,\rho} \pi_{\rho'}^* \\
&\geq g^2 \pi_{\rho'} G_\rho(D)^4 \pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] G_{\lambda,\rho} \pi_{\rho'}^*,
\end{aligned}
$$

where the first step holds because $[G_{\lambda,\rho}, \Gamma] = 0$ and $(\mathcal{H} \oplus \mathcal{H}) \ominus (\mathcal{H} \oplus \mathcal{H})_{\rho'} \subset \operatorname{Ker}(G_{\lambda,\rho})$, while the first, as well as the last, inequality follows from $G_\rho(D)^2 \leq G_{\lambda,\rho}^2$. For the special case of $\lambda = 0$, one has $G_{0,\rho} = \pi_{\rho'}^* \pi_{\rho'}$ and therefore a better estimate

$$
\begin{aligned}
&(\pi_{\rho'} G_{0,\rho} H G_{0,\rho} \pi_{\rho'}^*)^2 \\
&\quad \geq g^2 \pi_{\rho'} G_\rho(D)^2 \pi_{\rho'}^* + \pi_{\rho'} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] \pi_{\rho'}^*.
\end{aligned}
$$

Furthermore, by spectral calculus of D, one has the bound

$$
\kappa^2 \pi_{\rho'} D^2 \pi_{\rho'}^* \geq g^2 \pi_{\rho'} (\mathbf{1} - G_\rho(D)^2) \pi_{\rho'}^*,
$$

because the bound holds for spectral parameters in $[\frac{1}{2}\rho, \rho']$ due to (10.10) and because $\mathbf{1} - G_\rho(D)^2 \leq \mathbf{1}$, while it holds trivially on $[0, \frac{1}{2}\rho]$. Since

$$
\mathbf{1} - G_\rho(D)^2 + G_\rho(D)^4 \geq \frac{3}{4}\mathbf{1},
$$

it thus follows

$$
\begin{aligned}
L_{\kappa,\rho,\rho'}(\lambda)^2 \geq \frac{3}{4} g^2 \mathbf{1}_{\rho'} &+ \pi_{\rho'} G_{\lambda,\rho} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] G_{\lambda,\rho} \pi_{\rho'}^* \\
&- \kappa \pi_{\rho'} G_{\lambda,\rho} [D, H \oplus H] \Gamma G_{\lambda,\rho} \pi_{\rho'}^*,
\end{aligned}
$$

and in the special case $\lambda = 0$,

$$
\begin{aligned}
L_{\kappa,\rho,\rho'}(\lambda)^2 \geq g^2 \mathbf{1}_{\rho'} &+ \pi_{\rho'} G_{\lambda,\rho} [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] G_{\lambda,\rho} \pi_{\rho'}^* \\
&- \kappa \pi_{\rho'} G_{\lambda,\rho} [D, H \oplus H] \Gamma G_{\lambda,\rho} \pi_{\rho'}^*.
\end{aligned}
$$

Finally, the error term is bounded using the tapering estimate (10.12):

$$
\begin{aligned}
&\| [G_\rho(D)(H \oplus H), [G_\rho(D), H \oplus H]] - \kappa [D, (H \oplus H)] \Gamma \| \\
&\quad \leq \left(\frac{16}{\rho} \| G_\rho(D)(H \oplus H) \| + \kappa \right) \| [D, H \oplus H] \| \\
&\quad < \left(\frac{8}{g} \|H\| + 1 \right) \kappa \| [D, H \oplus H] \| \\
&\quad \leq \frac{9}{g} \|H\| \kappa \| [D, H \oplus H] \| \\
&\quad \leq \frac{3}{4} g^2,
\end{aligned}
$$

where the second step used the second inequality in (10.10), as well as $\|G_\rho(D)\| \le 1$, the third one took advantage of $\|H\| \ge g$, and the last inequality came from the first inequality in (10.10). Putting all together, one infers $L_{\kappa,\rho,\rho'}(\lambda)^2 > 0$ and $L_{\kappa,\rho,\rho'}(0)^2 \ge \frac{1}{4}g^2\mathbf{1}_{\rho'}$.

Next, let us show that

$$\mathrm{Sig}(L^{\mathrm{ev}}_{\kappa,\rho}) = \mathrm{Sig}(L^{\mathrm{ev}}_{\kappa',\rho'}),$$

for pairs κ, ρ and κ', ρ' in the permitted range of parameters. Without loss of generality, let $\rho \le \rho'$. As $L_{\kappa,\rho}$ is continuous in κ, it is sufficient to consider the case $\kappa = \kappa'$. Thus one needs to show

$$\mathrm{Sig}\big(L_{\kappa,\rho,\rho}(0)\big) = \mathrm{Sig}\big(L_{\kappa,\rho,\rho'}(0)\big),$$

when $\rho \le \rho'$ and (10.10) is true for κ and ρ. Clearly, $L_{\kappa,\rho,\rho'}(\lambda)$ is continuous in λ and it was shown above that $L_{\kappa,\rho,\rho'}(\lambda)$ is also invertible for all $\lambda \in [0,1]$, so it suffices to prove

$$\mathrm{Sig}\big(L_{\kappa,\rho,\rho}(1)\big) = \mathrm{Sig}\big(L_{\kappa,\rho,\rho'}(1)\big). \tag{10.13}$$

Consider

$$L_{\kappa,\rho,\rho'}(1) = \kappa\pi_{\rho'}D\pi^*_{\rho'} + \pi_{\rho'}G_\rho(D)\big((-H)\oplus H\big)G_\rho(D)\pi^*_{\rho'}.$$

Now D commutes with $\pi^*_{\rho'}\pi_{\rho'}$ so that $L_{\kappa,\rho,\rho'}(1)$ decomposes into a direct sum. Let next $\pi_{\rho',\rho} = \pi_{\rho'} \ominus \pi_\rho$ be the surjective partial isometry onto $(\mathcal{H}\oplus\mathcal{H})_{\rho'} \ominus (\mathcal{H}\oplus\mathcal{H})_\rho$. Then

$$L_{\kappa,\rho,\rho'}(1) = L_{\kappa,\rho,\rho}(1) \oplus \pi_{\rho',\rho}\kappa D\pi^*_{\rho',\rho}.$$

The signature of $\pi_{\rho',\rho}D\pi^*_{\rho',\rho}$ vanishes so that (10.13) follows.

It remains to show (10.11), for which $\kappa > 0$ can be chosen as small as needed and ρ as large as needed. Let us consider the odd increasing differentiable function $F_1 : \mathbb{R} \to \mathbb{R}$ given by

$$F_1(x) = \begin{cases} -2, & x < -2, \\ -x^3 - 4x^2 - 4x - 2, & x \in [-2,-1], \\ x, & x \in [-1,1], \\ -x^3 + 4x^2 - 4x + 2, & x \in [1,2], \\ 2, & x > 2. \end{cases}$$

The Fourier transform $\widehat{F_1'}$ of the derivative F_1' can be computed explicitly to be

$$\widehat{F_1'}(p) = \frac{1}{\pi}\left(\frac{-4\cos(2p)}{p^2} + \frac{-2\cos(p)}{p^2} + \frac{6\sin(2p)}{p^3} + \frac{-6\sin(p)}{p^3}\right).$$

Hence one has an L^1-norm bound $\|\widehat{F_1'}\|_1 \leq \frac{28}{\pi}$. Let us scale to $F_\rho : \mathbb{R} \to \mathbb{R}$ given by

$$F_\rho(x) = \rho F_1\left(\frac{x}{\rho}\right). \tag{10.14}$$

Hence F_ρ is an odd increasing differentiable function with $F_\rho(x) = x$ for $x \leq \rho$ and $F_\rho(x) = 2\rho = -F_\rho(-x)$ for $|x| \geq 2\rho$. Furthermore, the L^1-norm of the Fourier transform of the derivative is still bounded by $\frac{28}{\pi}$. Again by either Theorem 3.2.32 in [39] or Lemma 10.15 in [92], one has the bounds

$$\big\|[F_\rho(D), H \oplus H]\big\| \leq \frac{28}{\pi}\big\|[D, H \oplus H]\big\|. \tag{10.15}$$

Moreover, $F_\rho(D)$ anticommutes with Γ, hence is of the form

$$F_\rho(D) = \begin{pmatrix} 0 & (D_0')^* \\ D_0' & 0 \end{pmatrix}.$$

By Theorem 5.7.3,

$$\begin{aligned} \operatorname{Ind}(PFP + \mathbf{1} - P) &= \operatorname{Sf}\big(t \in [0,1] \mapsto (1-t)H + tF^* HF\big) \\ &= \operatorname{Sf}\big(t \in [0,1] \mapsto (1-t)FHF^* + tH\big), \end{aligned}$$

where item (vi) of Theorem 4.2.1 was used. For self-adjoint bounded Fredholm operators H_0 and H_1 such that the linear path $t \in [0,1] \mapsto (1-t)H_0 + tH_1$ connecting them is within the bounded Fredholm operators, the spectral flow of this path is from now on denoted by

$$\operatorname{Sf}(H_0, H_1) = \operatorname{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big).$$

Using (v) of Theorem 4.2.1, one obtains

$$\operatorname{Ind}(PFP + \mathbf{1} - P) = \operatorname{Sf}\left(\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}\begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix}, \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix}\right).$$

One has

$$\begin{aligned} \big(-H \otimes \Gamma + t\kappa F_\rho(D)\big)^2 &= \begin{pmatrix} H^2 + (t\kappa)^2 |D_0'|^2 & t\kappa [H, D_0']^* \\ t\kappa [H, D_0'] & H^2 + (t\kappa)^2 |(D_0')^*|^2 \end{pmatrix} \\ &\geq \big(g^2 - t\kappa \big\|[F_\rho(D), H \oplus H]\big\|\big)\mathbf{1}, \end{aligned}$$

for $t \in [0,1]$. By (10.15), the linear path connecting the operator $-H \otimes \Gamma$ to $-H \otimes \Gamma + \kappa F_\rho(D)$ is within the invertibles for κ sufficiently small. As $[H, F]$ is compact, the linear path connecting

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix} (-H \otimes \Gamma) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix} \quad \text{to } -H \otimes \Gamma + t\kappa F_\rho(D)$$

is within the Fredholm operators for all $t \in [0,1]$. The homotopy invariance of the spectral flow, see Theorem 4.2.2, implies

$$\operatorname{Ind}(PFP + \mathbf{1} - P) = \operatorname{Sf}\left(\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix}, \begin{pmatrix} -H & \kappa (D_0')^* \\ \kappa D_0' & H \end{pmatrix} \right).$$

Next one directly checks that

$$s \in [0, \kappa\rho] \mapsto \begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} -H & s \\ s & H \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}^* = \begin{pmatrix} -H & sF^* \\ sF & FHF^* \end{pmatrix}$$

is a path of invertibles. Let us also show that

$$\begin{aligned} A(s,t) &= t \begin{pmatrix} -H & \kappa (D_0')^* \\ \kappa D_0' & H \end{pmatrix} + (1-t) \begin{pmatrix} -H & sF^* \\ sF & FHF^* \end{pmatrix} \\ &= \begin{pmatrix} -H & t\kappa (D_0')^* + (1-t)sF^* \\ t\kappa D_0' + (1-t)sF & H - (1-t)[H,F]F^* \end{pmatrix} \end{aligned}$$

is Fredholm for all $(s,t) \in [0,\kappa\rho] \times [0,1]$. Because $[H,F]$ is compact, it is sufficient to show that

$$B(s,t) = \begin{pmatrix} -H & t\kappa (D_0')^* + (1-t)sF^* \\ t\kappa D_0' + (1-t)sF & H \end{pmatrix}$$

is Fredholm. One can replace $D|D|^{-1}$ by $\frac{1}{2\rho} F_\rho(D)$ as

$$\operatorname{Ran}\left(\frac{1}{2\rho} F_\rho(D) - D|D|^{-1} \right) \subset (\mathcal{H} \oplus \mathcal{H})_{2\rho}$$

is finite dimensional, so that $\frac{1}{2\rho} F_\rho(D) - D|D|^{-1}$ is compact. Therefore it is sufficient to show that

$$C(s,t) = -H \otimes \Gamma + t\kappa F_\rho(D) + (1-t)s \frac{1}{2\rho} F_\rho(D)$$

is Fredholm. Now

$$\begin{aligned} C(s,t)^2 &= (H \otimes \Gamma)^2 + \left(t\kappa F_\rho(D) + (1-t)s \frac{1}{2\rho} F_\rho(D) \right)^2 \\ &\quad - \left[t\kappa F_\rho(D) + (1-t)s \frac{1}{2\rho} F_\rho(D), H \oplus H \right] \Gamma \\ &\geq \left(g^2 - \left(t\kappa + (1-t)s \frac{1}{2\rho} \right) \| [F_\rho(D), H \oplus H] \| \right) \mathbf{1} \end{aligned}$$

$$\geq \left(g^2 - \left(\kappa + \frac{\kappa}{2}\right)\|[F_\rho(D), H \oplus H]\|\right)\mathbf{1}$$
$$\geq \left(g^2 - \frac{42\kappa}{\pi}\|[D, H \oplus H]\|\right)\mathbf{1},$$

where the last step follows form (10.15). Therefore $C(s,t)$ is invertible and $A(s,t)$ is Fredholm for all $(s,t) \in [0,\kappa\rho] \times [0,1]$ and κ sufficiently small. This implies by the homotopy invariance of the spectral flow

$$\operatorname{Ind}(PFP + \mathbf{1} - P) = \operatorname{Sf}\left(\begin{pmatrix} -H & \kappa\rho F^* \\ \kappa\rho F & FHF^* \end{pmatrix}, \begin{pmatrix} -H & \kappa(D_0')^* \\ \kappa D_0' & H \end{pmatrix}\right)$$
$$= \operatorname{Sf}\left(\begin{pmatrix} -H & \kappa\rho F^* \\ \kappa\rho F & FHF^* \end{pmatrix}, L^{\kappa,\rho}\right)$$

for

$$L^{\kappa,\rho} = \begin{pmatrix} -H & \kappa(D_0')^* \\ \kappa D_0' & H \end{pmatrix}.$$

For $(\mathcal{H} \oplus \mathcal{H})_{\rho^c} = (\mathcal{H} \oplus \mathcal{H}) \ominus (\mathcal{H} \oplus \mathcal{H})_\rho$, we denote the surjective partial isometry onto $(\mathcal{H} \oplus \mathcal{H})_{\rho^c}$ by π_{ρ^c}, and for any operator B on $\mathcal{H} \oplus \mathcal{H}$ set $B_{\rho^c} = \pi_{\rho^c} B (\pi_{\rho^c})^*$. Then one has $F_\rho(D) = F_\rho(D)_\rho \oplus F_\rho(D)_{\rho^c}$ and $F_\rho(D)_\rho = D_\rho$. Moreover, $(L^{\kappa,\rho})_\rho = L^{\mathrm{ev}}_{\kappa,\rho}$. Next we show that the linear path $t \in [0,1] \mapsto L^{\kappa,\rho}(t)$ for

$$L^{\kappa,\rho}(t) = \begin{pmatrix} (L^{\kappa,\rho})_\rho & 0 \\ 0 & (L^{\kappa,\rho})_{\rho^c} \end{pmatrix} + t \begin{pmatrix} 0 & \pi_\rho(-H \oplus H)(\pi_{\rho^c})^* \\ \pi_{\rho^c}(-H \oplus H)(\pi_\rho)^* & 0 \end{pmatrix}$$

is within the invertibles. First, $(L^{\kappa,\rho})_{\rho^c}$ can be bounded from below using (10.10):

$$((-H \otimes \Gamma + \kappa F_\rho(D))_{\rho^c})^2$$
$$= ((-H \otimes \Gamma)_{\rho^c})^2 + \kappa^2 (F_\rho(D)_{\rho^c})^2 - \kappa [F_\rho(D)_{\rho^c}, (H \otimes \mathbf{1})_{\rho^c}]\Gamma_{\rho^c}$$
$$\geq (\kappa^2\rho^2 - \kappa\|[F_\rho(D)_{\rho^c}, (H \otimes \mathbf{1})_{\rho^c}]\|)\mathbf{1}_{\rho^c}$$
$$\geq \left(\kappa^2\rho^2 - \frac{28\kappa}{\pi}\|[D, H \otimes \mathbf{1}]\|\right)\mathbf{1}_{\rho^c}$$
$$\geq \frac{1}{2}\kappa^2\rho^2 \mathbf{1}_{\rho^c},$$

where the third step follows from (10.15). Now $L^{\kappa,\rho}(t)$ is given by

$$L^{\kappa,\rho}(t) = |(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}|^{\frac{1}{2}} \left(G + t \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\right) |(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}|^{\frac{1}{2}},$$

where G is a diagonal unitary with respect to the direct sum $\mathcal{H} \oplus \mathcal{H} = (\mathcal{H} \oplus \mathcal{H})_\rho \oplus (\mathcal{H} \oplus \mathcal{H})_{\rho^c}$ and

$$B = \left|(L^{\kappa,\rho})_\rho\right|^{-\frac{1}{2}} \pi_\rho(-H \oplus H)(\pi_{\rho^c})^* \left|(L^{\kappa,\rho})_{\rho^c}\right|^{-\frac{1}{2}}.$$

The off-diagonal entries satisfy

$$\|B\| \le \frac{\sqrt[4]{8}\|H\|}{\sqrt{\kappa\rho g}},$$

thus their norm is smaller than 1 for ρ sufficiently large. Because $L^{\kappa,\rho} - ((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})$ is finite dimensional and therefore compact, the homotopy invariance of the spectral flow then implies

$$\mathrm{Ind}(PFP + \mathbf{1} - P) = \mathrm{Sf}\left(\begin{pmatrix} -H & \kappa\rho F^* \\ \kappa\rho F & FHF^* \end{pmatrix}, (L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\right).$$

The path

$$s \in [0,1] \mapsto A(s) = \begin{pmatrix} -sH & \kappa\rho F^* \\ \kappa\rho F & sFHF^* \end{pmatrix}$$

is within the invertibles for ρ sufficiently large. As $tA(s)_{\rho^c} + (1-t)(L^{\kappa,\rho})_{\rho^c}$ is invertible for all $(s,t) \in [0,1] \times [0,1]$ and ρ sufficiently large,

$$tA(s) + (1-t)((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})$$

is Fredholm for all $(s,t) \in [0,1] \times [0,1]$, so that again by the homotopy invariance of the spectral flow

$$\begin{aligned} \mathrm{Ind}(PFP + \mathbf{1} - P) &= \mathrm{Sf}(\kappa\rho D|D|^{-1}, ((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})) \\ &= \mathrm{Sf}(\kappa\rho(D|D|^{-1})_\rho, (L^{\kappa,\rho})_\rho) \\ &\quad + \mathrm{Sf}(\kappa\rho(D|D|^{-1})_{\rho^c}, (L^{\kappa,\rho})_{\rho^c}), \end{aligned}$$

where item (v) of Theorem 4.2.1 was used. The second summand vanishes because the linear path

$$t \in [0,1] \mapsto (1-t)\kappa\rho(D|D|^{-1})_{\rho^c} + t(L^{\kappa,\rho})_{\rho^c}$$

lies in the invertibles for ρ sufficiently large. As $(L^{\kappa,\rho})_\rho = L^{\mathrm{ev}}_{\kappa,\rho}$,

$$\mathrm{Ind}(PFP + \mathbf{1} - P) = \mathrm{Sf}(\kappa\rho(D|D|^{-1})_\rho, L^{\mathrm{ev}}_{\kappa,\rho}) = \frac{1}{2}(\mathrm{Sig}(L^{\mathrm{ev}}_{\kappa,\rho}) - \mathrm{Sig}(D_\rho)),$$

and because $\Gamma D \Gamma = -D$, the signature of D_ρ vanishes and the claim follows. □

10.4 Spectral localizer for odd index pairings

This section states and proves the equivalent to Theorem 10.3.1 for the odd index pairings described in Section 10.1. Hence let A be an invertible operator on $\mathcal{H}$ and associate to it the invertible chiral operator H on $\mathcal{H} \oplus \mathcal{H}$ as in (10.5). Further, let D_0 be an odd unbounded Fredholm module for A and associated to it is the odd Dirac operator D^{od}, see (10.4). The aim is to provide a finite-volume expression for the index pairing given in Theorem 10.1.9. The odd spectral localizer is now defined as the operator

$$L_\kappa^{\mathrm{od}} = \begin{pmatrix} \kappa D_0 & A \\ A^* & -\kappa D_0 \end{pmatrix}, \tag{10.16}$$

acting on $\mathcal{H}\oplus\mathcal{H}$ where $\kappa > 0$ is a tuning parameter. For the definition of the finite-volume approximations, let us set $\mathcal{H}_\rho = \mathrm{Ran}(\chi(|D_0| \leq \rho))$ and $(\mathcal{H} \oplus \mathcal{H})_\rho = \mathrm{Ran}(\chi(|D^{\mathrm{od}}| \leq \rho))$ for $\rho > 0$. Note that $(\mathcal{H}\oplus\mathcal{H})_\rho = \mathcal{H}_\rho\oplus\mathcal{H}_\rho$. As D^{od} has compact resolvent, each $\mathcal{H}_\rho$ and $(\mathcal{H}\oplus\mathcal{H})_\rho$ is finite dimensional. Let $\pi_\rho : \mathcal{H} \to \mathcal{H}_\rho$ denote the surjective partial isometry onto $\mathcal{H}_\rho$ with $\mathrm{Ker}(\pi_\rho) = (\mathcal{H}_\rho)^\perp$ and such that $\pi_\rho|_{\mathcal{H}_\rho}$ is the identity on $\mathcal{H}_\rho$. By abuse of notation, the surjective partial isometry onto $(\mathcal{H}\oplus\mathcal{H})_\rho$ is also denoted by $\pi_\rho : \mathcal{H}\oplus\mathcal{H} \to (\mathcal{H}\oplus\mathcal{H})_\rho$. As in Section 10.3, for any operator B on $\mathcal{H}$ or $\mathcal{H} \oplus \mathcal{H}$, we set $B_\rho = \pi_\rho B \pi_\rho^*$ which is an operator on $\mathcal{H}_\rho$ or $(\mathcal{H} \oplus \mathcal{H})_\rho$. With these notations, the finite-volume odd spectral localizer on $\mathcal{H}_\rho \oplus \mathcal{H}_\rho$ is defined by

$$L_{\kappa,\rho}^{\mathrm{od}} = \begin{pmatrix} \kappa D_{0,\rho} & A_\rho \\ A_\rho^* & -\kappa D_{0,\rho} \end{pmatrix}.$$

The following theorem goes back to [128] (at least with slightly stronger assumptions on the constants κ and ρ). For the proof by spectral flow, we will essentially follow [130], with some improvements stemming from [75]

Theorem 10.4.1. *Let $g = \|A^{-1}\|^{-1}$ be the gap of the invertible operator A. Suppose that*

$$\kappa \leq \frac{g^3}{12\|A\|\|[D_0, A]\|}, \qquad \frac{2g}{\kappa} < \rho. \tag{10.17}$$

Then the matrix $L_{\kappa,\rho}^{\mathrm{od}}$ satisfies the bound $(L_{\kappa,\rho}^{\mathrm{od}})^2 \geq \frac{g^2}{4}\mathbf{1}_\rho$. In particular, $L_{\kappa,\rho}^{\mathrm{od}}$ is invertible and thus has a well-defined signature $\mathrm{Sig}(L_{\kappa,\rho}^{\mathrm{od}})$. *It is independent of κ and ρ satisfying* (10.17), *and*

$$\mathrm{Ind}(EUE + \mathbf{1} - E) = \frac{1}{2}\,\mathrm{Sig}(L_{\kappa,\rho}^{\mathrm{od}}). \tag{10.18}$$

Proof. For sake of notation simplicity, let us denote $D = D^{\mathrm{od}}$. The proof that the signature of $L_{\kappa,\rho}^{\mathrm{od}}$ is independent of κ and ρ satisfying (10.17) is essentially the same as in the proof of Theorem 10.3.1. In particular, using the same function G_ρ one now has

$$\|[G_\rho(D_0), A]\| \le \frac{8}{\rho}\|[D_0, A]\|.$$

With D and H as in (10.4) and (10.5), respectively, one has

$$DH + HD = \begin{pmatrix} 0 & [D_0, A] \\ [D_0, A]^* & 0 \end{pmatrix}$$

and, due to $G_\rho(-D_0) = G_\rho(D_0)$, also

$$\|[G_\rho(D), H]\| \le \frac{8}{\rho}\|[D_0, A]\|. \tag{10.19}$$

Even though essentially the same as in the proof of Theorem 10.3.1, let us spell out in details how to connect the radii ρ and $\rho' \ge \rho$. Let us introduce

$$L_{\kappa,\rho,\rho'}(\lambda) = \kappa\pi_{\rho'} D\pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho} H G_{\lambda,\rho}\pi_{\rho'}^*,$$

acting on $\mathcal{H}_{\rho'} \oplus \mathcal{H}_{\rho'}$ where $0 \le \lambda \le 1$ and

$$G_{\lambda,\rho} = (1-\lambda)\pi_{\rho'}^*\pi_{\rho'} + \lambda G_\rho(D).$$

Also (10.17) is supposed to hold for the pair κ, ρ and for the pair κ, ρ'. Notice that $L_{\kappa,\rho,\rho'}(0) = L_{\kappa,\rho'}^{\mathrm{od}}$. The first goal is to show that $L_{\kappa,\rho,\rho'}(\lambda)$ is always invertible and that its square is bounded from below by $\frac{g^2}{4}\mathbf{1}_{\rho'}$ when $\lambda = 0$. The square of $L_{\kappa,\rho,\rho'}(\lambda)$ is

$$\begin{aligned} &L_{\kappa,\rho,\rho'}(\lambda)^2 \\ &\quad = \kappa^2\pi_{\rho'} D^2\pi_{\rho'}^* + (\pi_{\rho'} G_{\lambda,\rho} H G_{\lambda,\rho}\pi_{\rho'}^*)^2 + \kappa\pi_{\rho'} G_{\lambda,\rho}(DH + HD)G_{\lambda,\rho}\pi_{\rho'}^*, \end{aligned}$$

where $\pi_{\rho'} D\pi_{\rho'}^*\pi_{\rho'} = \pi_{\rho'} D$ was used. The second summand is bounded from below as follows:

$$\begin{aligned} &(\pi_{\rho'} G_{\lambda,\rho} H G_{\lambda,\rho}\pi_{\rho'}^*)^2 \\ &\quad = \pi_{\rho'} G_{\lambda,\rho} H G_{\lambda,\rho}^2 H G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad \ge \pi_{\rho'} G_{\lambda,\rho} H G_\rho(D)^2 H G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad = \pi_{\rho'} G_{\lambda,\rho} G_\rho(D) H^2 G_\rho(D) G_{\lambda,\rho}\pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho}[G_\rho(D)H, [G_\rho(D), H]]G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad \ge g^2\pi_{\rho'} G_{\lambda,\rho}^2 G_\rho(D)^2\pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho}[G_\rho(D)H, [G_\rho(D), H]]G_{\lambda,\rho}\pi_{\rho'}^* \\ &\quad \ge g^2\pi_{\rho'} G_\rho(D)^4\pi_{\rho'}^* + \pi_{\rho'} G_{\lambda,\rho}[G_\rho(D)H, [G_\rho(D), H]]G_{\lambda,\rho}\pi_{\rho'}^*, \end{aligned}$$

where the first step holds because $(\mathcal{H} \oplus \mathcal{H}) \ominus (\mathcal{H} \oplus \mathcal{H})_{\rho'} \subset \mathrm{Ker}(G_{\lambda,\rho})$, while the first, as well as the last, inequality follows from $G_\rho(D)^2 \le G_{\lambda,\rho}^2$. For the special case of $\lambda = 0$, one has $G_{0,\rho} = \pi_{\rho'}^*\pi_{\rho'}$ and therefore a better estimate

$$(\pi_{\rho'} G_{0,\rho} H G_{0,\rho} \pi_{\rho'}^*)^2 \geq g^2 \pi_{\rho'} G_\rho(D)^2 \pi_{\rho'}^* + \pi_{\rho'} [G_\rho(D) H, [G_\rho(D), H]] \pi_{\rho'}^*.$$

Furthermore, by spectral calculus of D, one has the bound

$$\kappa^2 \pi_{\rho'} D^2 \pi_{\rho'}^* \geq g^2 \pi_{\rho'} (\mathbf{1} - G_\rho(D)^2) \pi_{\rho'}^*,$$

because the bound holds for spectral parameters in $[\frac{1}{2}\rho, \rho']$ due to (10.17) using that $\mathbf{1} - G_\rho(D)^2 \leq \mathbf{1}$, while it holds trivially on $[0, \frac{1}{2}\rho]$. Since

$$\mathbf{1} - G_\rho(D)^2 + G_\rho(D)^4 \geq \frac{3}{4}\mathbf{1},$$

it thus follows

$$L_{\kappa,\rho,\rho'}(\lambda)^2 \geq \frac{3}{4} g^2 \mathbf{1}_{\rho'} + \pi_{\rho'} G_{\lambda,\rho}([G_\rho(D)H, [G_\rho(D), H]] + \kappa(DH + HD)) G_{\lambda,\rho} \pi_{\rho'}^*,$$

and in the special case $\lambda = 0$,

$$L_{\kappa,\rho,\rho'}(0)^2 \geq g^2 \mathbf{1}_{\rho'} + \pi_{\rho'} G_{\lambda,\rho}([G_\rho(D)H, [G_\rho(D), H]] + \kappa(DH + HD)) G_{\lambda,\rho} \pi_{\rho'}^*.$$

Finally, the error term is bounded using the tapering estimate (10.19):

$$\begin{aligned}
&\|[G_\rho(D)H, [G_\rho(D), H]] + \kappa(DH + HD)\| \\
&\quad \leq \left(\frac{16}{\rho}\|G_\rho(D)H\| + \kappa\right)\|[A, D_0]\| \\
&\quad < \left(\frac{8}{g}\|A\| + 1\right)\kappa\|[A, D_0]\| \\
&\quad \leq \frac{9}{g}\|A\|\kappa\|[A, D_0]\| \\
&\quad \leq \frac{3}{4} g^2,
\end{aligned}$$

where the second step used the second inequality in (10.17), as well as $\|G_\rho(D)\| \leq 1$, the third one took advantage of $\|A\| \geq g$, and finally the last inequality came from the first inequality in (10.17). Putting all together, one infers $L_{\kappa,\rho,\rho'}(\lambda)^2 > 0$ and $L_{\kappa,\rho,\rho'}(0)^2 \geq \frac{1}{4} g^2 \mathbf{1}_{\rho'}$.

Next, let us show that

$$\mathrm{Sig}(L^{\mathrm{od}}_{\kappa,\rho}) = \mathrm{Sig}(L^{\mathrm{od}}_{\kappa',\rho'}),$$

for pairs κ, ρ and κ', ρ' in the permitted range of parameters. Without loss of generality, let $\rho \leq \rho'$. As $L_{\kappa,\rho}$ is continuous in κ, it is sufficient to consider the case $\kappa = \kappa'$. Thus one needs to show

$$\mathrm{Sig}(L_{\kappa,\rho,\rho}(0)) = \mathrm{Sig}(L_{\kappa,\rho,\rho'}(0)),$$

when $\rho \le \rho'$ and (10.17) is true for κ and ρ. Clearly, $L_{\kappa,\rho,\rho'}(\lambda)$ is continuous in λ, so it suffices to prove

$$\mathrm{Sig}\big(L_{\kappa,\rho,\rho}(1)\big) = \mathrm{Sig}\big(L_{\kappa,\rho,\rho'}(1)\big).$$

Consider

$$L_{\kappa,\rho,\rho'}(1) = \kappa\pi_{\rho'}D\pi_{\rho'}^* + \pi_{\rho'}G_\rho(D)HG_\rho(D)\pi_{\rho'}^*.$$

Now D commutes with $\pi_{\rho'}^*\pi_{\rho'}$ so that $L_{\kappa,\rho,\rho'}(1)$ decomposes into a direct sum. Further let $\pi_{\rho',\rho} = \pi_{\rho'} \ominus \pi_\rho$ be the surjective partial isometry onto $(\mathcal{H}\oplus\mathcal{H})_{\rho'} \ominus (\mathcal{H}\oplus\mathcal{H})_\rho$. Then

$$L_{\kappa,\rho,\rho'}(1) = L_{\kappa,\rho,\rho}(1) \oplus \pi_{\rho',\rho}\kappa D\pi_{\rho',\rho}^*.$$

The signature of $\pi_{\rho',\rho}D\pi_{\rho',\rho}^*$ vanishes so that

$$\mathrm{Sig}\big(L_{\kappa,\rho,\rho'}(1)\big) = \mathrm{Sig}\big(L_{\kappa,\rho,\rho}(1)\big).$$

It remains to show (10.18), for which $\kappa > 0$ can be chosen as small as needed and ρ as large as needed. Let again F_1 and F_ρ be as in the proof of Theorem 10.3.1. Then one has, again similar to (10.15),

$$\big\|[F_\rho(D_0), A]\big\| \le \frac{28}{\pi}\big\|[D_0, A]\big\|, \quad \big\|[F_\rho(D_0), U]\big\| \le \frac{28}{\pi}\big\|[D_0, U]\big\|, \tag{10.20}$$

where $[D_0, U]$ is bounded by Theorem 3.3.6 in [163]. Because $F_\rho(D_0) - 2\rho(2E - \mathbf{1})$ is finite dimensional and $[E, U]$ is compact, it follows that $U^*F_\rho(D_0)U - F_\rho(D)$ is compact. By construction, $\chi(F_\rho(D_0) \ge 0) = E$, thus by Theorem 5.7.3,

$$\begin{aligned}\mathrm{Ind}(EUE + \mathbf{1} - E) &= -\,\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)F_\rho(D_0) + tU^*F_\rho(D_0)U\big)\\ &= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)U^*F_\rho(D_0)U + tF_\rho(D_0)\big),\end{aligned}$$

where items (iii) and (iv) of Theorem 4.2.1 were used. As in the proof of Theorem 10.3.1, for self-adjoint bounded Fredholm operators H_0 and H_1 such that the straight-line path $t \in [0,1] \mapsto (1-t)H_0 + tH_1$ connecting them is within the bounded Fredholm operators, the spectral flow of this path is denoted by

$$\mathrm{Sf}(H_0, H_1) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big).$$

Using (v) and (vi) of Theorem 4.2.1, one has

$$\begin{aligned}&\mathrm{Ind}(EUE + \mathbf{1} - E)\\ &= \mathrm{Sf}\left(\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix}, \begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}^*\right).\end{aligned}$$

The homotopy invariance of the spectral flow, see Theorem 4.2.4, implies

$$\begin{aligned}&\mathrm{Ind}(EUE+\mathbf{1}-E)\\&\quad=\mathrm{Sf}\left(\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix},\begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix}\kappa F_\rho(D_0) & \mathbf{1}\\ \mathbf{1} & -\kappa F_\rho(D_0)\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}^*\right),\end{aligned}\tag{10.21}$$

because

$$s\in[0,1]\mapsto\begin{pmatrix}\kappa F_\rho(D_0) & s\mathbf{1}\\ s\mathbf{1} & -\kappa F_\rho(D_0)\end{pmatrix}$$

is a norm-continuous path of invertibles and the linear path connecting

$$\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix}\quad\text{to}\quad\begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix}\kappa F_\rho(D_0) & s\mathbf{1}\\ s\mathbf{1} & -\kappa F_\rho(D_0)\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & \mathbf{1}\end{pmatrix}^*$$

is within the Fredholm operators for all $s\in[0,1]$ as $[F_\rho(D),U]$ is compact. Multiplying out (10.21) shows

$$\mathrm{Ind}(EUE+\mathbf{1}-E)=\mathrm{Sf}\left(\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix},\begin{pmatrix}\kappa UF_\rho(D_0)U^* & U\\ U^* & -\kappa F_\rho(D_0)\end{pmatrix}\right).$$

For κ sufficiently small, the linear path from

$$\begin{pmatrix}\kappa UF_\rho(D_0)U^* & U\\ U^* & -\kappa F_\rho(D_0)\end{pmatrix}\quad\text{to}\quad\begin{pmatrix}\kappa F_\rho(D_0) & U\\ U^* & -\kappa F_\rho(D_0)\end{pmatrix}$$

is within the invertibles because of the bound (10.20). As $[F_\rho(D_0),U]$ is compact, the homotopy invariance of the spectral flow implies

$$\mathrm{Ind}(EUE+\mathbf{1}-E)=\mathrm{Sf}\left(\begin{pmatrix}\kappa F_\rho(D_0) & 0\\ 0 & -\kappa F_\rho(D_0)\end{pmatrix},\begin{pmatrix}\kappa F_\rho(D_0) & U\\ U^* & -\kappa F_\rho(D_0)\end{pmatrix}\right).$$

For κ sufficiently small,

$$s\in[0,1]\mapsto\begin{pmatrix}\kappa F_\rho(D_0) & U|A|^s\\ (U|A|^s)^* & -\kappa F_\rho(D_0)\end{pmatrix}$$

is a norm-continuous path of invertibles. Using that $[F_\rho(D_0),U|A|^s]$ is compact for all $s\in[0,1]$, one directly checks that

$$t\in[0,1]\mapsto\begin{pmatrix}\kappa F_\rho(D_0) & tU|A|^s\\ t(U|A|^s)^* & -\kappa F_\rho(D_0)\end{pmatrix}$$

is a norm-continuous paths of bounded Fredholm operators for all $s\in[0,1]$. Then again by the homotopy invariance of the spectral flow,

$$\operatorname{Ind}(EUE + \mathbf{1} - E) = \operatorname{Sf}\left(\begin{pmatrix} \kappa F_\rho(D_0) & 0 \\ 0 & -\kappa F_\rho(D_0) \end{pmatrix}, \begin{pmatrix} \kappa F_\rho(D_0) & A \\ A^* & -\kappa F_\rho(D_0) \end{pmatrix}\right)$$
$$= \operatorname{Sf}\big(\kappa F_\rho(D), L^{\kappa,\rho}\big)$$

for

$$L^{\kappa,\rho} = \begin{pmatrix} \kappa F_\rho(D_0) & A \\ A^* & -\kappa F_\rho(D_0) \end{pmatrix}.$$

For $\mathcal{H}_{\rho^c} = \mathcal{H} \ominus \mathcal{H}_\rho$, we denote the surjective partial isometry onto $\mathcal{H}_{\rho^c}$ and $\mathcal{H}_{\rho^c} \oplus \mathcal{H}_{\rho^c}$ by π_{ρ^c}. For any operator B on $\mathcal{H}$ or $\mathcal{H} \oplus \mathcal{H}$, let us define $B_{\rho^c} = \pi_{\rho^c} B (\pi_{\rho^c})^*$. One clearly has $F_\rho(D_0) = F_\rho(D_0)_\rho \oplus F_\rho(D_0)_{\rho^c}$ and $F_\rho(D_0)_\rho = (D_0)_\rho$, and similarly for D. Moreover, $(L^{\kappa,\rho})_\rho = L^{\mathrm{od}}_{\kappa,\rho}$. Next we show that the linear path

$$t \in [0,1] \mapsto L^{\kappa,\rho}(t) = \begin{pmatrix} (L^{\kappa,\rho})_\rho & 0 \\ 0 & (L^{\kappa,\rho})_{\rho^c} \end{pmatrix} + t \begin{pmatrix} 0 & \pi_\rho H (\pi_{\rho^c})^* \\ \pi_{\rho^c} H (\pi_\rho)^* & 0 \end{pmatrix}$$

is within the invertibles. Let us first check that

$$(L^{\kappa,\rho})_{\rho^c} = \begin{pmatrix} \kappa F_\rho(D_0)_{\rho^c} & A_{\rho^c} \\ A^*_{\rho^c} & -\kappa F_\rho(D_0)_{\rho^c} \end{pmatrix}$$

is invertible. One has

$$\begin{aligned} \big((L^{\kappa,\rho})_{\rho^c}\big)^2 &= \begin{pmatrix} \kappa^2 F_\rho(D_0)^2_{\rho^c} + A_{\rho^c}(A_{\rho^c})^* & \kappa(F_\rho(D_0)A_{\rho^c} - A_{\rho^c}F_\rho(D_0)) \\ \kappa(F_\rho(D_0)A_{\rho^c} - A_{\rho^c}F_\rho(D_0))^* & \kappa^2 F_\rho(D_0)^2_{\rho^c} + (A_{\rho^c})^* A_{\rho^c} \end{pmatrix} \\ &\geq (\kappa^2\rho^2 - \kappa \| [F_\rho(D_0), A] \|) \mathbf{1}_{\rho^c} \\ &\geq \left(\kappa^2\rho^2 - \kappa \frac{28}{\pi} \| [D_0, A] \| \right) \mathbf{1}_{\rho^c} \\ &\geq \frac{1}{2} \kappa^2 \rho^2 \mathbf{1}_{\rho^c}, \end{aligned}$$

where the third step follows from (10.20) and the last from (10.17). Hence $L^{\kappa,\rho}(t)$ is given by

$$\big|(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\big|^{\frac{1}{2}} \left(G + t \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \right) \big|(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\big|^{\frac{1}{2}}$$

where G is a diagonal unitary with respect to the direct sum $\mathcal{H} \oplus \mathcal{H} = (\mathcal{H} \oplus \mathcal{H})_\rho \oplus (\mathcal{H} \oplus \mathcal{H})_{\rho^c}$ and

$$B = \big|(L^{\kappa,\rho})_\rho\big|^{-\frac{1}{2}} \pi_\rho H (\pi_{\rho^c})^* \big|(L^{\kappa,\rho})_{\rho^c}\big|^{-\frac{1}{2}}.$$

The off-diagonal entries satisfy

$$\|B\| \leq \frac{\sqrt[4]{8}\|H\|}{\sqrt{\kappa\rho g}},$$

thus their norm is smaller than 1 for ρ sufficiently large. Because $L^{\kappa,\rho} - ((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})$ is finite dimensional and therefore compact, the homotopy invariance of the spectral flow implies

$$\begin{aligned}\operatorname{Ind}(EUE + \mathbf{1} - E) &= \operatorname{Sf}\big(\kappa F_\rho(D), (L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\big)\\ &= \operatorname{Sf}\big(\kappa F_\rho(D)_\rho, (L^{\kappa,\rho})_\rho\big) + \operatorname{Sf}\big(\kappa F_\rho(D)_{\rho^c}, (L^{\kappa,\rho})_{\rho^c}\big).\end{aligned}$$

Now $(\kappa F_\rho(D)_{\rho^c})^2 \geq \kappa^2\rho^2$ so that the path $t \in [0,1] \mapsto t\kappa F_\rho(D)_{\rho^c} + (1-t)(L^{\kappa,\rho})_{\rho^c}$ consists of invertibles for ρ sufficiently large and its spectral flow vanishes by item (i) of Theorem 4.2.1. Therefore, using $F_\rho(D)_\rho = D_\rho$ and $(L^{\kappa,\rho})_\rho = L^{\mathrm{od}}_{\kappa,\rho}$, one concludes

$$\operatorname{Ind}(EUE + \mathbf{1} - E) = \operatorname{Sf}\big(\kappa F_\rho(D)_\rho, L^{\mathrm{od}}_{\kappa,\rho}\big) = \frac{1}{2}\big(\operatorname{Sig}(L^{\mathrm{od}}_{\kappa,\rho}) - \operatorname{Sig}(D_\rho)\big)$$

by Definition 1.1.3. As $D_\rho = \operatorname{diag}(D_{0,\rho}, -D_{0,\rho})$ fulfills $\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} D_\rho \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = -D_\rho$, its signature vanishes. This implies the claim. □

10.5 The η-invariant of the spectral localizer

Section 7.2 provided the definition of the η-invariant, as well as the motivation behind it, as a measure of the spectral asymmetry of an invertible self-adjoint operator, see Definition 7.2.1. Here the latter operator will be the spectral localizer L_κ associated to an index pairing. It is the first aim of this section to show that the η-invariant of the spectral localizer is well defined under suitable supplementary assumptions and that it can be computed as the finite-volume half-signature, as expected. Furthermore, a connection between η-invariant and spectral flow will be established and this provides yet another spectral flow approach to index theory.

In the following, it will be necessary to associate a spatial dimension to a Fredholm module. This will first be done via trace class property of the resolvent of the Dirac operator which, as shown in Lemma 10.5.3 below, is tightly connected to a short time asymptotics of the associated heat kernel. Later on we will also consider a spinorial dimension of the Dirac operator.

Definition 10.5.1. Let D be the Dirac operator of a Fredholm module, hence $D = D^{\mathrm{ev}}$ as in (10.1) or $D = D^{\mathrm{od}}$ as in (10.4). Then D is said to be (at least) of dimension d if, for every $\epsilon > 0$,

$$\operatorname{Tr}\big(|D|^{-d-\epsilon}\big) < \infty.$$

Then D and the associated Fredholm module is also called $(d+\epsilon)$-summable.

Note that every $(d+\epsilon)$-summable Fredholm module is also θ-summable in the sense of Definition 10.2.2. Indeed, θ-summability is a considerably weaker condition.

Example 10.5.2. Let D be the Dirac operator on the torus $\mathbb{T}^d$ given in Examples 10.1.3 and 10.1.8. Then $D^2 = -\sum_{j=1}^d \partial_j^2$ is the Laplacian. By Fourier transform, one sees that the eigenvalues of D^2 are $|n|^2 = \sum_{j=1}^d n_j^2$ for $n = (n_1, \dots, n_d)$. Eliminating $n = 0$ (by going to 2×2 matrices and adding a mass as in Remark 10.1.2), one then readily checks that D is of dimension d in the sense of Definition 10.5.1.

Lemma 10.5.3. *Let D be the Dirac operator of a Fredholm module satisfying*

$$\mathrm{Tr}\big(e^{-tD^2}\big) \le Ct^{-\frac{d}{2}}. \tag{10.22}$$

Then D is of dimension d, namely $(d+\epsilon)$-summable for all $\epsilon > 0$.

Proof. Let us first use functional calculus to rewrite

$$|D|^{-d-\epsilon} = \frac{1}{\Gamma(\frac{d+\epsilon}{2})} \int_0^\infty dt\, t^{\frac{d+\epsilon}{2}-1} e^{-tD^2}.$$

Hence splitting $e^{-tD^2} = e^{-\frac{t}{2}D^2} e^{-\frac{t}{2}D^2}$ and using that $D^2 \ge g^2$ for some positive g, one deduces from the hypothesis

$$\mathrm{Tr}\big(|D|^{-d-\epsilon}\big) \le \frac{C}{\Gamma(\frac{d+\epsilon}{2})} \int_0^\infty dt\, t^{\frac{d+\epsilon}{2}-1} \left(\frac{t}{2}\right)^{-\frac{d}{2}} e^{-\frac{t}{2}g^2},$$

and thus the integral is finite. □

If (10.22) holds, then Lemma 7.2.3 applied to $H_0 = D$ and $V = 0$ implies that

$$\mathrm{Tr}\big(|D|^\alpha e^{-tD^2}\big) \le C_\alpha t^{-\frac{d+\alpha}{2}}, \quad \alpha \ge 0. \tag{10.23}$$

Let us now first note that Dirac operators have well-defined, albeit uninteresting η-invariants.

Proposition 10.5.4. *Let D be an even or odd Dirac operator given by* (10.1) *or* (10.4) *satisfying* (10.22). *Then the η-invariant exists and vanishes, $\eta(D) = 0$.*

Proof. We will use (10.23) for $\alpha = 1$. For both an even and odd Dirac operator D, one has $\mathrm{Tr}(|D|e^{-tD^2}) = \mathrm{Tr}(|D_0|e^{-t|D_0|^2} + |D_0^*|e^{-t|D_0^*|^2}) < \infty$ so that $D_0 e^{-t|D_0|^2}$ and $D_0^* e^{-t|D_0^*|^2}$ are trace class. Hence one can compute, for the odd Dirac operator,

$$\mathrm{Tr}\big(De^{-tD^2}\big) = \mathrm{Tr}\big(D_0 e^{-tD_0^2} - D_0 e^{-tD_0^2}\big) = 0,$$

and, for even Dirac operator given by (10.1), the diagonal vanishes, so that the trace also vanishes. Consequently, also the η-invariant given by (7.4) vanishes. □

For the proof of Proposition 10.5.4, merely the particular structure of the Dirac operator given as a 2×2 matrix in (10.1) and (10.4) was of importance. If D is the Dirac operator of a spin manifold, one has further expressions in D and its heat kernel that have a vanishing trace. Let us show this explicitly for the example of a Dirac operator on a torus already analyzed in Examples 10.1.3 and 10.1.8.

Example 10.5.5. Let D_0 on $L^2(\mathbb{T}^d, \mathbb{C}^{d'})$ be given in terms of the irreducible Clifford algebra representation $\gamma_1, \dots, \gamma_d$ just as in (10.2) and (10.6). Recall that $\gamma_1, \dots, \gamma_d$ are constructed iteratively using 2×2 Pauli matrices and therefore, in particular, have vanishing trace. Hence for a bounded operator A acting as the identity on $\mathbb{C}^{d'}$, one has

$$\mathrm{Tr}(D_0 e^{-t|D_0|^2} A) = \imath \sum_{j=1}^{d} \mathrm{Tr}(\gamma_j \partial_j e^{-t|D_0|^2} A) = 0,$$

because already the partial trace of $\mathbb{C}^{d'}$ vanishes. Using further algebraic properties of the Clifford algebra representation, it is possible to show that also other traces involving several heat kernels as factors vanish, but this is not further developed here. ◇

The aim in the following is to show that the η-invariant of the spectral localizer exists, at least for low-dimensional Fredholm modules. We will deal with both the even and odd case simultaneously and simply write L_κ instead of L_κ^{ev} and L_κ^{od}, and similarly D for D^{ev} and D^{od}.

Theorem 10.5.6. *Let $d = 1, 2, 3$ and suppose that the Dirac operator satisfies* (10.22). *Further, suppose that the first condition in* (10.10) *or* (10.17) *holds. Then the spectral localizer L_κ defined in* (10.9) *and* (10.16) *has a well-defined η-invariant in the following cases:*
(i) $d = 1$;
(ii) $d = 2$ *and D_0 normal;*
(iii) $d = 3$ *and* $\mathrm{Tr}(D_0 e^{-tD_0^2} A) = 0$ *for any $t > 0$ and bounded operator A acting as the identity on the spinorial part $\mathbb{C}^{d'}$ of the Hilbert space.*

The first task of the proof will be to deduce heat kernel estimates for L_κ from those for D.

Lemma 10.5.7. *Suppose that* (10.22) *holds. Set $V = L_\kappa - \kappa D$. Then there are $C'_\alpha > 0$ such that*

$$\|(L_\kappa)^\alpha e^{-tL_\kappa^2}\|_1 \leq C'_\alpha e^{t\|V\|^2} t^{-\frac{d+\alpha}{2}}, \quad \alpha \geq 0. \tag{10.24}$$

Moreover, for $r \geq 1$, there are $C'_{\alpha,r} > 0$ such that

$$\|(L_\kappa)^\alpha e^{-tL_\kappa^2}\|_{\frac{1}{r}} \leq C'_{\alpha,r} e^{t\|V\|^2} t^{-\frac{d}{2}r-\frac{\alpha}{2}}, \quad \alpha > 0.$$

Proof. This follows directly from Lemma 7.2.3 applied to $H_0 = \kappa D$ and $V = L_\kappa - \kappa D$, combined with the bound (10.22). □

For the remainder of the proof, the square of the localizer will be used. Let us write

$$L_\kappa^2 = \Delta + \mathcal{V},$$

where $\Delta = \kappa^2 D^2$ and in the even, respectively odd, case

$$\mathcal{V} = \begin{pmatrix} H^2 & \kappa[D_0^*, H] \\ \kappa[H, D_0] & H^2 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} AA^* & \kappa[D_0, A] \\ \kappa[D_0, A]^* & A^*A \end{pmatrix}.$$

Then $\Delta > 0$ and, by the hypothesis in Theorem 10.5.6, $\mathcal{V}$ is a bounded operator. Let us note that Δ is even with respect to $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in the grading of $\mathcal{V}$, namely $\Delta\sigma_3 = \sigma_3\Delta$. Due to the boundedness of $\mathcal{V}$, DuHamel's formula holds [163]:

$$e^{-tL_\kappa^2} = e^{-t\Delta} - t \int_0^1 dr\, e^{-(1-r)t\Delta} \mathcal{V} e^{-rtL_\kappa^2}. \tag{10.25}$$

Proof of Theorem 10.5.6. We will use the representation (7.3) of the η-function $\eta_s(L_\kappa)$ in terms of the heat kernel of L_κ^2 and split it into three summands

$$\eta(L_\kappa, s) = \frac{1}{\Gamma(\frac{s+1}{2})} \big(\eta'(L_\kappa) + \eta''(L_\kappa) + \eta'''(L_\kappa)\big)$$

with

$$\eta'(L_\kappa) = \int_0^1 dt\, t^{\frac{s-1}{2}}\, \mathrm{Tr}\big(\kappa D e^{-tL_\kappa^2}\big),$$

$$\eta''(L_\kappa) = \int_0^1 dt\, t^{\frac{s-1}{2}}\, \mathrm{Tr}\big((L_\kappa - \kappa D) e^{-tL_\kappa^2}\big),$$

$$\eta'''(L_\kappa) = \int_1^\infty dt\, t^{\frac{s-1}{2}}\, \mathrm{Tr}\big(L_\kappa e^{-tL_\kappa^2}\big).$$

It will be shown below that it is relatively straightforward to bound $\eta'''(L_\kappa)$ by using that L_κ is invertible. It turns out to be more challenging to bound the two other terms because the bounds on the heat kernel given in Lemma 10.5.7 are insufficient due to the nonintegrable singularity at $t = 0$. Beneath these two terms, $\eta'(L_\kappa)$ seems to be more singular due to the presence of the unbounded operator D.

Let us start out with $\eta'(L_\kappa)$. Replacing DuHamel's formula (10.25) leads to

$$\eta'(L_\kappa) = \kappa \int_0^1 dt\, t^{\frac{s-1}{2}} \left(\mathrm{Tr}\big(De^{-t\Delta}\big) + t \int_0^1 dr\, \mathrm{Tr}\big(De^{-(1-r)t\Delta} \mathcal{V} e^{-rtL_\kappa^2}\big) \right)$$

$$= \kappa \int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\, \mathrm{Tr}\big(De^{-(1-r)t\Delta} \mathcal{V} e^{-rtL_\kappa^2}\big), \tag{10.26}$$

because

$$\mathrm{Tr}\big(De^{-t\Delta}\big) = \mathrm{Tr}\big(De^{-t\kappa^2 D^2}\big) = 0.$$

For d odd, this holds due to the symmetry of the spectrum of $D = D_0 \otimes \sigma_3$, or because D and Δ are odd and even with respect to σ_1, respectively. For d even, it follows directly from the fact that D is off-diagonal and $e^{-t\Delta}$ diagonal.

For $d = 1$, one can bound (10.26) directly by using the Cauchy–Schwarz inequality as follows:

$$\begin{aligned}
\big|\mathrm{Tr}\big(\kappa De^{-(1-r)t\Delta} \mathcal{V} e^{-rtL_\kappa^2}\big)\big| &\le \mathrm{Tr}\big(\Delta e^{-2(1-r)t\Delta}\big)^{\frac{1}{2}}\, \mathrm{Tr}\big(\mathcal{V}^* \mathcal{V} e^{-2rtL_\kappa^2}\big)^{\frac{1}{2}} \\
&\le \|\mathcal{V}\|\, \mathrm{Tr}\big(\Delta e^{-2(1-r)t\Delta}\big)^{\frac{1}{2}}\, \mathrm{Tr}\big(e^{-2rtL_\kappa^2}\big)^{\frac{1}{2}} \\
&\le C\big((1-r)t\big)^{-\frac{3}{4}} (rt)^{-\frac{1}{4}},
\end{aligned} \tag{10.27}$$

due to Lemma 10.5.7, so that

$$\eta'(L_\kappa) \le \int_0^1 dt\, t^{\frac{s-1}{2}} \int_0^1 dr\, C(1-r)^{-\frac{3}{4}} r^{-\frac{1}{4}} < \infty,$$

as long as $s > -1$, so in particular for $s = 0$.

For $d > 1$, one has to further expand the heat kernel of L_κ using the DuHamel's formula. Hence let us again substitute (10.25) into (10.26):

$$\begin{aligned}
\eta'(L_\kappa) = &-\kappa \int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\, \mathrm{Tr}\big(De^{-(1-r)t\Delta} \mathcal{V} e^{-rt\Delta}\big) \\
&- \kappa \int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\, (rt) \int_0^1 dr'\, \mathrm{Tr}\big(De^{-(1-r)t\Delta} \mathcal{V} e^{-(1-r')rt\Delta} \mathcal{V} e^{-r'rtL_\kappa^2}\big) \\
= &-\kappa \int_0^1 dt\, t^{\frac{s+1}{2}}\, \mathrm{Tr}\big(De^{-t\Delta} \mathcal{V}\big) \\
&- \kappa \int_0^1 dt\, t^{\frac{s+3}{2}} \int_0^1 dr\, r \int_0^1 dr'\, \mathrm{Tr}\big(De^{-(1-r)t\Delta} \mathcal{V} e^{-(1-r')rt\Delta} \mathcal{V} e^{-r'rtL_\kappa^2}\big).
\end{aligned} \tag{10.28}$$

The lowest-order term has to be dealt with separately for an even and odd Fredholm module. For d even, one finds using the normality of D_0 that

$$\mathrm{Tr}(De^{-t\Delta}\mathcal{V}) = \kappa\,\mathrm{Tr}\big((D_0^* e^{-t\kappa^2 D_0 D_0^*}[H, D_0] + D_0 e^{-t\kappa^2 D_0^* D_0}[D_0^*, H]\big)\big),$$

which vanishes due to the normality of D_0. For odd d, one finds

$$\mathrm{Tr}(De^{-t\Delta}\mathcal{V}) = \mathrm{Tr}(D_0 e^{-t\kappa^2 D_0^2}(A^*A - AA^*)),$$

which vanishes because the spinorial degrees of freedom of D_0 have a vanishing trace due to the hypothesis for $d = 3$. Hence remains to bound the double integral in (10.28). This will be possible for dimension $d = 2$ and $d = 3$, but not for $d > 3$ for which again DuHamel's formula has to be replaced. For $d = 1$, it was sufficient to bound the integrand of the remainder by the Cauchy–Schwarz inequality, see (10.27). Here one rather has to use the multiple Hölder inequality and then the bound of Lemma 10.5.7:

$$\begin{aligned}
&\big|\mathrm{Tr}(De^{-(1-r)t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V}e^{-r'rtL_\kappa^2})\big| \\
&\quad = \big|\mathrm{Tr}(e^{-r'rtL_\kappa^2}D\Delta^{-\alpha}\Delta^{\alpha}e^{-(1-r)t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V})\big| \\
&\quad \le \|\Delta^{\frac{1}{2}-\alpha}e^{-r'rtL_\kappa^2}\|_{\frac{1}{q_1}}\|\Delta^{\alpha}e^{-(1-r)t\Delta}\|_{\frac{1}{q_2}}\|e^{-(1-r')rt\Delta}\|_{\frac{1}{q_3}}\|\mathcal{V}\|^2\kappa^{-1} \\
&\quad \le C(r'rt)^{-(\frac{d}{2}q_1+\frac{1}{2}-\alpha)}((1-r)t)^{-(\frac{d}{2}q_2+\alpha)}((1-r')rt)^{-\frac{d}{2}q_3} \\
&\quad = Ct^{-(\frac{d}{2}+\frac{1}{2})}r^{-(\frac{d}{2}(q_1+q_3)+(\frac{1}{2}-\alpha))}(1-r)^{-(\frac{d}{2}q_2+\alpha)}(r')^{-(\frac{d}{2}q_1+\frac{1}{2}-\alpha)}(1-r')^{-\frac{d}{2}q_3},
\end{aligned}$$

where $q_1 + q_2 + q_3 = 1$. Note that the bound in t is sufficient to bound the integral over t in (10.28) as long as $d \le 3$. Now to insure integrability in r at 0 and 1, as well as in r' at 0 and 1, one needs respectively

$$\frac{d}{2}(q_1 + q_3) - \alpha < \frac{3}{2}, \quad \frac{d}{2}q_2 + \alpha < 1, \quad \frac{d}{2}q_1 + \frac{1}{2} - \alpha < 1, \quad \frac{d}{2}q_3 < 1.$$

One can hence choose $\alpha = \frac{1}{4}$ and $q_1 = q_2 = \frac{1}{4}$ so that $q_3 = \frac{1}{2}$ to obtain

$$\begin{aligned}
&\big|\mathrm{Tr}(De^{-(1-r)t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V}e^{-r'rtL_\kappa^2})\big| \\
&\quad \le Ct^{-(\frac{d}{2}+\frac{1}{2})}r^{-\frac{3d+2}{8}}((1-r)r')^{-(\frac{d}{4}+\frac{1}{8})}(1-r')^{-\frac{d}{4}}.
\end{aligned}$$

Replacing this into (10.28) shows that all integrals converge for $d \le 3$ and $s = 0$.

Next turning to $\eta''(L_\kappa)$, one starts out as in (10.26) to find

$$\eta''(L_\kappa) = \int_0^1 dt\, t^{\frac{s-1}{2}}\left(\mathrm{Tr}((L_\kappa - \kappa D)e^{-t\Delta}) - t\int_0^1 dr\,\mathrm{Tr}((L_\kappa - \kappa D)e^{-(1-r)t\Delta}\mathcal{V}e^{-rtL_\kappa^2})\right).$$

The first summand vanishes. For odd d, this results from the fact that $L_\kappa - \kappa D$ is off-diagonal while $e^{-t\Delta}$ is diagonal. For even d, $(L_\kappa - \kappa D)e^{-t\Delta}$ is diagonal, but, due to the normality of D_0, one diagonal entry is minus the second one so that the trace vanishes. Hence

$$\eta''(L_\kappa) = -\int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\, \mathrm{Tr}\big((L_\kappa - \kappa D)e^{-(1-r)t\Delta}\mathcal{V}e^{-rtL_\kappa^2}\big). \tag{10.29}$$

Now set $H' = L_\kappa - \kappa D$ which for odd d means $H = H'$ and for even d rather $H' = -H \otimes \sigma_3$. For dimension $d = 2$, it is sufficient to bound the integrand with the Cauchy–Schwarz inequality and Lemma 10.5.7:

$$\big|\mathrm{Tr}\big(H'e^{-(1-r)t\Delta}\mathcal{V}e^{-rtL_\kappa^2}\big)\big| \le \|H'\|\|\mathcal{V}\|C\big((1-r)rt^2\big)^{-\frac{d}{4}}$$

because replacing in (10.29) shows that $\eta''(L_\kappa)$ is finite for $s = 0$. For $d \ge 3$, it is again necessary to replace DuHamel's formula

$$\begin{aligned}\eta''(L_\kappa) = &-\int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\, \mathrm{Tr}\big(H'e^{-(1-r)t\Delta}\mathcal{V}e^{-rt\Delta}\big)\\ &-\int_0^1 dt\, t^{\frac{s+1}{2}} \int_0^1 dr\,(rt) \int_0^1 dr'\, \mathrm{Tr}\big(H'e^{-(1-r)t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V}e^{-r'rtL_\kappa^2}\big).\end{aligned}$$

For $d = 3$, the integrand in the leading term vanishes because

$$\mathrm{Tr}\big(He^{-t\Delta}\mathcal{V}e^{-t'\Delta}\big) = \mathrm{Tr}\big(A^*e^{-t\kappa^2D_0^2}[D_0, A]e^{-t'\kappa^2D_0^2} + Ae^{-t\kappa^2D_0^2}[A^*, D_0]e^{-t'\kappa^2D_0^2}\big)$$

and the trace vanishes due to the spinorial degrees of freedom of D_0. Then the second summand can be bounded as the term in (10.28) (actually it is less singular here).

For $\eta'''(L_\kappa)$ and hence large t, the estimate (10.24) is of little help. It has to be boosted by using the gap of L_κ. Suppose that $L_\kappa^2 \ge \epsilon$, a lower bound that holds for $\epsilon = \frac{g^2}{2}$. Then, for any $\alpha \in (0, 1)$, by Cauchy–Schwarz inequality,

$$\begin{aligned}\mathrm{Tr}\big(L_\kappa e^{-tL_\kappa^2}\big)^2 &\le \mathrm{Tr}\big(L_\kappa^2 e^{-2\alpha tL_\kappa^2}\big)\,\mathrm{Tr}\big(e^{-2(1-\alpha)tL_\kappa^2}\big)\\ &\le (2\alpha t)^{-1}\,\mathrm{Tr}\big(e^{-\alpha tL_\kappa^2}\big)\big\|e^{-2(1-2\alpha)tL_\kappa^2}\big\|\,\mathrm{Tr}\big(e^{-2\alpha tL_\kappa^2}\big),\end{aligned}$$

where the bound $xe^{-xt} \le t^{-1}e^{-\frac{xt}{2}}$ for $x, t > 0$ was used. Hence with (10.24),

$$\begin{aligned}\mathrm{Tr}\big(L_\kappa e^{-tL_\kappa^2}\big) &\le (2\alpha t)^{-\frac{1}{2}}e^{-(1-2\alpha)t\epsilon}\,\mathrm{Tr}\big(e^{-\alpha tL_\kappa^2}\big)\\ &\le (2\alpha t)^{-\frac{1}{2}}e^{-(1-2\alpha)t\epsilon}e^{\alpha t\|\mathcal{V}\|^2}C(\alpha t)^{-\frac{d}{2}}.\end{aligned}$$

Choosing $\alpha \le \frac{1}{4}\min\{1, \frac{\epsilon}{\|\mathcal{V}\|^2}\}$, one infers that, for some constant C''' depending on ϵ,

$$\mathrm{Tr}(L_\kappa e^{-tL_\kappa^2}) \le C''' e^{-\frac{t\epsilon}{4}}.$$

Hence also $\eta'''(L_\kappa)$ is bounded, actually for all s. □

Remark 10.5.8. Let us briefly indicate how to address the existence of the η-invariant for the spectral localizer of pairings with a Dirac operator that satisfies (10.23) for some $d > 3$, namely to extend Theorem 10.5.6 to this case. First of all, the contribution $\eta'''(L_\kappa)$ in the proof of Theorem 10.5.6 can be dealt with in the same manner, but the expressions $\eta'(L_\kappa)$ and $\eta''(L_\kappa)$ require more care. Let us focus on $\eta'(L_\kappa)$. Starting from (10.28) it is then necessary to replace once again DuHamel's formula to obtain

$$\begin{aligned}\eta'(L_\kappa) = &\int_0^1 dt\, t^{\frac{s+3}{2}} \int_0^1 dr\, r \int_0^1 dr'\, \kappa\, \mathrm{Tr}\big(De^{-(1-r(1-r'))t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V}\big)\\ &+ \int_0^1 dt\, t^{\frac{s+5}{2}} \int_0^1 dr\, r^2 \int_0^1 dr'\, r' \int_0^1 dr''\\ &\cdot \kappa\, \mathrm{Tr}\big(De^{-(1-r)t\Delta}\mathcal{V}e^{-(1-r')rt\Delta}\mathcal{V}e^{-(1-r'')r'rt\Delta}\mathcal{V}e^{-r''r'rtL_\kappa^2}\big).\end{aligned}$$

Again the first summand has to be shown to vanish for d even and d odd separately, by imposing supplementary conditions on D_0. Such conditions should, in particular, hold for the example of the Dirac operator on the torus, see Example 10.5.5. When this is achieved, a multiple Hölder inequality then allows us to control the term with the triple integral in dimension $d \le 5$. For $d > 5$, yet another iteration is needed. The term $\eta''(L_\kappa)$ can then be handled in a similar manner. Just as in Example 10.5.5, no further algebraic details are provided here. ◇

Next the general connection between the η-invariants and spectral flow as described in Section 7.2 will be applied to the particular case of a spectral localizer. This shows that the spectral asymmetry of the spectral localizers is acquired by a spectral flow while the Hamiltonian (namely K-theoretic part of the index pairing) is added to the free spectral localizer which is essentially given by the Dirac operator. For $d = 1$, this connection was first proved in [128].

Theorem 10.5.9. *Suppose that all hypothesis of Theorem* 10.5.6 *hold. Consider the path* $\lambda \in [0,1] \mapsto L_\kappa(\lambda) = (1-\lambda)\kappa D + \lambda L_\kappa$ *of self-adjoint operators with compact resolvents. Then*

$$\eta(L_\kappa) = 2\,\mathrm{Sf}(\lambda \in [0,1] \mapsto L_\kappa(\lambda)).$$

Proof. Because $\eta(L_\kappa(0)) = \eta(D) = 0$, the claim follows from Theorem 7.2.2 once it is shown that

$$\lim_{\epsilon\downarrow 0} \frac{\epsilon^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \int_0^1 d\lambda\, \mathrm{Tr}\big(\partial_\lambda L_\kappa(\lambda) e^{-\epsilon L_\kappa(\lambda)^2}\big) = 0. \tag{10.30}$$

One hence needs to control $\mathrm{Tr}(\partial_\lambda L_\kappa(\lambda) e^{-\epsilon L_\kappa(\lambda)^2})$. This can be done exactly as the bound on the term $\eta''(L_\kappa)$ in the proof of Theorem 10.5.6, the only difference being the supplementary integral in $\eta''(L_\kappa)$. Further details are not spelled out. □

The previous result implies that the η-invariant of the spectral localizer is an integer, provided the conditions of Theorem 10.5.6 hold. Finally, let us note that it is actually equal to the finite-volume signature and hence the value of the index pairing.

Proposition 10.5.10. *Suppose that all hypothesis of Theorem* 10.5.6 *hold. If, moreover, ρ is as in* (10.10) *or* (10.17)*, then*

$$\eta(L_\kappa) = \mathrm{Sig}(L_{\kappa,\rho}).$$

Proof. By the definition of the finite-dimensional spectral flow, see Definition 1.1.3, one has

$$\mathrm{Sig}(L_{\kappa,\rho}) = \mathrm{Sig}(L_{\kappa,\rho}) - \mathrm{Sig}(D_\rho) = 2\,\mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa D_\rho + \lambda L_{\kappa,\rho}\big).$$

Comparing this to Theorem 10.5.9 shows that it is sufficient to prove

$$\mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa D_\rho + \lambda L_{\kappa,\rho}\big) = \mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa D + \lambda L_\kappa\big).$$

Let us first show that $(1-\lambda)\kappa D_{\rho^c} + \lambda L_{\kappa,\rho^c} \in \mathbb{L}((\mathcal{H}\oplus\mathcal{H})_{\rho^c})$ is invertible for all $\lambda \in [0,1]$ and ρ sufficiently large. For the even spectral localizer, one directly checks

$$\begin{aligned}
&\big((1-\lambda)\kappa D_{\rho^c} + \lambda L_{\kappa,\rho^c}\big)^2 \\
&\quad = \kappa^2 D_{\rho^c}^2 + \lambda^2 (L_{\kappa,\rho^c} - \kappa D_{\rho^c})^2 + \lambda\kappa \begin{pmatrix} 0 & D_0^* H - H D_0^* \\ H D_0 - D_0 H & 0 \end{pmatrix} \\
&\quad \geq \kappa^2\rho^2 - \kappa \big\| [D_0, H] \big\|.
\end{aligned}$$

Thus for ρ sufficiently large, one has

$$\big((1-\lambda)\kappa D_{\rho^c} + \lambda L_{\kappa,\rho^c}\big)^2 \geq \frac{1}{2}\kappa^2\rho^2. \tag{10.31}$$

A similar argument shows that (10.31) holds for the odd spectral localizer if ρ is sufficiently large. Then $(1-\lambda)\kappa D_{\rho^c} + \lambda L_{\kappa,\rho^c}$ is invertible for all $\lambda \in [0,1]$, and therefore

$$\mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa D_{\rho^c} + \lambda L_{\kappa,\rho^c}\big) = 0.$$

Hence the additivity of the spectral flow, see item (v) of Theorem 7.1.7, implies

$$\begin{aligned}
&\mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa D_\rho + \lambda L_{\kappa,\rho}\big) \\
&\quad = \mathrm{Sf}\big(\lambda \in [0,1] \mapsto (1-\lambda)\kappa (D_\rho \oplus D_{\rho^c}) + \lambda (L_{\kappa,\rho} \oplus L_{\kappa,\rho^c})\big).
\end{aligned}$$

The linear homotopy $h : [0,1] \times [0,1] \to \mathbb{L}(\mathcal{H} \oplus \mathcal{H})$ defined by

$$h(\lambda, s) = (1-\lambda)\kappa D + \lambda(L_{\kappa,\rho} \oplus L_{\kappa,\rho^c}) + s\lambda\big(\pi_{\rho^c}(L_\kappa - \kappa D)(\pi_\rho)^* + \pi_\rho(L_\kappa - \kappa D)(\pi_{\rho^c})^*\big)$$

connects

$$\lambda \in [0,1] \mapsto h(\lambda, 0) = (1-\lambda)\kappa(D_\rho \oplus D_{\rho^c}) + \lambda(L_{\kappa,\rho} \oplus L_{\kappa,\rho^c})$$

to the path

$$\lambda \in [0,1] \mapsto h(\lambda, 0) = (1-\lambda)\kappa D + \lambda L_\kappa.$$

Moreover, $h(0,s) = D$ is invertible and an argument similar to that in the proofs of Theorems 10.3.1 and 10.4.1 shows that

$$h(1, s) = (L_{\kappa,\rho} \oplus L_{\kappa,\rho^c}) + s\big(\pi_{\rho^c}(L_\kappa - \kappa D)(\pi_\rho)^* + \pi_\rho(L_\kappa - \kappa D)(\pi_{\rho^c})^*\big)$$

is invertible for all $s \in [0,1]$ for ρ sufficiently large. Thus the claim follows from the homotopy invariance of the spectral flow, see Theorem 7.1.8, provided it is shown that h is gap-continuous. To show the gap-continuity, note that

$$(\lambda, s) \in [0,1] \times [0,1] \mapsto H(\lambda, s) = h(\lambda, s) - \kappa D$$

is bounded and norm-continuous. By Theorem 6.3.12, the gap metric on $\mathbb{L}_{\text{sa}}(\mathcal{H} \oplus \mathcal{H})$ is equivalent to the metric d_G'' defined in (6.19). For $(\lambda, s), (\lambda', s') \in [0,1] \times [0,1]$, one has

$$\begin{aligned} d_G''(h(\lambda,s) - h(\lambda',s')) &= 2\big\|(h(\lambda,s) + \imath\mathbf{1})^{-1} - (h(\lambda',s') + \imath\mathbf{1})^{-1}\big\| \\ &= 2\big\|(h(\lambda,s) + \imath\mathbf{1})^{-1}(h(\lambda',s') - h(\lambda,s))(h(\lambda',s') + \imath\mathbf{1})^{-1}\big\|, \end{aligned}$$

due to the resolvent identity. Because $\|(h(\lambda,s) + \imath\mathbf{1})^{-1}\| \le 1$ for $(\lambda,s) \in [0,1] \times [0,1]$ by the spectral radius theorem, one concludes that h is gap-continuous. □

11 Spectral flow in semifinite von Neumann algebras

In this chapter the theory of Fredholm operators in a von Neumann algebra $\mathcal{N}$ with respect to a semifinite, normal, faithful trace $\mathcal{T}$ is developed and then used to introduce a spectral flow for paths of self-adjoint $\mathcal{T}$-Fredholm operators, generalizing the more conventional spectral flow studied in prior chapters. This semifinite spectral flow is, in general, not integer-valued and measures, e. g., how much possibly absolutely continuous spectral density flows through 0 from left to right. The notion of semifinite spectral flow goes back to the work of Perera [144] and Phillips [148], and there are numerous later contributions [26, 197, 110, 93]. In Section 11.1, some basic facts about von Neumann algebras and traces thereon are reviewed. All of these facts can be found in the textbooks [72, 189]. Following the Appendix of [149] and [40, 41], $\mathcal{T}$-Fredholm operators (often also called Breuer–Fredholm operators) are introduced and then the generalization of Atkinson's theorem is proved. In Section 11.2, this is generalized to skew-corner Fredholm operators (also called $(P \cdot Q)$-Fredholm operators for orthogonal projections $P, Q \in \mathcal{N}$), following [54]. Sections 11.1 and 11.2 both provide considerably more detailed arguments than the references [149, 54] which at several places are sketchy or incomplete. Section 11.3 discusses semifinite Fredholm pairs of projections and generalizes many of the results of Chapter 5. The spectral flow for paths of self-adjoint $\mathcal{T}$-Fredholm operators is then defined in Section 11.4 by closely following [26]. Several basic properties of the semifinite spectral flow are shown. They are all generalizations of results from Chapter 4. Finally, Section 11.5 is about index formulas for the semifinite spectral flow, and Section 11.6 generalizes the concept and results on the spectral localizer from Chapter 10. This is based on [170].

11.1 Fredholm operators in semifinite von Neumann algebras

Let $\mathcal{N}$ be a subset of $\mathbb{B}(\mathcal{H})$. Its commutant $\mathcal{N}'$ is the algebra

$$\mathcal{N}' = \{B \in \mathbb{B}(\mathcal{H}) : AB = BA \ \forall A \in \mathcal{N}\}.$$

If $\mathcal{N}$ is self-adjoint (namely invariant under taking adjoints), this also holds for $\mathcal{N}'$. The bicommutant $\mathcal{N}'' = (\mathcal{N}')'$ of $\mathcal{N}$ is the commutant of $\mathcal{N}'$.

Definition 11.1.1. A unital self-adjoint subalgebra $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$ is called a von Neumann algebra if it coincides with its bicommutant, namely $\mathcal{N} = \mathcal{N}''$.

Let us recall that the weak operator topology on $\mathbb{B}(\mathcal{H})$ is the coarsest topology for which all functions

$$f_{\phi,\psi} : \mathbb{B}(\mathcal{H}) \to \mathbb{C}, \quad A \mapsto \langle \phi | A\psi \rangle, \quad \phi, \psi \in \mathcal{H},$$

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are continuous. Moreover, the strong operator topology on $\mathbb{B}(\mathcal{H})$ is the topology of pointwise convergence, namely the coarsest topology for which all maps

$$f_\phi : \mathbb{B}(\mathcal{H}) \to \mathcal{H}, \quad A \mapsto A\phi, \quad \phi \in \mathcal{H},$$

are continuous. The von Neumann bicommutant theorem, see [72, 189], states the following:

Theorem 11.1.2. *Let $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$ be a unital self-adjoint subalgebra, then the closure of $\mathcal{N}$ in the weak and strong operator topology both coincide with $\mathcal{N}''$.*

In particular, every von Neumann algebra is closed with respect to the weak and strong operator topology. For a von Neumann algebra $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$, let

$$\mathcal{N}_+ = \{A \in \mathcal{N} : A \geq 0\}$$

denote the set of positive semidefinite elements in $\mathcal{N}$. Also recall that the set $\mathbb{P}(\mathcal{N})$ of all projections in $\mathcal{N}$ is a complete lattice, namely given a family $(P_i)_{i\in I}$ of orthogonal projections in $\mathcal{N}$, also $\inf_{i\in I} P_i$ is an orthogonal projection in $\mathcal{N}$ onto the closed subspace $\bigcap_{i\in I} \mathrm{Ran}(P_i)$. Then $\sup_{i\in I} P_i = \mathbf{1} - \inf_{i\in I}(\mathbf{1} - P_i)$ is also in $\mathcal{N}$. Now all notions needed for the definition of semifinite, faithful, and normal traces are available.

Definition 11.1.3. Let $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra. A map $\mathcal{T} : \mathcal{N}_+ \to [0, \infty]$ is called a trace if it is positive additive and unitarily invariant, namely satisfies

$$\mathcal{T}(a_0 A_0 + a_1 A_1) = a_0 \mathcal{T}(A_0) + a_1 \mathcal{T}(A_1),$$

for all $A_0, A_1 \in \mathcal{N}_+$ and $a_0, a_1 \in \mathbb{R}_+$, and

$$\mathcal{T}(U^* A U) = \mathcal{T}(A)$$

for all unitary elements $U \in \mathcal{N}$ and all positive elements $A \in \mathcal{N}_+$.

(i) A trace $\mathcal{T} : \mathcal{N}_+ \to [0, \infty]$ is called semifinite if for every orthogonal projection $P \in \mathcal{N}$ there exists an increasing net of orthogonal projections $(P_i)_{i\in I}$ in $\mathcal{N}$ such that $\mathcal{T}(P_i) < \infty$ and such that $P = \sup_{i\in I} P_i$.
(ii) A trace $\mathcal{T} : \mathcal{N}_+ \to [0, \infty]$ is said to be faithful if $\mathcal{T}(A) = 0$ implies $A = 0$ for all $A \in \mathcal{N}_+$.
(iii) A semifinite trace $\mathcal{T} : \mathcal{N}_+ \to [0, \infty]$ is called normal if for any increasing net $(A_i)_{i\in I}$ with $A_i \in \mathcal{N}_+$ with supremum $A = \sup_{i\in I} A_i \in \mathcal{N}_+$, one has

$$\mathcal{T}(A) = \sup_{i\in I} \mathcal{T}(A_i).$$

If $A \in \mathcal{N}$ can be decomposed as $A = \sum_{n=1}^{N} a_n A_n$ for some $N \in \mathbb{N}$ with $a_n \in \mathbb{C}$ and $A_n \in \mathcal{N}_+$ such that $\mathcal{T}(A_n) < \infty$ for all $n = 1, \dots, N$, then A is called $\mathcal{T}$-finite and its trace is defined as

$$\mathcal{T}(A) = \sum_{n=1}^{N} a_n \mathcal{T}(A_n).$$

One directly checks that this is independent of the decomposition of A. Moreover, the set of $\mathcal{T}$-finite elements of $\mathcal{N}$ is a two-sided ideal in $\mathcal{N}$ and (see Chapter 6 of Part 1 in [72], or Section V.2 in [189]) the trace is cyclic in the sense that

$$\mathcal{T}(AB) = \mathcal{T}(BA),$$

whenever $A, B \in \mathcal{N}$ and A is $\mathcal{T}$-finite. One can readily check that the canonical trace on $\mathcal{H}$ is a semifinite, faithful, and normal trace on the von Neumann algebra $\mathbb{B}(\mathcal{H})$.

From now on let $\mathcal{N}$ be a von Neumann algebra with a semifinite faithful normal trace $\mathcal{T}$. We introduce the notion of compact operators and of Fredholm operators with respect to $\mathcal{T}$. Let $\mathcal{K}$ denote the norm-closure of the smallest algebraic ideal in $\mathcal{N}$ containing the $\mathcal{T}$-finite projections. This is called the ideal of $\mathcal{T}$-compact operators in $\mathcal{N}$ (in part of the literature, this is denoted by $\mathcal{K}_{\mathcal{T}}$ to stress the dependence on $\mathcal{T}$). It is known that any projection in $\mathcal{K}$ is $\mathcal{T}$-finite. Associated to $\mathcal{K}$ is a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{N} \to \mathcal{N}/\mathcal{K} \to 0.$$

The quotient $\mathcal{Q} = \mathcal{N}/\mathcal{K}$ is called the Calkin algebra and the quotient map is denoted by π.

In this chapter a lot of orthogonal projections appear. They are denoted by P, Q, R, and F. Thus, in this chapter Q is an orthogonal projection and not a symmetry, and F is an orthogonal projection and not an arbitrary element of $\mathbb{B}(\mathcal{H})$. Moreover, the following notation will be used:

Notation. *For any subspace $\mathcal{E}$ of $\mathcal{H}$, the orthogonal projection onto the closure of $\mathcal{E}$ is denoted by $P_{\mathcal{E}}$.*

Hence $\mathrm{Ran}(P_{\mathcal{E}}) = \overline{\mathcal{E}}$. For example, with this notation one has, for any orthogonal projection P,

$$P = P_{\mathrm{Ran}(P)} = P_{\mathrm{Ker}(P)^{\perp}} = \mathbf{1} - P_{\mathrm{Ker}(P)}.$$

As in this chapter all projections are orthogonal, we drop the specification “orthogonal” and simply speak of projections.

Definition 11.1.4. An operator $T \in \mathcal{N}$ is $\mathcal{T}$-Fredholm (or Breuer–Fredholm relative to $\mathcal{T}$) if $P_{\mathrm{Ker}(T)}$ is $\mathcal{T}$-finite and there is a $\mathcal{T}$-finite projection $P \in \mathcal{N}$ such that $\mathrm{Ran}(\mathbf{1}-P) \subset \mathrm{Ran}(T)$. Its $\mathcal{T}$-index is defined by

$$\mathcal{T}\text{-Ind}(T) = \mathcal{T}(P_{\mathrm{Ker}(T)}) - \mathcal{T}(P_{\mathrm{Ker}(T^*)}).$$

The set of $\mathcal{T}$-Fredholm operators will be denoted by $\mathbb{F}(\mathcal{N}, \mathcal{T})$.

Note that $\mathrm{Ran}(\mathbf{1} - P) \subset \mathrm{Ran}(T)$ implies $\mathrm{Ker}(T^*) = \mathrm{Ran}(T)^\perp \subset \mathrm{Ran}(P)$ and therefore $\mathcal{T}(\mathrm{Ker}(T^*))$ is $\mathcal{T}$-finite and the $\mathcal{T}$-index of T is a real number.

Remark 11.1.5. If $\mathcal{N} = \mathbb{B}(\mathcal{H})$ is equipped with the canonical trace Tr, which is a semifinite, faithful, normal trace, then the condition in Definition 11.1.4 implies that $\mathrm{Ran}(T)$ is a subspace of finite codimension and hence, in particular, a closed subspace. Therefore $\mathbb{F}(\mathcal{N}, \mathcal{T})$ is precisely the set $\mathbb{FB}(\mathcal{H})$ of bounded Fredholm operators studied in Section 3.2. Let us stress that in general Fredholm operators in the sense of Definition 11.1.4 need not have a closed range. ◇

The following result generalizes Atkinson's theorem, which is made up of parts of Theorem 3.2.4, Corollary 3.3.2, and Theorem 3.3.4.

Theorem 11.1.6. *Let $\mathcal{N} \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra with a semifinite faithful normal trace $\mathcal{T}$.*
(i) *Let $K \in \mathcal{K}$ be a $\mathcal{T}$-compact operator in $\mathcal{N}$. Then $\mathbf{1} - K$ is $\mathcal{T}$-Fredholm and*

$$\mathcal{T}\text{-Ind}(\mathbf{1} - K) = 0.$$

(ii) *$T \in \mathcal{N}$ is a $\mathcal{T}$-Fredholm operator if and only if the image $\pi(T)$ of T in $\mathcal{Q}$ is invertible.*
(iii) *If T and S are $\mathcal{T}$-Fredholm operators in $\mathcal{N}$, then so are T^* and ST. Their $\mathcal{T}$-indices fulfill*

$$\mathcal{T}\text{-Ind}(T^*) = -\mathcal{T}\text{-Ind}(T) \quad \textit{and} \quad \mathcal{T}\text{-Ind}(ST) = \mathcal{T}\text{-Ind}(S) + \mathcal{T}\text{-Ind}(T).$$

The proof is based on several lemmas and will take up the remainder of this section, except for Corollary 11.1.13 below that contains important supplementary information on semifinite Fredholm operators.

Lemma 11.1.7. *For $A \in \mathcal{N}$, the range projection $P_{\mathrm{Ran}(A)}$ is in $\mathcal{N}$, and one has*

$$\mathcal{T}(P_{\mathrm{Ran}(A)}) = \mathcal{T}(P_{\mathrm{Ran}(A^*)}).$$

Proof. Let $A = V|A|$ be the polar decomposition of A in the sense of von Neumann. Let us first check that V is an element of $\mathcal{N}''$ and therefore also of $\mathcal{N}$. First of all, as $A, A^* \in \mathcal{N}$, also $A^*A \in \mathcal{N}$ and, by writing the square root as a series, also $|A| \in \mathcal{N}$. Thus for any $B \in \mathcal{N}'$, one has $0 = [B, A] = [B, V]|A| + V[B, |A|]$ so that $[B, V]|A| = 0$. As B leaves $\mathrm{Ker}(A) = \mathrm{Ker}(V)$ invariant, this implies that $[B, V] = 0$, namely $V \in \mathcal{N}''$. One then also has $VV^* = P_{\mathrm{Ran}(A)} \in \mathcal{N}$ and $V^*V = P_{\mathrm{Ran}(A^*)} \in \mathcal{N}$. If $\mathcal{T}(VV^*) < \infty$, then also $V = VV^*V$ is $\mathcal{T}$-finite and therefore $\mathcal{T}(V^*V) < \infty$. Hence one can assume that both $\mathcal{T}(P_{\mathrm{Ran}(A)})$ and $\mathcal{T}(P_{\mathrm{Ran}(A^*)})$ are $\mathcal{T}$-finite because otherwise the claim is trivial. If the trace class condition holds, then the cyclicity of the trace implies

$$\mathcal{T}(P_{\mathrm{Ran}(A^*)}) = \mathcal{T}(V^*V) = \mathcal{T}(VV^*) = \mathcal{T}(P_{\mathrm{Ran}(A)}),$$

concluding the proof. □

The following is also known as the semifinite parallelogram law, see Proposition V.1.6 in [189].

Lemma 11.1.8. *For projections* $P, Q \in \mathcal{N}$*, one has* $\inf(P,Q) \in \mathcal{N}$ *and*

$$\mathcal{T}(P - \inf(P, \mathbf{1} - Q)) = \mathcal{T}(Q - \inf(Q, \mathbf{1} - P)).$$

Proof. As $\mathrm{Ker}(PQ) = \mathrm{Ker}(Q) \oplus (\mathrm{Ran}(Q) \cap \mathrm{Ker}(P))$, one has

$$\begin{aligned} P_{\mathrm{Ran}(QP)} &= P_{\mathrm{Ker}(PQ)^\perp} \\ &= \mathbf{1} - P_{\mathrm{Ker}(PQ)} \\ &= \mathbf{1} - P_{\mathrm{Ker}(Q)} - P_{\mathrm{Ran}(Q)\cap\mathrm{Ker}(P)} \\ &= Q - \inf(Q, \mathbf{1} - P). \end{aligned}$$

Because $P_{\mathrm{Ran}(QP)}$ is in $\mathcal{N}$ by Lemma 11.1.7, this implies that $\inf(Q, \mathbf{1} - P)$ is in $\mathcal{N}$. Replacing P by $\mathbf{1} - P$ shows $\inf(P,Q) \in \mathcal{N}$ for all projections $P, Q \in \mathcal{N}$. Similarly, as one has $\mathrm{Ker}(QP) = \mathrm{Ker}(P) \oplus (\mathrm{Ran}(P) \cap \mathrm{Ker}(Q))$, it follows that

$$P_{\mathrm{Ran}(PQ)} = P_{\mathrm{Ker}(QP)^\perp} = P - \inf(P, \mathbf{1} - Q).$$

The claim now follows from Lemma 11.1.7. □

Lemma 11.1.9. *For every* $T \in \mathcal{N}$*, there is a nondecreasing sequence* $(P_n)_{n\in\mathbb{N}}$ *of projections in* $\mathcal{N}$ *such that*

(i) $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ *for all* $n \in \mathbb{N}$,

(ii) $\sup_n P_n = P_{\mathrm{Ran}(T)}$.

Proof. Because $\mathrm{Ran}(T) \supset \mathrm{Ran}(TT^*)$ and $\overline{\mathrm{Ran}(T)} = \mathrm{Ker}(T^*)^\perp = \mathrm{Ker}(TT^*)^\perp = \overline{\mathrm{Ran}(TT^*)}$, it is sufficient to prove the lemma for $T \geq 0$. Then let the projection-valued spectral resolution of T be denoted by $(E_\lambda)_{\lambda\in\mathbb{R}}$, namely

$$T = \int_0^\infty \lambda \, dE_\lambda.$$

One has $E_\lambda \in \mathcal{N}$ for all $\lambda \in [0, \infty)$ by Appendix 1 of [72]. Moreover,

$$P_{\mathrm{Ker}(T)} = E_0, \quad P_{\mathrm{Ran}(T)} = \mathbf{1} - E_0 = \sup_{\epsilon>0}(\mathbf{1} - E_\epsilon).$$

For $\epsilon > 0$, the restriction of T to the range of $\mathbf{1} - E_\epsilon$ is denoted by T_ϵ. Then the operator $T_\epsilon : \mathrm{Ran}(\mathbf{1} - E_\epsilon) \to \mathrm{Ran}(\mathbf{1} - E_\epsilon)$ is invertible with inverse

$$T_\epsilon^{-1} = \int_\epsilon^\infty \frac{1}{\lambda} dE_\lambda.$$

Therefore $\mathrm{Ran}(\mathbf{1} - E_\epsilon) \subset \mathrm{Ran}(T)$. Setting $P_n = \mathbf{1} - E_{\frac{1}{n}}$, the sequence $(P_n)_{n\in\mathbb{N}}$ satisfied the properties stated in the lemma. □

Corollary 11.1.10. *For every $\mathcal{T}$-Fredholm operator $T \in \mathbb{F}(\mathcal{N}, \mathcal{T})$, there is a nondecreasing sequence $(P_n)_{n\in\mathbb{N}}$ of projections in $\mathcal{N}$ such that*
(i) $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ *for all $n \in \mathbb{N}$,*
(ii) $\sup_n P_n = P_{\mathrm{Ran}(T)}$,
(iii) $\mathcal{T}(\mathbf{1} - P_1) < \infty$.

Proof. Because T is $\mathcal{T}$-Fredholm, there exists a $\mathcal{T}$-finite projection $P \in \mathcal{N}$ such that $\mathrm{Ran}(\mathbf{1} - P) \subset \mathrm{Ran}(T)$. Applying Lemma 11.1.9 to PT, one gets a nondecreasing sequence $(P'_n)_{n\in\mathbb{N}}$ of projections with $\mathrm{Ran}(P'_n) \subset \mathrm{Ran}(PT)$ and $\sup_n P'_n = P_{\mathrm{Ran}(PT)}$. As $T = (\mathbf{1}-P)T + PT$ and $\mathrm{Ran}((\mathbf{1}-P)T) \subset \mathrm{Ran}(T)$ by hypothesis, one has $\mathrm{Ran}(PT) \subset \mathrm{Ran}(T)$. Therefore setting $P_n = (\mathbf{1} - P) + P'_n$, one has $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ for all $n \in \mathbb{N}$ and $\sup_n P_n = (\mathbf{1} - P) + P_{\mathrm{Ran}(PT)} = P_{\mathrm{Ran}(T)}$. Moreover, $\mathcal{T}(\mathbf{1} - P_1) \le \mathcal{T}(P) < \infty$. □

Lemma 11.1.11. *Let $(P_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of projections in $\mathcal{N}$ and Q a projection in $\mathcal{N}$. If $\mathcal{T}(\sup_n P_n) < \infty$, then*

$$\mathcal{T}\Big(\inf\Big(\sup_n P_n, Q\Big)\Big) = \mathcal{T}\Big(\sup_n(\inf(P_n, Q))\Big).$$

Proof. The supremum of the sequence is denoted by $P_\infty = \sup_n P_n$. For $\phi \in \mathcal{H}$, the limit $\lim_{n\to\infty} \|P_n\phi\| = \sup_n \|P_n\phi\|$ exists so that $P_\infty = \text{s-}\lim_{n\to\infty} P_n \in \mathcal{N}$. By Lemma 11.1.8,

$$\mathcal{T}(\mathbf{1} - Q - \inf(\mathbf{1} - Q, \mathbf{1} - P_n)) = \mathcal{T}(P_n - \inf(P_n, Q)) \tag{11.1}$$

for all $n \in \mathbb{N}$ and

$$\mathcal{T}(\mathbf{1} - Q - \inf(\mathbf{1} - Q, \mathbf{1} - P_\infty)) = \mathcal{T}(P_\infty - \inf(P_\infty, Q)). \tag{11.2}$$

As $P_n \le P_\infty$ for all $n \in \mathbb{N}$,

$$\mathbf{1} - Q - \inf(\mathbf{1} - Q, \mathbf{1} - P_n) \le \mathbf{1} - Q - \inf(\mathbf{1} - Q, \mathbf{1} - P_\infty).$$

Thus, by (11.1) and (11.2),

$$\mathcal{T}(P_n - \inf(P_n, Q)) \le \mathcal{T}(P_\infty - \inf(P_\infty, Q)).$$

Using the additivity (and thus monotonicity) of the trace, one gets

$$0 \le \mathcal{T}\big(\inf(P_\infty, Q) - \inf(P_n, Q)\big) \le \mathcal{T}(P_\infty - P_n).$$

By the normality of the trace,

$$\lim_{n\to\infty} \mathcal{T}(P_\infty - P_n) = 0,$$

and therefore

$$\lim_{n\to\infty} \mathcal{T}\big(\inf(P_\infty, Q) - \inf(P_n, Q)\big) = 0.$$

Hence

$$\begin{aligned}\mathcal{T}\big(\inf(P_\infty, Q)\big) &= \lim_{n\to\infty} \mathcal{T}\big(\inf(P_n, Q)\big)\\ &= \sup_n \mathcal{T}\big(\inf(P_n, Q)\big)\\ &= \mathcal{T}\Big(\sup_n\big(\inf(P_n, Q)\big)\Big),\end{aligned}$$

where the last step follows form the normality of the trace. □

Lemma 11.1.12. *For $\mathcal{T}$-Fredholm operators $S, T \in \mathbb{F}(\mathcal{N}, \mathcal{T})$, one has*

$$\mathcal{T}(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}) = \mathcal{T}\big(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})\big).$$

Proof. One has, with $\oplus$ and $\ominus$ being orthogonal direct sums and differences,

$$\begin{aligned}\mathrm{Ker}(TP_{\mathrm{Ker}(ST)} - TP_{\mathrm{Ker}(T)}) &= \mathrm{Ker}(TP_{\mathrm{Ker}(ST)})\\ &= \mathrm{Ker}(ST)^\perp \oplus \mathrm{Ker}(T)\\ &= \big(\mathrm{Ker}(ST) \ominus \mathrm{Ker}(T)\big)^\perp,\end{aligned}$$

where in the second equality we used $\mathrm{Ker}(AP) = \mathrm{Ker}(P) \oplus (\mathrm{Ran}(P) \cap \mathrm{Ker}(A))$ for any operator $A \in \mathbb{B}(\mathcal{H})$ and any orthogonal projection $P \in \mathbb{B}(\mathcal{H})$, here applied to $A = T$ and $P = P_{\mathrm{Ker}(ST)}$ so that $\mathrm{Ker}(P) = \mathrm{Ker}(ST)^\perp$ and $\mathrm{Ran}(P) = \mathrm{Ker}(ST) \supset \mathrm{Ker}(T)$. Therefore

$$\begin{aligned}\overline{\mathrm{Ran}\big((P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)})T^*\big)} &= \mathrm{Ker}(TP_{\mathrm{Ker}(ST)} - TP_{\mathrm{Ker}(T)})^\perp\\ &= \mathrm{Ker}(ST) \ominus \mathrm{Ker}(T).\end{aligned}$$

This implies

$$P_{\mathrm{Ran}((P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)})T^*)} = P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}. \tag{11.3}$$

On the other hand, using

$$\mathrm{Ran}(TP_{\mathrm{Ker}(ST)}) = \{T\phi : ST\phi = 0\} = \mathrm{Ran}(T) \cap \mathrm{Ker}(S),$$

one has

$$\overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)} - TP_{\mathrm{Ker}(T)})} = \overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)})}$$

$$= \overline{\mathrm{Ran}(T) \cap \mathrm{Ker}(S)}$$
$$\subset \overline{\mathrm{Ran}(T)} \cap \mathrm{Ker}(S),$$

and therefore

$$P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}))} \leq \inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)}). \tag{11.4}$$

Let $(P_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of projections in $\mathcal{N}$ as in Corollary 11.1.10, namely such that $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ for all $n \in \mathbb{N}$ and such that $\sup_n P_n = P_{\mathrm{Ran}(T)}$ and such that the trace of $\mathbf{1} - P_1$ is finite. Then

$$\mathrm{Ran}(P_n) \cap \mathrm{Ker}(S) \subset \mathrm{Ran}(T) \cap \mathrm{Ker}(S) \subset \overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)} - TP_{\mathrm{Ker}(T)})}$$

for all $n \in \mathbb{N}$ and therefore

$$\inf(P_n, P_{\mathrm{Ker}(S)}) \leq P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}))}. \tag{11.5}$$

Setting $P_0 = \inf(P_1, \mathbf{1} - P_{\mathrm{Ker}(S)})$, one has

$$\begin{aligned}
\mathcal{T}(\mathbf{1} - P_0) &= \mathcal{T}(\mathbf{1} - \inf(P_1, \mathbf{1} - P_{\mathrm{Ker}(S)})) \\
&= \mathcal{T}(P_{\mathrm{Ker}(S)} + \mathbf{1} - P_{\mathrm{Ker}(S)} - \inf(P_1, \mathbf{1} - P_{\mathrm{Ker}(S)})) \\
&= \mathcal{T}(P_{\mathrm{Ker}(S)}) + \mathcal{T}(\mathbf{1} - P_1 - \inf(P_{\mathrm{Ker}(S)}, \mathbf{1} - P_1)) \\
&= \mathcal{T}(P_{\mathrm{Ker}(S)}) + \mathcal{T}(\mathbf{1} - P_1) \\
&< \infty,
\end{aligned}$$

where in the third step Lemma 11.1.8 was used. Moreover, $P_0 P_{\mathrm{Ker}(S)} = 0$ and $P_0 \leq P_n$ for all $n \in \mathbb{N}$. It follows that

$$\inf(P_n, P_{\mathrm{Ker}(S)}) = \inf(P_n - P_0, P_{\mathrm{Ker}(S)})$$

for all $n \in \mathbb{N}$ and, as $\mathrm{Ran}(P_0) \subset \mathrm{Ran}(T)$,

$$\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)}) = \inf(P_{\mathrm{Ran}(T)} - P_0, P_{\mathrm{Ker}(S)}).$$

As $\mathcal{T}(\sup_n(P_n - P_0)) \leq \mathcal{T}(\mathbf{1} - P_0) < \infty$, Lemma 11.1.11 applies and one gets, together with the above,

$$\begin{aligned}
\mathcal{T}\Big(\sup_n(\inf(P_n, P_{\mathrm{Ker}(S)}))\Big) &= \mathcal{T}\Big(\sup_n(\inf(P_n - P_0, P_{\mathrm{Ker}(S)}))\Big) \\
&= \mathcal{T}\Big(\inf\Big(\sup_n(P_n - P_0), P_{\mathrm{Ker}(S)}\Big)\Big) \\
&= \mathcal{T}(\inf(P_{\mathrm{Ran}(T)} - P_0, P_{\mathrm{Ker}(S)})) \\
&= \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})).
\end{aligned}$$

Using the normality of the trace, (11.4) and (11.5), this implies

$$\mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})) = \mathcal{T}(P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}))}).$$

Finally, using first (11.3) and then Lemma 11.1.7 implies

$$\begin{aligned}\mathcal{T}(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}) &= \mathcal{T}(P_{\mathrm{Ran}((P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)})T^*)}) \\ &= \mathcal{T}(P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}))}) \\ &= \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})).\end{aligned}$$

This concludes the proof. □

Proof of Theorem 11.1.6. Let us first suppose that the range projection $P_{\mathrm{Ran}(K)}$ of K is $\mathcal{T}$-finite. By Lemma 11.1.7, one then has

$$\mathcal{T}(P_{\mathrm{Ran}(K^*)}) = \mathcal{T}(P_{\mathrm{Ran}(K)}) < \infty.$$

For

$$Q = \sup(P_{\mathrm{Ran}(K)}, P_{\mathrm{Ran}(K^*)}) \in \mathcal{N},$$

one has $Q \leq P_{\mathrm{Ran}(K)} + P_{\mathrm{Ran}(K^*)}$ and therefore $\mathcal{T}(Q) \leq \mathcal{T}(P_{\mathrm{Ran}(K)}) + \mathcal{T}(P_{\mathrm{Ran}(K^*)}) < \infty$. As $\mathrm{Ran}(K) \subset \mathrm{Ran}(Q)$ and $\mathrm{Ran}(K^*) \subset \mathrm{Ran}(Q)$,

$$(\mathbf{1} - Q)(\mathbf{1} - K) = \mathbf{1} - Q, \quad (\mathbf{1} - Q)(\mathbf{1} - K^*) = \mathbf{1} - Q.$$

As $\mathrm{Ker}(Q) \subset \mathrm{Ker}(K)$ and $\mathrm{Ker}(Q) \subset \mathrm{Ker}(K^*)$ (or alternatively by taking adjoints), one also has

$$(\mathbf{1} - K)(\mathbf{1} - Q) = \mathbf{1} - Q, \quad (\mathbf{1} - K^*)(\mathbf{1} - Q) = \mathbf{1} - Q.$$

This implies, in particular,

$$P_{\mathrm{Ker}(\mathbf{1}-K)} \leq Q, \quad P_{\mathrm{Ker}(\mathbf{1}-K^*)} \leq Q.$$

Therefore $P_{\mathrm{Ker}(\mathbf{1}-K)}$ and $P_{\mathrm{Ker}(\mathbf{1}-K^*)}$ are $\mathcal{T}$-finite. On the other hand, $\mathrm{Ran}(\mathbf{1}-Q) \subset \mathrm{Ran}(\mathbf{1}-K)$ and $\mathcal{T}(Q)$ is finite by assumption. Putting all together, this shows that $\mathbf{1}-K$ is $\mathcal{T}$-Fredholm. It remains to show that its $\mathcal{T}$-index vanishes. By Lemma 11.1.7,

$$\mathcal{T}(P_{\mathrm{Ran}(Q-K)}) = \mathcal{T}(P_{\mathrm{Ran}(Q-K^*)}). \tag{11.6}$$

Again using $\mathrm{Ran}(K^*) \subset \mathrm{Ran}(Q)$ and $\mathrm{Ker}(Q) \subset \mathrm{Ker}(K^*)$, one has

$$\overline{\mathrm{Ran}(Q - K)} = \mathrm{Ker}(Q - K^*)^{\perp} = (\mathrm{Ker}(Q) \oplus \mathrm{Ker}(\mathbf{1} - K^*))^{\perp},$$

and therefore

$$P_{\mathrm{Ran}(Q-K)} = \mathbf{1} - (P_{\mathrm{Ker}(Q)} + P_{\mathrm{Ker}(\mathbf{1}-K^*)}) = Q - P_{\mathrm{Ker}(\mathbf{1}-K^*)}.$$

Analogously,

$$P_{\mathrm{Ran}(Q-K^*)} = Q - P_{\mathrm{Ker}(\mathbf{1}-K)}.$$

Using the linearity of $\mathcal{T}$ on $\mathcal{T}$-finite operators and (11.6), this implies

$$\mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K)}) = \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K^*)}). \tag{11.7}$$

One concludes that the $\mathcal{T}$-index of $\mathbf{1} - K$ vanishes.

For a $\mathcal{T}$-compact operator $K \in \mathcal{K}$ with infinite range projection, there exists by definition an operator $K_0 \in \mathcal{K}$ such that $\|K - K_0\| < 1$ and such that the range projection $P_{\mathrm{Ran}(K_0)}$ of K_0 is $\mathcal{T}$-finite. Then

$$S = \mathbf{1} - (K - K_0)$$

is invertible. As $(S - K_0)P_{\mathrm{Ker}(S-K_0)} = 0$ and therefore

$$(\mathbf{1} - K_0S^{-1})SP_{\mathrm{Ker}(S-K_0)} = 0,$$

one has

$$\mathrm{Ran}(SP_{\mathrm{Ker}(S-K_0)}) \subset \mathrm{Ker}(\mathbf{1} - K_0S^{-1}).$$

As $\mathcal{T}(P_{\mathrm{Ran}(SP_{\mathrm{Ker}(S-K_0)})}) = \mathcal{T}(P_{\mathrm{Ran}(P_{\mathrm{Ker}(S-K_0)}S^*)}) = \mathcal{T}(P_{\mathrm{Ker}(S-K_0)})$ by Lemma 11.1.7, this implies

$$\mathcal{T}(P_{\mathrm{Ker}(S-K_0)}) \le \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})}).$$

Similarly, $(\mathbf{1} - K_0S^{-1})SS^{-1}P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})} = 0$ implies

$$\mathrm{Ran}(S^{-1}P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})}) \subset \mathrm{Ker}(S - K_0)$$

and therefore, using again Lemma 11.1.7,

$$\mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})}) \le \mathcal{T}(P_{\mathrm{Ker}(S-K_0)}).$$

Putting all together, one can conclude that

$$\mathcal{T}(P_{\mathrm{Ker}(S-K_0)}) = \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})}) < \infty.$$

As $\mathrm{Ker}(S^* - K_0^*) = \mathrm{Ker}(\mathbf{1} - (S^*)^{-1}K_0^*)$ and the range projection of K_0S^{-1} is $\mathcal{T}$-finite, one can use (11.7) to conclude

$$\begin{aligned}\mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K)}) &= \mathcal{T}(P_{\mathrm{Ker}(S-K_0)})\\ &= \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K_0S^{-1})})\\ &= \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-(S^{-1})^*K_0^*)})\\ &= \mathcal{T}(P_{\mathrm{Ker}(S^*-K_0^*)})\\ &= \mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K^*)}) < \infty.\end{aligned}$$

Moreover, $P_{\mathrm{Ran}((S^*)^{-1}K_0^*)}$ is $\mathcal{T}$-finite and

$$\begin{aligned}(\mathbf{1}-K)S^{-1}(\mathbf{1}-P_{\mathrm{Ran}((S^*)^{-1}K_0^*)}) &= (S-K_0)S^{-1}(\mathbf{1}-P_{\mathrm{Ran}((S^*)^{-1}K_0^*)})\\ &= (\mathbf{1}-K_0S^{-1})P_{\mathrm{Ker}(K_0S^{-1})}\\ &= P_{\mathrm{Ker}(K_0S^{-1})}\\ &= \mathbf{1}-P_{\mathrm{Ran}((S^*)^{-1}K_0^*)}.\end{aligned}$$

This shows that $\mathrm{Ran}(\mathbf{1}-P_{\mathrm{Ran}((S^*)^{-1}K_0^*)}) \subset \mathrm{Ran}(\mathbf{1}-K)$, showing that indeed $\mathbf{1}-K$ is $\mathcal{T}$-Fredholm because $P_{\mathrm{Ker}(\mathbf{1}-K)}$ is $\mathcal{T}$-finite. By the above equality, its $\mathcal{T}$-index indeed vanishes.

To show (ii), let us first suppose that $\pi(T)$ is invertible. Then there is $S \in \mathcal{N}$ such that

$$\pi(S)\pi(T) = \pi(T)\pi(S) = \pi(\mathbf{1}),$$

or equivalently

$$ST = \mathbf{1}-K_1, \quad TS = \mathbf{1}-K_2, \tag{11.8}$$

for $\mathcal{T}$-compact operators $K_1, K_2 \in \mathcal{K}$. Thus $\mathrm{Ker}(T) \subset \mathrm{Ker}(\mathbf{1}-K_1)$ by the first equation in (11.8) and therefore $P_{\mathrm{Ker}(T)} \le P_{\mathrm{Ker}(\mathbf{1}-K_1)}$. As $\mathbf{1}-K_1$ is $\mathcal{T}$-Fredholm by the first part of this theorem and therefore $\mathcal{T}(P_{\mathrm{Ker}(\mathbf{1}-K_1)}) < \infty$, one can conclude that $\mathcal{T}(P_{\mathrm{Ker}(T)})$ is finite. The second equation in (11.8) implies that $\mathrm{Ran}(\mathbf{1}-K_2)$ is a subset of $\mathrm{Ran}(T)$. Since $\mathbf{1}-K_2$ is $\mathcal{T}$-Fredholm, there is a $\mathcal{T}$-finite projection $P \in \mathcal{N}$ such that the range of $\mathbf{1}-P$ is a subset of the range of $\mathbf{1}-K_2$. Consequently,

$$\mathrm{Ran}(\mathbf{1}-P) \subset \mathrm{Ran}(T)$$

and T is $\mathcal{T}$-Fredholm. For the converse, assume that T is $\mathcal{T}$-Fredholm. Then there is a $\mathcal{T}$-finite projection $P \in \mathcal{N}$ such that $\mathrm{Ran}(\mathbf{1}-P) \subset \mathrm{Ran}(T)$. By Lemma 11.1.12,

$$\mathcal{T}(P_{\mathrm{Ker}((\mathbf{1}-P)T)}) - \mathcal{T}(P_{\mathrm{Ker}(T)}) = \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P)).$$

In particular, $P_{\mathrm{Ker}((\mathbf{1}-P)T)}$ is $\mathcal{T}$-finite. As $\mathrm{Ran}(\mathbf{1}-P) \subset \mathrm{Ker}(T^*)^\perp$, one has

$$\mathrm{Ker}(T^*(\mathbf{1}-P)) = \mathrm{Ker}(\mathbf{1}-P)$$

and, due to $\operatorname{Ran}(\mathbf{1}-P) \subset \operatorname{Ran}(T)$, one has

$$\operatorname{Ran}\big((\mathbf{1}-P)T\big) = \operatorname{Ran}(\mathbf{1}-P).$$

Therefore $\operatorname{Ran}(T^*(\mathbf{1}-P)) = \overline{\operatorname{Ran}(T^*(\mathbf{1}-P))} = \operatorname{Ker}((\mathbf{1}-P)T)^{\perp}$ and

$$(\mathbf{1}-P)TT^*(\mathbf{1}-P)$$

maps $\operatorname{Ran}(\mathbf{1}-P)$ bijectively onto $\operatorname{Ran}(\mathbf{1}-P)$. Thus there is $T' \in \mathcal{N}$ such that

$$(\mathbf{1}-P)TT^*(\mathbf{1}-P)T' = \mathbf{1}-P,$$

and thus

$$\pi(T)\pi(T^*T') = \pi\big((\mathbf{1}-P)TT^*(\mathbf{1}-P)T'\big) = \pi(\mathbf{1}-P) = \pi(\mathbf{1}).$$

On the other hand,

$$\operatorname{Ker}\big((\mathbf{1}-P)T(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})\big) = \operatorname{Ker}(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})$$

and

$$\operatorname{Ran}\big((\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})T^*(\mathbf{1}-P)\big) = \operatorname{Ran}(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)}),$$

so that

$$(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})T^*(\mathbf{1}-P)T(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})$$

maps $\operatorname{Ran}(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})$ bijectively onto $\operatorname{Ran}(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})$. Hence there is $T'' \in \mathcal{N}$ such that

$$T''(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})T^*(\mathbf{1}-P)T(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)}) = \mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)}.$$

As above,

$$\begin{aligned}\pi(T''T^*)\pi(T) &= \pi\big(T''(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})T^*(\mathbf{1}-P)T(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})\big)\\ &= \pi(\mathbf{1}-P_{\operatorname{Ker}((\mathbf{1}-P)T)})\\ &= \pi(\mathbf{1}).\end{aligned}$$

One concludes that $\pi(T)$ is invertible.

The fact that for $\mathcal{T}$-Fredholm operators $T, S \in \mathcal{N}$ also T^* and TS are $\mathcal{T}$-Fredholm directly follows from (ii). Also the first equation in (iii) is obvious. To prove the second equation, let us note that, by Lemma 11.1.12,

$$\mathcal{T}(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}) = \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})) \tag{11.9}$$

and

$$\mathcal{T}(P_{\mathrm{Ker}((ST)^*)} - P_{\mathrm{Ker}(S^*)}) = \mathcal{T}(\inf(P_{\mathrm{Ran}(S^*)}, P_{\mathrm{Ker}(T^*)})). \tag{11.10}$$

By Lemma 11.1.8,

$$\mathcal{T}(P_{\mathrm{Ker}(S)}) - \mathcal{T}(\inf(\mathbf{1} - P_{\mathrm{Ker}(T^*)}, P_{\mathrm{Ker}(S)})) = \mathcal{T}(P_{\mathrm{Ker}(T^*)}) - \mathcal{T}(\inf(\mathbf{1} - P_{\mathrm{Ker}(S)}, P_{\mathrm{Ker}(T^*)})),$$

or equivalently

$$\mathcal{T}(P_{\mathrm{Ker}(S)}) - \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})) = \mathcal{T}(P_{\mathrm{Ker}(T^*)}) - \mathcal{T}(\inf(P_{\mathrm{Ran}(S^*)}, P_{\mathrm{Ker}(T^*)})). \tag{11.11}$$

Equations (11.9), (11.10), and (11.11) imply

$$\mathcal{T}(P_{\mathrm{Ker}(S)}) - \mathcal{T}(P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}) = \mathcal{T}(P_{\mathrm{Ker}(T^*)}) - \mathcal{T}(P_{\mathrm{Ker}((ST)^*)} - P_{\mathrm{Ker}(S^*)}),$$

which shows that

$$\begin{aligned}\mathcal{T}\text{-Ind}(TS) &= \mathcal{T}(P_{\mathrm{Ker}(ST)}) - \mathcal{T}(P_{\mathrm{Ker}((ST)^*)}) \\ &= \mathcal{T}(P_{\mathrm{Ker}(S)}) + \mathcal{T}(P_{\mathrm{Ker}(T)}) - \mathcal{T}(P_{\mathrm{Ker}(S^*)}) - \mathcal{T}(P_{\mathrm{Ker}(T^*)}) \\ &= \mathcal{T}\text{-Ind}(T) + \mathcal{T}\text{-Ind}(S),\end{aligned}$$

concluding the argument. □

Theorem 11.1.6 implies the following generalization of item (i) and (ii) of Theorem 3.3.4.

Corollary 11.1.13. (i) *With respect to the norm topology, the set* $\mathbb{F}(\mathcal{N}, \mathcal{T})$ *is open in* $\mathcal{N}$ *and the index map* $T \mapsto \mathcal{T}\text{-Ind}(T)$ *is locally constant.*
(ii) *If* $T \in \mathcal{N}$ *is* $\mathcal{T}$*-Fredholm and* $K \in \mathcal{K}$ *is* $\mathcal{T}$*-compact, then* $T + K$ *is also* $\mathcal{T}$*-Fredholm and* $\mathcal{T}\text{-Ind}(T + K) = \mathcal{T}\text{-Ind}(T)$.

Proof. As the set of invertible elements in the Calkin algebra $\mathcal{Q}$ is open with respect to the norm topology and the Calkin quotient map $\pi : \mathcal{N} \to \mathcal{Q}$ is continuous, item (ii) of Theorem 11.1.6 implies that $\mathbb{F}(\mathcal{N}, \mathcal{T})$ is open in $\mathcal{N}$ with respect to the norm topology. For $T \in \mathbb{F}(\mathcal{N}, \mathcal{T})$ and $S \in \mathcal{N}$ such that

$$ST = \mathbf{1} + K_1, \quad TS = \mathbf{1} + K_2$$

with $\mathcal{T}$-compact operators $K_1, K_2 \in \mathcal{K}$, let $A \in \mathcal{N}$ be such that $\|A\| < \|S\|^{-1}$. Then

$$S(T + A) = \mathbf{1} + SA + K_1, \quad (T + A)S = \mathbf{1} + AS + K_2,$$

and $\mathbf{1}+SA$ and $\mathbf{1}+AS$ are invertible with inverse given by a Neumann series and therefore $(\mathbf{1}+SA)^{-1}, (\mathbf{1}+AS)^{-1} \in \mathcal{N}$. Then

$$(\mathbf{1}+SA)^{-1}S(T+A) = \mathbf{1} + (\mathbf{1}+SA)^{-1}K_1$$

and

$$(T+A)S(\mathbf{1}+AS)^{-1} = \mathbf{1} + K_2(\mathbf{1}+AS)^{-1},$$

where $(\mathbf{1}+SA)^{-1}K_1$ and $K_2(\mathbf{1}+AS)^{-1}$ are $\mathcal{T}$-compact. By item (ii) of Theorem 11.1.6, one concludes that $T+A \in \mathbb{F}(\mathcal{N},\mathcal{T})$ is $\mathcal{T}$-Fredholm, and item (i) of the same theorem implies

$$0 = \mathcal{T}\text{-Ind}(ST) = \mathcal{T}\text{-Ind}(S) + \mathcal{T}\text{-Ind}(T)$$

and

$$\begin{aligned} 0 &= \mathcal{T}\text{-Ind}\big((\mathbf{1}+SA)^{-1}S(T+A)\big) \\ &= \mathcal{T}\text{-Ind}\big((\mathbf{1}+SA)^{-1}\big) + \mathcal{T}\text{-Ind}(S) + \mathcal{T}\text{-Ind}(T+A). \end{aligned}$$

As $(\mathbf{1}+SA)^{-1}$ is invertible and therefore $\mathcal{T}\text{-Ind}((\mathbf{1}+SA)^{-1}) = 0$, one can conclude that $\mathcal{T}\text{-Ind}(T) = -\mathcal{T}\text{-Ind}(S) = \mathcal{T}\text{-Ind}(T+A)$. This shows that the index map $T \mapsto \mathcal{T}\text{-Ind}(T)$ is locally constant. The first part of claim (ii), namely that $T+K$ is $\mathcal{T}$-Fredholm for all $T \in \mathbb{F}(\mathcal{N},\mathcal{T})$ and $K \in \mathcal{K}$, directly follows from item (ii) of Theorem 11.1.6. Moreover, because $t \in [0,1] \mapsto T+tK$ is a norm-continuous path of $\mathcal{T}$-Fredholm operators, one has by the first part of this corollary that $t \in [0,1] \mapsto \mathcal{T}\text{-Ind}(T+tK)$ is constant and therefore $\mathcal{T}\text{-Ind}(T+K) = \mathcal{T}\text{-Ind}(T)$. □

11.2 $(P \cdot Q)$-Fredholm operators

For the definition of the semifinite spectral flow in Section 11.4 below, an extension of the concepts and results of Section 11.1 to skew-corners is needed and will be described in this section.

Definition 11.2.1. Associated to projections $P, Q \in \mathcal{N}$, the skew-corner $P\mathcal{N}Q$ consists of operators in $\mathcal{N}$ vanishing on $\text{Ran}(Q)^{\perp}$ and mapping $\text{Ran}(Q)$ to $\text{Ran}(P)$. Moreover, $P\mathcal{K}Q$ denotes the set of $\mathcal{T}$-compact operators from $P\mathcal{N}Q$.

Let us point out that $T^* \in Q\mathcal{N}P$ if and only if $T \in P\mathcal{N}Q$. Note also that $P\mathcal{N}Q$ is an algebra only if $P = Q$. The most basic example is $PQ \in P\mathcal{N}Q$ and this special case will be further discussed in Section 11.3.

Definition 11.2.2. Let $P, Q \in \mathcal{N}$ be projections and $T \in P\mathcal{N}Q$. Then T is called $(P \cdot Q)$-Fredholm if $\inf(P_{\text{Ker}(T)}, Q)$ and $\inf(P_{\text{Ker}(T^*)}, P)$ are $\mathcal{T}$-finite projections and there exists a $\mathcal{T}$-finite projection $F \in \mathcal{N}$ such that $F \leq P$ and $\mathrm{Ran}(P - F) \subset \mathrm{Ran}(T)$. Its (semifinite skew-corner) index is then defined as

$$\mathcal{T}\text{-Ind}_{(P\cdot Q)}(T) = \mathcal{T}(\inf(P_{\mathrm{Ker}(T)}, Q)) - \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)).$$

If $P = Q = \mathbf{1}$, this reduces to the $\mathcal{T}$-index of Definition 11.1.4. The following is a generalization of Theorem 11.1.6 and Corollary 11.1.13.

Theorem 11.2.3. *Let $P, Q, R \in \mathcal{N}$ be projections and $T \in P\mathcal{N}Q$.*
(i) *T is $(P \cdot Q)$-Fredholm if and only if there exists $S \in Q\mathcal{N}P$ with $TS - P \in P\mathcal{K}P$ and $ST - Q \in Q\mathcal{K}Q$.*
(ii) *If T is $(P \cdot Q)$-Fredholm and $S \in R\mathcal{N}P$ is $(R \cdot P)$-Fredholm, then ST is $(R \cdot Q)$-Fredholm with index*

$$\mathcal{T}\text{-Ind}_{(R\cdot Q)}(ST) = \mathcal{T}\text{-Ind}_{(R\cdot P)}(S) + \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T).$$

(iii) *The set of $(P\cdot Q)$-Fredholm operators is open in $P\mathcal{N}Q$ with respect to the norm topology and the index map $T \mapsto \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T)$ is locally constant.*
(iv) *If T is $(P \cdot Q)$-Fredholm, then $T + K$ is $(P \cdot Q)$-Fredholm for all $K \in P\mathcal{K}Q$ and one has $\mathcal{T}\text{-Ind}_{(P\cdot Q)}(T + K) = \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T)$.*

The following proposition is the first preparation for the proof of Theorem 11.2.3, and it is of independent interest.

Proposition 11.2.4. *Let $P, Q \in \mathcal{N}$ be projections and let $T \in P\mathcal{N}Q$ be $(P \cdot Q)$-Fredholm. Then T^* is $(Q \cdot P)$-Fredholm with index*

$$\mathcal{T}\text{-Ind}_{(Q\cdot P)}(T^*) = -\mathcal{T}\text{-Ind}_{(P\cdot Q)}(T).$$

Let $T = V|T|$ be the polar decomposition of T in the sense of von Neumann, then V is $(P \cdot Q)$-Fredholm with index given by

$$\mathcal{T}\text{-Ind}_{(P\cdot Q)}(V) = \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T).$$

Moreover, $|T|$ is $(Q \cdot Q)$-Fredholm with vanishing index $\mathcal{T}\text{-Ind}_{(Q\cdot Q)}(|T|) = 0$.

Proof. Let us first recall that $VV^* = P_{\mathrm{Ran}(T)}$ and $V^*V = P_{\mathrm{Ran}(T^*)}$. Furthermore,

$$\mathrm{Ran}(V) = \overline{\mathrm{Ran}(T)}, \quad \mathrm{Ran}(V^*) = \overline{\mathrm{Ran}(T^*)},$$

and

$$\mathrm{Ker}(V) = \mathrm{Ker}(T), \quad \mathrm{Ker}(V^*) = \mathrm{Ker}(T^*).$$

Since T is $(P \cdot Q)$-Fredholm,

$$\mathcal{T}\big(\inf(P_{\mathrm{Ker}(T)},Q)\big)=\mathcal{T}\big(\inf(P_{\mathrm{Ker}(V)},Q)\big)$$

and

$$\mathcal{T}\big(\inf(P_{\mathrm{Ker}(T^*)},P)\big)=\mathcal{T}\big(\inf(P_{\mathrm{Ker}(V^*)},P)\big)$$

are finite. Moreover, there is as $\mathcal{T}$-finite projection $F\in\mathcal{N}$ such that $F\le P$ and such that $\mathrm{Ran}(P-F)\subset\mathrm{Ran}(T)\subset\mathrm{Ran}(V)$. Altogether, this shows that V is $(P\cdot Q)$-Fredholm with index $\mathcal{T}\text{-}\mathrm{Ind}_{(P\cdot Q)}(V)=\mathcal{T}\text{-}\mathrm{Ind}_{(P\cdot Q)}(T)$.

Furthermore, as $T^*\in Q\mathcal{N}P$ and $\mathrm{Ran}(V)\subset\mathrm{Ran}(P)$ so that $V^*V=V^*PV$,

$$Q=P_{\mathrm{Ran}(T^*)}+\inf(P_{\mathrm{Ker}(T)},Q)=V^*FV+V^*(P-F)V+\inf(P_{\mathrm{Ker}(T)},Q),$$

where $\mathcal{T}(V^*FV)=\mathcal{T}(FVV^*)\le\mathcal{T}(F)$ is finite and $\inf(P_{\mathrm{Ker}(T)},Q)$ is $\mathcal{T}$-finite by assumption. Therefore $\widetilde{F}=V^*FV+\inf(P_{\mathrm{Ker}(T)},Q)\le Q$ is a $\mathcal{T}$-finite projection. One has

$$\mathrm{Ran}\big(Q-\widetilde{F}\big)\subset\mathrm{Ran}\big(V^*(P-F)\big)\subset\mathrm{Ran}\big(V^*T\big)=\mathrm{Ran}(|T|)=\mathrm{Ran}\big(T^*\big),$$

where in the second step $\mathrm{Ran}(P-F)\subset\mathrm{Ran}(T)$ was used. As $\inf(P_{\mathrm{Ker}(|T|)},Q)=\inf(P_{\mathrm{Ker}(T)},Q)$ is $\mathcal{T}$-finite by assumption and $|T|\in Q\mathcal{N}Q$ is self-adjoint, this implies that $|T|$ is $(Q\cdot Q)$-Fredholm with vanishing index. Finally, the existence of $\widetilde{F}$, combined with the fact that $\mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)},P))$ and $\mathcal{T}(\inf(P_{\mathrm{Ker}((T^*)^*)},Q))$ are finite because T is $(P\cdot Q)$-Fredholm, implies that T^* is indeed $(Q\cdot P)$-Fredholm with index as stated. □

The following generalization of Lemma 11.1.12 is the key element for the proof of item (ii) of Theorem 11.2.3.

Lemma 11.2.5. *Let $P,Q,R\in\mathcal{N}$ be projections and let $T\in P\mathcal{N}Q$ be $(P\cdot Q)$-Fredholm and $S\in R\mathcal{N}P$ be $(R\cdot P)$-Fredholm. Then*

$$\mathcal{T}\big(\inf(P_{\mathrm{Ker}(ST)},Q)-\inf(P_{\mathrm{Ker}(T)},Q)\big)=\mathcal{T}\big(\inf\big(P_{\mathrm{Ran}(T)},\inf(P_{\mathrm{Ker}(S)},P)\big)\big).$$

Proof. The proof implements the properties of skew-corner operators in the proof of Lemma 11.1.12. First, let us note that $\inf(P_{\mathrm{Ker}(ST)},Q)=P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)}$ as well as $\inf(P_{\mathrm{Ker}(T)},Q)=P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}$, and $\inf(P_{\mathrm{Ker}(S)},P)=P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}$. One has, by the same argument as in the proof of Lemma 11.1.12,

$$\begin{aligned}
&\mathrm{Ker}\big(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)}-TP_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}\big)\\
&\quad=\mathrm{Ker}\big(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)}\big)\\
&\quad=\big(\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)\big)^{\perp}\oplus\big(\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)\big)\\
&\quad=\big((\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q))\ominus(\mathrm{Ker}(T)\cap\mathrm{Ran}(Q))\big)^{\perp}\\
&\quad=\big(\mathrm{Ker}(ST)\ominus\mathrm{Ker}(T)\big)^{\perp}
\end{aligned}$$

and therefore

$$\begin{aligned}&\overline{\mathrm{Ran}\big((P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)})T^*\big)}\\&\quad = \mathrm{Ker}(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - TP_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)})^{\perp} = \mathrm{Ker}(ST) \ominus \mathrm{Ker}(T).\end{aligned}$$

As $\mathrm{Ran}(T^*) \subset \mathrm{Ran}(Q)$, one thus has

$$\begin{aligned}P_{\mathrm{Ran}((P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)})T^*)} &= P_{\mathrm{Ker}(ST)} - P_{\mathrm{Ker}(T)}\\&= P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}.\end{aligned}\tag{11.12}$$

Furthermore,

$$\begin{aligned}\overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - TP_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)})} &= \overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)})}\\&= \overline{\mathrm{Ran}(T) \cap \mathrm{Ker}(S)}\\&\subset \overline{\mathrm{Ran}(T)} \cap \mathrm{Ker}(S)\end{aligned}$$

and therefore

$$\begin{aligned}P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}))} &\leq \inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)})\\&= \inf(P_{\mathrm{Ran}(T)\cap\mathrm{Ran}(P)}, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}).\end{aligned}\tag{11.13}$$

Let $(P_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence of projections in $\mathcal{N}$ such that $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ for all $n \in \mathbb{N}$, $\sup_n P_n = P_{\mathrm{Ran}(T)}$, and the trace of $P - P_1$ is finite. Such a sequence can be constructed as in Corollary 11.1.10 as will be shown next. Because T is (P,Q)-Fredholm, there is a $\mathcal{T}$-finite projection $F \leq P$ such that $\mathrm{Ran}(P - F) \subset \mathrm{Ran}(T)$. By Lemma 11.1.9, there is a nondecreasing sequence $(P'_n)_{n\in\mathbb{N}}$ of projections such that $\mathrm{Ran}(P'_n) \subset \mathrm{Ran}(FT)$ and $\sup_n P'_n = P_{\mathrm{Ran}(FT)}$. In particular, P'_n is orthogonal to $P - F$. Then $P_n = (P - F) + P'_n$ is a nondecreasing sequence of projections in $\mathcal{N}$ such that $\mathrm{Ran}(P_n) \subset \mathrm{Ran}(T)$ for all $n \in \mathbb{N}$, $\sup_n P_n = P_{\mathrm{Ran}(T)}$, and $\mathcal{T}(P - P_1) \leq \mathcal{T}(F)$ is finite. Now, using the sequence $(P_n)_{n\in\mathbb{N}}$, one has

$$\mathrm{Ran}(P_n) \cap \mathrm{Ker}(S) \subset \mathrm{Ran}(T) \cap \mathrm{Ker}(S) \subset \overline{\mathrm{Ran}(TP_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - TP_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)})}$$

for all $n \in \mathbb{N}$, and therefore

$$\begin{aligned}\inf(P_n, P_{\mathrm{Ker}(S)}) &= \inf(P_n, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})\\&\leq P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}))}.\end{aligned}\tag{11.14}$$

Setting $P_0 = \inf(P_1, P - P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})$, one has

$$\begin{aligned}\mathcal{T}(\mathbf{1} - P_0) &= \mathcal{T}\big(P - \inf(P_1, P - P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})\big)\\&= \mathcal{T}\big(P - P_1 + P_1 - \inf(P_1, P - P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})\big)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{T}(P - P_1) + \mathcal{T}(P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)} - \inf(P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}, P - P_1)) \\
&\le \mathcal{T}(P - P_1) + \mathcal{T}(P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}) \\
&< \infty,
\end{aligned}
$$

where in the third step Lemma 11.1.8 was used. As $P_0 \le P_n$ for all $n \in \mathbb{N}$ and $P_0 P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)} = 0$, one concludes that

$$
\inf(P_n, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}) = \inf(P_n - P_0, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})
$$

for all $n \in \mathbb{N}$, and

$$
\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)}) = \inf(P_{\mathrm{Ran}(T)} - P_0, P_{\mathrm{Ker}(S)}).
$$

As $\mathcal{T}(\sup_n(P_n - P_0)) \le \mathcal{T}(P - P_0) < \infty$, Lemma 11.1.11 applies and one gets

$$
\begin{aligned}
\mathcal{T}\Big(\sup_n(\inf(P_n, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}))\Big) &= \mathcal{T}\Big(\sup_n(\inf(P_n - P_0, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}))\Big) \\
&= \mathcal{T}\Big(\inf\Big(\sup_n(P_n - P_0), P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)}\Big)\Big) \\
&= \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})).
\end{aligned}
$$

Combining this with the normality of $\mathcal{T}$, as well as (11.13) and (11.14), implies

$$
\mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})) = \mathcal{T}(P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}))}).
$$

Finally, using first (11.12) and then Lemma 11.1.7 shows

$$
\begin{aligned}
\mathcal{T}(P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}) &= \mathcal{T}(P_{\mathrm{Ran}((P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)})T^*)}) \\
&= \mathcal{T}(P_{\mathrm{Ran}(T(P_{\mathrm{Ker}(ST)\cap\mathrm{Ran}(Q)} - P_{\mathrm{Ker}(T)\cap\mathrm{Ran}(Q)}))}) \\
&= \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, P_{\mathrm{Ker}(S)\cap\mathrm{Ran}(P)})),
\end{aligned}
$$

concluding the proof. □

Proof of Theorem 11.2.3. For the proof of (i), let us first assume that there is $S \in Q\mathcal{N}P$ such that $TS = P + K_1$ with $K_1 \in P\mathcal{K}P$ and $ST = Q + K_2$ with $K_2 \in Q\mathcal{K}Q$. Then by item (i) of Theorem 11.1.6, $TS + \mathbf{1} - P = \mathbf{1} + K_1$ and $ST + \mathbf{1} - Q = \mathbf{1} + K_2$ are $\mathcal{T}$-Fredholm as $K_1, K_2 \in \mathcal{K}$ are $\mathcal{T}$-compact. Thus $P_{\mathrm{Ker}(ST+\mathbf{1}-Q)} = \inf(P_{\mathrm{Ker}(ST)}, Q)$ is $\mathcal{T}$-finite and as $\mathrm{Ker}(T) \subset \mathrm{Ker}(ST)$ also $\mathcal{T}(\inf(P_{\mathrm{Ker}(T)}, Q)) \le \mathcal{T}(\inf(P_{\mathrm{Ker}(ST)}, Q))$ is finite. As $TS + \mathbf{1} - P$ is $\mathcal{T}$-Fredholm, there is a $\mathcal{T}$-finite projection $\widetilde{F} \in \mathcal{N}$ such that $\mathrm{Ran}(\mathbf{1} - \widetilde{F}) \subset \mathrm{Ran}(TS + \mathbf{1} - P)$. Then $F = \inf(P, \widetilde{F})$ is $\mathcal{T}$-finite and $F \le P$, as well as $\mathrm{Ran}(P - F) \subset \mathrm{Ran}(TS) \subset \mathrm{Ran}(T)$. Moreover, the projection $\inf(P_{\mathrm{Ker}(T^*)}, P) = \inf(\mathbf{1} - P_{\mathrm{Ran}(T)}, P) = P - P_{\mathrm{Ran}(T)} \le F$ is $\mathcal{T}$-finite. This shows that T is $(P \cdot Q)$-Fredholm.

Conversely, assume that T is $(P \cdot Q)$-Fredholm. Then there is a $\mathcal{T}$-finite projection $F \le P$ such that $\mathrm{Ran}(P - F) \subset \mathrm{Ran}(T)$. Therefore

$$(P-F)T|_{\mathrm{Ker}((P-F)T)^\perp} : \mathrm{Ker}\big((P-F)T\big)^\perp \to \mathrm{Ran}\big((P-F)T\big) = \mathrm{Ran}(P-F)$$

is invertible with bounded inverse

$$\big((P-F)T|_{\mathrm{Ker}((P-F)T)^\perp}\big)^{-1} : \mathrm{Ran}(P-F) \to \mathrm{Ker}\big((P-F)T\big)^\perp.$$

The trivial extension of $((P-F)T|_{\mathrm{Ker}((P-F)T)^\perp})^{-1}$ to an operator from $Q\mathcal{N}P$ is then denoted by $S \in Q\mathcal{N}P$. Moreover, $P-F$ is $(P \cdot P)$-Fredholm as $\inf(P_{\mathrm{Ker}(P-F)}, P) = F$ is $\mathcal{T}$-finite and $P-F$ is a projection (with closed range). Thus by Lemma 11.2.5,

$$\begin{aligned}\mathcal{T}\big(\inf(P_{\mathrm{Ker}((P-F)T)}, Q)\big) &= \mathcal{T}\big(\inf(P_{\mathrm{Ker}(T)}, Q)\big) + \mathcal{T}\big(\inf(P_{\mathrm{Ran}(T)}, \inf(P_{\mathrm{Ker}(P-F)}, P))\big)\\ &\le \mathcal{T}\big(\inf(P_{\mathrm{Ker}(T)}, Q)\big) + \mathcal{T}(F),\end{aligned}$$

which is finite by assumption. Then, due to $SF = 0$ and $ST = SPT$,

$$\begin{aligned} ST &= S(P-F)T\\ &= P_{\mathrm{Ker}((P-F)T)^\perp}\\ &= Q - \inf(P_{\mathrm{Ker}((P-F)T)}, Q)\\ &= Q - K_1,\end{aligned}$$

where $K_1 = \inf(P_{\mathrm{Ker}((P-F)T)}, Q) \in Q\mathcal{K}Q$ because it is a $\mathcal{T}$-finite projection. Also

$$TS = (P-F)TS + FTS = P - F + FTS = P - K_2,$$

where $K_2 = F - FTS \in P\mathcal{K}P$ because $F \le P$ is $\mathcal{T}$-finite. This shows (i).

To show (ii), note that by assumptions and (i) there is $T_0 \in Q\mathcal{N}P$ with

$$T_0T = Q + K_1, \quad TT_0 = P + K_2,$$

where $K_1 \in Q\mathcal{K}Q$ and $K_2 \in P\mathcal{K}P$ are $\mathcal{T}$-compact. Similarly, there is $S_0 \in P\mathcal{N}R$ such that

$$S_0S = P + K_3, \quad SS_0 = R + K_4,$$

where $K_3 \in P\mathcal{K}P$ and $K_4 \in R\mathcal{K}R$ are $\mathcal{T}$-compact. Then

$$T_0S_0ST = T_0(P+K_3)T = T_0T + T_0K_3T = Q + K_1 + T_0K_3T,$$

where $K_1 + T_0K_3T \in Q\mathcal{K}Q$ is $\mathcal{T}$-compact, and

$$STT_0S_0 = S(P+K_2)S_0 = SS_0 + SK_2S_0 = R + K_4 + SK_2S_0,$$

where $K_4 + SK_2S_0 \in R\mathcal{K}R$ is $\mathcal{T}$-compact. Item (i) implies that ST is $(R \cdot Q)$-Fredholm. By Proposition 11.2.4, one, moreover, concludes that $S^* \in P\mathcal{N}R$ is $(P \cdot R)$-Fredholm, $T^* \in Q\mathcal{N}P$ is $(Q \cdot P)$-Fredholm, and thus $(ST)^* = T^*S^* \in Q\mathcal{N}R$ is $(Q \cdot R)$-Fredholm. By Lemma 11.2.5,

$$\mathcal{T}(\inf(P_{\mathrm{Ker}(ST)}, Q) - \inf(P_{\mathrm{Ker}(T)}, Q)) = \mathcal{T}(\inf(P_{\mathrm{Ran}(T)}, \inf(P_{\mathrm{Ker}(S)}, P))) \tag{11.15}$$

and

$$\mathcal{T}(\inf(P_{\mathrm{Ker}((ST)^*)}, R) - \inf(P_{\mathrm{Ker}(S^*)}, R)) = \mathcal{T}(\inf(P_{\mathrm{Ran}(S^*)}, \inf(P_{\mathrm{Ker}(T^*)}, P))). \tag{11.16}$$

Next by Lemma 11.1.8 (in the second step),

$$\begin{aligned}
&\mathcal{T}(\inf(P_{\mathrm{Ker}(S)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(S)}, P), P - \inf(P_{\mathrm{Ker}(T^*)}, P)))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(S)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(S)}, P), \mathbf{1} - \inf(P_{\mathrm{Ker}(T^*)}, P)))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(T^*)}, P), \mathbf{1} - \inf(P_{\mathrm{Ker}(S)}, P)))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(T^*)}, P), P - \inf(P_{\mathrm{Ker}(S)}, P))).
\end{aligned}$$

Substituting the identities $P - \inf(P_{\mathrm{Ker}(T^*)}, P) = P_{\mathrm{Ran}(T)}$ and $P - \inf(P_{\mathrm{Ker}(S)}, P) = P_{\mathrm{Ran}(S^*)}$, one concludes that

$$\begin{aligned}
&\mathcal{T}(\inf(P_{\mathrm{Ker}(S)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(S)}, P), P_{\mathrm{Ran}(T)}))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)) - \mathcal{T}(\inf(\inf(P_{\mathrm{Ker}(T^*)}, P), P_{\mathrm{Ran}(S^*)})).
\end{aligned}$$

Using equations (11.15) and (11.16), this implies

$$\begin{aligned}
&\mathcal{T}(\inf(P_{\mathrm{Ker}(S)}, P)) - \mathcal{T}(\inf(P_{\mathrm{Ker}(ST)}, Q) - \inf(P_{\mathrm{Ker}(T)}, Q))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)) - \mathcal{T}(\inf(P_{\mathrm{Ker}((ST)^*)}, R) - \inf(P_{\mathrm{Ker}(S^*)}, R)),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\mathcal{T}\text{-Ind}_{(R\cdot Q)}(ST)\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(ST)}, Q)) - \mathcal{T}(\inf(P_{\mathrm{Ker}((ST)^*)}, R))\\
&\quad = \mathcal{T}(\inf(P_{\mathrm{Ker}(S)}, P)) + \mathcal{T}(\inf(P_{\mathrm{Ker}(T)}, Q)) - \mathcal{T}(\inf(P_{\mathrm{Ker}(T^*)}, P)) - \mathcal{T}(\inf(P_{\mathrm{Ker}(S^*)}, R))\\
&\quad = \mathcal{T}\text{-Ind}_{(R\cdot P)}(S) + \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T),
\end{aligned}$$

concluding the argument for (ii).

To show (iii), item (i) is used. If $T \in P\mathcal{N}Q$ is $(P \cdot Q)$-Fredholm, there is $S \in Q\mathcal{N}P$ such that

$$ST = Q + K_1, \quad TS = P + K_2,$$

for $\mathcal{T}$-compact operators $K_1 \in Q\mathcal{K}Q$ and $K_2 \in P\mathcal{K}P$. Let next $A \in P\mathcal{N}Q$ be such that $\|A\| < \|S\|^{-1}$. Then

$$S(T + A) = Q + SA + K_1, \quad (T + A)S = P + AS + K_2,$$

and $Q+SA : \operatorname{Ran}(Q) \to \operatorname{Ran}(Q)$ and $P+AS : \operatorname{Ran}(P) \to \operatorname{Ran}(P)$ are invertible with inverse given by a Neumann series, and therefore $(Q+SA)^{-1} \in Q\mathcal{N}Q$ and also $(P+AS)^{-1} \in P\mathcal{N}P$. Then

$$(Q+SA)^{-1}S(T+A) = Q + (Q+SA)^{-1}K_1$$

and

$$(T+A)S(P+AS)^{-1} = P + K_2(P+AS)^{-1},$$

where $(Q+SA)^{-1}K_1 \in Q\mathcal{K}Q$ and $K_2(P+AS)^{-1} \in P\mathcal{K}P$ are $\mathcal{T}$-compact. By item (i), $T+A$ is $(P \cdot Q)$-Fredholm and the set of $(P \cdot Q)$-Fredholm operators is open in $P\mathcal{N}Q$ with respect to the norm topology. Then

$$\mathcal{T}\text{-Ind}_{(Q\cdot Q)}\big(Q + (Q+SA)^{-1}K_1\big) = \mathcal{T}\text{-Ind}\big(\mathbf{1} + (Q+SA)^{-1}K_1\big) = 0$$

by item (i) of Theorem 11.1.6, and analogously

$$\mathcal{T}\text{-Ind}_{(P\cdot P)}\big(P + K_2(P+AS)^{-1}\big) = \mathcal{T}\text{-Ind}\big(\mathbf{1} + K_2(P+AS)^{-1}\big) = 0.$$

Item (ii) implies

$$0 = \mathcal{T}\text{-Ind}_{(Q\cdot Q)}(ST) = \mathcal{T}\text{-Ind}_{(Q\cdot P)}(S) + \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T)$$

and

$$\begin{aligned} 0 &= \mathcal{T}\text{-Ind}_{(Q\cdot Q)}\big((Q+SA)^{-1}S(T+A)\big) \\ &= \mathcal{T}\text{-Ind}_{(Q\cdot Q)}\big((\mathbf{1}+SA)^{-1}\big) + \mathcal{T}\text{-Ind}_{(Q\cdot P)}(S) + \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T+A). \end{aligned}$$

Because $(Q+SA)^{-1}$ is invertible and therefore $\mathcal{T}\text{-Ind}_{(Q\cdot Q)}((\mathbf{1}+SA)^{-1}) = 0$, one concludes that $\mathcal{T}\text{-Ind}_{(P\cdot Q)}(T) = -\mathcal{T}\text{-Ind}_{(Q\cdot P)}(S) = \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T+A)$. This shows that the index map $T \mapsto \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T)$ is locally constant.

Finally, let us show (iv). That $T+K$ is $(P\cdot Q)$-Fredholm if $T \in P\mathcal{N}Q$ is $(P\cdot Q)$-Fredholm and $K \in P\mathcal{K}Q$ is $\mathcal{T}$-compact directly follows from item (i). As $t \in [0,1] \mapsto T + tK$ is a norm-continuous path of $(P \cdot Q)$-Fredholm operators, $t \in [0,1] \mapsto \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T+tK)$ is constant by item (iii) and therefore $\mathcal{T}\text{-Ind}_{(P\cdot Q)}(T+K) = \mathcal{T}\text{-Ind}_{(P\cdot Q)}(T)$. □

11.3 Semifinite Fredholm pairs of projections

In this section, the most prominent $(P \cdot Q)$-Fredholm operators are considered, namely the semifinite generalization of Fredholm pairs of projections as introduced in Definition 5.2.1 and studied abundantly in Chapter 5. Recall that in this chapter all projections are orthogonal.

Definition 11.3.1. Two projections $P, Q \in \mathcal{N}$ form a semifinite Fredholm pair in $(\mathcal{N}, \mathcal{T})$ if and only if $QP \in Q\mathcal{N}P$ is a $(Q \cdot P)$-Fredholm operator. The (semifinite) index of the pair of projections is denoted by

$$\mathcal{T}\text{-Ind}(P, Q) = \mathcal{T}\text{-Ind}_{(Q \cdot P)}(QP).$$

In many works [148, 54, 26], the semifinite index of a Fredholm pair of projections is called the essential codimension. This terminology seems to go back to the work of Brown, Douglas, and Filmore [42] (in the case $\mathcal{T} = \mathrm{Tr}$, however), who apparently were unaware of Kato's earlier work [112] which used the term index of a Fredholm pair. As Definition 11.3.1 is a direct generalization of the concept introduced in Definition 5.2.1 we stick with Kato's terminology, thus deviating from [148, 54, 26]. Let us also comment that there is a difference of sign with respect to the works [54, 26]. Spelling out the definition of the index, one obtains

$$\begin{aligned} \mathcal{T}\text{-Ind}(P, Q) &= \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-Q) \cap \mathrm{Ran}(P)}) - \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-P) \cap \mathrm{Ran}(Q)}) \\ &= \mathcal{T}(P_{\mathrm{Ker}(Q) \cap \mathrm{Ran}(P)}) - \mathcal{T}(P_{\mathrm{Ker}(P) \cap \mathrm{Ran}(Q)}), \end{aligned} \tag{11.17}$$

which is again in complete analogy with Proposition 5.3.2. Most of the results of Chapter 5 directly transpose to the semifinite context. Here we first focus on those which are relevant for the definition and analysis of the semifinite spectral flow in the next section and give detailed proofs of them. First of all, from (11.17) one obviously concludes

$$\mathcal{T}\text{-Ind}(P, Q) = -\mathcal{T}\text{-Ind}(Q, P).$$

Moreover, the following criterion is the generalization of Corollary 5.3.13, and it is crucial for the definition of the spectral flow in the next section.

Proposition 11.3.2. *If $P, Q \in \mathcal{N}$ are projections, then QP is $(Q \cdot P)$-Fredholm if and only if $\|\pi(Q - P)\| < 1$.*

Proof. Let us first suppose that $\|\pi(Q - P)\| < 1$. Then

$$\|\pi(Q) - \pi(QPQ)\| \leq \|\pi(Q - P)\| < 1.$$

Therefore there is a $\mathcal{T}$-compact operator $K \in Q\mathcal{N}Q$ such that $\|Q - QPQ + K\| < 1$ and therefore $Q - (Q - QPQ + K) = QPQ - K : \mathrm{Ran}(Q) \to \mathrm{Ran}(Q)$ is invertible. Thus $QPQ - K$ is $(Q \cdot Q)$-Fredholm, and item (iv) of Theorem 11.2.3 implies that QPQ is $(Q \cdot Q)$-Fredholm. Thus $\inf(P_{\mathrm{Ker}(QPQ)}, Q)$ is $\mathcal{T}$-finite and as $\mathrm{Ker}(PQ) \subset \mathrm{Ker}(QPQ)$ this implies that $\inf(Q_{\mathrm{Ker}((QP)^*)}, Q) = \inf(P_{\mathrm{Ker}(PQ)}, Q) \leq \inf(P_{\mathrm{Ker}(QPQ)}, Q)$ is $\mathcal{T}$-finite. Exchanging the roles of P and Q implies that also $\mathcal{T}(\inf(P_{\mathrm{Ker}(QP)}, P))$ is finite. Moreover, $\mathrm{Ran}(QPQ) \subset \mathrm{Ran}(QP)$ and because QPQ is $(Q \cdot Q)$-Fredholm there is a $\mathcal{T}$-finite projection $F \leq Q$ such that $\mathrm{Ran}(Q - F) \subset \mathrm{Ran}(QPQ) \subset \mathrm{Ran}(QP)$. This shows that QP is $(Q \cdot P)$-Fredholm.

Conversely, assume that QP is $(Q \cdot P)$-Fredholm. Then by Proposition 11.2.4 and item (ii) of Theorem 11.2.3, QPQ is $(Q \cdot Q)$-Fredholm with vanishing index. Thus $\pi(Q)\pi(P)\pi(Q)$ is an invertible element of $\pi(Q)\mathcal{Q}\pi(Q)$. Hence $\epsilon\pi(Q) < \pi(Q)\pi(P)\pi(Q) \le \pi(Q)$ for $\epsilon > 0$ sufficiently small, and one concludes $\|\pi(Q) - \pi(Q)\pi(P)\pi(Q)\| < 1$. As $(QP)^* = PQ$ is $(P \cdot Q)$-Fredholm, exchanging the roles of P and Q shows that $\|\pi(P) - \pi(P)\pi(Q)\pi(P)\| < 1$. As

$$\big(\pi(Q) - \pi(P)\big)^3 = \big(\pi(Q) - \pi(Q)\pi(P)\pi(Q)\big) - \big(\pi(P) - \pi(P)\pi(Q)\pi(P)\big)$$

and as $\pi(Q) - \pi(Q)\pi(P)\pi(Q)$ and $\pi(P) - \pi(P)\pi(Q)\pi(P)$ are positive semidefinite,

$$-\big(\pi(P) - \pi(P)\pi(Q)\pi(P)\big) \le \big(\pi(Q) - \pi(P)\big)^3 \le \pi(Q) - \pi(Q)\pi(P)\pi(Q).$$

This shows that

$$\begin{aligned}\big\|\big(\pi(Q) - \pi(P)\big)^3\big\| &\le \max\big\{\big\|\pi(P) - \pi(P)\pi(Q)\pi(P)\big\|, \big\|\pi(Q) - \pi(Q)\pi(P)\pi(Q)\big\|\big\} \\ &< 1\end{aligned}$$

and, as $\pi(Q) - \pi(P)$ is self-adjoint, one can conclude that

$$\big\|\pi(Q) - \pi(P)\big\| = \big\|\big(\pi(Q) - \pi(P)\big)^3\big\|^{\frac{1}{3}} < 1.$$

Thus the claim is shown. □

Moreover, a concatenation property similar to that for the index of Fredholm pairs of orthogonal projections holds; compare with Proposition 5.3.15.

Proposition 11.3.3. *Let $P_1, P_2, P_3 \in \mathcal{N}$ be projections such that one has $\|\pi(P_1 - P_2)\| < \frac{1}{2}$ and $\|\pi(P_2 - P_3)\| < \frac{1}{2}$. Then*

$$\mathcal{T}\text{-Ind}(P_1, P_3) = \mathcal{T}\text{-Ind}(P_1, P_2) + \mathcal{T}\text{-Ind}(P_2, P_3).$$

Proof. Let us first note that $\|\pi(P_1 - P_3)\| \le \|\pi(P_1 - P_2)\| + \|\pi(P_2 - P_3)\| < 1$. Therefore P_iP_j is $(P_i \cdot P_j)$-Fredholm for all $i, j \in \{1, 2, 3\}$. Then by item (ii) of Theorem 11.2.3,

$$\begin{aligned}\mathcal{T}\text{-Ind}(P_1, P_2) + \mathcal{T}\text{-Ind}(P_2, P_3) &= \mathcal{T}\text{-Ind}_{(P_2 \cdot P_1)}(P_2P_1) + \mathcal{T}\text{-Ind}_{(P_3 \cdot P_2)}(P_3P_2) \\ &= \mathcal{T}\text{-Ind}_{(P_3 \cdot P_1)}(P_3P_2P_1).\end{aligned}$$

As $\mathcal{T}\text{-Ind}_{(P_3 \cdot P_1)}(P_3P_1) = -\mathcal{T}\text{-Ind}_{(P_1 \cdot P_3)}((P_3P_1)^*)$ by Proposition 11.2.4, one concludes, invoking again Theorem 11.2.3(ii),

$$\begin{aligned}&\mathcal{T}\text{-Ind}(P_1, P_2) + \mathcal{T}\text{-Ind}(P_2, P_3) - \mathcal{T}\text{-Ind}(P_1, P_3) \\ &\quad = \mathcal{T}\text{-Ind}_{(P_3 \cdot P_1)}(P_3P_2P_1) + \mathcal{T}\text{-Ind}_{(P_1 \cdot P_3)}\big((P_3P_1)^*\big) \\ &\quad = \mathcal{T}\text{-Ind}_{(P_3 \cdot P_3)}\big(P_3P_2P_1(P_3P_1)^*\big)\end{aligned}$$

$$= \mathcal{T}\text{-Ind}_{(P_3 \cdot P_3)}(P_3P_2P_1P_3).$$

Therefore it is sufficient to show $\mathcal{T}\text{-Ind}_{(P_3 \cdot P_3)}(P_3P_2P_1P_3) = 0$. As

$$\begin{aligned}\|\pi(P_3P_2P_1P_3) - \pi(P_3)\| &\le \|\pi(P_2P_1) - \pi(P_3)\| \\ &\le \|\pi(P_2P_1) - \pi(P_2)\| + \|\pi(P_2) - \pi(P_3)\| \\ &\le \|\pi(P_1) - \pi(P_2)\| + \|\pi(P_2) - \pi(P_3)\| < 1,\end{aligned}$$

there is a $\mathcal{T}$-compact operator $K \in P_3\mathcal{K}P_3$ such that

$$\|P_3P_2P_1P_3 + K - P_3\| < 1.$$

This implies that $P_3P_2P_1P_3 + K - P_3 + P_3 : \text{Ran}(P_3) \to \text{Ran}(P_3)$ is invertible and therefore $\mathcal{T}\text{-Ind}_{(P_3 \cdot P_3)}(P_3P_2P_1P_3) = \mathcal{T}\text{-Ind}_{(P_3 \cdot P_3)}(P_3P_2P_1P_3 + K) = 0$. □

As already mentioned above, most of the results of Chapter 5 directly transpose to the semifinite context. Let us conclude this section by pointing out the most important ones, leaving it to the interested reader to extend the list. Most proofs transpose directly and are thus not spelled out here. First of all, one has the following generalization of Proposition 5.2.7.

Proposition 11.3.4. *Let $t \in [0,1] \mapsto P(t) \in \mathcal{N}$ and $t \in [0,1] \mapsto Q(t) \in \mathcal{N}$ be norm-continuous paths of projections such that $(P(t), Q(t))$ is a semifinite Fredholm pair for every $t \in [0,1]$. Then $t \in [0,1] \mapsto \mathcal{T}\text{-Ind}(P(t), Q(t))$ is constant.*

The proof is the same as the one of Proposition 5.2.7 and therefore omitted. The arguments leading to Proposition 5.2.10 and Theorem 5.2.11 transpose to the semifinite setting and one gets the following result. Let us note that there are further (slight) generalizations of these results in [56].

Proposition 11.3.5. *Let $P, Q \in \mathcal{N}$ be projections and $n \in \mathbb{N}$ such that*

$$(P - PQP)^n \quad \text{and} \quad (Q - QPQ)^n$$

are $\mathcal{T}$-finite. Then (P, Q) is a semifinite Fredholm pair in $(\mathcal{N}, \mathcal{T})$ and, for all $m \ge n$, one has

$$\mathcal{T}\text{-Ind}(P, Q) = \mathcal{T}\big((P - PQP)^m\big) - \mathcal{T}\big((Q - QPQ)^m\big).$$

Proposition 11.3.6. *Let (P, Q) be a semifinite Fredholm pair of projections in $\mathcal{N}$. If then $(P - Q)^{2n+1} \in \mathcal{N}$ is $\mathcal{T}$-finite for some integer $n \ge 0$, then for all $k \ge n$,*

$$\mathcal{T}\text{-Ind}(P, Q) = \mathcal{T}\big((P - Q)^{2k+1}\big).$$

Proposition 5.3.1 generalizes in the sense that projections $P, Q \in \mathcal{N}$ form a semifinite Fredholm pair (P, Q) if and only if

$$PQP + \mathbf{1} - P \quad \text{and} \quad QPQ + \mathbf{1} - Q$$

are $\mathcal{T}$-Fredholm operators. Then

$$\mathcal{T}\text{-Ind}(P, Q) = \mathcal{T}(P_{\text{Ker}(PQP+\mathbf{1}-P)}) - \mathcal{T}(P_{\text{Ker}(QPQ+\mathbf{1}-Q)}).$$

For the following generalization of Theorem 5.3.8, we provided a simpler proof.

Theorem 11.3.7. *Two projections $P, Q \in \mathcal{N}$ form a semifinite Fredholm pair if and only if*

$$P - Q - \mathbf{1} \quad \textit{and} \quad P - Q + \mathbf{1}$$

are $\mathcal{T}$-Fredholm. Then

$$\mathcal{T}\text{-Ind}(P, Q) = \mathcal{T}(P_{\text{Ker}(P-Q-\mathbf{1})}) - \mathcal{T}(P_{\text{Ker}(P-Q+\mathbf{1})}).$$

Proof. By Proposition 11.3.2, (P, Q) is a semifinite Fredholm pair if and only if $\|\pi(P-Q)\| < 1$. Then $\pi(P - Q - \mathbf{1})$ and $\pi(P - Q + \mathbf{1})$ are invertible with inverse given by a Neumann series. Therefore $P - Q - \mathbf{1}$ and $P - Q + \mathbf{1}$ are $\mathcal{T}$-Fredholm by item (ii) of Theorem 11.1.6. Conversely, assume that $P - Q - \mathbf{1}$ and $P - Q + \mathbf{1}$ are $\mathcal{T}$-Fredholm. Then $\pi(P - Q - \mathbf{1})$ and $\pi(P-Q+\mathbf{1})$ are invertible and therefore $\text{spec}(\pi(P-Q)) \subset (-1, 1)$. Then the spectral radius theorem in the C^*-algebra $\mathcal{Q}$ implies $\|\pi(P - Q)\| < 1$.

For the computation of the index, let us note that, because the kernel of a sum of two nonnegative operators is given by the intersection of their kernels,

$$\begin{aligned}
\text{Ker}(P - Q - \mathbf{1}) &= \text{Ker}\big(Q + (\mathbf{1} - P)\big) \\
&= \text{Ker}(Q) \cap \text{Ker}(\mathbf{1} - P) \\
&= \text{Ker}(Q) \cap \text{Ran}(P).
\end{aligned}$$

Similarly, $\text{Ker}(P - Q + \mathbf{1}) = \text{Ker}(P) \cap \text{Ran}(Q)$. Comparing this to (11.17) implies the last claim. ☐

Proposition 5.3.19 transposes as follows to the semifinite setting:

Proposition 11.3.8. *Let (P, Q) be a pair of orthogonal projections in $\mathcal{N}$ satisfying the bound $\|P - Q\| < 1$. Then there exists a path $t \in [0, 1] \mapsto P_t$ of orthogonal projections connecting $P_0 = P$ with $P_1 = Q$ such that (P, P_t) is a semifinite Fredholm pair for all $t \in [0, 1]$.*

As the proof is the same as that leading to Proposition 5.3.19, it is not spelled out again. Then the argument leading to Proposition 5.3.18 shows the following:

Proposition 11.3.9. *Let (P, Q) be a semifinite Fredholm pair of projections in $\mathcal{N}$. Then there exists a path $t \in [0, 1] \mapsto Q(t)$ of orthogonal projections such that $(P, Q(t))$ is a semifinite Fredholm pair for all $t \in [0, 1]$ with $Q(1) = Q$ and such that $P - Q(0)$ is $\mathcal{T}$-compact.*

In view of the last two propositions, one may expect that also a generalization of Proposition 5.3.23 holds in the semifinite setting. This is, however, not true as the following example shows.

Example 11.3.10. This example shows that the set of semifinite Fredholm pairs with vanishing index is, in general, not connected. Let $\mathcal{N} = L^\infty(\mathbb{R})$ and $\mathcal{T}$ be the Lebesgue integral. Set

$$P(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

and

$$P'(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Then, clearly, $P, P' \in \mathcal{N}$ are projections and (P, P) and (P', P') are Fredholm pairs with vanishing semifinite indices. As $\mathcal{N}$ is commutative and therefore $\|P - Q\| = 1$ for all projections $Q \in \mathcal{N}$ with $Q \neq P$, there is no norm-continuous path of projections in $\mathcal{N}$ connecting P to P'. Thus there is no continuous path of Fredholm pairs of projections connecting (P, P) and (P', P'). ◇

11.4 Definition and basic properties of the spectral flow

Denote by $\mathbb{F}_{\mathrm{sa}}(\mathcal{N}, \mathcal{T}) \subset \mathcal{N}$ the space of self-adjoint $\mathcal{T}$-Fredholm operators in $\mathcal{N}$. Then let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\mathrm{sa}}(\mathcal{N}, \mathcal{T})$ be a norm-continuous path. Hence $\pi(H_t)$ is invertible in $\mathcal{Q}$ for all $t \in [0,1]$. Thus the path $t \in [0,1] \mapsto \chi(\pi(H_t) \geq 0)$ is norm-continuous in $\mathcal{Q}$. Therefore there is a partition $0 = t_0 < t_1 < \cdots < t_M = 1$ such that for $m \in \{1, \ldots, M\}$,

$$\|\chi(\pi(H_t) \geq 0) - \chi(\pi(H_{t'}) \geq 0)\| < \frac{1}{2} \quad \text{for all } t, t' \in [t_{m-1}, t_m]. \tag{11.18}$$

It is now natural to consider the associated projections $P_t = \chi(H_t \geq 0)$ and interpret the bound in (11.18) as a Fredholm property in the sense of Proposition 11.3.2. This is possible due to the following technical fact.

Lemma 11.4.1. *For a self-adjoint operator $H = H^* \in \mathcal{N}$ such that $\pi(H)$ is invertible, one has $\pi(\chi(H \geq 0)) = \chi(\pi(H \geq 0))$.*

Proof. First recall that von Neumann algebras are invariant under measurable functional calculus so that $\chi(H \geq 0) \in \mathcal{N}$. As $\pi(H)$ is invertible, $\chi(\pi(H) \geq 0)$ is a well-defined element of the C^*-algebra $\mathcal{Q}$ and there is an $\epsilon > 0$ such that $[-\epsilon, \epsilon] \cap \operatorname{spec}(\pi(H)) = \emptyset$. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f_1(x) = \begin{cases} 0, & \text{for } x \le 0, \\ \frac{1}{\epsilon}x, & \text{for } x \in (0,\epsilon), \\ 1, & \text{for } x \ge \epsilon, \end{cases} \qquad f_2(x) = \begin{cases} 0, & \text{for } x \le -\epsilon, \\ 1 + \frac{1}{\epsilon}x, & \text{for } x \in (-\epsilon,0), \\ 1, & \text{for } x \ge 0. \end{cases}$$

As $f_1 \le \chi_{[0,\infty)} \le f_2$ on $\mathbb{R}$ and all three functions are equal on $\operatorname{spec}(\pi(H))$,

$$\begin{aligned} \chi(\pi(H \ge 0)) = f_1(\pi(H)) = \pi(f_1(H)) \le \pi(\chi(H \ge 0)) \\ \le \pi(f_2(H)) = f_2(\pi(H)) = \chi(\pi(H \ge 0)), \end{aligned}$$

and therefore $\pi(\chi(H \ge 0)) = \chi(\pi(H \ge 0))$. □

Now the definition of the semifinite spectral flow is a generalization that uses the formulation of the spectral flow in terms of a sum of indices of Fredholm pairs as given in Proposition 5.8.2. Let us stress that this approach uses infinite-dimensional projections and does *not* proceed as the definition of the spectral flow in Section 4.1 where merely compact (and hence finite-dimensional) projections are used. Actually, it is not clear whether it is in general possible to give a formulation of the semifinite spectral flow in terms of $\mathcal{T}$-compact projections. In the following definition, we again include the boundary terms so that the definition slightly deviates from [148, 26, 110]. As before, this assures that the semifinite spectral flow is antisymmetric under sign change.

Definition 11.4.2. Let $t \in [0,1] \mapsto H_t$ be a norm-continuous path in $\mathbb{F}_{sa}(\mathcal{N},\mathcal{T})$ and furthermore let $0 = t_0 < t_1 < \cdots < t_M = 1$ be a partition such that (11.18) holds. Setting $P_m = P_{t_m} = \chi(H_{t_m} \ge 0)$ for $m = 0,\dots,M$, the semifinite spectral flow of the path $t \in [0,1] \mapsto H_t$ is defined as the real number

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\mathcal{T}(P_{\operatorname{Ker}(H_0)}) + \sum_{m=1}^{M} \mathcal{T}\text{-Ind}(P_m, P_{m-1}) - \frac{1}{2}\mathcal{T}(P_{\operatorname{Ker}(H_1)}).$$

Let us note that $\mathcal{T}\text{-Ind}(P_m, P_{m-1})$ is indeed well defined by Proposition 11.3.2 and the above lemma. As in Chapter 4, the first task is prove that the semifinite spectral flow is well defined.

Proposition 11.4.3. *Let $t \in [0,1] \mapsto H_t$ be norm-continuous paths in $\mathbb{F}_{sa}(\mathcal{N},\mathcal{T})$. The definition of the spectral flow* $\operatorname{Sf}(t \in [0,1] \mapsto H_t)$ *is independent of the choice of the partition satisfying* (11.18).

Proof. As the boundary terms in Definition 11.4.2 are independent of the choice of the partition, they can and will be neglected in the remainder of the argument. Now let $0 = t_0 < t_1 < \cdots < t_M = 1$ be a partition such that $\|\chi(\pi(H_t) \ge 0) - \chi(\pi(H_{t'}) \ge 0)\| < \frac{1}{2}$ for all $m \in \{1,\dots,M\}$ and for all $t, t' \in [t_{m-1}, t_m]$ and let $0 = s_0 < s_1 < \cdots < s_{M+1} = 1$ be a second partition such that there is an $m \in \{1,\dots,M\}$ such that $s_i = t_i$ for $i \in \{0,\dots,m-1\}$, $s_m = t_*$, and $s_i = t_{i-1}$ for $i \in \{m+1,\dots,M+1\}$. Then

$$\begin{aligned}\sum_{j=1}^{M+1} \mathcal{T}\text{-Ind}(P_{s_j}, P_{s_{j-1}}) &= \sum_{j=1}^{m-1} \mathcal{T}\text{-Ind}(P_{s_j}, P_{s_{j-1}}) + \mathcal{T}\text{-Ind}(P_{s_m}, P_{s_{m-1}}) \\ &\quad + \mathcal{T}\text{-Ind}(P_{s_{m+1}}, P_{s_m}) + \sum_{j=m+2}^{M+1} \mathcal{T}\text{-Ind}(P_{s_j}, P_{s_{j-1}}) \\ &= \sum_{i=1}^{m-1} \mathcal{T}\text{-Ind}(P_{t_i}, P_{t_{i-1}}) + \mathcal{T}\text{-Ind}(P_{t_*}, P_{t_{m-1}}) \\ &\quad + \mathcal{T}\text{-Ind}(P_{t_m}, P_{t_*}) + \sum_{i=m+1}^{M} \mathcal{T}\text{-Ind}(P_{t_i}, P_{t_{i-1}}).\end{aligned}$$

As $\|\pi(P_{t_{m-1}}) - \pi(P_{t_*})\| < \frac{1}{2}$ and $\|\pi(P_{t_*}) - \pi(P_{t_m})\| < \frac{1}{2}$, one has, by Proposition 11.3.3,

$$\mathcal{T}\text{-Ind}(P_{t_*}, P_{t_{m-1}}) + \mathcal{T}\text{-Ind}(P_{t_m}, P_{t_*}) = \mathcal{T}\text{-Ind}(P_{t_m}, P_{t_{m-1}}).$$

Therefore

$$\sum_{j=1}^{M+1} \mathcal{T}\text{-Ind}(P_{s_j}, P_{s_{j-1}}) = \sum_{i=1}^{M} \mathcal{T}\text{-Ind}(P_{t_i}, P_{t_{i-1}}).$$

Iterating this procedure shows that the definition of the spectral flow is independent of the choice of the partition. □

Example 11.4.4. This example provides an example of nontrivial semifinite spectral flow. Let $\mathcal{N} = L^\infty(\mathbb{R})$ and $\mathcal{T}$ be the Lebesgue integral. Set

$$H_0(x) = \begin{cases} -1, & x \le -1, \\ x, & x \in [-1, 1], \\ 1, & x \ge 1. \end{cases}$$

For any $s > 0$, $t > 0$, and $x \in \mathbb{R}$, set

$$H_t(r) = H_0(x + ts).$$

Then $t \in [0,1] \mapsto H_t$ is a norm-continuous path of self-adjoint elements from $\mathcal{N} = L^\infty(\mathbb{R})$. Moreover, $\chi(H_t \ge 0) = \chi_{[-ts,\infty)}$ is a projection that is not finite. However, the pair of projections $(\chi(H_t \ge 0), \chi(H_0 \ge 0)) = (\chi_{[-ts,\infty)}, \chi_{[0,\infty)})$ is $\mathcal{T}$-Fredholm because the difference is a finite projection $\chi_{[-ts,0)}$ with $\mathcal{T}(\chi_{[-ts,0)}) = ts$. Therefore

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathcal{T}\text{-Ind}\big(\chi(H_1 \ge 0), \chi(H_0 \ge 0)\big) = s.$$

Note that $\pi(H_t)$ is constant and has spectrum $\{-1, 1\}$ in this example, while $\mathrm{spec}(H_t) = [-1, 1]$ for all t. However, the spectrum of H_t in $(-1, 1)$ is continuous and *thin*. It moves through 0 and this flow of continuous spectrum is measured by the semifinite spectral flow. ◇

Next let us state and prove the main properties of the spectral flow. This generalizes the most important results of Section 4.2.

Theorem 11.4.5. *Let $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\text{sa}}(\mathcal{N},\mathcal{T})$ be a continuous path.*

(i) *If $t \in [0,1] \mapsto \chi(H_t \geq 0)$ is norm-continuous, then*

$$\text{Sf}\big(t \in [0,1] \mapsto H_t\big) = \frac{1}{2}\mathcal{T}(P_{\text{Ker}(H_0)}) - \frac{1}{2}\mathcal{T}(P_{\text{Ker}(H_1)}).$$

(ii) *If $t \in [0,1] \mapsto H_t$ and $t \in [0,1] \mapsto H'_t \in \mathbb{F}_{\text{sa}}(\mathcal{N},\mathcal{T})$ have the same endpoints and are connected by a norm-continuous homotopy within $\mathbb{F}_{\text{sa}}(\mathcal{N},\mathcal{T})$, then*

$$\text{Sf}\big(t \in [0,1] \mapsto H_t\big) = \text{Sf}\big(t \in [0,1] \mapsto H'_t\big).$$

(iii) *The spectral flow has a concatenation property, namely if $t \in [1,2] \mapsto H_t \in \mathbb{F}_{\text{sa}}(\mathcal{N},\mathcal{T})$ is a second continuous path, composable to the first one in the sense that the endpoint of the first path is the initial point of the second path, then*

$$\text{Sf}\big(t \in [0,2] \mapsto H_t\big) = \text{Sf}\big(t \in [0,1] \mapsto H_t\big) + \text{Sf}\big(t \in [1,2] \mapsto H_t\big).$$

Proof. (i) As $t \in [0,1] \mapsto \chi(H_t \geq 0)$ is norm-continuous and $[0,1]$ is compact, there is $\delta > 0$ such that $\|\chi(H_t \geq 0) - \chi(H_{t'} \geq 0)\| < 1$ for $t, t' \in [0,1]$ such that $|t - t'| < \delta$. Then $\chi(H_t \geq 0)\chi(H_{t'} \geq 0) : \text{Ran}(\chi(H_{t'} \geq 0)) \to \text{Ran}(\chi(H_t \geq 0))$ is invertible, thus $\mathcal{T}\text{-Ind}(\chi(H_{t'} \geq 0), \chi(H_t \geq 0)) = 0$. Choosing a partition $0 = t_0 < t_1 < \cdots < t_M = 1$ such that $\|\chi(\pi(H_t \geq 0)) - \chi(\pi(H_{t'} \geq 0))\| < \frac{1}{2}$ for $m \in \{1,\dots,M\}$ and $t, t' \in [t_{m-1}, t_m]$ and such that $|t_{m-1} - t_m| < \delta$ for all $m \in \{1,\dots,M\}$ shows the first claim. The proofs of item (ii) is essentially identical to the proofs of Theorem 4.2.2 and therefore not spelled out again. Item (ii) directly follows from Definition 11.4.2. □

The next result concerns natural additivity properties of the semifinite spectral flow.

Theorem 11.4.6. *Let $P \in \mathcal{N} \subset \mathbb{B}(\mathcal{H})$ be a projection and further let $t \in [0,1] \mapsto H_t \in \mathcal{N}$ and $t \in [0,1] \mapsto H'_t \in \mathcal{N}$ be two paths such that one has $PH_tP = H_t \in \mathbb{F}_{\text{sa}}(P\mathcal{N},\mathcal{T})$ and $(\mathbf{1} - P)H'_t(\mathbf{1} - P) = H'_t \in \mathbb{F}_{\text{sa}}((\mathbf{1} - P)\mathcal{N},\mathcal{T})$ for all $t \in [0,1]$. Then*

$$\text{Sf}\big(t \in [0,1] \mapsto H_t \oplus H'_t\big) = \text{Sf}\big(t \in [0,1] \mapsto H_t\big) + \text{Sf}\big(t \in [0,1] \mapsto H'_t\big),$$

where $\oplus$ is in the grading of $\mathcal{H} = P\mathcal{H} \oplus (\mathbf{1} - P)\mathcal{H}$. The same equality holds for the direct sum $t \in [0,1] \mapsto H_t \oplus H'_t \in \mathbb{F}_{\text{sa}}(\mathcal{N} \oplus \tilde{\mathcal{N}}, \mathcal{T} \oplus \tilde{\mathcal{T}})$ of two paths $t \in [0,1] \mapsto H_t \in \mathbb{F}_{\text{sa}}(\mathcal{N},\mathcal{T})$ and $t \in [0,1] \mapsto H'_t \in \mathbb{F}_{\text{sa}}(\tilde{\mathcal{N}},\tilde{\mathcal{T}})$.

Proof. It is sufficient so show

$$\begin{aligned}&\mathcal{T}\text{-Ind}(\chi((H_{t_m}\oplus H'_{t_m})\geq 0),\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0))\\&\quad=\mathcal{T}\text{-Ind}(\chi(H_{t_m}\geq 0),\chi(H_{t_{m-1}}\geq 0))+\mathcal{T}\text{-Ind}(\chi(H'_{t_m}\geq 0),\chi(H'_{t_{m-1}}\geq 0))\end{aligned}$$

for all $t_m, t_{m-1}\in[0,1]$. Let us note that

$$\chi((H_s\oplus H'_s)\geq 0)=\inf(\chi(H_s\geq 0),P)+\inf(\chi(H'_s\geq 0),\mathbf{1}-P)$$

for all $s\in[0,1]$. Therefore

$$\begin{aligned}&\mathcal{T}(\inf(\mathbf{1}-\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0),\chi((H_{t_m}\oplus H'_{t_m})\geq 0)))\\&\quad=\mathcal{T}(\inf(\mathbf{1}-(\inf(\chi(H_{t_{m-1}}\geq 0),P)+\inf(\chi(H'_{t_{m-1}}\geq 0),\mathbf{1}-P)),\\&\qquad\inf(\chi(H_{t_m}\geq 0),P)+\inf(\chi(H'_{t_m}\geq 0),\mathbf{1}-P)))\\&\quad=\mathcal{T}(\inf(P-\inf(\chi(H_{t_{m-1}}\geq 0),P)+(\mathbf{1}-P)-\inf(\chi(H'_{t_{m-1}}\geq 0),\mathbf{1}-P),\\&\qquad\inf(\chi(H_{t_m}\geq 0),P)+\inf(\chi(H'_{t_m}\geq 0),\mathbf{1}-P)))\\&\quad=\mathcal{T}(\inf(P-\inf(\chi(H_{t_{m-1}}\geq 0),P),\inf(\chi(H_{t_m}\geq 0),P)))\\&\qquad+\mathcal{T}(\inf((\mathbf{1}-P)-\inf(\chi(H'_{t_{m-1}}\geq 0),\mathbf{1}-P),\inf(\chi(H'_{t_m}\geq 0),\mathbf{1}-P)))\\&\quad=\mathcal{T}(\inf(\mathbf{1}-\chi(H_{t_{m-1}}\geq 0),\chi(H_{t_m}\geq 0)))\\&\qquad+\mathcal{T}(\inf(\mathbf{1}-\chi(H'_{t_{m-1}}\geq 0),\chi(H'_{t_m}\geq 0))),\end{aligned}$$

where the last step follows as $\mathbf{1}-P\leq\chi(H_{t_{m-1}}\geq 0)$ and $P\leq\chi(H'_{t_{m-1}}\geq 0)$. Analogously, one has

$$\begin{aligned}&\mathcal{T}(\inf(\mathbf{1}-\chi((H_{t_m}\oplus H'_{t_m})\geq 0),\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0)))\\&\quad=\mathcal{T}(\inf(\mathbf{1}-\chi(H_{t_m}\geq 0),\chi(H_{t_m-1}\geq 0)))\\&\qquad+\mathcal{T}(\inf(\mathbf{1}-\chi(H'_{t_m}\geq 0),\chi(H'_{t_{m-1}}\geq 0))).\end{aligned}$$

Thus by (11.17),

$$\begin{aligned}&\mathcal{T}\text{-Ind}(\chi((H_{t_m}\oplus H'_{t_m})\geq 0),\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0))\\&\quad=\mathcal{T}(\inf(\mathbf{1}-\chi((H_{t_m}\oplus H'_{t_m})\geq 0),\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0)))\\&\qquad-\mathcal{T}(\inf(\mathbf{1}-\chi((H_{t_{m-1}}\oplus H'_{t_{m-1}})\geq 0),\chi((H_{t_m}\oplus H'_{t_m})\geq 0)))\\&\quad=\mathcal{T}(\inf(\mathbf{1}-\chi(H_{t_m}\geq 0),\chi(H_{t_{m-1}}\geq 0)))\\&\qquad+\mathcal{T}(\inf(\mathbf{1}-\chi(H'_{t_m}\geq 0),\chi(H'_{t_{m-1}}\geq 0)))\\&\qquad-\mathcal{T}(\inf(\mathbf{1}-\chi(H_{t_{m-1}}\geq 0),\chi(H_{t_m}\geq 0)))\\&\qquad-\mathcal{T}(\inf(\mathbf{1}-\chi(H'_{t_{m-1}}\geq 0),\chi(H'_{t_m}\geq 0)))\end{aligned}$$

$$= \mathcal{T}\text{-Ind}\big(\chi(H_{t_m} \geq 0), \chi(H_{t_{m-1}} \geq 0)\big) + \mathcal{T}\text{-Ind}\big(\chi(H'_{t_m} \geq 0), \chi(H'_{t_{m-1}} \geq 0)\big),$$

implying the first claim. The second follows in the same way. □

The attentive reader may note that this and the following sections do not contain information about a semifinite spectral flow for essentially $\mathcal{T}$-gapped unitaries, namely for paths $t \in [0,1] \mapsto U_t \in \mathcal{N}$ such that $-1 \notin \text{spec}(\pi(U_t))$. Indeed, a general definition of spectral flow for such paths does not seem to be available in the literature. The reason is that there is no index formulation for the spectral flow of essentially gapped unitaries, similar to Proposition 5.8.2. Under supplementary hypothesis, however, one can define a semifinite spectral flow for essentially $\mathcal{T}$-gapped unitaries. For instance, if the path has a gap, say at 1 throughout, one can readily adapt Definition 11.4.2. Furthermore, if the path is closed and differentiable with a derivative that is $\mathcal{T}$-trace class, one can use formulas as those in Proposition 4.5.10 as a definition. A density argument then also allows extending the definition to continuous closed paths. This is carried out in [197]. Based on this, it is subsequently possible (however, no further details are provided here) to define a semifinite Bott–Maslov and Conley–Zehnder indices for closed paths by transposing the formalism of Chapter 9.

11.5 Index formulas for semifinite spectral flow

This section generalizes some of the index formulas for the spectral flow to the semifinite setting. Such results go back to the work of Phillips [148]. A *KK*-theoretic proof is given in [110]. Let us begin with a generalization of Theorem 5.7.3. The index pairing of a projection $P \in \mathcal{N}$ with a unitary $F \in \mathcal{N}$ having a $\mathcal{T}$-compact commutator $[P, F] \in \mathcal{K}$ is given by the $\mathcal{T}$-Fredholm operator

$$T = PFP + \mathbf{1} - P \tag{11.19}$$

and its index $\mathcal{T}\text{-Ind}(T)$.

Theorem 11.5.1. *Let $t \in [0,1] \mapsto H_t \in \mathcal{N}$ be a norm-continuous path of self-adjoint operators with invertible endpoints H_0 and H_1 such that $H_t - H_0$ is $\mathcal{T}$-compact for all $t \in [0,1]$ and $H_1 = F^* H_0 F$ for a unitary $F \in \mathcal{N}$. If $P = \chi(H_0 \leq 0)$, then $PFP + \mathbf{1} - P$ is $\mathcal{T}$-Fredholm, or equivalently PFP is $(P \cdot P)$-Fredholm, with index*

$$\mathcal{T}\text{-Ind}(PFP + \mathbf{1} - P) = \mathcal{T}\text{-Ind}_{(P \cdot P)}(PFP) = \text{Sf}\big(t \in [0,1] \mapsto H_t\big).$$

In particular, one has, for the linear path connecting $\mathbf{1} - 2P$ and $F^*(\mathbf{1} - 2P)F$,

$$\mathcal{T}\text{-Ind}(PFP + \mathbf{1} - P) = \text{Sf}\big(t \in [0,1] \mapsto (1-t)(\mathbf{1} - 2P) + tF^*(\mathbf{1} - 2P)F\big).$$

Proof. First, $H_1 - H_0 = F^* H_0 F - H_0 = F^*[H_0, F] \in \mathcal{K}$ is $\mathcal{T}$-compact by assumption. Thus $[H_0, F] = FF^*[H_0, F] \in \mathcal{K}$ is $\mathcal{T}$-compact. Therefore, e. g., the argument in the proof of Proposition 5.3.17, also $[P, F]$ is $\mathcal{T}$-compact. As

$$(PFP + \mathbf{1} - P)(PF^*P + \mathbf{1} - P) = PFPF^*P + \mathbf{1} - P = \mathbf{1} + P[F, P]F^*P$$

and

$$(PF^*P + \mathbf{1} - P)(PFP + \mathbf{1} - P) = PF^*PFP + \mathbf{1} - P = \mathbf{1} + PF^*[P, F]P,$$

where $P[F, P]F^*P$ and $PF^*[P, F]P$ are $\mathcal{T}$-compact, $PFP + \mathbf{1} - P$ is $\mathcal{T}$-Fredholm by item (ii) of Theorem 11.1.6. Analogously,

$$(PFP)(PF^*P) = P + P[F, P]F^*P, \quad (PF^*P)(PFP) = P + PF^*[P, F]P$$

implies that PFP is $(P \cdot P)$-Fredholm by item (i) of Theorem 11.2.3. Its index is

$$\begin{aligned}
\mathcal{T}\text{-Ind}(PFP + \mathbf{1} - P) &= \mathcal{T}(P_{\mathrm{Ker}(PFP+\mathbf{1}-P)}) - \mathcal{T}(P_{\mathrm{Ker}(PF^*P+\mathbf{1}-P)}) \\
&= \mathcal{T}(\inf(P_{\mathrm{Ker}(PFP)}, P)) - \mathcal{T}(\inf(P_{\mathrm{Ker}(PF^*P)}, P)) \\
&= \mathcal{T}\text{-Ind}_{(P\cdot P)}(PFP).
\end{aligned}$$

To show that this index equals the spectral flow of the path $t \in [0, 1] \mapsto H_t$, let us first note that $t \in [0, 1] \mapsto \pi(H_t)$ is constant as $H_t - H_0$ is $\mathcal{T}$-compact for all $t \in [0, 1]$ by assumption. Therefore $t \in [0, 1] \mapsto \chi(\pi(H_t \geq 0))$ is constant and thus it is sufficient to consider the trivial partition $t_0 = 0 \leq t_1 = 1$ in the definition of the spectral flow. Then

$$\begin{aligned}
\mathrm{Sf}(t \in [0, 1] \mapsto H_t) &= \mathcal{T}\text{-Ind}(\chi(H_1 \geq 0), \chi(H_0 \geq 0)) \\
&= \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-\chi(H_0\geq 0))\cap \mathrm{Ran}(\chi(H_1\geq 0))}) - \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-\chi(H_1\geq 0))\cap \mathrm{Ran}(\chi(H_0\geq 0))}).
\end{aligned}$$

Using $\chi(H_0 \geq 0) = \mathbf{1} - P$ as H_0 is invertible and analogously $\chi(H_1 \geq 0) = \mathbf{1} - F^*PF$, one gets

$$\begin{aligned}
&\mathrm{Sf}(t \in [0, 1] \mapsto H_t) \\
&\quad = \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-\chi(H_0\geq 0))\cap \mathrm{Ran}(\chi(H_1\geq 0))}) - \mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-\chi(H_1\geq 0))\cap \mathrm{Ran}(\chi(H_0\geq 0))}) \\
&\quad = \mathcal{T}(P_{\mathrm{Ran}(P)\cap \mathrm{Ran}(\mathbf{1}-F^*PF)}) - \mathcal{T}(P_{\mathrm{Ran}(F^*PF)\cap \mathrm{Ran}(\mathbf{1}-P)}) \\
&\quad = \mathcal{T}(P_{\mathrm{Ran}(P)\cap \mathrm{Ker}(F^*PF)}) - \mathcal{T}(FP_{\mathrm{Ran}(F^*PF)\cap \mathrm{Ran}(\mathbf{1}-P)}F^*) \\
&\quad = \mathcal{T}(P_{\mathrm{Ran}(P)\cap \mathrm{Ker}(PF)}) - \mathcal{T}(P_{\mathrm{Ran}(P)\cap \mathrm{Ran}(\mathbf{1}-FPF^*)}) \\
&\quad = \mathcal{T}(P_{\mathrm{Ker}(PFP+\mathbf{1}-P)}) - \mathcal{T}(P_{\mathrm{Ker}(PF^*P+\mathbf{1}-P)}) \\
&\quad = \mathcal{T}\text{-Ind}(PFP + \mathbf{1} - P),
\end{aligned}$$

concluding the argument. □

As to other generalization of formulas from Chapter 5, let us note that [56] contains a semifinite version of Theorem 5.2.11. Let us here rather turn to a semifinite signature as introduced in [170].

Definition 11.5.2. Let $H \in \mathcal{N}$ be self-adjoint with support projection $\mathrm{supp}(H) = \chi(H \neq 0)$ that is $\mathcal{T}$-finite. Then the $\mathcal{T}$-signature of H is defined by

$$\mathcal{T}\text{-Sig}(H) = \mathcal{T}\big(\mathrm{sgn}(H)\big), \quad \mathrm{sgn}(H) = \chi(H > 0) - \chi(H < 0).$$

Note that $\chi(H > 0) \leq \chi(H \neq 0)$ and $\chi(H < 0) \leq \chi(H \neq 0)$ are $\mathcal{T}$-finite and therefore the signature of H is a well-defined real number. The following generalizes Sylvester's law of inertia.

Proposition 11.5.3. *Let $H \in \mathcal{N}$ be self-adjoint with a $\mathcal{T}$-finite support projection. Further, let $A \in \mathcal{N}$ be invertible. Then*

$$\mathcal{T}\text{-Sig}\big(A^* HA\big) = \mathcal{T}\text{-Sig}(H).$$

Proof. Decomposing into positive and negative part $H = H_+ - H_-$, it is enough to prove the statement for $H \geq 0$. In that case, one has

$$\mathcal{T}\text{-Sig}(H) = \mathcal{T}\big(\mathrm{supp}(H)\big), \quad \mathcal{T}\text{-Sig}\big(A^* HA\big) = \mathcal{T}\big(\mathrm{supp}(A^* HA)\big),$$

with the respective support projections. Let $HA = V|HA|$ be the polar decomposition of HA in the sense of von Neumann. Then

$$\mathrm{Ran}(V) = \overline{\mathrm{Ran}(HA)} = \overline{\mathrm{Ran}(H)}, \quad \mathrm{Ran}(V^*) = \overline{\mathrm{Ran}\big(A^* H\big)} = \overline{\mathrm{Ran}\big(A^* HA\big)}.$$

As H and $A^* HA$ are self-adjoint, their support projections are given by $\mathrm{supp}(H) = VV^*$ and $\mathrm{supp}(A^* HA) = V^* V$. Hence the claim follows from $\mathcal{T}(VV^*) = \mathcal{T}(V^* V)$. □

There is a connection of the spectral flow to the signature of the endpoints of the considered path, generalizing the finite dimensional result stated in Proposition 1.1.1.

Proposition 11.5.4. *Let $t \in [0,1] \mapsto H_t$ be a norm-continuous path of self-adjoints in $\mathcal{N}$ such that the support projections satisfy $\mathrm{supp}(H_t) \leq P$ for all t and a single $\mathcal{T}$-finite projection $P \in \mathcal{N}$. Then $t \in [0,1] \mapsto H'_t = PH_tP + \mathbf{1} - P$ is a norm-continuous path in $\mathbb{F}_{\mathrm{sa}}(\mathcal{N}, \mathcal{T})$ and*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H'_t\big) = \frac{1}{2}\big(\mathcal{T}\text{-Sig}(H_1) - \mathcal{T}\text{-Sig}(H_0)\big).$$

Proof. Since $\pi(H'_t) = \mathbf{1}$, H'_t is Fredholm of all $t \in [0,1]$ by item (ii) of Theorem 11.1.6. The norm-continuity of the path $t \in [0,1] \mapsto H'_t$ is obvious. Because $t \in [0,1] \mapsto \pi(H'_t)$ is constant, the two-point partition $t_0 = 0 \leq t_1 = 1$ is sufficiently fine and hence the spectral flow is given by

$$\begin{aligned}
&\mathrm{Sf}(t \in [0,1] \mapsto H'_t) \\
&\quad = \mathcal{T}\text{-Ind}\big(\chi(H'_1 \geq 0), \chi(H'_0 \geq 0)\big) + \frac{1}{2}\big(\mathcal{T}(P_{\mathrm{Ker}(PH_0P+\mathbf{1}-P)}) - \mathcal{T}(P_{\mathrm{Ker}(PH_1P+\mathbf{1}-P)})\big) \\
&\quad = \mathcal{T}\text{-Ind}\big(\chi(H'_1 \geq 0), \chi(H'_0 \geq 0)\big) + \frac{1}{2}\big(\mathcal{T}(PP_{\mathrm{Ker}(H_0)}) - \mathcal{T}(PP_{\mathrm{Ker}(H_1)})\big),
\end{aligned}$$

because $PP_{\mathrm{Ker}(H_t)} = P_{\mathrm{Ker}(PH_tP+\mathbf{1}-P)}$. Since P and H_t commute for all $t \in [0,1]$, one has

$$\chi(H'_t \geq 0) = \chi(PH_tP \oplus (\mathbf{1}-P) \geq 0) = P\chi(H_t \geq 0)P \oplus (\mathbf{1}-P),$$

and hence

$$\begin{aligned}
\mathcal{T}(P_{\mathrm{Ran}(\mathbf{1}-\chi(H'_0\geq 0))\cap \mathrm{Ran}(\chi(H'_1\geq 0))}) &= \mathcal{T}(P_{\mathrm{Ran}(P(\mathbf{1}-\chi(H_0\geq 0))P)\cap \mathrm{Ran}(P\chi(H_1\geq 0)P)}) \\
&= \mathcal{T}(P_{\mathrm{Ran}(P-P\chi(H_0\geq 0))\cap \mathrm{Ran}(\chi(H_1\geq 0))}) \\
&= \mathcal{T}\big(P\chi(H_1 \geq 0)\big) - \mathcal{T}\big(P\inf\big(\chi(H_0 \geq 0), \chi(H_1 \geq 0)\big)\big).
\end{aligned}$$

Switching 0 and 1 and taking the difference leads to

$$\begin{aligned}
&\mathrm{Sf}(t \in [0,1] \mapsto H'_t) \\
&\quad = \mathcal{T}\big(P\chi(H_1 \geq 0) - P\chi(H_0 \geq 0)\big) + \frac{1}{2}\big(\mathcal{T}(PP_{\mathrm{Ker}(H_0)}) - \mathcal{T}(PP_{\mathrm{Ker}(H_1)})\big).
\end{aligned}$$

Finally, noting that $P\chi(H_t \geq 0) = \chi(H_t > 0) + PP_{\mathrm{Ker}(H_t)}$ and thus

$$\mathcal{T}(PP_{\mathrm{Ker}(H_t)}) + \mathcal{T}\big(\chi(H_t > 0)\big) = \mathcal{T}\big(P\chi(H_t \geq 0)\big) = \mathcal{T}(P) - \mathcal{T}\big(\chi(H_t < 0)\big),$$

one obtains

$$\begin{aligned}
2\,\mathrm{Sf}(t \in [0,1] \mapsto H'_t) &= \mathcal{T}\big(\chi(H_1 > 0)\big) + \mathcal{T}(P) - \mathcal{T}\big(\chi(H_1 < 0)\big) \\
&\qquad - \big(\mathcal{T}\big(\chi(H_0 > 0)\big) + \mathcal{T}(P) - \mathcal{T}\big(\chi(H_0 < 0)\big)\big) \\
&= \mathcal{T}\text{-Sig}(H_1) - \mathcal{T}\text{-Sig}(H_0).
\end{aligned}$$

Dividing this by 2 implies the claim. □

The signature also has an additional invariance property that is somewhat inconvenient to express in terms of the spectral flow:

Proposition 11.5.5. *If $t \in [0,1] \mapsto H_t \in \mathcal{N}$ is a norm-continuous path of self-adjoints all of which have $\mathcal{T}$-finite support projections and such that for every $t \in [0,1]$ there is an open interval $(-\delta,\delta)$ around 0 such that $(-\delta,\delta) \cap \mathrm{spec}(H_t) \subset \{0\}$, then for all $t, t' \in [0,1]$,*

$$\mathcal{T}\text{-Sig}(H_t) = \mathcal{T}\text{-Sig}(H_{t'}).$$

Proof. As $[0,1]$ is compact, the spectra $\mathrm{spec}(H_t) \setminus \{0\}$ have a common gap $(-\delta,\delta)$ and hence one can choose continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(H_t > 0) = f(H_t), \quad \chi(H_t < 0) = g(H_t), \quad \forall t \in [0,1].$$

Therefore the paths $t \mapsto \chi(H_t > 0)$ and $t \mapsto \chi(H_t < 0)$ are actually norm-continuous paths of projections. Since projections that are close in norm are unitarily equivalent by Lemma 8.3.3, this implies that the signature is constant along the path. □

11.6 Semifinite spectral localizer

In this section the results from Chapter 10 are generalized to $\mathcal{T}$-Fredholm operators. If the index pairing (11.19) results from a pairing between a K-theory class and a semifinite spectral triple (also called an unbounded semifinite Fredholm module), it can be computed in terms of the spectral localizer, as shown in [170]. Other than in [170], we here suppose that the Dirac operator of the spectral triple has $\mathcal{T}$-compact resolvent. Using techniques of [51, 50], one can also deal with the so-called nonunital case where the resolvent is only relatively compact.

As in Chapter 10, odd and even pairings have to be distinguished. Let us begin with odd index pairings and their spectral localizer. Suppose given an invertible operator $A \in \mathcal{N}$. Associated with A is its unitary phase $U = A|A|^{-1}$, as well as

$$H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}. \tag{11.20}$$

Then H is an element of the von Neumann algebra $\mathcal{N} \otimes \mathbb{C}^{2\times 2}$ and $\mathcal{T} \otimes \mathrm{Tr}$ is a semifinite faithful normal trace on $\mathcal{N} \otimes \mathbb{C}^{2\times 2}$ (e. g., Proposition V.2.14 in [189]). The self-adjoint and invertible Dirac operator is supposed to be of the form

$$D^{\mathrm{od}} = \begin{pmatrix} D_0 & 0 \\ 0 & -D_0 \end{pmatrix},$$

and to have a $\mathcal{T} \otimes \mathrm{Tr}$-compact resolvent. By proceeding as in Remark 10.1.2, one sees that it is no restriction to assume the invertibility of the Dirac operator. Furthermore, it will be assumed as in Definition 10.1.7 that A leaves the domain $\mathcal{D}(D_0)$ of D_0 invariant and that the (densely defined) commutator $[A, D_0]$ extends to a bounded operator. Moreover, it is assumed that D_0 is affiliated to $\mathcal{N}$ or equivalently that D^{od} is affiliated to $\mathcal{N} \otimes \mathbb{C}^{2\times 2}$ in the sense that it commutes with every unitary in $(\mathcal{N} \otimes \mathbb{C}^{2\times 2})'$. All these conditions together imply that D^{od} specifies an unbounded semifinite spectral triple of the algebra generated by A in the sense of [51, 53, 54].

Then for $E = \chi(D_0 \geq 0)$, one can check as in Theorem 10.1.4 that the commutator $[E, U]$ is $\mathcal{T}$-compact. Hence

$$T = EUE + \mathbf{1} - E \tag{11.21}$$

is a $\mathcal{T}$-Fredholm operator. The odd spectral localizer is defined as the operator

$$L_\kappa^{\mathrm{od}} = \begin{pmatrix} \kappa D_0 & A \\ A^* & -\kappa D_0 \end{pmatrix}, \tag{11.22}$$

acting on $\mathcal{H} \oplus \mathcal{H}$ where $\kappa > 0$ is a tuning parameter. Let us set

$$\mathcal{H}_\rho = \mathrm{Ran}(\chi(|D_0| \le \rho)), \quad (\mathcal{H} \oplus \mathcal{H})_\rho = \mathrm{Ran}(\chi(|D^{\mathrm{od}}| \le \rho)),$$

for $\rho > 0$. Let $\pi_\rho = P_{\mathcal{H}_\rho} : \mathcal{H} \to \mathcal{H}$ denote the projection onto $\mathcal{H}_\rho$. By abuse of notation, the projection $P_{(\mathcal{H}\oplus\mathcal{H})_\rho} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is also denoted by $\pi_\rho : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$. As D^{od} has $\mathcal{T} \otimes \mathrm{Tr}$-compact resolvent, each π_ρ is $\mathcal{T} \otimes \mathrm{Tr}$-finite or $\mathcal{T}$-finite, respectively. For any operator $B \in \mathbb{B}(\mathcal{H})$ or $B \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$, set $B_\rho = \pi_\rho B \pi_\rho$. With these notations, the finite volume odd spectral localizer on $\mathcal{H}_\rho \oplus \mathcal{H}_\rho$ is defined by

$$L_{\kappa,\rho}^{\mathrm{od}} = \begin{pmatrix} \kappa D_{0,\rho} & A_\rho \\ A_\rho^* & -\kappa D_{0,\rho} \end{pmatrix}.$$

Theorem 11.6.1. *Let $g = \|A^{-1}\|^{-1}$ be the gap of the invertible operator A. Suppose that*

$$\kappa \le \frac{g^3}{12\|A\|\|[D_0, A]\|}, \quad \frac{2g}{\kappa} < \rho. \tag{11.23}$$

Then $L_{\kappa,\rho}^{\mathrm{od}}$ satisfies the bound $(L_{\kappa,\rho}^{\mathrm{od}})^2 \ge \frac{g^2}{4}\pi_\rho$. In particular, $L_{\kappa,\rho}^{\mathrm{od}} + \mathbf{1} - \pi_\rho$ is invertible. As $\mathrm{supp}(L_{\kappa,\rho}^{\mathrm{od}}) \le \pi_\rho$ *and π_ρ is* $\mathcal{T}\otimes\mathrm{Tr}$*-finite, it has a well-defined* $\mathcal{T}\otimes\mathrm{Tr}$*-signature* $\mathcal{T}\otimes\mathrm{Tr}\text{-}\mathrm{Sig}(L_{\kappa,\rho}^{\mathrm{od}})$. *This signature is independent of κ and ρ satisfying* (11.23)*, and*

$$\mathcal{T}\text{-}\mathrm{Ind}(EUE + \mathbf{1} - E) = \frac{1}{2}\mathcal{T} \otimes \mathrm{Tr}\text{-}\mathrm{Sig}(L_{\kappa,\rho}^{\mathrm{od}}). \tag{11.24}$$

The proof of Theorem 11.6.1 is essentially identical to that of Theorem 10.4.1. Moreover, the necessary modifications are very similar to those made in the proof of the main result in the even case (Theorem 11.6.2 below). Therefore the proof is not spelled out. Full details are provided in [170].

Let us next describe the semifinite spectral localizer in the even case. Let us consider an invertible operator $H = H^* \in \mathcal{N}$. The self-adjoint, invertible, even Dirac operator D^{ev} is again assumed to be affiliated to $\mathcal{N}$. Moreover, there is a symmetry $\Gamma \in \mathcal{N}$ such that D^{ev} anticommutes with Γ and H commutes with Γ. Without loss of generality, it is assumed that Γ is of the form $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. In this basis, the even Dirac operator is of the form

$$D^{\mathrm{ev}} = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}.$$

It is supposed that D^{ev} has a $\mathcal{T}$-compact resolvent, H leaves the domain of D^{ev} invariant, and the commutator $[D^{\mathrm{ev}}, H]$ extends to a bounded operator. Moreover, H is of the form

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}.$$

Furthermore, it is supposed that with respect to the grading induced by Γ, $\mathcal{N} \subset \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$ is of the form $\mathcal{N}' \otimes \mathbb{C}^{2\times 2}$ for some von Neumann algebra $\tilde{\mathcal{N}} \subset \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $\mathcal{T}$ is of the form $\mathcal{T} = \mathcal{T}' \otimes \mathrm{Tr}$ for some semifinite faithful normal trace $\mathcal{T}'$ on $\tilde{\mathcal{N}}$. Let $F = D_0|D_0|^{-1} \in \mathcal{N}'$ denote the unitary phase of D_0. Then

$$[H, D^{\mathrm{ev}}|D^{\mathrm{ev}}|^{-1}] = \begin{pmatrix} 0 & H_+F^* - F^*H_- \\ H_-F - FH_+ & 0 \end{pmatrix}$$

is $\mathcal{T}$-compact, again by following the proof of Theorem 10.1.4. Therefore $H_+F^* - F^*H_-$ and $H_-F - FH_+$ are $\mathcal{T}'$-compact. Thus setting $P_\pm = \chi(H_\pm \leq 0)$, the operator P_-FP_+ is $(P_- \cdot P_+)$-Fredholm with index

$$\begin{aligned}
\mathcal{T}\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) &= \mathcal{T}'(P_{\mathrm{Ker}(P_-FP_+)\cap\mathrm{Ran}(P_+)}) - \mathcal{T}'(P_{\mathrm{Ker}(P_+F^*P_-)\cap\mathrm{Ran}(P_-)}) \\
&= \mathcal{T}'(P_{\mathrm{Ker}(P_-F)\cap\mathrm{Ran}(P_+)}) - \mathcal{T}'(P_{\mathrm{Ker}(P_+F^*)\cap\mathrm{Ran}(P_-)}) \\
&= \mathcal{T}'(P_{\mathrm{Ker}(F^*P_-F)\cap\mathrm{Ran}(P_+)}) - \mathcal{T}'(P_{\mathrm{Ker}(FP_+F^*)\cap\mathrm{Ran}(P_-)}) \\
&= \mathcal{T}'(P_{\mathrm{Ker}(P_-)\cap\mathrm{Ran}(FP_+F^*)}) - \mathcal{T}'(P_{\mathrm{Ran}(\mathbf{1}-FP_+F^*)\cap\mathrm{Ran}(P_-)}) \\
&= \mathcal{T}'(P_{\mathrm{Ran}(\mathbf{1}-P_-)\cap\mathrm{Ran}(FP_+F^*)}) - \mathcal{T}'(P_{\mathrm{Ran}(\mathbf{1}-FP_+F^*)\cap\mathrm{Ran}(P_-)}) \\
&= \mathcal{T}'\text{-Ind}(FP_+F^*, P_-) \\
&= \mathrm{Sf}(F(\mathbf{1} - 2P_+)F^*, \mathbf{1} - 2P_-) \\
&= \mathrm{Sf}(FH_+F^*, H_-).
\end{aligned}$$

The even spectral localizer is defined as the operator

$$L_\kappa^{\mathrm{ev}} = \begin{pmatrix} -H_+ & \kappa D_0^* \\ \kappa D_0 & H_- \end{pmatrix}, \tag{11.25}$$

that is affiliated to $\mathcal{N}$ where $\kappa > 0$ is a tuning parameter. To construct finite-volume restrictions of the spectral localizer, let us now set $(\mathcal{H} \oplus \mathcal{H})_\rho = \mathrm{Ran}(\chi(|D^{\mathrm{ev}}| \leq \rho))$ for a radius $\rho > 0$. Let $\pi_\rho = P_{(\mathcal{H}\oplus\mathcal{H})_\rho} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ denote the projection onto $(\mathcal{H} \oplus \mathcal{H})_\rho$. As D^{ev} has $\mathcal{T}$-compact resolvent, each π_ρ is $\mathcal{T}$-finite. For any operator $B \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$, set $B_\rho = \pi_\rho B \pi_\rho$. With these notations, the finite-volume even spectral localizer is defined by

$$L_{\kappa,\rho}^{\mathrm{ev}} = \begin{pmatrix} -H_+ & \kappa D_0^* \\ \kappa D_0 & H_- \end{pmatrix}_\rho .$$

Theorem 11.6.2. *Let $g = \|H^{-1}\|^{-1}$ be the gap of the invertible self-adjoint operator H. Suppose that*

$$\kappa \le \frac{g^3}{12\|H\|\|[D,H]\|}, \qquad \frac{2g}{\kappa} < \rho. \tag{11.26}$$

Then $(L^{\rm ev}_{\kappa,\rho})^2 \ge \frac{g^2}{4}\pi_\rho$. *In particular,* $L^{\rm ev}_{\kappa,\rho} + \mathbf{1} - \pi_\rho$ *is invertible. As* $\operatorname{supp}(L^{\rm ev}_{\kappa,\rho}) \le \pi_\rho$ *is* $\mathcal{T}$*-finite, it has a well-defined* $\mathcal{T}$*-signature* $\mathcal{T}\text{-Sig}(L^{\rm ev}_{\kappa,\rho})$ *which is independent of* κ *and* ρ *satisfying* (11.26)*, and*

$$\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) = \frac{1}{2}\mathcal{T}\text{-Sig}(L^{\rm ev}_{\kappa,\rho}).$$

Proof. Even though similar to the proof of Theorem 10.3.1, most arguments are given in detail. Again one starts out by showing that the $\mathcal{T}$-signature of $L^{\rm ev}_{\kappa,\rho}$ is independent of κ and ρ satisfying (11.26). The proof will use an even and differentiable tapering function $G_\rho : \mathbb{R} \to [0,1]$ with three properties:

(i) $G_\rho(x) = 1$ for $|x| \le \frac{\rho}{2}$;
(ii) $G_\rho(x) = 0$ for $|x| \ge \rho$;
(iii) The Fourier transform $\widehat{G'_\rho} : \mathbb{R} \to \mathbb{R}$, $\widehat{G'_\rho}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath px} G'_\rho(x)dx$ of the derivative G'_ρ has an L^1-norm bounded by $\frac{8}{\rho}$.

This function is constructed in the proof of Theorem 10.3.1, and, by the argument given there, one now has

$$\|[G_\rho(D), H]\| \le \frac{8}{\rho}\|[D,H]\|, \tag{11.27}$$

where $D = D^{\rm ev}$. To connect radii ρ and $\rho' \ge \rho$, let us consider the operator

$$L_{\kappa,\rho,\rho'}(\lambda) = \kappa\pi_{\rho'} D\pi_{\rho'} + \pi_{\rho'} G_{\lambda,\rho}((-H_+) \oplus H_-)G_{\lambda,\rho}\pi_{\rho'},$$

where $0 \le \lambda \le 1$ and

$$G_{\lambda,\rho} = (1-\lambda)\pi_{\rho'} + \lambda G_\rho(D).$$

Also (11.26) is supposed to hold for the pair κ, ρ and thus also for the pair κ, ρ'. Notice that $L_{\kappa,\rho,\rho'}(0) = L^{\rm ev}_{\kappa,\rho'}$. The first goal is to show that $L_{\kappa,\rho,\rho'}(\lambda)$ is invertible for all $\lambda \in [0,1]$ and that its square is bounded from below by $\frac{g^2}{4}\pi_{\rho'}$ when $\lambda = 0$. The square of $L_{\kappa,\rho,\rho'}(\lambda)$ is

$$\begin{aligned} L_{\kappa,\rho,\rho'}(\lambda)^2 = \kappa^2\pi_{\rho'} D^2\pi_{\rho'} &+ \big(\pi_{\rho'} G_{\lambda,\rho}((-H_+) \oplus H_-)G_{\lambda,\rho}\pi_{\rho'}\big)^2 \\ &- \kappa\pi_{\rho'} G_{\lambda,\rho}[D,H]\Gamma G_{\lambda,\rho}\pi_{\rho'}, \end{aligned}$$

where $D\pi_{\rho'} = \pi_{\rho'} D$ was used and $\Gamma = \operatorname{diag}(\mathbf{1}, -\mathbf{1})$. The second summand is bounded from below as follows:

$$\begin{aligned}
&\big(\pi_{\rho'}G_{\lambda,\rho}((-H_+)\oplus H_-)G_{\lambda,\rho}\pi_{\rho'}\big)^2 \\
&\quad= \pi_{\rho'}G_{\lambda,\rho}HG^2_{\lambda,\rho}HG_{\lambda,\rho}\pi_{\rho'} \\
&\quad\geq \pi_{\rho'}G_{\lambda,\rho}HG_\rho(D)^2HG_{\lambda,\rho}\pi_{\rho'} \\
&\quad= \pi_{\rho'}G_{\lambda,\rho}G_\rho(D)H^2G_\rho(D)G_{\lambda,\rho}\pi_{\rho'} \\
&\qquad+ \pi_{\rho'}G_{\lambda,\rho}[G_\rho(D)H,[G_\rho(D),H]]G_{\lambda,\rho}\pi_{\rho'} \\
&\quad\geq g^2\pi_{\rho'}G^2_{\lambda,\rho}G_\rho(D)^2\pi_{\rho'} + \pi_{\rho'}G_{\lambda,\rho}[G_\rho(D)H,[G_\rho(D),H]]G_{\lambda,\rho}\pi_{\rho'} \\
&\quad\geq g^2\pi_{\rho'}G_\rho(D)^4\pi_{\rho'} + \pi_{\rho'}G_{\lambda,\rho}[G_\rho(D)H,[G_\rho(D),H]]G_{\lambda,\rho}\pi_{\rho'},
\end{aligned}$$

where the first step holds because $[G_{\lambda,\rho},\Gamma] = 0$ and $(\mathcal{H}\oplus\mathcal{H})\ominus(\mathcal{H}\oplus\mathcal{H})_{\rho'} \subset \mathrm{Ker}(G_{\lambda,\rho})$, the first, as well as the last, inequality follows from $G_\rho(D)^2 \leq G^2_{\lambda,\rho}$. For the special case of $\lambda = 0$, one has $G_{0,\rho} = \pi_{\rho'}$ and therefore a better estimate

$$\begin{aligned}
&\big(\pi_{\rho'}G_{0,\rho}((-H_+)\oplus H_-)G_{0,\rho}\pi_{\rho'}\big)^2 \\
&\quad\geq g^2\pi_{\rho'}G_\rho(D)^2\pi_{\rho'} + \pi_{\rho'}[G_\rho(D)H,[G_\rho(D),H]]\pi_{\rho'}.
\end{aligned}$$

Furthermore, by spectral calculus of D, one has the bound

$$\kappa^2\pi_{\rho'}D^2\pi_{\rho'} \geq g^2\pi_{\rho'}(\mathbf{1} - G_\rho(D)^2)\pi_{\rho'},$$

because the bound holds for spectral parameters in $[\frac{1}{2}\rho,\rho']$ due to (11.26) where it was used that $\mathbf{1} - G_\rho(D)^2 \leq \mathbf{1}$ holds, while it holds trivially on $[0,\frac{1}{2}\rho]$. Since

$$\mathbf{1} - G_\rho(D)^2 + G_\rho(D)^4 \geq \frac{3}{4}\mathbf{1},$$

it thus follows that

$$\begin{aligned}
L_{\kappa,\rho,\rho'}(\lambda)^2 \geq{}& \frac{3}{4}g^2\pi_{\rho'} + \pi_{\rho'}G_{\lambda,\rho}[G_\rho(D)H,[G_\rho(D),H]]G_{\lambda,\rho}\pi_{\rho'} \\
&- \kappa\pi_{\rho'}G_{\lambda,\rho}[D,H]\Gamma G_{\lambda,\rho}\pi_{\rho'},
\end{aligned}$$

and in the special case $\lambda = 0$,

$$\begin{aligned}
L_{\kappa,\rho,\rho'}(\lambda)^2 \geq{}& g^2\pi_{\rho'} + \pi_{\rho'}G_{\lambda,\rho}[G_\rho(D)H,[G_\rho(D),H]]G_{\lambda,\rho}\pi_{\rho'} \\
&- \kappa\pi_{\rho'}G_{\lambda,\rho}[D,H]\Gamma G_{\lambda,\rho}\pi_{\rho'}.
\end{aligned}$$

Finally, the error term is bounded using the tapering estimate (11.27):

$$\begin{aligned}
&\big\|[G_\rho(D)H,[G_\rho(D),H]] - \kappa[D,H]\Gamma\big\| \\
&\quad\leq \left(\frac{16}{\rho}\|G_\rho(D)H\| + \kappa\right)\|[D,H]\|
\end{aligned}$$

$$
\begin{aligned}
&< \left(\frac{8}{g}\|H\| + 1\right)\kappa\big\|[D, H]\big\| \\
&\leq \frac{9}{g}\|H\|\kappa\big\|[D, H]\big\| \\
&\leq \frac{3}{4}g^2,
\end{aligned}
$$

where the second step used the second inequality in (11.26), as well as $\|G_\rho(D)\| \leq 1$, the third one $\|H\| \geq g$, and finally the last inequality came from the first inequality in (11.26). Putting all together, one infers $L_{\kappa,\rho,\rho'}(\lambda)^2 > \epsilon\pi_{\rho'}$ for some $\epsilon > 0$ and $L_{\kappa,\rho,\rho'}(0)^2 \geq \frac{1}{4}g^2\pi_{\rho'}$.

Next, let us show that

$$
\mathcal{T}\text{-Sig}(L^{\text{ev}}_{\kappa,\rho}) = \mathcal{T}\text{-Sig}(L^{\text{ev}}_{\kappa',\rho'}),
$$

for pairs κ, ρ and κ', ρ' in the permitted range of parameters. Without loss of generality, let $\rho \leq \rho'$. As $L_{\kappa,\rho}$ is continuous in κ, it is sufficient to consider the case $\kappa = \kappa'$. Thus one needs to show

$$
\mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho}(0)\big) = \mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho'}(0)\big),
$$

when $\rho \leq \rho'$ and (10.10) is true for κ and ρ. Clearly, $L_{\kappa,\rho,\rho'}(\lambda)$ is continuous in λ, so it suffices to prove

$$
\mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho}(1)\big) = \mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho'}(1)\big).
$$

Consider

$$
L_{\kappa,\rho,\rho'}(1) = \kappa\pi_{\rho'}D\pi_{\rho'} + \pi_{\rho'}G_\rho(D)\big((-H_+) \oplus H_-\big)G_\rho(D)\pi_{\rho'}.
$$

Now D commutes with $\pi_{\rho'}$ so that $L_{\kappa,\rho,\rho'}(1)$ decomposes into a direct sum. Let furthermore $\pi_{\rho',\rho} = \pi_{\rho'} - \pi_\rho$ be the projection onto $(\mathcal{H} \oplus \mathcal{H})_{\rho'} \ominus (\mathcal{H} \oplus \mathcal{H})_\rho$. Then

$$
L_{\kappa,\rho,\rho'}(1) = L_{\kappa,\rho,\rho}(1) \oplus \pi_{\rho',\rho}\kappa D\pi^*_{\rho',\rho}.
$$

The signature of $\pi_{\rho',\rho}D\pi_{\rho',\rho}$ vanishes because $\Gamma\pi_{\rho',\rho}D\pi_{\rho',\rho}\Gamma = -\pi_{\rho',\rho}D\pi_{\rho',\rho}$ so that

$$
\mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho'}(1)\big) = \mathcal{T}\text{-Sig}\big(L_{\kappa,\rho,\rho}(1)\big).
$$

It remains to show (11.24), for which $\kappa > 0$ can be chosen as small as needed and ρ as large as needed. For that purpose, let F_ρ be the function constructed in the proof of Theorem 10.3.1, namely F_ρ is an odd increasing differentiable function with $F_\rho(x) = x$ for $|x| \leq \rho$ and $F_\rho(x) = 2\rho = -F_\rho(-x)$ for $x \geq 2\rho$. Furthermore, the L^1-norm of the Fourier transform of the derivative is still bounded by $\frac{28}{\pi}$ so that

$$
\big\|[F_\rho(D), H]\big\| \leq \frac{28}{\pi}\big\|[D, H]\big\|. \tag{11.28}
$$

Moreover, $F_\rho(D)$ anticommutes with Γ, hence is of the form

$$F_\rho(D) = \begin{pmatrix} 0 & (D_0')^* \\ D_0' & 0 \end{pmatrix}.$$

By the above,

$$\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) = \mathrm{Sf}(t \in [0,1] \mapsto (1-t)FH_+F^* + tH_-).$$

Using Theorem 11.4.6, one has

$$\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) = \mathrm{Sf}\left(\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}\begin{pmatrix} -H_+ & 0 \\ 0 & H_+ \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix}, \begin{pmatrix} -H_+ & 0 \\ 0 & H_- \end{pmatrix}\right),$$

where the right-hand side denotes the semifinite spectral flow along the straight-line path between the arguments. One has

$$\begin{aligned}((-H_+ \oplus H_-) + t\kappa F_\rho(D))^2 &= \begin{pmatrix} H_+^2 + (t\kappa)^2|D_0'|^2 & t\kappa((D_0')^*H_- - H_+(D_0')^*) \\ t\kappa(H_-D_0' - D_0'H_+) & H_-^2 + (t\kappa)^2|(D_0')^*|^2 \end{pmatrix} \\ &\geq (g^2 - t\kappa\|[F_\rho(D), H]\|)\mathbf{1},\end{aligned}$$

for $t \in [0,1]$. By (11.28), the straight-line path connecting $-H_+ \oplus H_-$ to $-H_+ \oplus H_- + \kappa F_\rho(D)$ is within the invertibles for κ sufficiently small. As $FH_+F^* - H_-$ is $\mathcal{T}'$-compact, the linear path connecting

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}(-H_+ \oplus H_+)\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix} \quad \text{to } -H_+ \oplus H_- + t\kappa F_\rho(D)$$

is within the Fredholm operators for all $t \in [0,1]$. The homotopy invariance of the spectral flow implies

$$\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) = \mathrm{Sf}\left(\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}\begin{pmatrix} -H_+ & 0 \\ 0 & H_+ \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F^* \end{pmatrix}, \begin{pmatrix} -H_+ & \kappa(D_0')^* \\ \kappa D_0' & H_- \end{pmatrix}\right).$$

Next one directly checks that

$$s \in [0, \kappa\rho] \mapsto \begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}\begin{pmatrix} -H_+ & s \\ s & H_+ \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & F \end{pmatrix}^* = \begin{pmatrix} -H_+ & sF^* \\ sF & FH_+F^* \end{pmatrix}$$

is a path of invertibles. Let us also show that

$$\begin{aligned}A(s,t) &= t\begin{pmatrix} -H_+ & \kappa(D_0')^* \\ \kappa D_0' & H_- \end{pmatrix} + (1-t)\begin{pmatrix} -H_+ & sF^* \\ sF & FH_+F^* \end{pmatrix} \\ &= \begin{pmatrix} -H_+ & t\kappa(D_0')^* + (1-t)sF^* \\ t\kappa D_0' + (1-t)sF & H_- - (1-t)(H_- - FH_+F^*) \end{pmatrix}\end{aligned}$$

is $\mathcal{T}$-Fredholm for all $(s,t) \in [0,\kappa\rho] \times [0,1]$. Because $H_- - FH_+F^*$ is $\mathcal{T}'$-compact, it is sufficient to show that

$$B(s,t) = \begin{pmatrix} -H_+ & t\kappa(D_0')^* + (1-t)sF^* \\ t\kappa D_0' + (1-t)sF & H_- \end{pmatrix}$$

is Fredholm. One can replace $D|D|^{-1}$ by $\frac{1}{2\rho}F_\rho(D)$ as $\operatorname{Ran}(\frac{1}{2\rho}F_\rho(D) - D|D|^{-1}) \subset (\mathcal{H}\oplus\mathcal{H})_{2\rho}$ is $\mathcal{T}$-finite, so that $\frac{1}{2\rho}F_\rho(D) - D|D|^{-1}$ is $\mathcal{T}$-compact. Therefore it is sufficient to show that

$$C(s,t) = -H_+ \oplus H_- + t\kappa F_\rho(D) + (1-t)s\frac{1}{2\rho}F_\rho(D)$$

is $\mathcal{T}$-Fredholm. Now

$$\begin{aligned} C(s,t)^2 &= (-H_+\oplus H_-)^2 + \left(t\kappa F_\rho(D) + (1-t)s\frac{1}{2\rho}F_\rho(D)\right)^2 \\ &\quad - \left[t\kappa F_\rho(D) + (1-t)s\frac{1}{2\rho}F_\rho(D), H\right]\Gamma \\ &\geq \left(g^2 - \left(t\kappa + (1-t)s\frac{1}{2\rho}\right)\|[F_\rho(D),H]\|\right)\mathbf{1} \\ &\geq \left(g^2 - \left(\kappa + \frac{\kappa}{2}\right)\|[F_\rho(D),H]\|\right)\mathbf{1} \\ &\geq \left(g^2 - \frac{42\kappa}{\pi}\|[D,H]\|\right)\mathbf{1}, \end{aligned}$$

where the last step follows form (11.28). Therefore $C(s,t)$ is invertible and $A(s,t)$ is $\mathcal{T}$-Fredholm for all $(s,t) \in [0,\kappa\rho] \times [0,1]$ and κ sufficiently small. This implies by the homotopy invariance of the spectral flow

$$\begin{aligned} \mathcal{T}'\text{-}\mathrm{Ind}_{(P_-\cdot P_+)}(P_-FP_+) &= \mathrm{Sf}\left(\begin{pmatrix} -H_+ & \kappa\rho F^* \\ \kappa\rho F & FH_+F^* \end{pmatrix}, \begin{pmatrix} -H_+ & \kappa(D_0')^* \\ \kappa D_0' & H_- \end{pmatrix}\right) \\ &= \mathrm{Sf}\left(\begin{pmatrix} -H_+ & \kappa\rho F^* \\ \kappa\rho F & FH_+F^* \end{pmatrix}, L^{\kappa,\rho}\right) \end{aligned}$$

for

$$L^{\kappa,\rho} = \begin{pmatrix} -H & \kappa(D_0')^* \\ \kappa D_0' & H \end{pmatrix}.$$

Setting $\pi_{\rho^c} = \mathbf{1} - \pi_\rho$, and $B_{\rho^c} = \pi_{\rho^c}B(\pi_{\rho^c})$ for any operator B on $\mathcal{H}\oplus\mathcal{H}$, one then has $F_\rho(D) = F_\rho(D)_\rho \oplus F_\rho(D)_{\rho^c}$ and $F_\rho(D)_\rho = D_\rho$. Moreover, $(L^{\kappa,\rho})_\rho = L^{\mathrm{ev}}_{\kappa,\rho}$. Next we show that the linear path $t \in [0,1] \mapsto L^{\kappa,\rho}(t)$ for

$$L^{\kappa,\rho}(t) = \begin{pmatrix} (L^{\kappa,\rho})_\rho & 0 \\ 0 & (L^{\kappa,\rho})_{\rho^c} \end{pmatrix} + t\begin{pmatrix} 0 & \pi_\rho(-H_+\oplus H_-)(\pi_{\rho^c})^* \\ \pi_{\rho^c}(-H_+\oplus H_-)(\pi_\rho)^* & 0 \end{pmatrix}$$

is within the invertibles. First, $(L^{\kappa,\rho})_{\rho^c}$ can be bounded from below using (11.26) as

$$\begin{aligned}
&\big(((-H_+ \oplus H_-) + \kappa F_\rho(D))_{\rho^c}\big)^2 \\
&\quad = \big((-H_+ \oplus H_-)_{\rho^c}\big)^2 + \kappa^2\big(F_\rho(D)_{\rho^c}\big)^2 - \kappa[F_\rho(D)_{\rho^c}, H_{\rho^c}]\Gamma_{\rho^c} \\
&\quad \geq \kappa^2\rho^2 - \kappa\|[F_\rho(D)_{\rho^c}, H_{\rho^c}]\|\pi_{\rho^c} \\
&\quad \geq \kappa^2\rho^2 - \frac{28\kappa}{\pi}\|[D,H]\|\pi_{\rho^c} \\
&\quad \geq \frac{1}{2}\kappa^2\rho^2\pi_{\rho^c},
\end{aligned}$$

where the third step follows from (11.28). Now $L^{\kappa,\rho}(t)$ is given by

$$L^{\kappa,\rho}(t) = \big|(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\big|^{\frac{1}{2}}\left(G + t\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\right)\big|(L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\big|^{\frac{1}{2}}$$

where G is a diagonal unitary with respect to the direct sum $\mathcal{H}\oplus\mathcal{H} = (\mathcal{H}\oplus\mathcal{H})_\rho \oplus (\mathcal{H}\oplus\mathcal{H})_{\rho^c}$ and

$$B = \big|(L^{\kappa,\rho})_\rho\big|^{-\frac{1}{2}}\pi_\rho(-H_+ \oplus H_-)\pi_{\rho^c}\big|(L^{\kappa,\rho})_{\rho^c}\big|^{-\frac{1}{2}}.$$

The off-diagonal entries satisfy

$$\|B\| \leq \frac{\sqrt[4]{8}\|H\|}{\sqrt{\kappa\rho g}},$$

thus their norm is smaller than 1 for ρ sufficiently large. Because $L^{\kappa,\rho} - ((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})$ is $\mathcal{T}$-finite and therefore $\mathcal{T}$-compact, the homotopy invariance of the spectral flow then implies

$$\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) = \mathrm{Sf}\left(\begin{pmatrix} -H_+ & \kappa\rho F^* \\ \kappa\rho F & FH_+F^* \end{pmatrix}, (L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c}\right).$$

The path

$$s \in [0,1] \mapsto A(s) = \begin{pmatrix} -sH_+ & \kappa\rho F^* \\ \kappa\rho F & sFH_+F^* \end{pmatrix}$$

is within the invertibles for ρ sufficiently large. As $tA(s)_{\rho^c} + (1-t)(L^{\kappa,\rho})_{\rho^c}$ is invertible for all $(s,t) \in [0,1] \times [0,1]$ and ρ sufficiently large,

$$tA(s) + (1-t)((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})$$

is $\mathcal{T}$-Fredholm for all $(s,t) \in [0,1] \times [0,1]$, so that

$$\begin{aligned}\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) &= \mathrm{Sf}\big(\kappa\rho D|D|^{-1}, ((L^{\kappa,\rho})_\rho \oplus (L^{\kappa,\rho})_{\rho^c})\big)\\ &= \mathrm{Sf}\big(\kappa\rho(D|D|^{-1})_\rho, (L^{\kappa,\rho})_\rho\big)\\ &\quad + \mathrm{Sf}\big(\kappa\rho(D|D|^{-1})_{\rho^c}, (L^{\kappa,\rho})_{\rho^c}\big).\end{aligned}$$

The second summand vanishes because the linear path

$$t \in [0,1] \mapsto (1-t)\kappa\rho(D|D|^{-1})_{\rho^c} + t(L^{\kappa,\rho})_{\rho^c}$$

lies in the invertibles for ρ sufficiently large. As $(L^{\kappa,\rho})_\rho = L^{\mathrm{ev}}_{\kappa,\rho}$, Theorem 11.4.6 implies

$$\begin{aligned}\mathcal{T}'\text{-Ind}_{(P_-\cdot P_+)}(P_-FP_+) &= \mathrm{Sf}\big(\kappa\rho(D|D|^{-1})_\rho, L^{\mathrm{ev}}_{\kappa,\rho}\big)\\ &= \frac{1}{2}\big(\mathcal{T}\text{-Sig}(L^{\mathrm{ev}}_{\kappa,\rho}) - \mathcal{T}\text{-Sig}(D_\rho)\big).\end{aligned}$$

As $\Gamma D\Gamma = -D$, the $\mathcal{T}$-signature of D_ρ vanishes and the claim follows. □

12 Spectral flow in bifurcation theory

The aim of this chapter is to explain the role of spectral flow in variational bifurcation theory which is a branch of nonlinear functional analysis that deals with the sudden appearance of critical points of families of functionals when a parameter varies. Throughout this chapter and in contrast to earlier chapters, the separable Hilbert space $\mathcal{H}$ is assumed to be real. The spectral flow of a path of self-adjoint Fredholm operators on $\mathcal{H}$ can verbatim be defined as in Definition 4.1.2, and it is readily seen that all properties of the spectral flow from Section 4.2 also hold on real Hilbert spaces. Alternatively, one can define the spectral flow by Definition 4.1.2 for the path of complexified operators on $\mathcal{H} \otimes \mathbb{C}$.

For a differentiable functional $f : \mathcal{H} \to \mathbb{R}$, the derivative $D_u f$ at some $u \in \mathcal{H}$ is an element of $\mathbb{B}(\mathcal{H}, \mathbb{R})$. By the Riesz representation theorem, the latter space is canonically isomorphic to $\mathcal{H}$, and this yields some $(\nabla f)(u)$ which is uniquely determined by

$$(D_u f)(v) = \big\langle (\nabla f)(u) \big| v \big\rangle, \quad v \in \mathcal{H}.$$

The element $(\nabla f)(u)$ of $\mathcal{H}$ is called the gradient of f at u. If f is twice continuously differentiable, the bounded symmetric bilinear form $D_u^2 f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ can canonically be identified with a bounded self-adjoint operator $H \in \mathbb{B}_{\text{sa}}(\mathcal{H})$ by

$$D_u^2 f(v, w) = \langle v | Hw \rangle, \quad v, w \in \mathcal{H}, \tag{12.1}$$

which is called the Hessian of f at u.

12.1 A primer of variational bifurcation theory

The main object of study in this chapter are continuous one-parameter families of C^2 -functionals $f : [a, b] \times \mathcal{H} \to \mathbb{R}$, i. e., each $f_t = f(t, \cdot) : \mathcal{H} \to \mathbb{R}$ is a C^2-functional such that $\nabla f_t(u)$ and $D_u^2 f_t$ depend continuously on $(t, u) \in [a, b] \times \mathcal{H}$. Let us consider the determining equations for a critical point u,

$$(\nabla f_t)(u) = 0, \tag{12.2}$$

where $\nabla f_t = \nabla_{\mathcal{H}} f_t$ is the gradient with respect to the argument from $\mathcal{H}$. The standing assumption in the following is that

$$(\nabla f_t)(0) = 0, \quad t \in [a, b], \tag{12.3}$$

i. e., $0 \in \mathcal{H}$ is a critical point of all functionals $f_t : \mathcal{H} \to \mathbb{R}$. The next definition will be crucial in this section.

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Definition 12.1.1. A parameter value $t^* \in [a,b]$ is called a bifurcation point for the equations (12.2) if, for every neighborhood $\mathcal{U} \subset [a,b] \times \mathcal{H}$ of $(t^*, 0)$, there is some $(t,u) \in \mathcal{U}$ such that $(\nabla f_t)(u) = 0$ and $u \neq 0$.

The presumably simplest finite-dimensional example is $f : [-1,1] \times \mathbb{R} \to \mathbb{R}$ given by $f(t,u) = \frac{u^2}{2}(\frac{u^2}{2} - t)$, where two branches of nonzero critical points appear as t increases past 0. This is called a continuous pitchfork bifurcation. The following more sophisticated infinite-dimensional example will be continued below.

Example 12.1.2. Recall that the Sobolev space $W_0^{1,2}([0,1], \mathbb{R}^N)$ consists of all absolutely continuous vector-valued functions on $[0,1]$ which have a square integrable derivative and vanish at 0 and 1. It is a Hilbert space with scalar product

$$\langle u|v\rangle = \int_0^1 \langle \dot{u}(s)|\dot{v}(s)\rangle_{\mathbb{R}^N}\, ds. \tag{12.4}$$

Now let $F : [0,1] \times \mathbb{R}^N \to \mathbb{R}$ be a C^2-function such that

$$\|D_u^2 F(s,\cdot)\| \le c(1 + \|u\|^r), \quad u \in \mathbb{R}^n,$$

for some constants $c, r > 0$. Then

$$f(t,u) = \frac{1}{2}\int_0^1 \langle \dot{u}(s)|\dot{u}(s)\rangle_{\mathbb{R}^n}\, ds - t\int_0^1 F(s,u(s))\, ds \tag{12.5}$$

is a continuous family of C^2-functionals on $\mathcal{H} = W_0^{1,2}([0,1], \mathbb{R}^N)$ (see [156, Proposition B.34]). The derivative of $f_t = f(t,\cdot)$ at some $u \in \mathcal{H}$ is, when evaluated on $v \in \mathcal{H}$, given by

$$(D_u f)v = \int_0^1 \langle \dot{u}(s)|\dot{v}(s)\rangle_{\mathbb{R}^N}\, ds - t\int_0^1 \langle (\nabla_u F)(s,u(s))|v(s)\rangle_{\mathbb{R}^N}\, ds.$$

Now classical regularity theory [156] shows that the critical points of $f_t : \mathcal{H} \to \mathbb{R}$ are the solutions of the boundary value problem

$$\begin{cases} -\ddot{u}(s) = t(\nabla_u F)(s,u(s)), & \text{for } s \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{12.6}$$

If $(\nabla_u F)(s,0) = 0$ for all $s \in [0,1]$, then $0 \in \mathcal{H}$ is a critical point of all f_t, and thus one is in the setting of (12.2) and (12.3). A bifurcation point of the family of functionals f yields values of the parameter t at which nontrivial solutions of (12.6) appear. ◇

The aim of bifurcation theory is to understand which parameter values $t^* \in [a,b]$ are bifurcation points. The first aim of this section is to give a necessary criterion for a parameter value to be a bifurcation point.

Theorem 12.1.3. *Let $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ be a continuous family of C^2-functionals that satisfies* (12.3) *and let $H_t \in \mathbb{B}_{sa}(\mathcal{H})$ be the Hessians of f_t at $0 \in \mathcal{H}$, namely*

$$\langle u|H_t v\rangle = (D_0^2 f_t)(u,v), \quad u,v \in \mathcal{H}. \tag{12.7}$$

If t^* is a bifurcation point for f, then H_{t^*} is not invertible.

Proof. This is a simple consequence of the implicit function theorem in Banach spaces (see, e. g., [69, § 15]). By assumption, the map

$$G : [a,b] \times \mathcal{H} \to \mathcal{H}, \quad G(t,u) = (\nabla f_t)(u),$$

is continuous and continuously differentiable with respect to u. Again set $G_t = G(t,\cdot)$. Now, if $H_{t^*} = D_0 G_{t^*}$ is invertible, then there are an open interval $J \subset [a,b]$ containing t^*, a neighborhood $V \subset \mathcal{H}$ of 0, and a differentiable map $g : J \to V$ such that $G(t,u) = 0$ for $(t,u) \in J \times V$ if and only if $u = g(t)$. As $G(t,0) = 0$ for all $t \in [a,b]$ by (12.3), it follows that the only solutions of $G(t,u) = 0$ in $J \times V$ are of the form $(t,0)$. Consequently, t^* is not a bifurcation point of f. □

It is readily seen that Theorem 12.1.3 is necessary, but not sufficient, for the existence of a bifurcation point. For example, let $\mathcal{H} = \mathbb{R}$, $I = [-1,1]$ and consider the functions $f(t,u) = t^2\frac{u^2}{2} + \frac{u^4}{4}$. Then $H_t = t^2$ is singular for $t = 0$, but none of the f_t has other critical points than $0 \in \mathcal{H}$. The following example shows that this can also happen with an affine parameter dependence, which will be of interest below.

Example 12.1.4. Consider on $\mathcal{H} = \mathbb{R}^2$ the family of functionals $f : [0,2] \times \mathcal{H} \to \mathbb{R}$ given by

$$f(t,u,v) = \frac{1}{2}(1-t)(v^2 - u^2) - t(u^3 v + v^3 u), \quad (u,v) \in \mathcal{H}.$$

Then

$$(\nabla f_t)(u,v) = (1-t)\begin{pmatrix} -u \\ v \end{pmatrix} - t\begin{pmatrix} 3u^2 v + v^3 \\ u^3 + 3v^2 u \end{pmatrix} \tag{12.8}$$

and

$$H_t = (1-t)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{12.9}$$

Thus H_t is singular for $t = 1$. However, if one multiplies in $(\nabla f_t)(u,v) = 0$ the first equation by v, the second by u, and adds the results, it follows that $t(u^4 + v^4 + 6u^2 v^2) = 0$, and hence

$u = v = 0$ if t is close to 1. Consequently, there is no bifurcation of critical points for f. The attentive reader may already note that the spectral flow $\mathrm{Sf}(t \in [0,2] \mapsto H_t)$ vanishes. ◇

Example 12.1.5. This is a continuation of Example 12.1.2, so all objects are as stated there. Let $A(s) = D_0^2 F_s$ be the Hessian matrix of $F_s = F(s,\cdot) : \mathbb{R}^N \to \mathbb{R}$ at the critical point $0 \in \mathbb{R}^N$. Then the second derivatives of the functionals $f_t : \mathcal{H} \to \mathbb{R}$ at $0 \in \mathcal{H}$ are given by (see [156, Proposition B.34])

$$\langle u|H_t v\rangle = \int_0^1 \langle \dot{u}(s)|\dot{v}(s)\rangle_{\mathbb{R}^N}\, ds - t\int_0^1 \langle A(s)u(s)|v(s)\rangle_{\mathbb{R}^N}\, ds, \quad u,v \in \mathcal{H}.$$

As the first term on the right-hand side is the scalar product on $\mathcal{H} = W_0^{1,2}([0,1],\mathbb{R}^N)$ (see (12.4)) and as $\mathcal{H}$ is compactly embedded into $L^2([0,1],\mathbb{R}^N)$, the self-adjoint operators H_t are compact perturbations of the identity and thus Fredholm of index 0. Moreover, it follows from standard regularity theory that the kernel of H_t consists of the solutions of the linear boundary value problem

$$\begin{cases} -\ddot{u}(s) = tA(s)u(s), & \text{for } s \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{12.10}$$

Consequently, by Theorem 12.1.3, a bifurcation can only occur at those $t \in \mathbb{R}$ for which these boundary value problems have a nontrivial solution. ◇

Let us now assume that $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ is a continuous family of C^2-functionals such that

$$\nabla f_t = \mathbf{1}_{\mathcal{H}} - t(K+R), \tag{12.11}$$

where K is a compact self-adjoint linear operator and $R(u) = o(\|u\|)$ as $u \to 0$. Note that $H_t = \mathbf{1}_{\mathcal{H}} - tK$, and thus Theorem 12.1.3 shows that $t = 0$ cannot be a bifurcation point of (12.2). Moreover, if t_0 is a bifurcation point, then $\frac{1}{t_0}$ is an eigenvalue of the compact self-adjoint operator K. The following classical theorem in variational bifurcation theory has its origin in the work of Krasnoselkii in the 1960s (see [117]). Various generalizations and alternative proofs appeared over the following decades. The presentation here follows the monograph [69, § 30].

Theorem 12.1.6. *Suppose that $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ satisfies* (12.3) *and* (12.11). *If $t_0 \in (a,b)$ is such that $\frac{1}{t_0}$ is an eigenvalue of K, then t_0 is a bifurcation point.*

Proof. As the theorem is also a direct consequence of Theorem 12.2.1 below, we only sketch the argument (see [69] for details). As already observed above, $t_0 = 0$ cannot be a bifurcation point. Thus one can consider instead of (12.11) the family of equations

$$\lambda u = Ku + R(u), \quad u \in \mathcal{H}, \tag{12.12}$$

where $\lambda = \frac{1}{t}$. Now assume that λ_0 is an eigenvalue of K and let P be the orthogonal projection onto the kernel of

$$T = \lambda_0 \mathbf{1}_{\mathcal{H}} - K.$$

Setting $\lambda = \lambda_0 + \mu$, $v = Pu$ and $z = (\mathbf{1} - P)u$, it follows that (12.12) is equivalent to the equations

$$z = \mu S z - S(\mathbf{1} - P)R(v + z), \quad \mu v = PR(v + z), \tag{12.13}$$

where S denotes the inverse of T, which is defined and bounded on $\operatorname{Ran}(T)$. Now the implicit function theorem yields a C^1-map $z = z(\mu, v)$ that solves the first equation in (12.13) for sufficiently small $|\mu|$ and $\|v\|$ and such that $z(\mu, 0) = 0$ for all μ. Plugging this into the second equation in (12.13) yields

$$\mu v = PR\big(v + z(\mu, v)\big). \tag{12.14}$$

Hence (12.12) is reduced to finite dimensions and it remains to show that 0 is a bifurcation point for the latter equation. This procedure is usually called Lyapunov–Schmidt reduction. Now consider the function

$$g(\mu, v) = \begin{cases} \mu - \frac{1}{\|v\|^2} \langle PR(v + z(\mu, v)) | v \rangle, & v \neq 0, \\ \mu, & v = 0, \end{cases}$$

which is continuous in a neighborhood of $(0, 0)$ and continuously differentiable in μ with a nonvanishing partial derivative at $(0, 0)$. The implicit function theorem yields a continuous map $\mu(v)$ defined in a neighborhood $B_r(0)$ of 0 in $\operatorname{Ker}(T)$ such that $g(\mu(v), v) = 0$, $v \in B_r(0)$, or equivalently

$$\mu(v)\|v\|^2 = \big\langle PR\big(v + z(\mu(v), v)\big)\big| v\big\rangle. \tag{12.15}$$

A careful analysis of the map $G : B_r(0) \to \operatorname{Ran}(T)$ given by $G(v) = z(\mu(v), v)$ shows that G is actually C^1 with $G(v) = o(\|v\|)$ and $D_v G = o(1)$. Next consider the functional

$$\Phi : B_r(0) \subset \operatorname{Ker}(T) \to \mathbb{R}, \quad \Phi(v) = \tilde{f}\big(v + G(v)\big),$$

where $\tilde{f} : \mathcal{H} \to \mathbb{R}$ is such that $\nabla \tilde{f} = K + R$, and the sets

$$M_\varepsilon = \big\{v \in \operatorname{Ker}(T) : \big\|v + G(v)\big\| = \varepsilon\big\}.$$

It is readily seen that M_ε is a compact submanifold of $\operatorname{Ker}(T)$ if ε is sufficiently small. Thus $\Phi|_{M_\varepsilon}$ has at least two critical points. Now, calculating the derivative of $\Phi|_{M_\varepsilon}$ and using (12.13) yields that if v is a critical point of $\Phi|_{M_\varepsilon}$, then there is some real $\lambda = \lambda(v)$ such that

$$\langle (\lambda_0 - \lambda)v + PR(v + G(v)) \big| h \rangle = -\langle (\mu(v) + \lambda_0 - \lambda)G(v) \big| (D_v G)h \rangle, \tag{12.16}$$

for all $h \in \mathrm{Ker}(T)$. For $h = v$, this implies by (12.15)

$$(\lambda_0 - \lambda + \mu(v))(\|v\|^2 + \langle G(v) \big| (D_v G)v \rangle) = 0.$$

As $G(v) = o(\|v\|)$ and $D_v G = o(1)$, it follows that

$$\lambda = \lambda_0 + \mu(v) \tag{12.17}$$

for $\|v\|$ sufficiently small. Moreover, note that, as $\mathrm{Ran}(G) \subset \mathrm{Ran}(T)$, one can conclude that $\|v + G(v)\|^2 = \|v\|^2 + \|G(v)\|^2$ and thus $\|v\| \le \varepsilon$ as $v \in M_\varepsilon$. Thus it is no loss of generality to require $\|v\|$ to be small. Plugging (12.17) into (12.16) shows that

$$\mu(v)v = PR(v + G(v)) = PR(v + z(\mu(v), v)),$$

and consequently $(\mu(v), v)$ is a solution of (12.14) and thus yields a solution of (12.12). Let us denote this solution by v_ε. As μ is continuous, $\mu(0) = 0$ and $\|v_\varepsilon\| \le \varepsilon$, it follows that $(\mu(v_\varepsilon), v_\varepsilon) \to (0, 0)$ as $\varepsilon \to 0$ and thus λ_0 is a bifurcation point for (12.12). □

Example 12.1.7. Let us further elaborate on Example 12.1.5. It was shown in Example 12.1.2 that if $t \in [a, b]$ is a bifurcation point of critical points for f in (12.5), then the boundary value problem (12.10) has a nontrivial solution. As ∇f is of the form (12.11), Theorem 12.1.6 shows that also the converse is true. Hence the problem of finding the bifurcation points of f is entirely reduced to finding nontrivial solutions of (12.10). Note that the motivation of studying bifurcations of f is to find parameter values where nontrivial solutions of the nonlinear differential equations (12.6) appear. ◇

The final example of this section shows that the assumption that H_t is a compact perturbation of the identity in Theorem 12.1.6 cannot be lifted.

Example 12.1.8. Consider on $\mathcal{H} = \mathbb{R}^2$ the family of functionals in Example 12.1.4. Here the gradients are due to (12.8) of the form

$$(\nabla f_t)(u) = Au - tKu + tR(u),$$

where $A = K = \mathrm{diag}(-1, 1)$ and R satisfies the required growth condition. As K is compact on the finite-dimensional space $\mathbb{R}^2$, the only difference to (12.11) is that A is not the identity. Recall that f has no bifurcation points by Example 12.1.4. ◇

12.2 The spectral flow in variational bifurcation theory

The main theorem of the previous section states that if $f : [a, b] \times \mathcal{H} \to \mathbb{R}$ is a continuous family of C^2-functionals such that the gradients of f_t are of the form (12.11), then $t \in (a, b)$

is a bifurcation point of f if and only if $\frac{1}{t}$ is an eigenvalue of K. On the other hand, Example 12.1.8 shows that the particular form of the operators in (12.11) is necessary for this result. To motivate the main theorem of this chapter, note that the Hessians of f_t at $0 \in \mathcal{H}$ in Theorem 12.1.6 are of the form $H_t = \mathbf{1}_{\mathcal{H}} - tK$. Clearly, $H_t \in \mathbb{FB}^+_{\mathrm{sa}}(\mathcal{H})$ and H_t can only be noninvertible for finitely many t in the compact interval $[a,b]$. If t_0 is such a parameter in the interior of $[a,b]$ and $\varepsilon > 0$ is such that H_t is invertible for $t \in [t_0-\varepsilon, t_0+\varepsilon] \setminus \{t_0\}$, then $\mathrm{Sf}(t \in [t_0-\varepsilon, t_0+\varepsilon] \mapsto H_t) \neq 0$. If, however, H_t are the Hessians of the functionals in Example 12.1.8, then H_t is given by (12.9) and here the spectral flow through $t = 1$ clearly vanishes.

The following theorem is the main result of this chapter and it is due to Fitzpatrick, Pejsachowicz, and Recht [84].

Theorem 12.2.1. *Let $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ be a C^2-map such that* (12.3) *holds. As in* (12.1), *let H_t denote the Hessian of f_t at $0 \in \mathcal{H}$ and assume that $H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ for all $t \in [a,b]$. If H_a, H_b are invertible and $\mathrm{Sf}(t \in [a,b] \mapsto H_t) \neq 0$, then there is a bifurcation of critical points for f.*

As shown in Proposition 4.3.1, $\mathrm{Sf}(t \in [a,b] \mapsto H_t) = \iota_-(H_a) - \iota_-(H_b)$ if the operators H_t are in the component $\mathbb{FB}^+_{\mathrm{sa}}(\mathcal{H})$ of $\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ and H_a, H_b are invertible. Thus the following corollary is an immediate consequence of the previous theorem (see [132, 180]).

Corollary 12.2.2. *If H_a, H_b are invertible, $H_t \in \mathbb{FB}^+_{\mathrm{sa}}(\mathcal{H})$, $t \in [a,b]$, and $\iota_-(H_a) \neq \iota_-(H_b)$, then there is a bifurcation of critical points of f.*

Note that Theorem 12.1.6 is an immediate consequence of this corollary.

The assumption on the invertibility of the endpoints in Theorem 12.2.1 cannot be lifted as can be seen by the following simple example. Consider $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $f_t(u) = -\frac{u^2}{2}(\frac{u^2}{2} - t + 1)$. Then $H_t = t - 1$ and thus $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2} \neq 0$ by Proposition 4.3.1. However, it is readily seen that 0 is the only critical point of the functionals f_t for $t \in [0,1]$ and thus there is no bifurcation.

Actually, the following theorem which is cited without proof shows that also the nonvanishing of the spectral flow cannot be lifted from the hypothesis of Theorem 12.2.1 in the following sense.

Theorem 12.2.3 ([4]). *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ be a path and $t_0 \in (0,1)$ such that H_t is invertible for $t \neq t_0$. If $\mathrm{Sf}(t \in [0,1] \mapsto H_t) = 0$, then there exist an open interval $J \subset [0,1]$ containing t_0, an open ball $B \subset \mathcal{H}$ and a continuous family $f : J \times B \to \mathbb{R}$ of C^2-functionals such that H_t are the Hessians of f_t at $0 \in \mathcal{H}$ and* (12.3) *holds for $t \in J$, but there is no bifurcation of critical points for f in J.*

Let us stress that the proof of this theorem uses concepts of differential geometry and transversality theory and is already highly nontrivial in finite dimensional Hilbert spaces. As explained at the beginning of this section, the assertion of Theorem 12.2.3 occurs for the functionals in Example 12.1.8.

The proof of Theorem 12.2.1 will be split into two main steps. The first step proves the theorem in the case that $\mathcal{H}$ is of finite dimension. Secondly, it is shown how the general case can be reduced to finite dimensions by using basic properties of the spectral flow and Theorem 5.9.3.

Step 1 (Proof in finite dimensions via Morse theory). Let $\mathcal{H}$ be of finite dimension and let $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ be C^2 and such that $\nabla f_t(0) = 0$ for all $t \in [a,b]$. As in Theorem 12.2.1, H_a and H_b are supposed to be invertible. The nontriviality of $\mathrm{Sf}([a,b] \ni t \mapsto H_t)$ is now equivalent to

$$\iota_-(H_a) \neq \iota_-(H_b). \tag{12.18}$$

Note that it is no loss of generality to assume that $f_t(0) = 0$ for all $t \in [a,b]$.

The maps $f_t : \mathcal{H} \to \mathbb{R}$ induce a family of flows $\varphi_t : \mathcal{D}_t \to \mathcal{H}$, where

$$\mathcal{D}_t = \{(s,u) \in \mathbb{R} \times \mathcal{H} : \alpha_t(u) < s < \omega_t(u)\}$$

is an open subset of $\mathbb{R} \times \mathcal{H}$ and $\varphi_t(\cdot,u) : (\alpha_t(u), \omega_t(u)) \to \mathcal{H}$ is the unique maximal solution of the initial value problem

$$\begin{cases} \partial_s \varphi_t(s,u) = -\nabla f_t(\varphi_t(s,u)), \\ \varphi_t(0,u) = u. \end{cases}$$

Note that

$$\partial_s f_t(\varphi_t(s,u)) = -\|\nabla f_t(\varphi_t(s,u))\|^2, \quad (s,u) \in \mathcal{D}_t, \tag{12.19}$$

and thus either $\varphi_t(s,u) = u$ for all s or $f_t(\varphi_t(\cdot,u))$ is decreasing.

To prove Theorem 12.2.1, it will henceforth be assumed by contradiction that there is no bifurcation point of $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ in $[a,b]$. Then there is $\rho > 0$ such that there are no critical points of f_t in $B_{2\rho}(0)$ for all $t \in [a,b]$. Now set $f_t^c = f_t^{-1}((-\infty, c])$ for $c \in \mathbb{R}$ and consider the dimensions $d_k(t)$ of the (singular) homology groups

$$H_k(f_t^0 \cap \overline{B}_\rho(0), (f_t^0 \setminus \{0\}) \cap \overline{B}_\rho(0); \mathbb{Z}_2) \tag{12.20}$$

with coefficients in $\mathbb{Z}_2$ for $k \in \mathbb{N}_0$. Note that these homology groups are actually vector spaces as $\mathbb{Z}_2$ is a field. Thus the dimension $d_k(t)$ is indeed defined. Moreover, by the excision property of homology ρ can be replaced by any smaller $0 < \rho' < \rho$ without affecting $d_k(t)$. Set

$$\delta = \inf\left\{\|\nabla f_t(u)\| : \frac{\rho}{2} \leq \|u\| \leq \rho,\ 0 \leq t \leq 1\right\} > 0,$$

and

$$\varepsilon = \frac{1}{4}\delta\rho, \quad B = \overline{B}_{\frac{\rho}{2}}(0) \cap f_t^{\varepsilon}.$$

Note that ε does not depend on t. The next claim is that, if $u \in B$, then either $\varphi_t(s,u)$ stays in $B_\rho(0)$ for all $s \in [0, \omega_t(u))$, or $\varphi_t(s,u)$ stays in $B_\rho(0)$ until $f_t(\varphi_t(s,u))$ is less than $-\varepsilon$. Indeed, if $u \in B$ is such that $\varphi_t(s,u)$ does not stay in $B_\rho(0)$ for all times $s \in [0, \omega_t(u))$, then there are minimal $0 \le s_1 < s_2 < \omega_t(u)$ such that $\frac{\rho}{2} \le \|\varphi_t(s,u)\| \le \rho$, $s_1 \le s \le s_2$ and $\|\varphi_t(s_1,u)\| = \frac{\rho}{2}$, as well as $\|\varphi_t(s_2,u)\| = \rho$. Consequently,

$$\begin{aligned} f_t(\varphi_t(s_2,u)) &\le f_t(\varphi_t(s_1,u)) - \delta \int_{s_1}^{s_2} \|\nabla f_t(\varphi_t(s,u))\| ds \\ &\le f_t(u) - \delta \int_{s_1}^{s_2} \|\dot{\varphi}_t(s,u)\| ds \\ &\le f_t(u) - \delta \|\varphi_t(s_1,u) - \varphi_t(s_2,u)\| \\ &\le \varepsilon - \delta(\|\varphi_t(s_2,u)\| - \|\varphi_t(s_1,u)\|) \\ &\le \varepsilon - 2\varepsilon = -\varepsilon, \end{aligned} \tag{12.21}$$

which shows the claim.

Now define X_t as the closure of

$$\{\varphi_t(s,u) : u \in B, 0 \le s < \omega_t(u)\}.$$

By construction, X_t is a closed neighborhood of 0 such that $\varphi_t(s,u) \in X_t$ for all $u \in X_t$ and all $s < \omega_t(u)$ (see [6, Remark 16.3(e)]). Moreover, $f_t^{-1}([-\varepsilon,\varepsilon]) \cap X_t \subset \overline{B}_\rho(0)$ by (12.21). In particular, 0 is the only critical point of f_t in $f_t^{-1}([-\varepsilon,\varepsilon]) \cap X_t$.

Set $X_t^c = X_t \cap f_t^c$ for $c \in \mathbb{R}$. It is a standard argument in Morse theory to use the flow φ_t and (12.19) for showing that X_t^0 is a strong deformation retract of X_t^ε, as well as $X_t^{-\varepsilon}$ is a strong deformation retract of $X_t^0 \setminus \{0\}$ (cf., e. g., [132, Lemma 8.3]), and that these deformations induce isomorphisms

$$H_k(X_t^\varepsilon, X_t^{-\varepsilon}; \mathbb{Z}_2) \cong H_k(X_t^0, X_t^0 \setminus \{0\}; \mathbb{Z}_2), \quad k \in \mathbb{N}_0. \tag{12.22}$$

If now $B' \subset f_t^{-1}([-\varepsilon,\varepsilon]) \cap X_t \subset \overline{B}_\rho(0)$ is a closed ball of positive radius about 0, then by the excision property of homology

$$\begin{aligned} H_k(X_t^0, X_t^0 \setminus \{0\}; \mathbb{Z}_2) &\cong H_k(X_t^0 \cap B', (X_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2) \\ &= H_k(f_t^0 \cap B', (f_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2). \end{aligned} \tag{12.23}$$

Note that the dimension of the latter is $d_k(t)$.

After these preliminaries, the next aim is to show that for any $t_0 \in [a,b]$ fixed, there is $\eta > 0$ such that $d_k(t) = d_k(t_0)$ for all $t \in [t_0 - \eta, t_0 + \eta] \cap [a,b]$ and all k. Let $\zeta > 0$ be such that

$$\overline{B}_{2\zeta}(0) \subset f_{t_0}^{-1}\left(\left[-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon\right]\right) \cap X_{t_0} \subset \overline{B}_\rho(0) \tag{12.24}$$

and $\psi : \mathcal{H} \to [0,1]$ a C^2-function such that $\psi(u) = 1$ if $\|u\| \le \frac{\zeta}{2}$, $\psi(u) = 0$ if $\|u\| \ge \zeta$ and

$$\gamma = \sup_{u \in \overline{B}_\zeta(0)} \|\nabla\psi(u)\| < \infty.$$

Set

$$\tilde{\delta} = \inf\left\{\|\nabla f_{t_0}(u)\| : \frac{\zeta}{2} \le \|u\| \le \zeta\right\} > 0,$$

and

$$\mu = \min\left\{\frac{\varepsilon}{3}, \frac{\tilde{\delta}}{2(1+\gamma)}\right\}.$$

Define $\tilde{f}_t : \mathcal{H} \to \mathbb{R}$ by

$$\tilde{f}_t(u) = f_{t_0}(u) + \psi(u)(f_t(u) - f_{t_0}(u)).$$

As f is C^2, there is $\eta > 0$ such that

$$\sup_{u \in \overline{B}_\zeta(0)} |f_t(u) - f_{t_0}(u)| + \sup_{u \in \overline{B}_\zeta(0)} \|\nabla f_t(u) - \nabla f_{t_0}(u)\| < \mu$$

for all $|t - t_0| < \eta$. Now, if $|t - t_0| < \eta$ and $\frac{\zeta}{2} \le \|u\| \le \zeta$, then

$$\begin{aligned} \|\nabla\tilde{f}_t(u)\| &\ge \|\nabla f_{t_0}(u)\| - \psi(u)\|\nabla f_t(u) - \nabla f_{t_0}(u)\| - \|\nabla\psi(u)\||f_t(u) - f_{t_0}(u)| \\ &\ge \tilde{\delta} - (1+\gamma)\mu \ge \frac{\tilde{\delta}}{2}, \end{aligned} \tag{12.25}$$

and, moreover,

$$|\tilde{f}_t(u) - f_{t_0}(u)| = \psi(u)|f_t(u) - f_{t_0}(u)| \le \mu \le \frac{\varepsilon}{3} \tag{12.26}$$

for $|t - t_0| < \eta$ and $u \in \overline{B}_\zeta(0)$.

As $\tilde{f}_t(u) = f_{t_0}(u)$ for $\|u\| \ge \zeta$, it follows from (12.24) and (12.26) that $\tilde{f}_t^{\pm\varepsilon} = f_{t_0}^{\pm\varepsilon}$ and thus $\tilde{f}_t^{-1}([-\varepsilon,\varepsilon]) \cap X_{t_0} = f_{t_0}^{-1}([-\varepsilon,\varepsilon]) \cap X_{t_0}$. Moreover, $\overline{B}_\zeta(0)$ is contained in the interior of X_{t_0}, showing that X_{t_0} is positively invariant for the flow of $\tilde{f}_t$. Finally, (12.25) implies that 0 is the only critical point of $\tilde{f}_t$ in $\overline{B}_\zeta(0)$. Hence, it follows by the same argument as in (12.22) and (12.23) that

$$H_k(X_{t_0}^{\varepsilon}, X_{t_0}^{-\varepsilon}; \mathbb{Z}_2) \cong H_k(\tilde{f}_t^0 \cap B', (\tilde{f}_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2)$$

and thus

$$H_k(f_{t_0}^0 \cap B', (f_{t_0}^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2) \cong H_k(\tilde{f}_t^0 \cap B', (\tilde{f}_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2) \tag{12.27}$$

if the radius of B' is sufficiently small.

Finally, as the isomorphism class of (12.20) does not depend on the radius, and as $\tilde{f}_t(u) = f_t(u)$ for all $\|u\| \le \frac{\zeta}{2}$, it follows that

$$H_k(f_t^0 \cap B', (f_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2) \cong H_k(\tilde{f}_t^0 \cap B', (\tilde{f}_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2)$$

and thus by (12.27)

$$H_k(f_t^0 \cap B', (f_t^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2) \cong H_k(f_{t_0}^0 \cap B', (f_{t_0}^0 \setminus \{0\}) \cap B'; \mathbb{Z}_2).$$

Consequently, $d_k(t) = d_k(t_0)$ for all $t \in [t_0 - \eta, t_0 + \eta] \cap [a, b]$ and all k. This in particular shows that $d_k(a) = d_k(b)$, $k \in \mathbb{N}_0$.

The final step of the proof for a finite-dimensional $\mathcal{H}$ links $d_k(a)$ and $d_k(b)$ to the Morse index of H_t for $t = a, b$. As 0 is a nondegenerate critical point, by the Morse lemma (see, e. g., [132, Theorem 8.3]) there is a homeomorphism h_t between neighborhoods of 0 in $\mathcal{H}$ such that $h_t(0) = 0$ and

$$f_t(h_t(u)) = \frac{1}{2}\langle H_t u | u \rangle.$$

Consequently, for $g_t = f_t \circ h_t$ and any sufficiently small closed ball B'' about 0 in $\mathcal{H}$,

$$H_k(f_t^0 \cap h_t(B''), (f_t^0 \setminus \{0\}) \cap h_t(B''); \mathbb{Z}_2) \cong H_k(g_t^0 \cap B'', (g_t^0 \setminus \{0\}) \cap B''; \mathbb{Z}_2).$$

As f_t is nondegenerate, it follows that $\mathcal{H}$ is the orthogonal sum of $\mathcal{H}_t^+$ and $\mathcal{H}_t^-$ such that g_t is positive definite on $\mathcal{H}_t^+$ and negative definite on $\mathcal{H}_t^-$. Define a deformation D of B'' by

$$D : [0, 1] \times B'' \to B'', \quad (s, u) \mapsto u^- + (1 - s)u^+,$$

where $u = u^- + u^+$ is the decomposition of u according to the splitting $\mathcal{H} = \mathcal{H}_t^- \oplus \mathcal{H}_t^+$. Then $g_t(D(s, u)) = g_t(u^-) + (1 - s)^2 g_t(u^+)$ which shows that $\mathcal{H}_t^- \cap B''$ is a deformation retract of $g_t^0 \cap B''$ and $\mathcal{H}_t^- \cap B'' \setminus \{0\}$ is a deformation retract of $g_t^0 \cap B'' \setminus \{0\}$ by the homotopy D. Now set $N = \dim \mathcal{H}_t^-$ which is the Morse index $\iota_-(H_t)$. Then, if $N \ge 1$,

$$\begin{aligned} H_k(g_t^0 \cap B'', (g_t^0 \setminus \{0\}) \cap B''; \mathbb{Z}_2) &\cong H_k(\mathcal{H}_t^- \cap B'', (\mathcal{H}_t^- \setminus \{0\}) \cap B''; \mathbb{Z}_2) \\ &\cong H_k(B^N, S^{N-1}; \mathbb{Z}_2), \end{aligned}$$

where B^N is the closed unit ball of dimension N in $\mathcal{H}_t^-$ and S^{N-1} is its boundary, and for $N = 0$

$$H_k(g_t^0 \cap B'', (g_t^0 \setminus \{0\}) \cap B''; \mathbb{Z}_2) \cong H_k(\{0\}, \emptyset; \mathbb{Z}_2).$$

Thus for $t = a, b$ the groups (12.20) are isomorphic to $\mathbb{Z}_2$ if $k = \iota_-(H_t)$ and trivial otherwise.

In summary, if there is no bifurcation, then $d_k(a) = d_k(b)$ for all k and thus the Morse indices of H_a and H_b coincide. This shows Theorem 12.2.1 under the additional assumption that $\mathcal{H}$ is of finite dimension.

Step 2 (Finite-dimensional reduction and proof in the general case).

For the proof of this step and thus Theorem 12.2.1, we closely follow [146]. First, note that it suffices to prove the statement for families of functionals $f : [a,b] \times \mathcal{H} \to \mathbb{R}$ such that $H_t \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$, $t \in [a,b]$. Indeed, the family of functionals $\overline{f} : [a,b] \times \mathcal{H}^3 \to \mathbb{R}$ given by

$$\overline{f}_t(w,u,v) = f_t(u) + \frac{1}{2}\|w\|^2 - \frac{1}{2}\|v\|^2$$

has the same bifurcation points of critical points as f. Moreover, the corresponding Hessians $\overline{H}_t$ are in $\mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ and

$$\mathrm{Sf}(t \in [a,b] \mapsto \overline{H}_t) = \mathrm{Sf}(t \in [a,b] \mapsto H_t)$$

by Theorem 4.2.1(v). Thus it will henceforth be assumed that $H_t \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$. By Theorem 5.9.3, there are paths $M : [a,b] \to \mathbb{G}(\mathcal{H})$, $K : [a,b] \to \mathbb{K}(\mathcal{H})$ and a symmetry Q such that $M_t^* H_t M_t = Q + K_t$ for all $t \in [a,b]$. Let us set

$$\tilde{f}_t(u) = f_t(M_t u)$$

and note that $\nabla\tilde{f}_t(u) = M_t^*(\nabla f_t)(M_t u)$, as well as $\tilde{H}_t = M_t^* H_t M_t$. As the operators M_t are invertible, the families f and $\tilde{f}$ have the same bifurcation points. Moreover, it follows from the homotopy invariance of the spectral flow in the form stated in Corollary 4.2.5 that $\mathrm{Sf}(t \in [a,b] \mapsto H_t) = \mathrm{Sf}(t \in [a,b] \mapsto M_t^* H_t M_t)$. Thus it is enough to prove Theorem 12.2.1 under the additional assumption that $H_t = Q + K_t$ for a symmetry Q and compact symmetric operators K_t, $t \in [a,b]$.

Let now $\mathcal{H}_\pm$ denote the eigenspaces of Q for the eigenvalues ± 1 and let $(e_k^\pm)_{k\in\mathbb{N}}$ be corresponding Hilbert bases. Moreover, let $\mathcal{H}_n$ be the span of $\{e_k^\pm : k = 1, \dots, n\}$ and denote by P_n the orthogonal projection onto $\mathcal{H}_n$. Note that P_n commutes with Q and thus $Q(\mathcal{H}_n) = \mathcal{H}_n$, as well as $Q(\mathcal{H}_n^\perp) = \mathcal{H}_n^\perp$.

Lemma 12.2.4. *There is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $(\mathbf{1} - P_n)H_t|_{\mathcal{H}_n^\perp} \in \mathbb{G}(\mathcal{H}_n^\perp)$ *for $t \in [a,b]$;*

(ii) $sH_t + (1-s)((\mathbf{1} - P_n)H_t(\mathbf{1} - P_n) + P_n H_t P_n) \in \mathbb{G}(\mathcal{H})$ *for $t = a, b$ and $s \in [0,1]$.*

Proof. Let us first note that

$$(\mathbf{1}-P_n)H_t|_{\mathcal{H}_n^\perp} = Q + (\mathbf{1}-P_n)K_t|_{\mathcal{H}_n^\perp}$$

is a compact perturbation of an invertible operator and thus a Fredholm operator of index 0. Therefore, to prove the first assertion, it is sufficient to show that $(\mathbf{1}-P_n)H_t|_{\mathcal{H}_n^\perp}$ is injective.

As $\|Qu\| = \|u\|$,

$$\big\|(\mathbf{1}-P_n)Qu\big\| = \|Qu\| = \|u\|, \quad u \in \mathcal{H}_n^\perp. \tag{12.28}$$

Moreover, since $t \in [a,b] \mapsto K_t$ is a continuous family of compact operators, the set $\{K_t(u) : t \in [a,b], \|u\| = 1\}$ is relatively compact. As $\mathbf{1}-P_n$ converges uniformly to 0 on compact subsets of $\mathcal{H}$, there exists $n_0 \in \mathbb{N}$ such that

$$\big\|(\mathbf{1}-P_n)K_t u\big\| \le \frac{1}{2}\|u\|, \quad u \in \mathcal{H},\ t \in [a,b],\ n \ge n_0.$$

Hence one obtains from (12.28)

$$\big\|(\mathbf{1}-P_n)H_t(u)\big\| = \big\|(\mathbf{1}-P_n)Qu + (\mathbf{1}-P_n)K_t u\big\| \ge \frac{1}{2}\|u\|, \quad u \in \mathcal{H}_n^\perp,$$

showing the injectivity of $(\mathbf{1}-P_n)H_t|_{\mathcal{H}_n^\perp}$.

To show (ii), let us note at first that by a direct calculation

$$\begin{aligned} &sH_t + (1-s)\big((\mathbf{1}-P_n)H_t(\mathbf{1}-P_n) + P_nH_tP_n\big) \\ &\quad = Q + sK_t + (1-s)\big((\mathbf{1}-P_n)K_t(\mathbf{1}-P_n) + P_nK_tP_n\big), \end{aligned}$$

which are all Fredholm operators of index 0. Let us now assume by contradiction that an n_0 as in the assertion does not exist. Consequently, there are sequences $(u_n)_{n\in\mathbb{N}}$, $\|u_n\| = 1$, and $(s_n)_{n\in\mathbb{N}}$ such that

$$Qu_n + s_nK_au_n + (1-s_n)\big((\mathbf{1}-P_n)K_a(\mathbf{1}-P_n)u_n + P_nK_aP_nu_n\big) = 0, \quad n \in \mathbb{N}.$$

As K_a is compact and P_n converges on compact subsets of $\mathcal{H}$ to the identity, one sees that there is a convergent subsequence of $(Q(u_n))_{n\in\mathbb{N}}$. Henceforth, let us denote this sequence by the same indices and assume as well that s_n converges to some $s^* \in [0,1]$. It follows from the invertibility of Q that $(u_n)_{n\in\mathbb{N}}$ converges to some $u \in \mathcal{H}$ of norm 1. Thus

$$\lim_{n\to\infty}(\mathbf{1}-P_n)K_a(\mathbf{1}-P_n)u_n = 0, \quad \lim_{n\to\infty} P_nK_aP_nu_n = K_au,$$

and so

$$H_au = Qu + K_au = Qu + s^*K_au + (1-s^*)K_au = 0,$$

in contradiction to the invertibility of H_a. Of course, the same argument applies to the invertible operator H_b. □

Let us now set $H_t^n = P_n H_t|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n$ and note that it follows from Lemma 12.2.4 combined with Theorem 4.2.1 and Proposition 4.3.1 that for $n \geq n_0$,

$$\mathrm{Sf}(t \in [a,b] \mapsto H_t) = \mathrm{Sf}(t \in [a,b] \mapsto H_t^n) = \iota_-(H_a^n) - \iota_-(H_b^n). \tag{12.29}$$

For reducing the nonlinear problem to finite dimensions, the following technical lemma is needed.

Lemma 12.2.5. *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{U} \subset \mathcal{H}$ an open neighborhood of $0 \in \mathcal{U}$, and $f : [a,b] \times \mathcal{U} \to \mathbb{R}$ a continuous family of C^2-functionals. Let $F(t,u) = (\nabla f_t)(u)$ and assume that $F(t,0) = 0$ for all $t \in [a,b]$. Suppose that there is an orthogonal decomposition $\mathcal{H} = \mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{X}$ is of finite dimension, and such that for*

$$F(t,u) = \big(F_1(t,x,y), F_2(t,x,y)\big) \in \mathcal{X} \oplus \mathcal{Y}, \quad u = (x,y) \in \mathcal{X} \oplus \mathcal{Y},$$

one has that $(D_y F_2)(t,0,0) : \mathcal{Y} \to \mathcal{Y}$ is invertible for all $t \in [a,b]$. Then:

(i) *There are an open ball $B_{\mathcal{X}} = B_\delta(0) \subset \mathcal{X}$ and a unique continuous family of C^1-maps $\eta : [a,b] \times B_{\mathcal{X}} \to \mathcal{Y}$ such that $\eta(t,0) = 0$ for all $t \in [a,b]$, and*

$$F_2\big(t,x,\eta(t,x)\big) = 0, \quad (t,x) \in [a,b] \times B_{\mathcal{X}}. \tag{12.30}$$

(ii) *Let the family of functionals $\overline{f} : [a,b] \times B_{\mathcal{X}} \to \mathbb{R}$ and the map $\overline{F} : [a,b] \times B_{\mathcal{X}} \to \mathcal{X}$ be defined by*

$$\overline{f}(t,x) = f\big(t,x,\eta(t,x)\big), \quad \overline{F}(t,x) = F_1\big(t,x,\eta(t,x)\big).$$

Then $\overline{f}$ is a continuous family of C^2-functionals on $B_{\mathcal{X}}$ and

$$\nabla \overline{f}(t,x) = \overline{F}(t,x), \quad (t,x) \in [a,b] \times B_{\mathcal{X}}. \tag{12.31}$$

Proof. Let us first consider the map $\overline{F}_2 : [a,b] \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ defined by

$$\overline{F}_2(t,x,y) = F_2\big(t,x,\big(D_y F_2(t,0,0)\big)^{-1} y\big),$$

and note that $D_y \overline{F}_2(t,0,0) = \mathbf{1}_{\mathcal{Y}}$. Obviously, a map η as in (12.30) exists for F_2 if and only if it exists for $\overline{F}_2$. Thus we can henceforth assume without loss of generality that

$$D_y F_2(t,0,0) = \mathbf{1}_{\mathcal{Y}}. \tag{12.32}$$

Now consider $C : [a,b] \times (\mathcal{X} \times \mathcal{Y}) \to \mathcal{X} \times \mathcal{Y}$ defined by

$$C(t,x,y) = y - F_2(t,x,y)$$

and note that $D_y C(t,0,0) = 0$ for all $t \in [a,b]$ by (12.32). As $D_y C(t,x,y)$ is continuous by assumption, there exists $\varepsilon > 0$ such that

$$\|D_yC(t,x,y)\| \le \frac{1}{2}, \quad (t,x,y) \in [a,b] \times \overline{B}_{\mathcal{X},\varepsilon} \times \overline{B}_{\mathcal{Y},\varepsilon},$$

where $\overline{B}_{\mathcal{X},\varepsilon}$ and $\overline{B}_{\mathcal{Y},\varepsilon}$ denote the closed balls of radius ε in $\mathcal{X}$ and $\mathcal{Y}$. Consequently,

$$\|C(t,x,y) - C(t,x,y')\| \le \frac{1}{2}\|y - y'\|, \quad (t,x) \in [a,b] \times \overline{B}_{\mathcal{X},\varepsilon},\ y,y' \in \overline{B}_{\mathcal{Y},\varepsilon}. \tag{12.33}$$

As $F_2(t,0,0) = 0$, there is $\delta < \varepsilon$ such that

$$\|C(t,x,0)\| = \|F_2(t,x,0)\| < \frac{\varepsilon}{2}, \quad (t,x) \in [a,b] \times \overline{B}_{\mathcal{X},\delta},$$

and one obtains from (12.33) that, for $(t,x,y) \in [a,b] \times \overline{B}_{\mathcal{X},\delta} \times \overline{B}_{\mathcal{Y},\varepsilon}$,

$$\|C(t,x,y)\| \le \|C(t,x,y) - C(t,x,0)\| + \|C(t,x,0)\| < \varepsilon.$$

Thus for each $(t,x) \in [a,b] \times \overline{B}_{\mathcal{X},\delta}$, the map $C(t,x,\cdot) : \overline{B}_{\mathcal{Y},\varepsilon} \to \overline{B}_{\mathcal{Y},\varepsilon}$ is a strict contraction and thus has a unique fixed point $\eta(t,x)$. As these fixed points depend continuously on parameters, one obtains a continuous map $\eta : [a,b] \times \overline{B}_{\mathcal{X},\delta} \to \mathcal{Y}$. Note that η satisfies (12.30) and thus for each fixed t, $\eta(t,\cdot)$ is the unique map that can be obtained from the classical implicit function theorem. The latter theorem also shows that

$$D_x\eta(t,x) = -\big(D_yF_2(t,x,\eta(t,x))\big)^{-1} D_xF_2(t,x,\eta(t,x)),$$

which implies that $D_x\eta(t,x)$ depends continuously on $(t,x) \in [a,b] \times \overline{B}_{\mathcal{X},\delta}$.

Finally, (12.31) is a direct consequence of the chain rule and (12.30). □

Let us now set $\mathcal{X} = \mathcal{H}_n$, $\mathcal{Y} = \mathcal{H}_n^\perp$ and consider the splitting $F = (F_1^n, F_2^n)$, where

$$F_1^n(t,u,v) = P_nF(t,u,v), \quad F_2^n(t,u,v) = (\mathbf{1} - P_n)F(t,u,v).$$

As $D_vF_2^n(t,0,0) = (\mathbf{1} - P_n)H_t|_{\mathcal{H}_n^\perp} : \mathcal{H}_n^\perp \to \mathcal{H}_n^\perp$ is an isomorphism for $n \ge n_0$ by Lemma 12.2.4, one obtains from Lemma 12.2.5 a family of functionals $\overline{f} : [a,b] \times B_n \to \mathbb{R}$ for some open ball $B_n \subset \mathcal{H}_n$ such that each bifurcation point of critical points of $\overline{f}$ is also a bifurcation point of f. Thus it suffices to show that $\overline{f}$ has a bifurcation of critical points from the trivial branch if (12.29) is nonzero. The following proposition is the final step in the proof of Theorem 12.2.1.

Proposition 12.2.6. *For the Hessians $\overline{H}_t^n$ of the functionals $\overline{f}_t$ at $0 \in \mathcal{H}_n$, there exists $n_1 \ge n_0$ such that for $n \ge n_1$ and $t = a,b$, $\overline{H}_t^n$ is invertible and*

$$\iota_-(\overline{H}_t^n) = \iota_-(H_t^n).$$

Proof. Let $\eta_t^n : B_n \to \mathcal{H}_n^\perp$ be the continuous family of C^1-maps as specified in Lemma 12.2.5 for the splitting $\mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_n^\perp$. Set $C_t^n = D_0\eta_t^n$. By differentiating (12.30) implicitly, it follows that

$$C_t^n = -\big(D_y F_2^n(t,0,0)\big)^{-1} D_x F_2^n(t,0,0) = -\big((\mathbf{1}-P_n)H_t|_{\mathcal{H}_n^\perp}\big)^{-1}(\mathbf{1}-P_n)H_t|_{\mathcal{H}_n}.$$

Now

$$(\mathbf{1}-P_n)H_t = Qu + (\mathbf{1}-P_n)K_t u, \quad u \in \mathcal{H}_n^\perp.$$

As $(\mathbf{1}-P_n)K_t$ converges uniformly to 0 on bounded sets, there is $k \in \mathbb{N}$ such that for $t = a, b$,

$$\|(\mathbf{1}-P_n)K_t u\| \le \frac{1}{2}\|u\|, \quad u \in \mathcal{H},\ n \ge k.$$

Consequently, as $\|Qu\| = \|u\|$ for all $u \in \mathcal{H}$,

$$\|(\mathbf{1}-P_n)H_t u\| \ge \frac{1}{2}\|u\|, \quad u \in \mathcal{H}_n^\perp,\ n \ge k,$$

which shows that

$$\big\|\big((\mathbf{1}-P_n)H_t|_{\mathcal{H}_n^\perp}\big)^{-1}\big\| \le \frac{1}{2}, \quad n \ge k,\ t = a, b.$$

Using once again that $H_t = Q + K_t$, this yields

$$\|C_t^n\| \le \frac{1}{2}\|(\mathbf{1}-P_n)(Q+K_t)|_{\mathcal{H}_n}\| \le \frac{1}{2}\|(\mathbf{1}-P_n)K_t|_{\mathcal{H}_n}\| \to 0 \quad \text{as } n \to \infty,$$

because $(\mathbf{1}-P_n)K_t$ converges to 0 in $\mathbb{B}(\mathcal{H})$ as $n \to \infty$ by the compactness of K_t. Moreover, there is a constant $c > 0$ such that $\|H_t^n u\| = \|P_n H_t u\| \ge c\|u\|$ for all $u \in \mathcal{H}_n$ and all $n \in \mathbb{N}$. Finally, it follows from the definition of $\overline{H}_t^n$ that

$$\overline{H}_t^n = P_n H_t(\mathbf{1} + C_t^n) = H_t^n + P_n H_t C_t^n,$$

and thus there is $n_1 \in \mathbb{N}$ such that $\overline{H}_t^n$ is invertible and has the same Morse index as H_t^n for $n \ge n_1$ and $t = a, b$. □

Proof of Theorem 12.2.1. This now follows from the finite-dimensional case, the previous Proposition 12.2.6, and (12.29). □

12.3 Applications to Hamiltonian systems

The aim of this section is to study bifurcation of periodic solutions of families of Hamiltonian systems of the form

$$\begin{cases} I\dot{u}(s) + \nabla_u F\big(t, s, u(s)\big) = 0, & s \in [0, 2\pi], \\ u(0) = u(2\pi), \end{cases} \tag{12.34}$$

where I is the $2N \times 2N$ symplectic matrix that was already introduced in Section 2.1, $F : [0,1] \times \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is 2π-periodic in s, satisfies a number of regularity assumptions stated below, and

$$\nabla_u F(t,s,0) = 0.$$

The function F is called the Hamiltonian. Its arguments are the exterior parameter t, the time variable is denoted by s, and finally, u are the phase space points. Usually, the Hamiltonian is denoted by the letter H which, however, here is reserved for the Hessian further down. The Hamiltonian system (12.34) is called semilinear because it is linear in the time derivative, but in general F is not linear in u. Furthermore, (12.34) is called autonomous if the Hamiltonian has no explicit dependence on time, namely $F(t,s,u) = F(t,u)$. Let us also note that in many situations the dependence on t is of the form $F(t,u) = tF(u)$ where then t is proportional to the period of periodic orbits that one is looking for. Examples of systems like (12.34) not only come from classical mechanics, but also geodesic equations can be written in such a Hamiltonian form.

Now, clearly, the constant function $u \equiv 0$ is a solution of (12.34) for all $t \in [0,1]$. One is then interested in finding bifurcation points at which new branches of solutions arise. One way to address this problem is to study the linearization of (12.34) at $u \equiv 0$. It is given by

$$\begin{cases} I\dot{u}(s) + (\nabla_u^2 F_{t,s,0})u(s) = 0, & s \in [0,2\pi], \\ u(0) = u(2\pi), \end{cases} \tag{12.35}$$

and thus of the form (7.6). Periodic solutions of this equation can be accessed by the oscillation theory techniques of Section 7.3, see also Section 2.5 for a discrete time setting. It is then an analytic issue to connect those solutions to solutions of the nonlinear problem (12.34). For the problem of finding periodic geodesics, this was the route followed by Bott [35]. Here another strategy will be followed: one first constructs a family f_t of functionals on a suitable Hilbert space of functions u for which the critical points are just the solutions of (12.34). For these functionals, one can then apply Theorem 12.2.1. This was firstly done by Fitzpatrick, Pejsachowicz, and Recht in [85], which is the main reference for this section.

Throughout the analysis, the following technical assumptions are supposed to hold [22]:

(H1) $F \in C^2([0,1] \times \mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$;

(H2) F is 2π-periodic in the s-variable;

(H3) There are $c > 0$ and $r > 2$ such that for all $(t,s,u) \in [0,1] \times \mathbb{R} \times \mathbb{R}^{2N}$,

$$\left| \frac{\partial^2 F}{\partial t^2}(t,s,u) \right| + \|(\nabla_u^2 F)(t,s,u)\| \le c(1 + \|u\|^{r-2});$$

(H4) $\nabla_u F(t,s,0) = 0$ for $(t,s) \in [0,1] \times \mathbb{R}$.

As a technical preliminary, let us begin by introducing the spaces that will be needed below to apply Theorem 12.2.1. The unit circle $\mathbb{S}^1$ will be identified with $\mathbb{R}/(2\pi\mathbb{Z}) \cong [0, 2\pi)$. Let us first recall that $L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ consists of all functions $u : [0, 2\pi) \to \mathbb{R}^{2N}$ such that

$$u(s) = c_0 + \sum_{k=1}^{\infty} (a_k \sin(ks) + b_k \cos(ks)), \tag{12.36}$$

where $c_0, a_k, b_k \in \mathbb{R}^{2N}$, $k \in \mathbb{N}$, and

$$\sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2) < \infty.$$

The scalar product on $L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ is given by

$$\langle u|v\rangle_{L^2} = 2\pi\langle c_0|\tilde{c}_0\rangle + \pi \sum_{k=1}^{\infty} (\langle a_k|\tilde{a}_k\rangle + \langle b_k|\tilde{b}_k\rangle),$$

where $\tilde{c}_0$ and $\tilde{a}_k$, $\tilde{b}_k$ denote the Fourier coefficients of $v \in L^2(\mathbb{S}^1, \mathbb{R}^{2N})$. The subset $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2N})$ of all functions $u \in L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ such that

$$\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty \tag{12.37}$$

is a Hilbert space in its own right with respect to the scalar product

$$\langle u|v\rangle_{W^{\frac{1}{2},2}} = 2\pi\langle c_0|\tilde{c}_0\rangle + \pi \sum_{k=1}^{\infty} k(\langle a_k|\tilde{a}_k\rangle + \langle b_k|\tilde{b}_k\rangle), \tag{12.38}$$

and the embedding

$$W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2N}) \hookrightarrow L^p(\mathbb{S}^1, \mathbb{R}^{2N}) \tag{12.39}$$

is compact for every $p \in [1, \infty)$ (e. g., [3, § 3.1]). Note that elements in $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2N})$ do not need to have continuous representatives. In contrast, replacing k by k^2 in (12.37) and (12.38), one obtains $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ and elements in this space can be represented by absolutely continuous functions having a square integrable first derivative.

The first aim is to construct a family of C^2-functionals $f : [0, 1] \times \mathcal{H} \to \mathbb{R}$ on the Hilbert space $\mathcal{H} = W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2N})$ such that the critical points of f_t are the weak solutions of (12.34). Let us first note that there is an orthogonal decomposition $\mathcal{H} = \mathcal{E}_+ \oplus \mathcal{E}_0 \oplus \mathcal{E}_-$, where

$$\mathcal{E}_0 = \{u \in \mathcal{H} : u \equiv c_0, c_0 \in \mathbb{R}^{2N}\}$$

and

$$\mathcal{E}_\pm = \left\{ u \in \mathcal{H} : u(s) = \sum_{k=1}^{\infty} (a_k \cos(ks) \mp I a_k \sin(ks)) \text{ with } a_k \in \mathbb{R}^{2N} \right\}.$$

Let $P_\pm$ denote the orthogonal projections in $\mathcal{H}$ onto $\mathcal{E}_\pm$. For u as in (12.36), one has explicitly

$$(P_+ u)(s) = \frac{1}{2} \sum_{k=1}^{\infty} ((a_k - I b_k) \sin(ks) + (I a_k + b_k) \cos(ks))$$

and

$$(P_- u)(s) = \frac{1}{2} \sum_{k=1}^{\infty} ((a_k + I b_k) \sin(ks) + (-I a_k + b_k) \cos(ks)).$$

Next let us define a bilinear form by

$$\Gamma : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \quad \Gamma(u, v) = \langle P_+ u - P_- u | v \rangle_{\mathcal{H}}. \tag{12.40}$$

For $u \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$, the definition of $\mathcal{E}_\pm$, and the definition of the scalar product in $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ allow verifying by a direct computation that

$$\Gamma(u, v) = \int_0^{2\pi} \langle I \dot{u} | v \rangle_{\mathbb{R}^{2N}} \, ds, \quad v \in \mathcal{H}. \tag{12.41}$$

Now consider the family of functionals

$$f : [0,1] \times \mathcal{H} \to \mathbb{R}, \quad f(t, u) = \frac{1}{2} \Gamma(u, u) + \int_0^{2\pi} F(t, s, u(s)) \, ds. \tag{12.42}$$

The following proposition shows, in particular, that the critical points of f_t are the weak solutions of the Hamiltonian system (12.34).

Proposition 12.3.1. *The map f defined by* (12.42) *is C^2 and*

$$\nabla f_t(u) = (P_+ - P_-) u + G(t, u),$$

where

$$\langle G(t, u) | v \rangle_{\mathcal{H}} = \int_0^{2\pi} \langle \nabla_u F(t, s, u(s)) | v(s) \rangle_{\mathbb{R}^{2N}} \, ds, \quad u, v \in \mathcal{H}. \tag{12.43}$$

Moreover, the Hessian H_t of f_t at $0 \in \mathcal{H}$ is given by

$$\langle u|H_t v\rangle_{\mathcal{H}} = \Gamma(u,v) + \int_0^{2\pi} \langle u(s)|A_t(s)v(s)\rangle_{\mathbb{R}^{2N}}\,ds, \quad u,v \in \mathcal{H}, \tag{12.44}$$

where

$$A_t(s) = \nabla_u^2 F(t,s,0), \quad (t,s) \in [0,1] \times \mathbb{R}.$$

Proof. The proof closely follows [205, Appendix A]. Let us first briefly recall that a map $G : \mathcal{X} \to \mathcal{Y}$ between Banach spaces $\mathcal{X}$, $\mathcal{Y}$ is Gateaux differentiable at $x_0 \in \mathcal{X}$ if for all $h \in \mathcal{X}$ the limit

$$dG(x_0;h) = \lim_{\tau\to 0} \frac{1}{\tau}(G(u+\tau h) - G(u))$$

exists. Moreover, G is (Fréchet) differentiable in x_0 if there is $D_{x_0}G \in \mathbb{B}(\mathcal{X},\mathcal{Y})$ such that

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|}\|G(x_0+h) - G(x_0) - D_{x_0}Gh\| = 0.$$

Of course, if G is differentiable, then G is Gateaux differentiable and $dG(x_0;h) = (D_{x_0}G)h$. Moreover, if G is everywhere Gateaux differentiable and for all x there is $A(x) \in \mathbb{B}(\mathcal{X},\mathcal{Y})$ depending continuously on x and such that $dG(x;h) = A(x)h$, then G is differentiable and $D_xG = A(x)$.

One only needs to discuss the second term in the definition of $f : [0,1] \times \mathcal{H} \to \mathbb{R}$ as the first is clearly smooth and its first and second derivatives are as stated in the proposition. Thus let us henceforth consider the family

$$g : [0,1] \times \mathcal{H} \to \mathbb{R}, \quad g(t,u) = \int_0^{2\pi} F(t,s,u(s))\,ds.$$

Note that by (H3) there is a constant $d > 0$ such that for all $(t,s,u) \in [0,1] \times \mathbb{R} \times \mathbb{R}^{2N}$,

$$\|(\nabla_u F)(t,s,u)\| \le d(1 + \|u\|^{r-1}). \tag{12.45}$$

Let us now first show that $g(t,\cdot) : \mathcal{H} \to \mathbb{R}$ is Gateaux differentiable for every fixed $t \in [0,1]$. Let $u,h \in \mathcal{H}$, $s \in \mathbb{S}^1$ and $0 < |\tau| < 1$. By the mean value theorem, there is $\lambda \in (0,1)$ such that

$$\begin{aligned}\frac{1}{|\tau|}\big|F(t,s,u(s)+\tau h(s)) - F(t,s,u(s))\big| &= \|(\nabla_u F)(t,s,u(s)+\lambda\tau h(s))h(s)\| \\ &\le d(1 + (\|u(s)\| + \|h(s)\|)^{r-1})\|h(s)\| \\ &\le d(1 + 2^{r-1}(\|u(s)\|^{r-1} + \|h(s)\|^{r-1}))\|h(s)\|,\end{aligned}$$

where (12.45) was used. As the embedding (12.39) is in particular continuous, $\|u(s)\|^{r-1}$ and $\|h(s)\|^{r-1}$ are in $L^2(\mathbb{S}^1, \mathbb{R}^{2N})$. Thus the right-hand side in the latter inequality is integrable by Hölder's inequality, and it follows from the dominated convergence theorem that

$$\lim_{\tau\to 0} \frac{1}{\tau}(g(t, u + \tau h) - g(t, u)) = \lim_{\tau\to 0} \int_0^{2\pi} \frac{1}{\tau}(F(t, s, u(s) + \tau h(s)) - F(t, s, u(s)))ds$$
$$= \int_0^{2\pi} \langle \nabla_u F(t, s, u(s)) | h(s) \rangle_{\mathbb{R}^{2N}} \, ds.$$

Let us now set $G : [0, 1] \times \mathcal{H} \to \mathcal{H}$ as in (12.43) and show that G continuously depends on $(t, u) \in [0, 1] \times \mathcal{H}$. If $h : [0, 1] \times [0, 2\pi] \times \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is continuous and satisfies the estimate

$$\|h(t, s, u)\| \le \tilde{c}(1 + \|u\|^{\frac{p}{q}}) \tag{12.46}$$

for some $\tilde{c} > 0$ and $1 \le p, q < \infty$, then $h(t, \cdot, u) \in L^q(\mathbb{S}^1, \mathbb{R}^{2N})$ for all $u \in L^p(\mathbb{S}^1, \mathbb{R}^{2N})$ and the superposition operator

$$[0, 1] \times L^p(\mathbb{S}^1, \mathbb{R}^{2N}) \to L^q(\mathbb{S}^1, \mathbb{R}^{2N}), \quad (t, u) \mapsto h(t, \cdot, u)$$

is continuous. A proof of this fact without a parameter t can be found in various textbooks in nonlinear analysis and is a rather straightforward application of the dominated convergence theorem (cf., e. g., [5, 205]). The above parametrized version only requires minor modifications of the argument. Now let r^* be the conjugate exponent of r, i. e., $\frac{1}{r} + \frac{1}{r^*} = 1$. Because then $\frac{r}{r^*} = r - 1$, it follows from (12.45) that the superposition operator $\nabla F : [0, 1] \times L^r(\mathbb{S}^1, \mathbb{R}^{2N}) \to L^{r^*}(\mathbb{S}^1, \mathbb{R}^{2N})$ is continuous. Finally, let $(t_n, u_n) \to (t, u) \in [0, 1] \times \mathcal{H}$. Then by the Hölder inequality,

$$\begin{aligned} |\langle G(t_n, u_n) - G(t, u) | v \rangle| &= \left| \int_0^{2\pi} \langle \nabla_u F(t_n, s, u_n(s)) - \nabla_u F(t, s, u(s)) | v(s) \rangle ds \right| \\ &\le \|\nabla_u F(t_n, \cdot, u_n) - \nabla_u F(t, \cdot, u)\|_{r^*} \|v\|_r \\ &\le c_r \|\nabla_u F(t_n, \cdot, u_n) - \nabla_u F(t, \cdot, u)\|_{r^*} \|v\|_{\mathcal{H}}, \end{aligned}$$

where $c_r > 0$ exists by the boundedness of the inclusion (12.39). Thus

$$\|G(t_n, u_n) - G(t, u)\| \le c_r \|\nabla_u F(t_n, \cdot, u_n) - \nabla_u F(t, \cdot, u)\|_{r^*} \to 0, \quad n \to \infty,$$

which shows the continuity of $G : [0, 1] \times \mathcal{H} \to \mathcal{H}$. In summary, it has been shown that the functionals $f_t : \mathcal{H} \to \mathbb{R}$ are continuously differentiable, their derivatives are given by (12.43), and they depend continuously on $(t, u) \in [0, 1] \times \mathcal{H}$.

Differentiation under the integral sign yields the partial derivative of g with respect to t, and its continuous dependence on $(t, u) \in [0,1] \times \mathcal{H}$ follows once again from (H3) and (12.46).

The assertion about the second derivatives of f can be obtained from (H3) by direct modifications of the above arguments. □

The following lemma shows that the Hessians H_t in Proposition 12.3.1 are actually Fredholm operators.

Lemma 12.3.2. *The Hessians H_t of f_t at $0 \in \mathcal{H}$ are of the form $H_t = A + K_t$, where $A \in \mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$ is a self-adjoint Fredholm operator and K is a continuous family of compact operators.*

Proof. At first, let us set $A = P_+ - P_-$, which clearly is an element of $\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$. Secondly, the maps

$$\beta_t : L^2(\mathbb{S}^1, \mathbb{R}^{2N}) \times L^2(\mathbb{S}^1, \mathbb{R}^{2N}) \to \mathbb{R}, \quad (u, v) \mapsto \int_0^{2\pi} \langle u(s) | A_t(s)v(s) \rangle_{\mathbb{R}^{2N}} \, ds$$

restrict to a continuous family of bounded bilinear forms on $\mathcal{H}$, and thus

$$\langle u | K_t v \rangle_{\mathcal{H}} = \beta_t(u, v), \quad u, v \in \mathcal{H},$$

defines a continuous family of bounded self-adjoint operators K_t on $\mathcal{H}$ such that $H_t = A + K_t$, $t \in [0,1]$. It remains to show the compactness of K_t. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ which weakly converge to some elements $u, v \in \mathcal{H}$. From the compactness of (12.39), it follows that they converge strongly in $L^2(\mathbb{S}^1, \mathbb{R}^{2N})$. Consequently, $\langle K_t u_n | v_n \rangle_{\mathcal{H}} = \beta_t(u_n, v_n)$ converges to $\beta_t(u, v) = \langle K_t u | v \rangle_{\mathcal{H}}$, which shows the compactness of K_t. □

The previous lemma implies that $t \in [0,1] \mapsto H_t$ is a continuous family of bounded self-adjoint Fredholm operators in $\mathcal{H}$. Thus $\mathrm{Sf}(t \in [0,1] \mapsto H_t)$ is well defined and, if it is nontrivial and H_0, H_1 are invertible, then there is a bifurcation of critical points for f by Theorem 12.2.1. Actually, by using relatively standard regularity arguments, this will next be improved.

Theorem 12.3.3. *If*

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) \neq 0$$

and the linear equations (12.35) *only have the trivial solution for $t = 0, 1$, then there are a sequence $(t_n)_{n\in\mathbb{N}}$ converging to some $t^* \in (0,1)$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset C^1(\mathbb{S}^1, \mathbb{R}^{2N})$ converging to* 0 *with respect to $\|u\|_{C^1} = \|u\|_\infty + \|\dot{u}\|_\infty$ such that u_n is a nontrivial periodic solution of* (12.34) *for t_n.*

Proof. The argument closely follows [156, § 6]. Let us first show that every critical point of f_t is in $C^1(\mathbb{S}^1, \mathbb{R}^{2N})$. If $u \in \mathcal{H}$ is a critical point of f_t, then one obtains from (12.41) that, for $v \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$,

$$\big\langle (\nabla_u f_t)(u) \big| v \big\rangle_{W^{\frac{1}{2},2}} = \int_0^{2\pi} \big(\langle u | I\dot{v} \rangle_{\mathbb{R}^{2N}} + \big\langle \nabla_u F(t,s,u) \big| v \big\rangle_{\mathbb{R}^{2N}} \big)\, ds = 0. \tag{12.47}$$

Plugging into this formula for v the $2N$ constant functions given by the standard basis of $\mathbb{R}^{2N}$ yields $\int_0^{2\pi} \nabla_u F(t,s,u)\, ds = 0$. By using a Fourier expansion, it can be shown that for all $v \in L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ such that $\frac{1}{2\pi}\int_0^{2\pi} v\, ds = 0$ and every $\xi \in \mathbb{R}^{2N}$, there is a unique $\tilde{u} \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ such that $\frac{1}{2\pi}\int_0^{2\pi} \tilde{u}\, ds = \xi$ and $\partial_s \tilde{u} = v$. If one applies this fact to $v = I\nabla F(t,s,u(t))$ and $\xi = \frac{1}{2\pi}\int_0^{2\pi} u\, ds$, one obtains a unique $w \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ such that

$$\int_0^{2\pi} w\, ds = \int_0^{2\pi} u\, ds, \quad \dot{w} = I\nabla_u F(t,\cdot,u). \tag{12.48}$$

Now let us take the scalar product of the latter equation with Iv for $v \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ and integrate by parts to obtain

$$\int_0^{2\pi} \big(\langle w | I\dot{v} \rangle_{\mathbb{R}^{2N}} + \big\langle \nabla_u F(t,s,u) \big| v \big\rangle_{\mathbb{R}^{2N}} \big)\, ds = 0.$$

Comparing this to (12.47) yields the L^2-orthogonality relation

$$\int_0^{2\pi} \langle u - w | I\dot{v} \rangle_{\mathbb{R}^{2N}}\, ds = 0, \quad v \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N}).$$

This shows that u and w can differ only by a constant function which is actually zero by the first equation in (12.48). Consequently, $u \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ satisfies

$$I\dot{u} + \nabla_u F(t,\cdot,u) = 0 \tag{12.49}$$

almost everywhere. As every function in $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2N})$ is continuous, it actually follows that $u \in C(\mathbb{S}^1, \mathbb{R}^{2N})$. Finally, (12.49) shows that $u \in C^1(\mathbb{S}^1, \mathbb{R}^{2N})$.

The same argument applied to (12.44) yields that the kernel of H_t consists of the classical solutions of (12.35). Thus if (12.35) has only the trivial solution for $t = 0, 1$, then H_0, H_1 are invertible.

It remains to show that the sequence u_n converges to 0 with respect to the C^1 norm. Let us first note that $u_n \to 0$ in $L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ as the embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2N})$ is continuous. It follows by (H3) and an elementary estimate as in the proof of Proposition 12.3.1

that $\nabla_u F(t_n,\cdot,u_n) \to 0$ in $L^2(\mathbb{S}^1,\mathbb{R}^{2N})$, and thus $\dot{u}_n \to 0$ in $L^2(\mathbb{S}^1,\mathbb{R}^{2N})$ as u_n satisfies (12.49). Since u_n and $\dot{u}_n$ tend to zero in $L^2(\mathbb{S}^1,\mathbb{R}^{2N})$, one sees that $u_n \to 0$ in $W^{1,2}(\mathbb{S}^1,\mathbb{R}^{2N})$. As the embedding $W^{1,2}(\mathbb{S}^1,\mathbb{R}^{2N}) \hookrightarrow C(\mathbb{S}^1,\mathbb{R}^{2N})$ is continuous, it follows that $u_n \to 0$ in $C(\mathbb{S}^1,\mathbb{R}^{2N})$. Finally, (12.49) shows that $\dot{u}_n \to 0$ in $C(\mathbb{S}^1,\mathbb{R}^{2N})$, and thus $u_n \to 0$ in $C^1(\mathbb{S}^1,\mathbb{R}^{2N})$ as claimed. □

Note that it will usually be relatively easy to figure out if the linear equations (12.35) have nontrivial solutions for $t = 0,1$. Moreover, by Proposition 7.3.1, the spectral flow in Theorem 12.3.3 is the Bott–Maslov index of a path of Lagrangians obtained from the fundamental solutions of (12.35). However, Theorem 12.3.3 can only be useful if there are appropriate ways to obtain the latter number. This is the topic of the remainder of this section.

Let us first consider the case that (12.34) is a higher-order perturbation of an autonomous system, i. e., we assume in addition to (H1)–(H4) that

(H5) $A_t = \nabla_u^2 F(t,s,0)$ does not depend on s.

Then (12.35) is autonomous and has a nontrivial solution if and only if 1 is an eigenvalue of the matrix $\exp(2\pi I A_t)$. The latter is equivalent to the existence of some $k \in \mathbb{Z}$ such that $k\imath$ is an eigenvalue of IA_t.

The next aim is to obtain an explicit formula for the spectral flow of $t \in [0,1] \mapsto H_t$ under the additional assumption (H5). Let us consider for $k \geq 0$ the spaces

$$\mathcal{V}_k = \{a\sin(ks) + b\cos(ks) : a,b \in \mathbb{R}^{2N}\} \subset \mathcal{H},$$

and note that it follows from (12.41) and (12.44) that each space $\mathcal{V}_k$ is invariant under the operators H_t. Let $e_1,\dots,e_N,e_{N+1},\dots,e_{2N}$ be the standard basis of $\mathbb{R}^{2N}$. This yields a basis of $\mathcal{V}_k$ for $k \in \mathbb{N}$ by

$$\{u_1^k,\dots,u_{2N}^k,v_1^k,\dots,v_{2N}^k\}, \tag{12.50}$$

where $u_i^k = \sin(ks)e_i$ and $v_i^k = \cos(ks)e_i$ for $i = 1,\dots,2N$. As $Ie_i = e_{i+N}$ for $i = 1,\dots,N$, it is readily seen from (12.41) and (12.44) that $H_t|_{\mathcal{V}_K}$, $k \in \mathbb{N}$, is given with respect to the basis (12.50) by the $4N \times 4N$-matrix

$$B_k(t) = \begin{pmatrix} \frac{1}{k}A_t & I \\ -I & \frac{1}{k}A_t \end{pmatrix}. \tag{12.51}$$

Moreover, $\{e_1,\dots,e_{2N}\}$ is a basis of $\mathcal{V}_0$ and $L_t|_{\mathcal{V}_0}$ is given by multiplication by A_t.

The aim is to find a decomposition $\mathcal{H} = \mathcal{X} \oplus \mathcal{Y}$ into closed subspaces that reduce the operators H_t and are such that $\dim(\mathcal{X}) < \infty$, as well as $H_t|_{\mathcal{Y}} \in \mathbb{G}(\mathcal{Y})$, $t \in [0,1]$.

Let $m_0 \in \mathbb{N}$ be such that $B_k(t)$ is invertible for all $k > m_0$ and $t \in [0,1]$. Then the operators $H_t|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_k$ are invertible as well for $k > m_0$. Let us now consider the spaces $\mathcal{X} = \bigoplus_{k=0}^{m_0} \mathcal{V}_k$ and $\mathcal{Y} = \mathcal{X}^\perp$. The operators H_t are reduced by the decomposition $\mathcal{H} = \mathcal{X} \oplus \mathcal{Y}$, and one obtains from Theorem 4.2.1(v)

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{X}}) + \mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{Y}}).$$

As $H_t|_{\mathcal{Y}} \in \mathbb{F}\mathbb{B}_{\mathrm{sa}}(\mathcal{Y})$ and $H_t|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_k$ is invertible for $k \geq m_0 + 1$, it follows that $H_t|_{\mathcal{Y}}$ is invertible for $t \in [0,1]$. Consequently, by Theorem 4.2.1(i),

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{X}}), \tag{12.52}$$

and so the spectral flow computation is reduced to finite dimensions. Moreover, one obtains from Theorem 4.2.1(i) and (v) that

$$\begin{aligned}\mathrm{Sf}(t \in [0,1] \mapsto H_t) &= \sum_{k=0}^{m_0} \mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{V}_k}) \\ &= \sum_{k=0}^{\infty} \mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{V}_k}),\end{aligned} \tag{12.53}$$

where it was used once again that $H_t|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_k$ is invertible for $k > m_0$.

Let us now consider $H_t|_{\mathcal{V}_k}$ for some $k = 0,1,2,\ldots$. As $\mathcal{V}_k$ is of finite dimension, there is a single $a > 0$ in (4.4) such that $\mathrm{spec}(H_t|_{\mathcal{V}_k}) \subset [-a,a]$ and all elements in $\mathrm{spec}(H_t|_{\mathcal{V}_k})$ are eigenvalues of finite multiplicity. Thus

$$\mathrm{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}}(P^{>}_{a,1} - P^{<}_{a,1} - P^{>}_{a,0} + P^{<}_{a,0}). \tag{12.54}$$

By Lemma 4.1.1, $\mathrm{Tr}(\chi_{[-a,a]}(H_t))$ is constant on $[0,1]$, and one obtains, as in the proof of Proposition 4.3.1,

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{H}}(P^{>}_{a,1}) - \mathrm{Tr}_{\mathcal{H}}(P^{>}_{a,0}) \\ &\quad = \mathrm{Tr}_{\mathcal{H}}(P^{<}_{a,0}) + \dim(\mathrm{Ker}(H_0)) - \mathrm{Tr}_{\mathcal{H}}(P^{<}_{a,1}) - \dim(\mathrm{Ker}(H_1)).\end{aligned}$$

Plugging this into (12.54) yields

$$\begin{aligned}\mathrm{Sf}(t \in [0,1] \mapsto H_t) &= \mathrm{Tr}_{\mathcal{H}}(P^{<}_{a,0}) - \mathrm{Tr}_{\mathcal{H}}(P^{<}_{a,1}) \\ &\quad + \frac{1}{2}(\dim(\mathrm{Ker}(H_0)) - \dim(\mathrm{Ker}(H_1))) \\ &= \iota_-(H_0|_{\mathcal{V}_k}) - \iota_-(H_1|_{\mathcal{V}_k}) \\ &\quad + \frac{1}{2}(\dim(\mathrm{Ker}(H_0)) - \dim(\mathrm{Ker}(H_1))).\end{aligned}$$

Let us now assume that H_0, H_1 are invertible. Then, with respect to the basis (12.50),

$$\begin{aligned}&\mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{V}_k}) = \iota_-(B_k(0)) - \iota_-(B_k(1)), \quad k \in \mathbb{N}, \\ &\mathrm{Sf}(t \in [0,1] \mapsto H_t|_{\mathcal{V}_0}) = \iota_-(A_0) - \iota_-(A_1).\end{aligned} \tag{12.55}$$

Moreover, the matrices $B_k(i)$, $i = 0, 1$, are invertible as well, and thus

$$\frac{1}{2}\operatorname{Sig}(B_k(i)) = 2N - \iota_-(B_k(i)).$$

Consequently, by (12.53),

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \iota_-(A_0) - \iota_-(A_1) + \frac{1}{2}\sum_{k=1}^{\infty}(\operatorname{Sig}(B_k(1)) - \operatorname{Sig}(B_k(0))).$$

As $B_k(i)$ converge to matrices of signature 0, the series $\sum_{k=1}^{\infty}\operatorname{Sig}(B_k(i))$, $i = 0, 1$, converge as actually only finitely many of their terms are nonzero. Thus the definition

$$\nu(A_i) = \iota_-(A_i) - \frac{1}{2}\sum_{k=1}^{\infty}\operatorname{Sig}(B_k(i)) \tag{12.56}$$

makes sense. The following theorem summarizes the above findings. It was proved along these lines by Fitzpatrick, Pejsachowicz, and Recht in [85] and recently obtained by other methods in [24, 107].

Theorem 12.3.4. *If* (H1)–(H5) *hold, no eigenvalue of the matrices IA_0, IA_1 is an integral multiple of the imaginary unit $\imath$ and*

$$\nu(A_0) \neq \nu(A_1), \tag{12.57}$$

then there are a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,1]$ converging to some $t^* \in (0,1)$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^1(\mathbb{S}^1, \mathbb{R}^{2N})$ converging to 0 such that u_n is a nontrivial 2π-periodic solution of (12.34) *for t_n.*

The previous discussion has shown that the spectral flow of the family H_t, $t \in [0,1]$, of the Hessians (12.44) can conveniently be computed by (12.56). The main advantage of the latter formula is that it provides a way to compute the spectral flow just from the coefficients of the linearized equation (12.35). In particular, neither eigenvalues nor eigenfunctions of the operators H_t need to be determined. Note that (H5) is vital for the arguments above as the spaces $\mathcal{V}_k$ will generally not reduce the operators H_t if S_t depends on s. Thus a simple formula as (12.57) cannot be expected to hold without assuming (H5). On the other hand, Theorem 12.3.3 only requires a nonvanishing spectral flow, and so it would be enough to have an estimate that yields its nontriviality. The following theorem is called the comparison principle of the spectral flow.

Theorem 12.3.5. *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ and $t \in [0,1] \mapsto H'_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ be two paths such that H_t and H'_t are Calkin equivalent for all $t \in [0,1]$. If*

$$H_0 \le H'_0, \quad H'_1 \le H_1,$$

then

$$\operatorname{Sf}(t \in [0,1] \mapsto H'_t) \le \operatorname{Sf}(t \in [0,1] \mapsto H_t).$$

Proof. Let us set $K_t = H'_t - H_t$ and define a homotopy

$$(s,t) \in [0,1] \times [0,1] \mapsto h_{(s,t)} \in \mathbb{FB}_{sa}(\mathcal{H})$$

by $h_{(s,t)} = H_t + sK_t$. It follows from Theorem 4.2.2 and Theorem 4.2.1(ii)–(iii) that

$$\begin{aligned}&\mathrm{Sf}(t \in [0,1] \mapsto H_t)\\ &\quad = \mathrm{Sf}(s \in [0,1] \mapsto h_{(s,0)}) + \mathrm{Sf}(t \in [0,1] \mapsto H'_t) - \mathrm{Sf}(s \in [0,1] \mapsto h_{(s,1)}),\end{aligned}$$

which shows the assertion as

$$\mathrm{Sf}(s \in [0,1] \mapsto h_{(s,0)}) \geq 0 \quad \text{and} \quad \mathrm{Sf}(s \in [0,1] \mapsto h_{(s,1)}) \leq 0 \tag{12.58}$$

by Theorem 4.2.6. □

Let us now consider the equations (12.34) under the assumptions (H1)–(H4) and define for $i = 0, 1$,

$$\alpha_i = \inf_{s\in[0,2\pi]} \inf_{\|u\|=1} \langle A_i(s)u|u\rangle_{\mathbb{R}^{2N}}, \quad \beta_i = \sup_{s\in[0,2\pi]} \sup_{\|u\|=1} \langle A_i(s)u|u\rangle_{\mathbb{R}^{2N}}.$$

Note that $\inf_{\|u\|=1}\langle A_i(s)u, u\rangle$ and $\sup_{\|u\|=1}\langle A_i(s)u, u\rangle$ are the smallest and the largest eigenvalue of the symmetric matrix $A_i(s)$, and

$$\alpha_i \mathbf{1}_{2N} \leq A_i(s) \leq \beta_i \mathbf{1}_{2N}, \quad x \in I. \tag{12.59}$$

Let us now assume that $\beta_0 < \alpha_1$ and consider the path $t \in [0,1] \mapsto H'_t \in \mathbb{FB}_{sa}(\mathcal{H})$ of self-adjoint Fredholm operators defined by

$$\langle H'_t u|v\rangle_{W^{\frac{1}{2},2}} = \Gamma(u,v) + (\beta_0 + t(\alpha_1 - \beta_0)) \int_0^{2\pi} \langle u|v\rangle_{\mathbb{R}^{2N}}\, ds,$$

where Γ is the bounded bilinear form in (12.40). It follows from Lemma 12.3.2 that $H_t - H'_t$ is compact for all $t \in [0,1]$. Moreover,

$$\begin{aligned}\langle (H_t - H'_t)u|v\rangle_{W^{\frac{1}{2},2}} &= \int_0^{2\pi} \langle (A_t(s) - (\beta_0 + t(\alpha_1 - \beta_0))\mathbf{1}_{2N})u|v\rangle_{\mathbb{R}^{2N}}\, ds\\ &= \begin{cases} \leq 0, & t = 0,\\ \geq 0, & t = 1,\end{cases}\end{aligned}$$

by (12.59), and hence it follows from Theorem 12.3.5 that

$$\mathrm{Sf}(t \in [0,1] \mapsto H'_t) \leq \mathrm{Sf}(t \in [0,1] \mapsto H_t).$$

Thus one obtains the existence of a bifurcation from Theorem 12.3.3 if $\mathrm{Sf}(t \in [0,1] \mapsto H'_t) > 0$. Note at first that the crossing form of H' at a crossing t^* is

$$\Gamma_{t^*} = \alpha_1 - \beta_0 > 0,$$

and consequently only the existence of a crossing is needed. Now, the kernel of H'_t consists of the solutions of

$$\begin{cases} I\dot{u}(s) + \mu(t)u(s) = 0, & s \in [0, 2\pi], \\ u(0) = u(2\pi), \end{cases} \tag{12.60}$$

where $\mu(t) = \beta_0 + t(\alpha_1 - \beta_0)$. The fundamental solution of this differential equation is

$$\cos(\mu(t)s)\mathbf{1}_{2N} + \sin(\mu(t)s)I,$$

which shows that there is a nontrivial solution of (12.60) if and only if $\mu(t) \in \mathbb{Z}$. Taking into account that t is in the unit interval, it follows that $\mathrm{Sf}(t \in [0,1] \mapsto H'_t) \geq 1$ if $(\beta_0, \alpha_1) \cap \mathbb{Z} \neq \emptyset$. If one repeats the above argument for

$$\langle H''_t u|v\rangle_{W^{\frac{1}{2},2}} = \Gamma(u,v) + \big(\alpha_0 + t(\beta_1 - \alpha_0)\big) \int_0^{2\pi} \langle u|v\rangle_{\mathbb{R}^{2N}}\, ds,$$

under the assumption $\beta_1 < \alpha_0$, it follows that $\mathrm{Sf}(t \in [0,1] \mapsto H''_t) \leq -1$ if one has $(\beta_1, \alpha_0) \cap \mathbb{Z} \neq \emptyset$. The following theorem summarizes these results.

Theorem 12.3.6. *If* (H1)–(H4) *hold,* (12.35) *has only the trivial solution and either*

$$(\beta_0, \alpha_1) \cap \mathbb{Z} \neq \emptyset, \quad \textit{or} \quad (\beta_1, \alpha_0) \cap \mathbb{Z} \neq \emptyset,$$

then there are a sequence $(t_n)_{n\in\mathbb{N}}$ *in* $[0,1]$ *converging to some* $t^* \in (0,1)$ *and a sequence* $(u_n)_{n\in\mathbb{N}} \subset C^1(\mathbb{S}^1, \mathbb{R}^{2N})$ *converging to* 0 *such that* u_n *is a nontrivial* 2π*-periodic solution of* (12.34) *for* t_n.

Note that, similar to Theorem 12.3.4, the previous theorem allows obtaining the existence of a bifurcation point for (12.34) from the coefficients of the linearized equations (12.35). Finally, applications of Theorem 12.3.5 to PDE can be found in [202]. Other methods to compute the spectral flow for applications in bifurcation theory of differential equations are discussed in [145, 198, 199, 203, 150].

A Collection of technical elements

A.1 Riesz projections

The following proposition resembles a few facts about Riesz projections associated to a bounded operator T on a Hilbert space. Let us stress that Riesz projection are not necessarily self-adjoint (and hence may also just be called Riesz idempotent).

Proposition A.1.1. *Let* $\Delta \subset \mathrm{spec}(T)$ *be a separated spectral subset, namely a closed subset which has trivial intersection with the closure of* $\mathrm{spec}(T)\backslash\Delta$. *Associated to* Δ *let* Γ *be a curve in* $\mathbb{C} \setminus \mathrm{spec}(T)$ *with winding number* 1 *around each point of* Δ *and* 0 *around all points of* $\mathrm{spec}(T) \setminus \Delta$. *The Riesz projection of* T *on* Δ *is defined as*

$$R_\Delta = \oint_\Gamma \frac{dz}{2\pi\imath}(z - T)^{-1}. \tag{A.1}$$

The range and kernel of R_Δ *are denoted by* $\mathcal{E}_\Delta = \mathrm{Ran}(R_\Delta)$ *and* $\mathcal{F}_\Delta = \mathrm{Ker}(R_\Delta)$. *If* $\Delta = \{\lambda\}$ *is an isolated point in* $\mathrm{spec}(T)$, *let us also use the notation* $R_\lambda = R_\Delta$, $\mathcal{E}_\lambda = \mathcal{E}_\Delta$, *and so on. The following properties hold:*

(i) R_Δ *is idempotent, namely an oblique projection, and* $\mathcal{E}_\Delta$ *and* $\mathcal{F}_\Delta$ *are closed subspaces. Moreover,* R_Δ *is independent of the choice of* Γ.

(ii) *Let* T *be invertible. Then* Γ *can, moreover, be chosen to have a vanishing winding number around* 0. *If* $(\Gamma)^{-1}$ *denotes the path of inverted complex points, one has*

$$R_\Delta = \oint_{(\Gamma)^{-1}} \frac{dz}{2\pi\imath}\left(z - T^{-1}\right)^{-1}. \tag{A.2}$$

(iii) *If* Δ *and* Δ' *are disjoint separated spectral subsets, then one has* $R_\Delta R_{\Delta'} = 0$ *as well as* $R_{\Delta\cup\Delta'} = R_\Delta + R_{\Delta'}$.

(iv) *For is a disjoint decomposition* $\mathrm{spec}(T) = \bigcup_{l=1}^L \Delta_l$ *in separated spectral subsets,* $\sum_{l=1}^L R_{\Delta_l} = \mathbf{1}$.

(v) $\mathcal{E}_\Delta$ *is invariant for* T *and* $\mathcal{F}_\Delta$ *is invariant for* T^*. *Moreover,* $\dim(\mathcal{E}_\Delta) = \dim(\mathcal{F}_\Delta^\perp)$.

(vi) *If* Φ_Δ *and* Ψ_Δ *are frames for* $\mathcal{E}_\Delta$ *and* $\mathcal{F}_\Delta^\perp$, *and* $\Psi_\Delta^*\Phi_\Delta$ *is invertible, then*

$$R_\Delta = \Phi_\Delta\left(\Psi_\Delta^*\Phi_\Delta\right)^{-1}\Psi_\Delta^*.$$

(vii) *The orthogonal projections on* $\mathcal{E}_\Delta$ *and* $\mathcal{F}_\Delta$ *are* $R_\Delta(R_\Delta^* R_\Delta)^{-1}R_\Delta^* = \Phi_\Delta\Phi_\Delta^*$ *and* $R_\Delta^*(R_\Delta R_\Delta^*)^{-1}R_\Delta = \Psi_\Delta\Psi_\Delta^*$, *respectively.*

(viii) *If* $\dim(\mathcal{E}_\lambda) < \infty$, *then* $\mathcal{E}_\lambda$ *is the span of the generalized eigenvectors of* T *to* λ.

(ix) *Let* f *be an analytic function on the convex closure of* $\mathrm{spec}(T)$. *Suppose that*

$$f\left(\mathrm{spec}(T) \cap \Delta\right) \cap f\left(\mathrm{spec}(T) \setminus \Delta\right) = \emptyset.$$

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Denote by $Q_{f(\Delta)}$ the Riesz projection of $f(T)$ on $f(\Delta)$, which is a separated spectral subset for $f(T)$. Then $Q_{f(\Delta)} = R_\Delta$.

Proof. (i) Let Γ' be another path with the same properties as Γ, but, moreover, encircling every point of Γ once. Then

$$\begin{aligned}(R_\Delta)^2 &= \oint_\Gamma \frac{dz}{2\pi\imath} \oint_{\Gamma'} \frac{d\xi}{2\pi\imath}(\xi - T)^{-1}(z-T)^{-1}\\ &= \oint_\Gamma \frac{dz}{2\pi\imath} \oint_{\Gamma'} \frac{d\xi}{2\pi\imath}\frac{1}{z-\xi}((\xi - T)^{-1} - (z-T)^{-1})\\ &= \oint_{\Gamma'} \frac{d\xi}{2\pi\imath}(\xi - T)^{-1} \oint_\Gamma \frac{dz}{2\pi\imath}\frac{1}{z-\xi} - \oint_\Gamma \frac{dz}{2\pi\imath}(z-T)^{-1} \oint_{\Gamma'} \frac{d\xi}{2\pi\imath}\frac{1}{z-\xi},\end{aligned}$$

where in the second equality the resolvent identity was used, and in the third we used Fubini's theorem. Now the first summand vanishes because the integral over Γ vanishes, while in the second summand the integral over ξ is equal to -1. This shows that $(R_\Delta)^2 = R_\Delta$. In a similar manner, all the other properties of the Riesz projection are derived.

(ii) As T is invertible, the integral over a sufficiently small circle vanishes and can thus always be added with the right orientation to assure that Γ has a vanishing winding number around 0. The change of variable $\xi = z^{-1}$ leads to

$$R_\Delta = \oint_{(\Gamma)^{-1}} \frac{d\xi}{2\pi\imath}\frac{-1}{\xi^2}(\xi^{-1} - T)^{-1} = \oint_{(\Gamma)^{-1}} \frac{d\xi}{2\pi\imath}\left((\xi - T^{-1})^{-1} - \frac{1}{\xi}\right).$$

Now the integral of the last summand vanishes because the winding number does by assumption.

Proofs of items (iii) to (viii) are standard and can be found in [112].

(ix) Let γ denote a curve in the resolvent set of $f(T)$ circling once around $f(\Delta)$. Then

$$Q_{f(\Delta)} = \oint_\gamma \frac{d\xi}{2\pi\imath}(\xi^{-1} - f(T))^{-1}.$$

Now let us write the operator function $(\xi^{-1} - f(T))^{-1}$ also by holomorphic functional calculus using a curve Γ around $\mathrm{spec}(T)$. But as Δ is a separated spectral subset of T, it is possible to choose Γ composed of two curves Γ_1 around Δ and Γ_2 around the remainder $\mathrm{spec}(T) \setminus \Delta$. Moreover, Γ_1 is chosen such that all of $f(\Gamma_1)$ is encircled once by γ, and Γ_2 such that $f(\Gamma_2)$ does not intersect γ, which is possible by hypothesis. Now

$$Q_{f(\Delta)} = \oint_\gamma \frac{d\xi}{2\pi\imath}\left(\oint_{\Gamma_1} + \oint_{\Gamma_2}\right)\frac{dz}{2\pi\imath}\frac{1}{\xi - f(z)}(z-T)^{-1}.$$

Exchanging the integrals gives

$$Q_{f(\Delta)} = \oint_{\Gamma_1} \frac{dz}{2\pi\imath}(z-T)^{-1} \oint_{\gamma} \frac{d\xi}{2\pi\imath} \frac{1}{\xi - f(z)} + \oint_{\Gamma_2} \frac{dz}{2\pi\imath}(z-T)^{-1} \oint_{\gamma} \frac{d\xi}{2\pi\imath} \frac{1}{\xi - f(z)}.$$

The first integral gives R_Δ, while the second vanishes because the contour is chosen such that there is no singularity encircled by γ for $z \in \Gamma_2$. □

A.2 Norm estimates on roots

The first inequality goes back to Haagerup, see [143].

Proposition A.2.1. *Let $U \in \mathbb{U}(\mathcal{H})$ be unitary and $T \in \mathbb{B}(\mathcal{H})$ be positive semidefinite, namely $T \geq 0$. Then for $\alpha \in [0,1]$,*

$$\|[U, T^\alpha]\| \leq \|[U,T]\|^\alpha.$$

Proof. Recall that the roots $x \in [0,\infty) \mapsto x^\alpha$ are Herglotz functions and are therefore operator monotone by Loewner's theorem. Thus,

$$\begin{aligned} UT^\alpha U^* &= (UTU^*)^\alpha \\ &= (UTU^* - T + T)^\alpha \\ &\leq (\|UTU^* - T\| + T)^\alpha \\ &\leq \|UTU^* - T\|^\alpha + T^\alpha, \end{aligned}$$

where the last inequality follows from the spectral theorem, applied to T, and the inequality $(x+y)^\alpha \leq x^\alpha + y^\alpha$ for positive numbers x and y. Hence

$$\|UT^\alpha U^* - T^\alpha\| \leq \|UTU^* - T\|^\alpha,$$

which is equivalent to the claim. □

The next bound can be found in [128].

Proposition A.2.2. *Let $A, B \in \mathbb{B}(\mathcal{H})$ satisfy $A \geq 0$, $B \geq 0$. Then for $\alpha \in [0,1]$,*

$$\|A^\alpha - B^\alpha\| \leq \|A - B\|^\alpha.$$

Proof. Let us apply Proposition A.2.1 to

$$T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad U = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

One finds

$$\left\| \begin{pmatrix} 0 & B^\alpha - A^\alpha \\ A^\alpha - B^\alpha & 0 \end{pmatrix} \right\| \le \left\| \begin{pmatrix} 0 & B - A \\ A - B & 0 \end{pmatrix} \right\|^\alpha ,$$

which proves the lemma. □

A.3 Definitions and facts from topology

Many places in the book use notions and results from topology, some rather basic and some fairly deep. This appendix collects these facts and provides references to the literature where more can be read up on them.

Definition A.3.1 ([79]). Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and, furthermore, let $f, F : X \to Y$ be continuous maps.

(i) Space $(X, \mathcal{O}_X)$ is contractible if there is a continuous map $h : X \times [0,1] \to X$ such that $h(x,1) = x$ for all $x \in X$ and $h(x,0) = x_{\text{ref}}$ for some reference point $x_{\text{ref}} \in X$. Then h is called a contraction.

(ii) Maps f and F are homotopic if there exists a continuous map $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = F(x)$ for all $x \in X$.

(iii) A continuous map $g : Y \to X$ is a homotopy inverse to f if $f \circ g : Y \to Y$ and $g \circ f : X \to X$ are homotopic to the identity. Then both f and g are called homotopy equivalences.

(iv) Spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homotopy equivalent if there exists a homotopy equivalence $f : X \to Y$.

(v) Let $(Y, \mathcal{O}_Y)$ be a topological subspace of $(X, \mathcal{O}_X)$. Then Y is a deformation retract of X if there is a homotopy $h : X \times [0,1] \to X$ such that for every $x \in X$ and $y \in Y$ $h(x,0) = x$, $h(x,1) \in Y$ and $h(y,1) = y$.

Section 8.6 applies a criterion of tom Dieck for homotopy equivalence [191]. It uses Dold's notion of a numerable cover of a topological space.

Definition A.3.2. Let $(X, \mathcal{O}_X)$ be a topological space. An open cover (X_τ) of X indexed by the index set $\mathcal{T}$ is called numerable if there is a locally finite partition of unity $(f_\tau)_{\tau\in\mathcal{T}}$ such that the closure of the support of f_τ is contained in X_τ, namely, $\text{supp}(f_\tau) \subset X_\tau$ for every $\tau \in \mathcal{T}$.

A Hausdorff space X is called paracompact if every open cover has a locally finite subcover. For such subcover one can then construct a locally finite partition of unity. Hence any open cover of a paracompact space is numerable. Let us also recall a theorem of Stone, stating that every metrizable space is paracompact.

Theorem A.3.3 ([191, Theorem 1]). *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and let $\phi : X \to Y$ be a continuous map. Let $(X_\tau)_{\tau\in\mathcal{T}}$, respectively $(Y_\tau)_{\tau\in\mathcal{T}}$, be numerable coverings of X, respectively Y, indexed by the same index set $\mathcal{T}$. Assume that $\phi(X_\tau) \subset Y_\tau$ for all $\tau \in \mathcal{T}$*

and that for every finite subset $\sigma \subset \mathcal{T}$ the restriction map $\phi_\sigma : \bigcap_{\tau\in\sigma} X_\tau \to \bigcap_{\tau\in\sigma} Y_\tau$ induced by ϕ is a homotopy equivalence. Then ϕ itself is a homotopy equivalence.

Next let us come to homotopy groups (e. g., [185, 188, 192, 103]). Recall that for a topological space X, the set of connected components is denoted by $\pi_0(X)$. It has no group structure. Then for $k \in \mathbb{N}$, the homotopy group $\pi_k(X, x_0)$ consists of the homotopy classes of base-point preserving homotopies of continuous maps $f : (\mathbb{S}^k, s_0) \to (X, x_0)$, where $s_0 \in \mathbb{S}^k$ and $x_0 \in X$ are any points. The group operation given by a suitable concatenation is abelian for $k \geq 2$, but not in the so-called fundamental group $\pi_1(X, x_0)$. For different choices of the base point s_0, these groups are isomorphic. If X is connected, this definition also is independent of the choice of x_0. If Y is another topological space and $F : X \to Y$ continuous, then the concatenation of elements in $\pi_n(X, x_0)$ by F yields an induced group homomorphism $F_* : \pi_n(X, x_0) \to \pi_n(Y, F(x_0))$ for all $n \in \mathbb{N}$.

Definition A.3.4. Two path-connected topological spaces X, Y are called weakly homotopy equivalent if there is some $x_0 \in X$ and a continuous map $F : X \to Y$ such that the induced maps $F_* : \pi_n(X, x_0) \to \pi_n(Y, F(x_0))$ are isomorphisms for all $n \in \mathbb{N}$.

The following theorem due to Whitehead explains the importance of the above notion.

Theorem A.3.5 ([188, Theorem 6.32]). *A map $F : X \to Y$ between path-connected CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence.*

Note that the statement of the previous theorem is still true if X and Y are homotopy equivalent to CW-complexes.

Theorem A.3.6. *Every metrizable Banach manifold is homotopy equivalent to a CW-complex.*

Proof. This follows as every metrizable Banach manifold is an absolute neighborhood retract (ANR) by [142, Theorem 5], and every ANR is homotopy equivalent to a CW-complex by [89, Theorem 5.2.1]. □

In linear spaces, it is also possible to prove the homotopy equivalence of an open subset to a CW-complex without assuming completeness, as shows the next result.

Theorem A.3.7. *Every open subset of a normed linear space is homotopy equivalent to a CW-complex.*

Proof. Milnor showed in [134, Lemma 4] that a paracompact space A is homotopy equivalent to a CW-complex if there is a neighborhood U of the diagonal in $A \times A$ and a map $\lambda : U \times [0, 1] \to A$ such that $\lambda(a, b, 0) = a$, $\lambda(a, b, 1) = b$ for all $(a, b) \in U$, $\lambda(a, a, t) = a$ for all $a \in A$, $t \in [0, 1]$, and there is an open cover $\{V_i\}_{i\in J}$ of A such that $V_i \times V_i \subset U$ and $\lambda(V_i \times V_i \times [0, 1]) = V_i$ for all $i \in J$.

Let now A be an open subset of a normed linear space. Then A is paracompact. For any $x \in X$, let $V_x \subset A$ be an open ball around x. Moreover, define $U \subset A \times A$ as the set of all (x,y) such that there is an open ball $B \subset A$ such that $x,y \in B$. Finally, define $\lambda : U \times [0,1] \to A$ by $\lambda(x,y,t) = (1-t)x + ty$. Note that, indeed, $\lambda(U \times [0,1]) \subset A$ by the definition of U. Moreover, $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$, for all $(x,y) \in U$, and $\lambda(x,x,t) = x$ for all $x \in A$, $t \in [0,1]$. Note that $V_x \times V_x \subset U$ as V_x is a ball in A. Finally, $V_x \subset \lambda(V_x \times V_x \times [0,1])$ by definition of λ and this inclusion actually is an equality as for $y_1, y_2 \in V_x$ and $t \in [0,1]$, $\lambda(y_1,y_2,t) \in V_x$. □

The main tool of homotopy theory used here is the long exact sequence of homotopy groups associated to a fiber bundle $\pi : X \to B$ where X is the total space, B the base space, π is surjective, and, moreover, every point $b \in B$ has a neighborhood $\mathcal{U}$ such that $\pi^{-1}(\mathcal{U})$ is homeomorphic to $\mathcal{U} \times F$ where the fiber F is another topological space and the homeomorphism is fiber-preserving. If B is connected (which will always be assumed here), then F can be chosen to be the same for all $b \in B$. The main fact is now that there is an exact sequence of homotopy groups

$$\cdots \to \pi_k(F) \to \pi_k(X) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to 0.$$

While all these maps can be constructed quite explicitly, their particular form is not needed here. At some places we will encounter a special form of fiber bundles, namely principal bundles which arise in the following situation: let F be a topological group acting continuously on X and let B be the space of orbits; if then $\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ is a local trivialization, the action is required to be equivariant in the sense that $\phi(g \cdot u) = g \cdot \phi(u)$ for $g \in F$ and $u \in \mathcal{U}$. The bundle structure theorem (e. g., [188, 192]) provides a convenient way to check that one has a principal bundle if X is a topological group with closed subgroup F so that the base $B = X/F$ is a homogeneous space. Let then $b_0 = F \in B$ and $\pi : X \to X/F$ defined by $\pi(x) = xF$. A local section for π is a continuous map $\rho : \mathcal{U} \to X$ defined on a neighborhood $\mathcal{U} \subset B$ of b_0 such that $\pi \circ \rho = \mathrm{id}$. If there exists such a local section, then $\pi : X \to X/F$ is a principal bundle with principal group F by the bundle structure theorem. Often, one encounters several principal bundles and then one of the spaces, X, B, or F, has already known homotopy groups by a previous result which is described next.

In a celebrated work [36] (see also [135]), Bott computed the (stable) homotopy groups of the general linear group $\mathrm{GL}(\infty,\mathbb{C}) = \bigcup_{n\geq 1} \mathrm{GL}(n,\mathbb{C})$ equipped with the inductive limit topology:

$$\pi_k\big(\mathrm{GL}(\infty,\mathbb{C})\big) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases} \tag{A.3}$$

In Section 8.6, yet another type of a fiber bundle appears that is now introduced a bit more detailed as it is far harder to find in the literature than the above types of bundle.

Definition A.3.8. A Hilbert bundle over the topological space X consists of a topological space $\mathcal{E}$, a surjective continuous map $\pi : \mathcal{E} \to X$, a Hilbert space $\mathcal{H}$, and a family $\{(U_j, \varphi_j)\}_{j \in J}$ where J is some index set, $\{U_j\}_{j \in J}$ is an open cover of X, and each map $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathcal{H}$ is a homeomorphism, such that
(i) $\mathcal{E}_x = \pi^{-1}(x)$ is a Hilbert space for every $x \in X$;
(ii) if $x \in U_j$, then $\varphi_j(\mathcal{E}_x) = \{x\} \times \mathcal{H}$ and φ_j is a linear operator when $\{x\} \times \mathcal{H}$ is identified with $\mathcal{H}$ by ignoring the first component;
(iii) if $\varphi_{j,x} : \mathcal{E}_x \to \mathcal{H}$ denotes the invertible operator described in the previous item and U_i is another element of the cover $\{U_j\}_{j \in J}$, then the map

$$x \in U_j \cap U_i \mapsto \varphi_{i,x} \circ \varphi_{j,x}^{-1} \in \mathbb{G}(\mathcal{H}) \tag{A.4}$$

is continuous.

Let us note that the latter item is redundant in case that $\mathcal{H}$ is of finite dimension, a situation that will never be assumed in this section. Moreover, we only consider Hilbert bundles over the space X and denote them as triples $(\mathcal{E}, \pi, \{(U_i, \varphi_i)\}_{i \in J})$. The set $\{(U_i, \varphi_i)\}_{i \in J}$ is called an atlas and it will now be shown that the families $\varphi_{i,x} \circ \varphi_{j,x}^{-1}$ of invertible operators in (A.4) play a crucial role in the classification of Hilbert bundles in the following sense.

Definition A.3.9. Let $(\mathcal{E}_1, \pi_1, \{(U_j^1, \varphi_j^1)\}_{j \in J})$ and $(\mathcal{E}_2, \pi_2, \{(U_i^2, \varphi_i^2)\}_{i \in I})$ be two Hilbert bundles. They are called isomorphic if there is a homeomorphism $F : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\pi_2 \circ F = \pi_1$ (i. e., $F(\mathcal{E}_{1,x}) = \mathcal{E}_{2,x}$, $x \in X$), each restriction $F_x = F|_{\mathcal{E}_{1,x}} : \mathcal{E}_{1,x} \to \mathcal{E}_{2,x}$ is linear (thus an isomorphism) and the maps

$$x \in U_i^1 \cap U_j^2 \mapsto \varphi_{j,x}^2 \circ F_x \circ (\varphi_{i,x}^1)^{-1} \in \mathbb{G}(\mathcal{H}) \tag{A.5}$$

are continuous.

The latter condition is again redundant if $\mathcal{H}$ is finite dimensional.

Definition A.3.10. A set of transition functions for X and $\mathbb{G}(\mathcal{H})$ consists of an open cover $\{U_i\}_{i \in J}$ of X and a collection of continuous maps $\tau_{i,j} : U_i \cap U_j \to \mathbb{G}(\mathcal{H})$, $i, j \in J$, such that

$$\tau_{k,j}(x) \circ \tau_{j,i}(x) = \tau_{k,i}(x), \quad x \in U_i \cap U_j \cap U_k, \tag{A.6}$$

and thus in particular $\tau_{ii}(x) = \mathrm{id}$, as well as $\tau_{j,i}(x) = \tau_{i,j}(x)^{-1}$. Two sets of transition functions $\{U_i^1, \tau_{i,j}^1\}_{i,j \in J}$ and $\{U_k^2, \tau_{k,l}^2\}_{k,l \in I}$ for X are called equivalent if there are continuous maps $\gamma_{k,i} : U_i^1 \cap U_k^2 \to \mathbb{G}(\mathcal{H})$, $k \in I$, $i \in J$, such that

$$\tau_{k,l}^2(x) = \gamma_{k,i}(x) \tau_{i,j}^1(x) \gamma_{l,j}(x)^{-1}, \quad x \in U_i^1 \cap U_j^1 \cap U_k^2 \cap U_l^2. \tag{A.7}$$

Note that if $(\mathcal{E}, \pi, \{(U_i, \varphi_i)\}_{i \in J})$ is a Hilbert bundle, then $\tau_{i,j}(x) = \varphi_{i,x} \circ \varphi_{j,x}^{-1} \in \mathbb{G}(\mathcal{H})$, $i, j \in J$, yields a set of transition functions for X.

Theorem A.3.11. *The map which associates to a Hilbert bundle the set of transition functions for X by* (A.4) *induces a bijection between the set of isomorphism classes of Hilbert bundles over X and the set of equivalence classes of transition functions for X.*

Proof. We denote by Γ the map as in the statement of the theorem. Furthermore, let $(\mathcal{E}_1, \pi_1, \{(U_j^1, \varphi_j^1)\}_{j\in J})$ and $(\mathcal{E}_2, \pi_2, \{(U_k^2, \varphi_k^2)\}_{k\in I})$ be isomorphic by a map $F : \mathcal{E}_1 \to \mathcal{E}_2$ and let $\{\tau_{i,j}^1\}_{i,j\in J}$ and $\{\tau_{k,l}^2\}_{k,l\in I}$ be the corresponding sets of transition functions. Define the map $\gamma_{k,i} : U_i^1 \cap U_k^2 \to \mathbb{G}(\mathcal{H})$, $i \in J$, $k \in I$ by setting

$$\gamma_{k,i}(x) = \varphi_{k,x}^2 \circ F_x \circ (\varphi_{i,x}^1)^{-1}.$$

Note that this is continuous by (A.5) from the definition of an isomorphism. Moreover, (A.7) follows by a straightforward computation and thus $\{\tau_{i,j}^1\}_{i,j\in J}$ and $\{\tau_{k,l}^2\}_{k,l\in I}$ are equivalent. Consequently, the map Γ is well-defined.

To show that Γ is surjective, let us construct for every set of transition functions $\{U_i, \tau_{i,j}\}_{i,j\in J}$ a suitable Hilbert bundle $(\mathcal{E}, \pi, \{(U_i, \varphi_i)\}_{i\in J})$ such that $\tau_{i,j} = \varphi_i \circ \varphi_j^{-1}$ on $U_i \cap U_j$. Consider on

$$\overline{\mathcal{E}} = \bigcup_{i\in J} U_i \times \mathcal{H} \times \{i\}$$

the relation $(x, u, i) \sim (y, v, j) \Leftrightarrow x = y$, $u = \tau_{i,j}(x)v$. Note that this is an equivalence relation by the cocycle condition (A.6). Let $\mathcal{E} = \overline{\mathcal{E}}/\sim$ with the quotient topology and let $\pi : \mathcal{E} \to X$ be defined by $\pi([x, u, i]) = x$, which is well-defined and continuous. Consider the map $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathcal{H}$ defined by $\varphi_i([x, u, i]) = (x, u)$ and its inverse $\varphi_i^{-1}(x, u) = [x, u, i]$. Note that both are continuous and hence φ_i is a homeomorphism. Moreover, $\pi(\varphi_i^{-1}(x, u)) = \pi([x, u, i]) = x$, $(x, u) \in U_i \times \mathcal{H}$ and thus φ_i is fiber-preserving. Finally, let us note that $\varphi_i^{-1}(x, \tau_{i,j}(x)u) = [x, \tau_{i,j}(x)u, i] = [x, u, j] = \varphi_j^{-1}(x, u)$ for $x \in U_i \cap U_j$ and $u \in \mathcal{H}$ and thus $\tau_{i,j}(x) = \varphi_{i,x} \circ \varphi_{j,x}^{-1}$. This firstly implies that the maps (A.4) are continuous and thus the construction indeed gives a Hilbert bundle. Secondly, it shows that the transition functions of the constructed bundle are given by the set of transition functions $\{U_i, \tau_{i,j}\}_{i,j\in J}$.

For the injectivity of Γ, let $(\mathcal{E}_1, \pi_1, \{(U_i^1, \varphi_i^1)\}_{i\in J})$ and $(\mathcal{E}_2, \pi_2, \{(U_k^2, \varphi_k^2)\}_{k\in I})$ be two Hilbert bundles with equivalent sets of transition functions $\{(U_i^1, \tau_{i,j}^1)\}_{i,j\in J}$ and $\{(U_k^2, \tau_{k,l}^2)\}_{k,l\in I}$. Let $\gamma_{k,i} : U_i^1 \cap U_k^2 \to \mathbb{G}(\mathcal{H})$ be the maps in (A.7). For $e \in \mathcal{E}_1$ such that $\pi_1(e) \in U_i^1 \cap U_k^2$ let us set $F(e) = (\varphi_k^2)^{-1}(x, \gamma_{k,i}(x)u)$, where $e = (\varphi_i^1)^{-1}(x, u)$. If also $\pi_1(e) \in U_j^1 \cap U_l^2$, then $e = (\varphi_j^1)^{-1}(x, \tau_{j,i}^1(x)u)$ and thus by (A.7)

$$(\varphi_l^2)^{-1}(x, \gamma_{l,j}(x)\tau_{j,i}^1(x)u) = (\varphi_l^2)^{-1}(x, \tau_{l,k}^2(x)\gamma_{k,i}(x)u) = (\varphi_k^2)^{-1}(x, \gamma_{k,i}(x)u),$$

which shows that one obtains a well-defined map $F : \mathcal{E}_1 \to \mathcal{E}_2$ in this way. Note that F is by construction fiber-preserving, and it is continuous as a local composition of continuous functions. Let us now consider a new family of maps $\tilde{\gamma}_{k,i} : U_i^1 \cap U_k^2 \to \mathbb{G}(\mathcal{H})$

by $\tilde{\gamma}_{k,i} = \gamma_{k,i}^{-1}$ and similarly define a map $G : \mathcal{E} \to \mathcal{E}$ by $G(e) = (\varphi_i^1)^{-1}(x, \tilde{\gamma}_{k,i}(x)u)$, where $e = (\varphi_k^2)^{-1}(x, u)$. Then for $x \in U_i^1 \cap U_k^2$ and $u \in \mathcal{H}$,

$$\begin{aligned}(G \circ F \circ (\varphi_i^1)^{-1})(x, u) &= G((\varphi_k^2)^{-1}(x, \gamma_{k,i}(x)u)) \\ &= (\varphi_i^1)^{-1}(x, \gamma_{k,i}(x)^{-1}\gamma_{k,i}(x)u) \\ &= (\varphi_i^1)^{-1}(x, u),\end{aligned}$$

which shows that $G \circ F$ is the identity on $\mathcal{E}$. Likewise it follows that $F \circ G$ is the identity and thus F is a homeomorphism. Finally, let us note that the condition (A.5) holds as it is by construction of F equivalent to the continuity of the maps $\gamma_{k,i}$. Thus $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic Hilbert bundles. □

If $\mathcal{E} = X \times \mathcal{H}$ and π is the canonical projection onto X, then $(\mathcal{E}, \pi, \{(X, \mathrm{id})\})$ is called *the* trivial bundle. More generally, a bundle $(\mathcal{E}, \pi, \{(U_i, \varphi_i)\}_{i\in J})$ is called trivial, if it is isomorphic to the trivial bundle.

Theorem A.3.12. *Every Hilbert bundle $(\mathcal{E}, \pi, \{(U_i, \varphi_i)\}_{i\in J})$ having an infinite dimensional model space $\mathcal{H}$ and a paracompact Hausdorff space X as base space is trivial.*

Proof. This follows by combining the previous theorem with some standard results from algebraic topology. First, sets of transition functions for a space X can be defined verbatim as in (A.4) for a general topological group G instead of $\mathbb{G}(\mathcal{H})$. The notion of equivalence in (A.7) is the same when replacing everywhere $\mathbb{G}(\mathcal{H})$ by G [188, Definition 11.6]. By [188, Theorem 11.16], the set of equivalence classes of transition functions is in one-to-one correspondence with the set of equivalence classes of G-principal bundles over X. Now by [192, Theorem 14.4.1], for every topological group G there is a topological space BG such that the set of isomorphism classes of G-principal bundles over a paracompact Hausdorff space X is in one-to-once correspondence with the set $[X, BG]$ of homotopy classes of maps from X into BG. This uniquely determines BG up to homotopy type [74, § 7]. It follows from [188, § 11.33], which is essentially based on Brown's representability theorem [188, Theorem 9.12], that BG is homotopy equivalent to a CW-complex. Finally, if G is contractible as a topological space, then BG is weakly contractible by [192, Example 14.4.7]. Thus Whitehead's Theorem A.3.5 implies that BG is a contractible topological space and, consequently, $[X, BG]$ consists of a singleton in this case.

Now by the above explanations and Theorem A.3.11, the set of isomorphism classes of Hilbert bundles over X is in one-to-one correspondence with $[X, B\mathbb{G}(\mathcal{H})]$. As $\mathbb{G}(\mathcal{H})$ is contractible by Kuiper's theorem, there is only one isomorphism class of Hilbert bundles over X. As the trivial bundle is a Hilbert bundle over X, every Hilbert bundle over X has to be isomorphic to it, namely every Hilbert bundle is trivial. □

Acronyms and notations

$\imath$	imaginary unit $\sqrt{-1}$
$\mathrm{supp}(f)$	support of a function f
$\lvert\cdot\rvert$	absolute value
$\mathrm{sgn}(x)$	sign of a nonzero real number $x \in \mathbb{R} \setminus \{0\}$
χ_I	indicator function of a set $I \subset X$
χ_x	indicator function of a point $x \in X$
$\mathbb{R}^N$	N-component vectors with real entries
$\mathbb{C}^N$	N-component vectors with complex entries
$\mathbb{C}^{N\times N}$	$N \times N$ matrices with complex entries
$\mathbb{S}^1$	unit circle
$\sigma_1, \sigma_2, \sigma_3$	2×2 Pauli matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathrm{diag}(A, B)$	block diagonal matrix built from matrices A and B
T^*	adjoint of a matrix T
$\mathrm{Sig}(H)$	signature of a self-adjoint matrix $H = H^*$
$\mathcal{H}, \mathcal{H}'$	separable Hilbert spaces
ϕ, ψ	vectors in a Hilbert space
$\langle\phi\vert\psi\rangle = \phi^*\psi$	scalar product in a Hilbert space
Φ, Ψ	frames for subspaces of a Hilbert space
$\lVert\cdot\rVert$	Hilbert space norm and operator norm
Tr	trace over a Hilbert space
$\mathcal{L}^p(\mathcal{H})$	Schatten ideal of p-trace class operator
$\lVert\cdot\rVert_{(p)}$	p-Schatten norm
$\mathcal{E}, \mathcal{E}', \mathcal{F}$	subspaces of Hilbert spaces
$\mathcal{E}^\perp$	orthogonal complement of $\mathcal{E}$ with respect to Hilbert space scalar product
$\mathbb{L}(\mathcal{H}, \mathcal{H}')$	set of densely defined, closed linear operators from $\mathcal{H}$ to $\mathcal{H}'$
$\mathbb{L}(\mathcal{H})$	set of densely defined, closed linear operators on $\mathcal{H}$
A, B, T	linear operators on $\mathcal{H}$
$\mathcal{D}(T)$	domain of a linear operator T
T^*	adjoint of $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$
H	self-adjoint linear operator on $\mathcal{H}$
$\mathbf{1}$	identity operator on $\mathcal{H}$
$\mathbf{1}_N$	identity matrix of size N
P	orthogonal projection on $\mathcal{H}$
Q	symmetry on $\mathcal{H}$, namely $Q = Q^* = Q^{-1}$
$\mathrm{spec}(T)$	spectrum of an operator T
$\mathrm{spec}_{\mathrm{dis}}(T)$	discrete spectrum of an operator T
$\mathrm{spec}_{\mathrm{ess}}(T)$	essential spectrum of a normal operator T
$\mathbb{B}(\mathcal{H}, \mathcal{H}')$	set of bounded operators from $\mathcal{H}$ to $\mathcal{H}'$
$\mathbb{B}_a(\mathcal{H}, \mathcal{H}')$	set of bounded operators T from $\mathcal{H}$ to $\mathcal{H}'$ with $\lVert T\rVert \le a$
$\mathbb{B}(\mathcal{H})$	set of bounded operators on $\mathcal{H}$, namely $\mathbb{B}(\mathcal{H}) = \mathbb{B}(\mathcal{H}, \mathcal{H})$
$\mathbb{B}_a(\mathcal{H})$	set of bounded operators T on $\mathcal{H}$ with $\lVert T\rVert \le a$
$\mathbb{L}_{\mathrm{sa}}(\mathcal{H})$	set of self-adjoint operators on $\mathcal{H}$
$\mathbb{B}_{\mathrm{sa}}(\mathcal{H})$	set of bounded self-adjoint operators on $\mathcal{H}$
$\mathbb{B}_{a,\mathrm{sa}}(\mathcal{H})$	set of bounded self-adjoint operators T on $\mathcal{H}$ with $\lVert T\rVert \le a$
$\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$	set of $T \in \mathbb{B}_{1,\mathrm{sa}}(\mathcal{H})$ with $\mathrm{Ker}(T \pm \mathbf{1}) = \{0\}$
$\mathbb{K}(\mathcal{H}, \mathcal{H}')$	set of compact operators from $\mathcal{H}$ to $\mathcal{H}'$
$\mathbb{K}(\mathcal{H})$	set of compact operators on $\mathcal{H}$

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$\mathbb{K}(\mathcal{H})^\sim$	unitization of $\mathbb{K}(\mathcal{H})$
$\mathbb{G}^c(\mathcal{H})$	invertibles in $\mathbb{K}(\mathcal{H})^\sim$ (multiplicative group)
$\mathbb{U}(\mathcal{H})$	set of unitary operators on $\mathcal{H}$
$\mathbb{U}^C(\mathcal{H})$	set of unitary operators in $\mathbb{K}(\mathcal{H})^\sim$
$\mathbb{U}^0(\mathcal{H})$	set of $U \in \mathbb{U}(\mathcal{H})$ with $\mathrm{Ker}(U - \mathbf{1}) = \{0\}$
$\mathbb{U}^{C,0}(\mathcal{H})$	intersection $\mathbb{U}^C(\mathcal{H}) \cap \mathbb{U}^0(\mathcal{H})$
$\mathbb{FU}(\mathcal{H})$	set of unitary operators U with $U + \mathbf{1} \in \mathbb{FB}(\mathcal{H})$, or $-1 \notin \mathrm{spec}_{\mathrm{ess}}(U)$
$\mathbb{U}_{\mathrm{sa}}(\mathcal{H})$	set of symmetries (self-adjoint unitaries)
$\mathbb{U}^*_{\mathrm{sa}}(\mathcal{H})$	set of proper symmetries
$\mathbb{Q}(\mathcal{H})$	Calkin algebra $\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ over $\mathcal{H}$
π	Calkin projection $\pi : \mathbb{B}(\mathcal{H}) \to \mathbb{Q}(\mathcal{H})$
$\mathbb{UQ}(\mathcal{H})$	set of unitaries in the Calkin algebra
$\mathbb{UQ}^*_{\mathrm{sa}}(\mathcal{H})$	set of proper symmetries in the Calkin algebra
$\mathbb{F}(\mathcal{H}, \mathcal{H}')$	set of Fredholm operators from $\mathcal{H}$ to $\mathcal{H}'$
$\mathbb{F}(\mathcal{H})$	set of Fredholm operators on $\mathcal{H}$
$\mathrm{Ind}(T)$	index of a Fredholm operator T
$j_\pm(H)$	Morse indices of a self-adjoint Fredholm operator H
$\mathbb{F}_{\mathrm{sa}}(\mathcal{H})$	set of self-adjoint Fredholm operators on $\mathcal{H}$
$\mathbb{F}^c_{\mathrm{sa}}(\mathcal{H})$	set of self-adjoint Fredholm operators with compact resolvent
$\mathbb{FB}(\mathcal{H}, \mathcal{H}')$	set of bounded Fredholm operators from $\mathcal{H}$ to $\mathcal{H}'$
$\mathbb{FB}(\mathcal{H})$	set of bounded Fredholm operators on $\mathcal{H}$
$\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$	set of bounded self-adjoint Fredholm operators on $\mathcal{H}$
$\mathbb{FB}^\pm_{\mathrm{sa}}(\mathcal{H})$	set of $T \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ with only positive/negative essential spectrum
$\mathbb{FB}^*_{\mathrm{sa}}(\mathcal{H})$	set of $T \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ with positive and negative essential spectrum
$\mathbb{FB}_{a,\mathrm{sa}}(\mathcal{H})$	set of self-adjoint Fredholm operators T on $\mathcal{H}$ with $\|T\| \le a$
$\mathbb{FB}^c_{1,\mathrm{sa}}(\mathcal{H})$	set of $T \in \mathbb{FB}_{a,\mathrm{sa}}(\mathcal{H})$ with $\mathrm{spec}_{\mathrm{ess}}(T) \subset \{-1, 1\}$
$\mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$	set of Fredholm operators in $\mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H})$
$\mathbb{FB}^{c,0}_{1,\mathrm{sa}}(\mathcal{H})$	intersection $\mathbb{FB}^c_{1,\mathrm{sa}}(\mathcal{H}) \cap \mathbb{FB}^0_{1,\mathrm{sa}}(\mathcal{H})$
$\mathbb{G}(\mathcal{H})$	set of invertible bounded operators on $\mathcal{H}$
$\mathcal{F}$	bounded transform of an unbounded operator
$\mathcal{C}$	Cayley transform of an unbounded self-adjoint operator
$\mathcal{G}$	transformation $\mathcal{C} \circ \mathcal{F}^{-1} : \mathbb{B}^0_{1,\mathrm{sa}}(\mathcal{H}) \to \mathbb{U}^0(\mathcal{H})$
d_N	metric on $\mathbb{B}(\mathcal{H})$ induced by the operator norm
d_G	gap metric on $\mathbb{L}(\mathcal{H})$
d_R	Riesz metric on $\mathbb{L}(\mathcal{H})$
d_E	extended gap metric on $\mathbb{B}_1(\mathcal{H})$
$\mathcal{O}_N$	operator norm topology on $\mathbb{B}(\mathcal{H})$
$\mathcal{O}_S$	strong operator topology on $\mathbb{L}(\mathcal{H})$
$\mathcal{O}_G$	gap topology on $\mathbb{L}(\mathcal{H})$
$\mathcal{O}_R$	Riesz topology on $\mathbb{L}(\mathcal{H})$
$\mathcal{O}_E$	extended gap topology on $\mathbb{B}_1(\mathcal{H})$
$\mathcal{O}_{SE}$	strong extended gap topology on $\mathbb{B}_1(\mathcal{H})$
$t \in [0, 1] \mapsto H_t$	path of self-adjoint Fredholm operators
$\mathcal{T}$	semifinite trace on a von Neumann algebra
$\mathbb{P}(\mathcal{H})$	proper orthogonal projections on $\mathcal{H}$
$\mathbb{FPP}(\mathcal{H})$	Fredholm pairs of proper projections (P_0, P_1) on $\mathcal{H}$
$\mathrm{Ind}(P_0, P_1)$	index of a Fredholm pair (P_0, P_1) of projections
P_{ref}	reference (Lagrangian) projection
$\mathbb{FP}(\mathcal{H})$	proper orthogonal projections P such that (P_{ref}, P) Fredholm pair

J	fundamental symmetry of a Krein space; always $J = \text{diag}(\mathbf{1}, -\mathbf{1})$
$(\mathcal{K}, J)$	separable Krein space with fundamental symmetry J
$\mathbb{U}(\mathcal{K}, J)$	set of J-unitary operators on Krein space $(\mathcal{K}, J)$
$\mathbb{P}(\mathcal{K}, J)$	J-Lagrangian projections on $\mathcal{K}$
$\mathbb{FPP}(\mathcal{K}, J)$	Fredholm pairs of J-Lagrangian projections (P_0, P_1) on $\mathcal{K}$
$\mathbb{FP}(\mathcal{K}, J)$	Fredholm J-Lagrangian Grassmannian with respect to P_{ref}
$\mathbb{U}^{\mathrm{C}}(\mathcal{K}, J)$	intersection of unitization $\mathbb{K}(\mathcal{K})^{\sim}$ of compact operators with $\mathbb{U}(\mathcal{K}, J)$
$\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$	set of bounded J-self-adjoint operators on Krein space $(\mathcal{K}, J)$
$S(T)$	scattering matrix associated to J-unitary T

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