

9 Bott–Maslov index via spectral flow

This chapter develops the theory of the Bott–Maslov and Conley–Zehnder indices in the framework of a complex infinite-dimensional Krein space. It hence generalizes many of the results of Chapter 2 by imposing suitable Fredholm conditions. Standard monographs on Krein spaces are [29, 20]. The infinite-dimensional Bott–Maslov index was introduced and studied by Swanson [187], Nicolaescu [138], Booß-Bavnbek and Furutani [30], Kirk and Lesch [113], and Furutani [90], see also [137, 33, 168, 203]. Apart from these fundamental references, other literature is cited in the text below. Works on the finite-dimensional case are already mentioned in Chapter 2. As an application of the infinite-dimensional theory, Section 9.7 develops oscillation theory for the bound states of a high-dimensional scattering setup. Let us note that numerous other applications can be found in the literature, in particular, most notably in Morse theory [138, 113].

9.1 Krein spaces and operators thereon

In this chapter, the separable complex Hilbert space $\mathcal{K}$ is supposed to be equipped with a proper symmetry $J = J^* = J^{-1} \in \mathbb{B}(\mathcal{K})$, namely one which has infinite-dimensional eigenspaces for the eigenvalues 1 and -1 . We will always assume to be in the spectral representation of J so that

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus J introduces a grading of $\mathcal{K}$, namely $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Because J is proper, both $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite dimensional and therefore they can be naturally identified with a separable Hilbert space $\mathcal{H}$, namely $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. One then calls the couple $(\mathcal{K}, J)$ a complex Krein space with fundamental symmetry J . Let us note that this excludes the class of infinite-dimensional Pontryagin spaces [29, 20] where one of the fibers $\mathcal{H}_+$ or $\mathcal{H}_-$ is finite-dimensional. However, later on (in particular, in Section 9.2) Pontryagin subspaces of a Krein space and their Krein signature will be relevant.

Definition 9.1.1. Let P be an orthogonal projection on a Krein space $(\mathcal{K}, J)$ and let Φ be a normalized frame for P , namely $P = \Phi\Phi^*$.

- (i) P is called J -invariant if and only if $PJ = JP$.
- (ii) P is called nondegenerate if $0 \notin \text{spec}(\Phi^*J\Phi)$.
- (iii) A nondegenerate orthogonal projection P is called a Pontryagin projection if and only if $\Phi^*J\Phi$ has only a finite number of positive eigenvalues or a finite number of negative eigenvalues. Then the Krein signature of a Pontryagin projection P is

$$\text{KSig}(P) = \text{Sig}(\Phi^*J\Phi) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

- (iv) A Pontryagin projection is called Krein-definite if $\Phi^* J \Phi$ is either positive or negative definite, or equivalently if the restricted quadratic form $J|_{\text{Ran}(P)}$ is either positive or negative definite.

Identifying orthogonal projections with their range, all these notions directly transpose to closed subspaces $\mathcal{E}$ of $\mathcal{K}$.

Note that every J -invariant projection is nondegenerate. In this section, Pontryagin spaces and their Krein signature will not play any role, yet. The focus here is rather on the analysis of linear operators on the Krein space that preserve J as a quadratic form.

Definition 9.1.2. A bounded invertible operator $T \in \mathbb{B}(\mathcal{K})$ on a Krein space $(\mathcal{K}, J)$ is called J -unitary if

$$T^* J T = J. \quad (9.1)$$

The set of J -unitary operators on $\mathcal{K}$ is denoted by $\mathbb{U}(\mathcal{K}, J)$ and it will be equipped with the norm metric d_N and the associated norm topology $\mathcal{O}_N$.

Let us stress that $\mathbb{U}(\mathcal{K}, J)$ does *not* denote the unitary operators on $\mathcal{K}$ viewed as a Hilbert space. These latter operators are simply denoted by $\mathbb{U}(\mathcal{K})$. Also note that the relation (9.1) alone does not imply that T is invertible. For example, set $\mathcal{K} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ in the grading of J and define $T = S \oplus S$ where S denotes the right-shift on $\ell^2(\mathbb{N})$. Then (9.1) holds but T is not invertible and hence not in $\mathbb{U}(\mathcal{K}, J)$. Many of the basic algebraic properties of J -unitaries transfer from the finite-dimensional case. In particular, the spectrum satisfies (2.16) and the Riesz projection of $T \in \mathbb{U}(\mathcal{K}, J)$ are those given in Proposition 2.2.2. Furthermore, $\mathbb{U}(\mathcal{K}, J)$ is clearly a subgroup of the set $\mathbb{G}(\mathcal{K})$ of invertible operators on $\mathcal{K}$. One can also rewrite the definition of $\mathbb{U}(\mathcal{K}, J)$ as follows.

Proposition 9.1.3. *The group $\mathbb{U}(\mathcal{K}, J)$ is invariant under taking adjoints. In the grading of J , one has*

$$\begin{aligned} \mathbb{U}(\mathcal{K}, J) &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{G}(\mathcal{K}) : A^* A - C^* C = \mathbf{1}, D^* D - B^* B = \mathbf{1}, A^* B = C^* D \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{G}(\mathcal{K}) : AA^* - BB^* = \mathbf{1}, DD^* - CC^* = \mathbf{1}, AC^* = BD^* \right\}, \end{aligned}$$

and in this representation A and D are invertible and satisfy $\|A^{-1}\| \leq 1$, $\|D^{-1}\| \leq 1$. Also $\|A^{-1}B\| < 1$, $\|D^{-1}C\| < 1$, $\|BD^{-1}\| < 1$, and $\|CA^{-1}\| < 1$.

Proof. Inverting $T^* J T = J$ shows $T^{-1} J (T^*)^{-1} = J$ so that $J = T J T^*$. The fact that A is invertible follows from $AA^* \geq \mathbf{1}$ and $A^* A \geq \mathbf{1}$. Analogously, one shows that D is invertible. Furthermore, $AA^* - BB^* = \mathbf{1}$ implies that $A^{-1} B (A^{-1} B)^* = \mathbf{1} - A^{-1} (A^{-1})^* < \mathbf{1}$, so that $\|A^{-1} B\| < 1$. The same argument leads to the other inequalities. $\square$

As in finite dimension (Proposition 2.2.5), the polar decomposition of J -unitary operators only involves J -unitary operators. In fact, the proof in finite dimension is based on Lemma 2.2.6 which directly generalizes to the Krein space framework.

Proposition 9.1.4. *Let $T \in \mathcal{U}(\mathcal{K}, J)$ have the polar decomposition $T = W|T|$, namely where $|T| = (T^*T)^{\frac{1}{2}}$ and W is unitary. Then $|T| \in \mathcal{U}(\mathcal{K}, J)$ and $W \in \mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$.*

Now let us turn to study the topology of $(\mathcal{U}(\mathcal{K}, J), \mathcal{O}_N)$ and some of its subspaces. As in the finite-dimensional case, one has the following.

Corollary 9.1.5. *The group $(\mathcal{U}(\mathcal{K}, J), \mathcal{O}_N)$ is path connected.*

Proposition 9.1.6. *The group $(\mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K}), \mathcal{O}_N)$ is contractible and given by*

$$\mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K}) = \{\text{diag}(V_+, V_-) \in \mathcal{U}(\mathcal{K}) : V_+, V_- \in \mathcal{U}(\mathcal{H})\}.$$

Proof. Let us first note that $\mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$ is the set of unitaries commuting with J . These are the, in the grading of J , diagonal unitaries, just as stated. Therefore the contractibility of $\mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$ follows as $\mathcal{U}(\mathcal{H})$ is contractible by Kuiper's theorem. $\square$

Let us next consider another subgroup of $\mathcal{U}(\mathcal{K}, J)$, namely the set of J -unitary operators that are compact perturbations of the identity,

$$\mathcal{U}^c(\mathcal{K}, J) = \{\mathbf{1} + K \text{ invertible} : K \in \mathcal{K}(\mathcal{K}), (\mathbf{1} + K)^* J (\mathbf{1} + K) = J\}.$$

This is the norm-closure of the finite-dimensional J -unitaries, under suitable embedding of the latter in $\mathcal{U}(\mathcal{K}, J)$. Proposition 9.1.4 directly implies the following result.

Corollary 9.1.7. *Let $T \in \mathcal{U}^c(\mathcal{K}, J)$ have the polar decomposition $T = W|T|$, then one has $|T| \in \mathcal{U}^c(\mathcal{K}, J)$ and $W \in \mathcal{U}^c(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$.*

The next result follows from Corollary 9.1.7 combined with (8.1).

Proposition 9.1.8. *The space $(\mathcal{U}^c(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K}), \mathcal{O}_N)$ is a deformation retract of the space $(\mathcal{U}^c(\mathcal{K}, J), \mathcal{O}_N)$. The homotopy groups of $(\mathcal{U}^c(\mathcal{K}, J), \mathcal{O}_N)$ are*

$$\pi_k(\mathcal{U}^c(\mathcal{K}, J)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Proof. Using the polar decomposition in $\mathcal{U}^c(\mathcal{K}, J)$ as given in Corollary 9.1.7 and deforming the radial part shows that $\mathcal{U}^c(\mathcal{K}, J)$ can be retracted to $\mathcal{U}^c(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$. Moreover, Proposition 9.1.6 shows that $\mathcal{U}^c(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K}) = \mathcal{U}^c(\mathcal{H}) \times \mathcal{U}^c(\mathcal{H})$ where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Therefore, the claim follows from (8.1). $\square$

Next let us come to the Lie algebra of $\mathcal{U}(\mathcal{K}, J)$.

Definition 9.1.9. A bounded operator $H \in \mathcal{B}(\mathcal{K})$ on a Krein space $(\mathcal{K}, J)$ is called J -self-adjoint if

$$JH^*J = H. \quad (9.2)$$

The set of J -self-adjoint bounded operators on $\mathcal{K}$ is denoted by $\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ and it is equipped with the norm metric d_N and the associated norm topology $\mathcal{O}_N$.

There is a close connection between J -self-adjoint operators and self-adjoint operators on $\mathcal{K}$. More precisely, H is J -self-adjoint if and only if $S = JH$ is self-adjoint. Let us note that $\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ is an $\mathbb{R}$ -vector space. Moreover, $\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ is the Lie algebra of $\mathcal{U}(\mathcal{K}, J)$ in the sense that

$$H \in \mathbb{B}_{\text{sa}}(\mathcal{K}, J) \implies e^{iH} \in \mathcal{U}(\mathcal{K}, J).$$

If $H \in \mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ is such that $H + i\mathbf{1} \in \mathcal{B}(\mathcal{K})$ is invertible, also the Cayley transform $\mathcal{C}(H) = (H - i\mathbf{1})(H + i\mathbf{1})^{-1}$ lies in $\mathcal{U}(\mathcal{K}, J)$. Finally, the set $\mathbb{B}_{\text{sa}}(\mathcal{K}) \cap \mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ is the real vector space which is the Lie algebra of $\mathcal{U}(\mathcal{K}) \cap \mathcal{U}(\mathcal{K}, J)$, namely $H \in \mathbb{B}_{\text{sa}}(\mathcal{K}) \cap \mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ implies $e^{iH} \in \mathcal{U}(\mathcal{K}) \cap \mathcal{U}(\mathcal{K}, J)$. Statements and formulas similar to those in Proposition 9.1.3 also hold for operators in the Lie algebra $\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$.

Proposition 9.1.10. *The $\mathbb{R}$ -vector space $\mathbb{B}_{\text{sa}}(\mathcal{K}, J)$ is invariant under taking adjoints. In the grading of J , one has*

$$\mathbb{B}_{\text{sa}}(\mathcal{K}, J) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A = A^*, D = D^*, B = -C^* \right\}.$$

Proof. The claim follows directly by writing out (9.2). □

There is another natural class of bounded operators on the Krein space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, namely for a given operator $B \in \mathcal{B}(\mathcal{H})$ one can set

$$H = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$

By construction, $H = H^*$ is self-adjoint and satisfies $JHJ = -H$, namely iH is J -self-adjoint. In a quantum-mechanical setting, the operator H is then called a Hamiltonian and the relation $JHJ = -H$ either a supersymmetry [67] or a chiral symmetry [152]. This motivates the following definition.

Definition 9.1.11. A self-adjoint operator $H \in \mathbb{B}_{\text{sa}}(\mathcal{K})$ satisfying $JHJ = -H$ is called chiral.

Chiral operators clearly have a spectral symmetry $\text{spec}(H) = -\text{spec}(H) \subset \mathbb{R}$. A particular example of a chiral operator is a chiral symmetry. For every chiral symmetry Q , there is a unitary operator U on $\mathcal{H}$ such that

$$Q = \begin{pmatrix} 0 & -U \\ -U^* & 0 \end{pmatrix}. \quad (9.3)$$

As will be discussed below, there is a tight connection between invertible chiral operators and J -Lagrangian projections, and this also explains why we choose to add the minus sign in (9.3).

9.2 J -isotropic subspaces

Definition 9.2.1. A closed subspace $\mathcal{E}$ of a Krein space $(\mathcal{K}, J)$ is called J -isotropic if J viewed as a hermitian sesquilinear form vanishes when restricted to $\mathcal{E}$. More explicitly, for all $\phi, \psi \in \mathcal{E}$, one has $\phi^* J \psi = 0$. Two closed subspaces $\mathcal{E}$ and $\mathcal{E}'$ are called J -orthogonal if and only if $\phi^* J \psi = 0$ for all $\phi \in \mathcal{E}$ and $\psi \in \mathcal{E}'$.

Let us note that any subspace on which J vanishes can be closed and its closure will then be J -isotropic so that it is natural to require closedness in Definition 9.2.1. Furthermore, every J -isotropic subspace is J -orthogonal to itself. As there is a bijection between closed subspaces of a Hilbert space and orthogonal projections (self-adjoint idempotents), the following definition is hence in line with the above.

Definition 9.2.2. An orthogonal projection P is called J -isotropic if $PJP = 0$. The set of J -isotropic projections will be denoted by $\mathbb{I}(\mathcal{K}, J)$. Two orthogonal projections P and P' are called J -orthogonal if and only if $PJP' = 0$.

One has the following characterization of J -isotropic projections.

Lemma 9.2.3. *An orthogonal projection P is J -isotropic if and only if*

$$P \leq J(1 - P)J.$$

Proof. First note that $J(1 - P)J$ is an orthogonal projection. Moreover, $P \leq J(1 - P)J$ is equivalent to $\text{Ran}(P) \subset \text{Ran}(J(1 - P)J)$ because if $P \leq J(1 - P)J$ holds and $\phi_1 \in \text{Ran}(P)$ is a vector in the range of P then one has

$$\|\phi_1\|^2 = \langle \phi_1 | P \phi_1 \rangle \leq \langle \phi_1 | J(1 - P)J \phi_1 \rangle \leq \|\phi_1\|^2 \|J(1 - P)J\| = \|\phi_1\|^2.$$

Therefore, as the Cauchy–Schwarz inequality is an equality in this case, $J(1 - P)J\phi_1 = \phi_1$ and ϕ_1 is in the range of $J(1 - P)J$. Conversely, assume that $\text{Ran}(P) \subset \text{Ran}(J(1 - P)J)$ holds. Then for $\phi = \phi_0 + \phi_1 \in \mathcal{H}$ with $\phi_0 \in \text{Ker}(P)$ and $\phi_1 \in \text{Ran}(P)$, one has $J(1 - P)J\phi_1 = \phi_1$ and therefore

$$\begin{aligned} \langle \phi | J(1 - P)J \phi \rangle &= \langle \phi_1 | J(1 - P)J \phi_1 \rangle + \langle \phi_0 | J(1 - P)J \phi_1 \rangle + \langle \phi_1 | J(1 - P)J \phi_0 \rangle + \langle \phi_0 | J(1 - P)J \phi_0 \rangle \\ &= \langle \phi_1 | \phi_1 \rangle + \langle \phi_0 | \phi_1 \rangle + \langle \phi_1 | \phi_0 \rangle + \langle \phi_0 | J(1 - P)J \phi_0 \rangle \end{aligned}$$

$$\begin{aligned}
 &= \langle \phi_1 | \phi_1 \rangle + \langle \phi_0 | J(1 - P)J\phi_0 \rangle \geq \langle \phi_1 | \phi_1 \rangle \\
 &= \langle \phi | P\phi \rangle,
 \end{aligned}$$

thus $P \leq J(1 - P)J$ follows. We show

$$P \leq J(1 - P)J \iff P = PJ(1 - P)JP = P - PJPJP. \quad (9.4)$$

Let us first suppose that $P = PJ(1 - P)JP$ holds. Then for $\phi_1 \in \text{Ran}(P)$, one obtains the equalities $\phi_1 = P\phi_1 = PJ(1 - P)J\phi_1 = J(1 - P)J\phi_1$ where the last step follows as $\|J(1 - P)J\phi_1\| \leq \|\phi_1\|$. This implies $\text{Ran}(P) \subset \text{Ran}(J(1 - P)J)$ and therefore $P \leq J(1 - P)J$. Conversely, $P \leq J(1 - P)J$ implies $\text{Ran}(P) \subset \text{Ran}(J(1 - P)J)$ and therefore $J(1 - P)J\phi_1 = \phi_1$ for $\phi_1 \in \text{Ran}(P)$. Thus $P\phi_1 = PJ(1 - P)JP\phi_1$ follows. As $P\phi_0 = 0 = PJ(1 - P)JP\phi_0$ for $\phi_0 \in \text{Ker}(P)$ is obvious, $P = PJ(1 - P)JP$ follows. This concludes the proof of (9.4). If P is J -isotropic, the right-hand side of (9.4) is obviously correct and therefore $P \leq J(1 - P)J$ holds. Conversely, $P \leq J(1 - P)J$ implies by (9.4) that $0 = PJPJP = (PJP)^2$, as PJP is self-adjoint, and this implies that P is J -isotropic. $\square$

Associated to a given J -unitary operator $T \in \mathcal{U}(\mathcal{K}, J)$ there are numerous J -isotropic subspaces. Recall that a subset $\Delta \subset \text{spec}(T)$ is called separated spectral subset if it is a closed subset and has trivial intersection with the closure of $\text{spec}(T) \setminus \Delta$.

Proposition 9.2.4. *Let $T \in \mathcal{U}(\mathcal{K}, J)$ and $\Delta, \Delta' \subset \text{spec}(T)$ be separated spectral subsets. Set $\overline{\Delta}^{-1} = \{z \in \mathbb{C} : \overline{z}^{-1} \in \Delta\}$.*

(i) *If $\Delta' \cap \overline{\Delta}^{-1} = \emptyset$, then the associated Riesz projections of T satisfy*

$$(R_\Delta)^* J R_{\Delta'} = 0.$$

(ii) *If $\Delta \cap \overline{\Delta}^{-1} = \emptyset$, then the range of the Riesz projection R_Δ is J -isotropic.*

(iii) *If $\Delta \cap \overline{\Delta}^{-1} = \emptyset$, then the projection on the cokernel of R_Δ is J -isotropic.*

(iv) *Suppose that $\text{spec}(T) = \Delta \cup \Delta'$ and*

$$\Delta = \overline{\Delta}^{-1}, \quad \Delta' = \overline{\Delta'}^{-1}, \quad \Delta \cap \Delta' = \emptyset.$$

Furthermore, let R_Δ only have finite-dimensional range. Then both $\text{Ran}(R_\Delta)$ and $\text{Ran}(R_{\Delta'})$ are nondegenerate.

Proof. First of all, let us note that Proposition 2.2.2 remains valid for infinite-dimensional Krein spaces by the same proof, namely the Riesz projections satisfy

$$(R_\Delta)^* = J R_{\overline{\Delta}^{-1}} J.$$

Therefore

$$(R_\Delta)^* J R_{\Delta'} = J R_{\overline{\Delta}^{-1}} J J R_{\Delta'} = J R_{\overline{\Delta}^{-1}} R_{\Delta'} = 0,$$

the latter by the assumption and the properties of the Riesz projections, see Appendix A.1. The claim (ii) is now a direct consequence and (iii) follows from the identity $R_\Delta J(R_\Delta)^* = 0$ obtained in a similar manner.

(iv) By item (i), $\text{Ran}(R_\Delta)$ and $\text{Ran}(R_{\Delta'})$ are J -orthogonal. Moreover, these two subspaces span all $\mathcal{K}$ due to $R_\Delta + R_{\Delta'} = \mathbf{1}$ which follows from Proposition A.1.1(iii). Suppose that $\text{Ran}(R_\Delta)$ is not nondegenerate. Then there exists a nonvanishing vector $\phi \in \text{Ran}(R_\Delta)$ such that $(\phi')^* J \phi = 0$ for all $\phi' \in \text{Ran}(R_\Delta)$ and hence ϕ is J -orthogonal to all vectors in $\mathcal{K}$. This is a contradiction to the fact that J is invertible. Now let Φ be a normalized frame with $\text{Ran}(\Phi) = \text{Ran}(R_{\Delta'})$. Then $\Phi \Phi^* J \Phi \Phi^* - J$ is finite dimensional. Hence the essential spectrum of $\Phi \Phi^* J \Phi \Phi^*$ is $\{-1, 1\}$. But $\text{spec}(\Phi^* J \Phi) \cup \{0\} = \text{spec}(\Phi \Phi^* J \Phi \Phi^*)$, unless $\Delta = \emptyset$. However, by the same argument as above, $\text{Ker}(\Phi^* J \Phi) = \{0\}$. Taking these facts together, one deduces that also $R_{\Delta'}$ is nondegenerate. $\square$

Remark 9.2.5. Results similar to Proposition 9.2.4 also hold for a J -self-adjoint operator H . One merely has to replace the spectral reflection on the unit circle $\mathbb{S}^1$ by a reflection on the real axis, namely by complex conjugation. For example, let $\Delta \subset \text{spec}(H)$ be a spectral subset such that $\Delta \cap \bar{\Delta} = \emptyset$ where the complex conjugate is $\bar{\Delta} = \{z \in \mathbb{C} : \bar{z} \in \Delta\}$. Then the range and cokernel of the Riesz projection R_Δ of H are J -isotropic subspaces. Several of the results below transfer in the same way, even though this will not be spelled out. The reader may consult [29, 175]. $\diamond$

Next let us note that for a J -isotropic orthogonal projection P , also $P + JPJ$ is an orthogonal projection. Its range is a J -invariant subspace, and so is therefore its orthogonal complement which will be denoted

$$\mathcal{F}_P = \text{Ran}(P + JPJ)^\perp = \text{Ker}(P) \cap \text{Ker}(JPJ).$$

Definition 9.2.6. A J -isotropic projection P is called semi-Fredholm if $\mathcal{F}_P$ is a Pontryagin space, and it is called Fredholm if $\mathcal{F}_P$ is finite dimensional.

Let us establish an elementary link between the Fredholm property of J -isotropic projections and Fredholm pairs of projections.

Proposition 9.2.7. A J -isotropic projection P is Fredholm if and only if $(\mathbf{1} - P, JPJ)$ is a Fredholm pair. Its index is given by

$$\text{Ind}(\mathbf{1} - P, JPJ) = \dim(\mathcal{F}_P).$$

Proof. The characterization of the Fredholm property given in Proposition 5.3.2 can be readily checked and

$$\begin{aligned} \text{Ind}(\mathbf{1} - P, JPJ) &= \dim(\text{Ran}(\mathbf{1} - P) \cap \text{Ker}(JPJ)) - \dim(\text{Ran}(JPJ) \cap \text{Ker}(\mathbf{1} - P)) \\ &= \dim(\text{Ker}(P) \cap \text{Ker}(JPJ)) - \dim(\text{Ran}(JPJ) \cap \text{Ran}(P)), \end{aligned}$$

which is indeed equal to $\dim(\mathcal{F}_P)$ because JPJ and P are orthogonal. $\square$

The invariant $\text{Ind}(\mathbf{1} - P, JPJ)$ is, however, not the only interesting integer that can be associated to a Fredholm J -isotropic projection. Even for any semi-Fredholm J -isotropic projection P , one can furthermore consider the Krein signature

$$\text{KSig}(\mathbf{1} - P - JPJ) = \text{Sig}(J|_{\mathcal{F}_P}) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

It turns out that these two quantities are related for the following class of J -isotropic subspaces.

Definition 9.2.8. A J -isotropic subspace $\mathcal{E}$ is called maximally J -isotropic if there is no J -isotropic subspace $\mathcal{F}$ with $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{E} \neq \mathcal{F}$. A projection P is called maximally J -isotropic if its range is maximally J -isotropic. The set of all Fredholm maximally J -isotropic projections will be denoted by

$$\mathbb{FI}(\mathcal{K}, J) = \{P \in \mathbb{P}(\mathcal{K}) : P \text{ maximally } J\text{-isotropic and Fredholm}\}.$$

It is equipped with the norm topology $\mathcal{O}_N$.

Proposition 9.2.9. A J -isotropic projection P is maximal if and only if $\mathbf{1} - (P + JPJ)$ is a Krein-definite Pontryagin projection. Moreover, for every maximally J -isotropic projection P , one has

$$\text{Ind}(\mathbf{1} - P, JPJ) = |\text{KSig}(\mathbf{1} - P - JPJ)|.$$

Proof. Note that $\mathbf{1} - (P + JPJ)$ is the projection onto $\mathcal{F}_P$. The J -isotropic projection P is not maximal if and only if there exists a nontrivial subspace of $\mathcal{F}_P$ that is J -orthogonal to itself. For any unit vector ϕ in this subspace, $P + \phi\phi^*$ is a J -isotropic projection. Then $J\phi$ and ϕ are linearly independent vectors from $\mathcal{F}_P$, and J restricted to the two-dimensional subspace spanned by ϕ and $J\phi$ has eigenvalues 1 and -1 , so that $\mathcal{F}_P$ is not Krein-definite. Conversely, if $\mathcal{F}_P$ is not Krein-definite, there is a unit vector $\phi \in \mathcal{F}_P$ that is J -orthogonal to itself. Then $P + \phi\phi^*$ is a J -isotropic projection and P is not maximal. The claim about $\text{Ind}(\mathbf{1} - P, JPJ)$ directly follows from Proposition 9.2.7. $\square$

For a finite-dimensional $\mathcal{H}$, every maximally J -isotropic subspace is of dimension $\dim(\mathcal{H})$ and is hence J -Lagrangian in the sense that $JPJ = \mathbf{1} - P$ holds for its range projection P , see Definition 9.3.1 below. However, in infinite dimension there are more maximally J -isotropic subspaces, namely maximally J -isotropic subspaces that are not J -Lagrangian (other than incorrectly stated in Section 2 of [168]). This shows the following example.

Example 9.2.10. Let us fix an orthonormal basis $\{b_l : l \in \mathbb{N}\}$ of $\text{Ker}(J - \mathbf{1})$ and an orthonormal basis $\{e_l : l \in \mathbb{N}\}$ of $\text{Ker}(J + \mathbf{1})$. For $k \in \mathbb{N}$, let us then define P_k as the projection onto $\text{Ran}(P_k) = \overline{\text{span}\{b_l + e_{l-k} : l \in \mathbb{N}, l > k\}}$. As $(\mathbf{1} - P_k)J(\mathbf{1} - P_k)b_l = b_l$ for all $l = 1, \dots, k$, one has $(\mathbf{1} - P_k)J(\mathbf{1} - P_k) \neq 0$. (Thus P_k is not J -Lagrangian.) One directly checks

that P_k is J -isotropic and, equivalently, $\text{Ran}(P_k)$ is J -isotropic. Because of the orthogonal decomposition

$$\mathcal{K} = \text{Ran}(P_k) \oplus \text{Ran}(JP_kJ) \oplus \overline{\text{span}\{b_1, \dots, b_k\}}, \quad (9.5)$$

there is no extension of $\text{Ran}(P_k)$ to a J -isotropic subspace, namely $\text{Ran}(P_k)$ is maximally J -isotropic. Along the same lines, it is also possible to construct an example of a maximally J -isometric projection P_∞ such that $\text{Ran}(P_\infty) \oplus \text{Ran}(JP_\infty J)$ has infinite codimension, by setting, e. g., $\text{Ran}(P_\infty) = \overline{\text{span}\{b_l + e_{2l} : l \in \mathbb{N}\}}$. $\diamond$

The maximally J -isotropic projections P_k constructed in Example 9.2.10 are Fredholm for $k < \infty$ and their Krein signature is $k > 0$. In the same manner, it is also possible to construct maximally J -isotropic projections with negative Krein signature. On the other hand, P_∞ is not Fredholm and has Krein signature $+\infty$.

Proposition 9.2.11. *The space $(\mathbb{FI}(\mathcal{K}, J), \mathcal{O}_N)$ has $\mathbb{Z}$ connected components labeled by the Krein signature, that is, the map $K : \pi_0(\mathbb{FI}(\mathcal{K}, J)) \rightarrow \mathbb{Z}$ given by*

$$K(P) = \text{KSig}(1 - P - JPJ)$$

is an isomorphism.

Proof. It is shown that every $P \in \mathbb{FI}(\mathcal{K}, J)$ with $\text{KSig}(1 - P - JPJ) = k$ is unitarily equivalent to a fixed maximally J -isotropic projection P_k with $\text{KSig}(1 - P_k - JP_kJ) = k$ via a unitary that commutes with J (Example 9.2.10 constructs such projections P_k for $k \geq 0$). From this unitary equivalence, one readily constructs the desired connecting path by taking a root of the unitary. For the construction of the unitary, it is convenient to use normalized frames (see Definition 5.1.5). Hence let Φ_k be a normalized frame for P_k , namely one has $P_k = \Phi_k \Phi_k^*$. Then $J\Phi_k$ is a normalized frame for JP_kJ . Further set $\Psi_k = (\Phi_k, J\Phi_k)^\perp$ which is then a normalized frame for the orthogonal projection onto the finite-dimensional space $\mathcal{F}_{P_k}$. Note that $J\Psi_k = \text{sgn}(k)\Psi_k$. Similarly, let $\Phi, J\Phi$ and Ψ be associated to P . Then one checks that $U = (\Phi, J\Phi, \Psi)(\Phi_k, J\Phi_k, \Psi_k)^*$ is well defined, unitary, commutes with J , and satisfies $UP_kU^* = P$. $\square$

Given a maximally J -isotropic projection P , one can set

$$T_P = \frac{1}{2}P + 2JPJ + (1 - P - JPJ), \quad (9.6)$$

and then readily checks that $T_P \in \mathcal{U}(\mathcal{K}, J)$ is J -unitary. The same holds, e. g., for the operator $zP + \bar{z}^{-1}JPJ + e^{i\varphi}(1 - P - JPJ)$ where z is a complex number with $|z| \in (0, 1)$ and $e^{i\varphi} \in \mathbb{S}^1$ a phase. Even further, one can spread out the spectrum on the unit circle within the class of J -unitary operators. On the other hand, it is impossible for these unit eigenvalues to leave the unit circle under any perturbation within the set of J -unitaries. Indeed, each such eigenvalue λ would lead to another eigenvalue $\bar{\lambda}^{-1}$ by (2.16), and a more detailed

elementary analysis shows that the J -inertia on the joint eigenspace has to be $(1, 0, 1)$ which is impossible because $\text{Ran}(\mathbf{1} - P - JPJ)$ is a J -definite subspace. This is the essence of Krein stability which associates a signature to each unit eigenvalue of a J -unitary operator, for details see, e. g., [168]. In Definition 9.2.12 below, the Krein signature is not associated to a single eigenvalue on the unit circle, but rather jointly to all eigenvalues on $\mathbb{S}^1$, which corresponds to taking the sum of all Krein signatures of unit eigenvalues. The example of T_p suggests the following natural situation in which Fredholm J -isotropic projections appear. This is relevant for applications, such as in [168, 175].

Definition 9.2.12. A J -unitary operator $T \in \mathbb{U}(\mathcal{K}, J)$ is said to be essentially $\mathbb{S}^1$ -gapped if it only has discrete spectrum (isolated eigenvalues of finite algebraic multiplicity) on $\mathbb{S}^1$. The total Krein signature of such an essentially $\mathbb{S}^1$ -gapped J -unitary T is

$$\text{KSig}(T) = \text{KSig}(P^\pm),$$

where $P^\pm$ denotes the finite-dimensional range projection of all eigenvalues on the unit circle $\mathbb{S}^1$.

Let us note that Proposition 9.2.4(iv) applies directly, in particular, to an essentially $\mathbb{S}^1$ -gapped operator $T \in \mathbb{U}(\mathcal{K}, J)$ if one chooses $\Delta = \mathbb{S}^1 \cap \text{spec}(T)$ and $\Delta' = \text{spec}(T) \setminus \Delta$. Thus $\text{Ran}(P^\pm) = \text{Ran}(R_\Delta)$ and $\text{Ran}(R_{\Delta'})$ are J -orthogonal and nondegenerate. As $P^\pm$ is finite dimensional, it is hence a Pontryagin projection with a well-defined Krein signature. Therefore $\text{KSig}(T)$ is well defined.

Proposition 9.2.13. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ be an essentially $\mathbb{S}^1$ -gapped J -unitary. Then let $R^<$ and $R^>$ be the Riesz projections of T associated to the spectral subsets $\text{spec}(T) \cap B_1(0)$ and $\text{spec}(T) \setminus \overline{B_1(0)}$, respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto the subspaces $\mathcal{E}^< = \text{Ran}(R^<)$ and $\mathcal{E}^> = \text{Ran}(R^>)$. Further let $P^\pm$ be the finite-dimensional range projection of all eigenvalues on the unit circle $\mathbb{S}^1$. The (total) Krein signature $\text{KSig}(T)$ of the essentially $\mathbb{S}^1$ -gapped J -unitary T is continuous in T . The projections $P^<$ and $P^>$ are Fredholm J -isotropic projections, which are maximal (namely in $\mathbb{FI}(\mathcal{K}, J)$) if the restriction $J|_{\text{Ran}(P^\pm)}$ of the quadratic form J to $\text{Ran}(P^\pm)$ is definite.*

Proof. The fact that $P^<$ and $P^>$ are J -isotropic follows from Proposition 9.2.4 applied to $\Delta = \text{spec}(T) \cap B_1(0)$. The Fredholm property follows directly from the hypothesis because $JP^<J$ is the orthogonal projection onto $\text{Ker}(R^>)^\perp$ and thus $\mathcal{F}_{P^<} = \text{Ker}(R^>) \cap \text{Ran}(R^<)^\perp$ is finite dimensional as $P^\pm$ is finite dimensional. The same argument shows that also $\mathcal{F}_{P^>} = \text{Ker}(R^<) \cap \text{Ran}(R^>)^\perp$ is finite dimensional. Now an eigenvalue λ of T can leave $\mathbb{S}^1$ only together with its reflected $\bar{\lambda}^{-1}$ (Krein collision). But on the span of the two corresponding eigenvectors, J has vanishing signature (this requires an addendum to the argument leading to Proposition 9.2.4, see [168]). Even though this process changes the projection $P^\pm$, it does therefore not change the Krein signature $\text{KSig}(T)$. Once J is definite on the range of $P^\pm$, no eigenvalue can leave the unit circle and neither $P^<$ nor $P^>$ can be enlarged, and are thus maximal. $\square$

Admittedly, the above proof of Proposition 9.2.13 is only a sketch of what is the heart of the celebrated Krein stability result [118]. The reader interested in further details is referred to [168, 175].

Remark 9.2.14. Once again, one can also introduce essentially $\mathbb{R}$ -gapped bounded J -self-adjoint operators, namely those bounded J -self-adjoints that only have discrete spectrum of finite multiplicity on $\mathbb{R}$. Then one can consider their global Krein signature. Using Riesz projections for the upper and lower half-plane, as well as a perturbative argument for the real eigenvalues of indefinite signature, one can show that the set of all essentially $\mathbb{R}$ -gapped bounded J -self-adjoint operators can be retracted to the set $\mathbb{FI}(\mathcal{K}, J)$, if the latter is identified with the J -self-adjoint operators $H = \iota P - \iota P J$ (this is similar to (9.6)). Moreover, it is possible to show by analytic Fredholm theory that the set of all essentially $\mathbb{R}$ -gapped bounded J -self-adjoint operators is equal to the set $\{H \in \mathbb{B}_{\text{sa}}(\mathcal{K}, J) : H - \lambda \mathbf{1} \in \mathbb{FI}(\mathcal{K}) \text{ for all } \lambda \in \mathbb{R}\}$. Detailed proofs can be found in [175]. Such a characterization with a Fredholm property is not possible for the essentially $\mathbb{S}^1$ -gapped J -unitaries, see [168] for a counterexample. It is likely also not true that the essentially $\mathbb{S}^1$ -gapped J -unitaries can be retracted to $\mathbb{FI}(\mathcal{K}, J)$. $\diamond$

9.3 J -Lagrangian subspaces

Definition 9.3.1. Projection $P = P^* = P^2 \in \mathbb{P}(\mathcal{K})$ is called J -Lagrangian if and only if $J P J = \mathbf{1} - P$. A closed subspace is called J -Lagrangian if its range projection is J -Lagrangian. The J -Lagrangian Grassmannian is defined as

$$\mathbb{P}(\mathcal{K}, J) = \{P = P^* = P^2 \in \mathbb{B}(\mathcal{K}) : J P J = \mathbf{1} - P\}.$$

It is equipped with the metric d_N and thus the norm topology $\mathcal{O}_N$.

A Fredholm maximally J -isotropic projection P is J -Lagrangian if and only if one has $\text{KSig}(\mathbf{1} - P - J P J) = 0$. Clearly, one can reformulate Definition 9.3.1 as

$$P \text{ } J\text{-Lagrangian} \iff P + J P J = \mathbf{1}.$$

The definition implies that $\mathbf{1} - P$ is J -Lagrangian if and only if P is J -Lagrangian. Furthermore, every J -Lagrangian projection P provides a chiral symmetry $Q = \mathbf{1} - 2P$, and vice versa. More generally, the negative spectral projection $P = \chi(H < 0)$ of an invertible chiral operator H is J -Lagrangian. Definition 9.3.1 can further be reformulated algebraically. In view of (9.3), every J -Lagrangian projection is of the form

$$P = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix}, \quad (9.7)$$

where U is a unitary on $\mathcal{H}$, see also Proposition 9.3.4 below. Let us next give another characterization of J -Lagrangian projections.

Lemma 9.3.2. *An orthogonal projection P is J -Lagrangian if and only if*

$$PJP = 0 \quad \text{and} \quad (1 - P)J(1 - P) = 0,$$

or alternatively if and only if the restrictions $J|_{\text{Ran}(P)}$ and $J|_{\text{Ran}(1-P)}$ of the quadratic form J vanish.

Proof. Multiplying $JPJ = 1 - P$ by P from the left and J from the right shows $PJP = 0$. Proceeding similarly with $P = J(1 - P)J$ shows $(1 - P)J(1 - P) = 0$. Conversely,

$$\begin{aligned} JPJ &= JPJP + JPJ(1 - P) \\ &= JPJ(1 - P) \\ &= J(P + 1 - P)J(1 - P) \\ &= (1 - P), \end{aligned}$$

showing the claimed equivalence. $\square$

The following result describes a natural situation in which J -Lagrangian subspaces arise. It is the infinite-dimensional analogue of Proposition 2.2.3.

Proposition 9.3.3. *Let $T \in \mathcal{U}(\mathcal{K}, J)$ satisfy $\text{spec}(T) \cap \mathbb{S}^1 = \emptyset$. Then let $R^<$ and $R^>$ be the Riesz projections of T associated to the separated spectral subsets $\text{spec}(T) \cap B_1(0)$ and $\text{spec}(T) \setminus B_1(0)$, respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto the subspaces $\mathcal{E}^< = \text{Ran}(R^<)$ and $\mathcal{E}^> = \text{Ran}(R^>)$. Then $P^<$ and $P^>$ are J -Lagrangian.*

Proof. This follows directly from Proposition 9.2.13. $\square$

Next let us consider the set of all J -Lagrangian subspaces. Due to (9.7), the J -Lagrangian Grassmannian $\mathbb{P}(\mathcal{K}, J)$ on $\mathcal{K}$ can naturally be identified with the unitary group on $\mathcal{H}$.

Proposition 9.3.4. *The stereographic projection $\Pi : \mathbb{P}(\mathcal{K}, J) \rightarrow \mathcal{U}(\mathcal{H})$ defined by*

$$\Pi(P) = U, \quad P = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix},$$

is a bijective isometry.

Proof. The stereographic projection is surjective because, for $U \in \mathcal{U}(\mathcal{H})$,

$$P = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{pmatrix} \in \mathbb{P}(\mathcal{K}, J)$$

is a Lagrangian projection and $\Pi(P) = U$. Moreover, the stereographic projection is injective as, for $P, P' \in \mathbb{P}(\mathcal{K}, J)$ with $U = \Pi(P)$ and $U' = \Pi(P')$, one has

$$\|P - P'\| = \frac{1}{2} \|\Pi(P) - \Pi(P')\| = \frac{1}{2} \|U - U'\|.$$

Therefore Π is injective. In conclusion, the stereographic projection is a bijection. The above identity also shows that it is bi-Lipshitz-continuous. $\square$

The spectral theory in $\mathbb{U}(\mathcal{H})$ is of importance for the intersection of two Lagrangian subspaces, as shows the following result which is at the heart of intersection theory of J -Lagrangian subspaces and hence of crucial relevance for the Bott–Maslov index introduced and analyzed in the next section.

Proposition 9.3.5. *Let P_0 and P_1 be J -Lagrangian projections with stereographic projections $U_0 = \Pi(P_0)$ and $U_1 = \Pi(P_1)$. One has*

$$\begin{aligned} \dim(\text{Ran}(P_0) \cap \text{Ker}(P_1)) &= \dim(\text{Ker}(U_1^* U_0 + \mathbf{1})) \\ &= \dim(\text{Ker}(U_1 U_0^* + \mathbf{1})), \end{aligned}$$

or alternatively

$$\dim(\text{Ran}(P_0) \cap \text{Ran}(JP_1J)) = \dim(\text{Ker}(U_1^* U_0 + \mathbf{1})).$$

Proof. A vector $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{K}$ with $\phi_1, \phi_2 \in \mathcal{H}$ is in the range of P_0 if and only if

$$P_0 \phi = \frac{1}{2} \begin{pmatrix} \phi_1 + U_0 \phi_2 \\ U_0^* \phi_1 + \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

which is equivalent to $U_0 \phi_2 = \phi_1$. Then

$$P_1 \phi = \frac{1}{2} \begin{pmatrix} \phi_1 + U_1 \phi_2 \\ U_1^* \phi_1 + \phi_2 \end{pmatrix} = 0$$

if and only if $-U_1^* \phi_1 = \phi_2$. In conclusion, $\phi \in \text{Ran}(P_0) \cap \text{Ker}(P_1)$ implies $\phi_2 = -U_1^* U_0 \phi_2$ and $\dim(\text{Ran}(P_0) \cap \text{Ker}(P_1)) \leq \dim(\text{Ker}(U_1^* U_0 + \mathbf{1}))$. Conversely, for $\phi_2 \in \text{Ker}(U_1^* U_0 + \mathbf{1})$, one has $U_0 \phi_2 = -U_1 \phi_2$ and therefore $\phi = \begin{pmatrix} U_0 \phi_2 \\ \phi_2 \end{pmatrix} \in \text{Ran}(P_0) \cap \text{Ker}(P_1)$. This implies that $\dim(\text{Ran}(P_0) \cap \text{Ker}(P_1)) \geq \dim(\text{Ker}(U_1^* U_0 + \mathbf{1}))$ and thus the claim follows. $\square$

A J -unitary operator $T \in \mathbb{U}(\mathcal{K}, J)$ sends a J -Lagrangian subspace $\mathcal{E}$ to a J -Lagrangian subspace $T\mathcal{E}$. Indeed, for all vectors $\psi_0 = T\phi_0 \in T\mathcal{E}$ and $\psi_1 = T\phi_1 \in T\mathcal{E}$, one deduces $\psi_0^* J \psi_1 = \phi_0^* T^* J T \phi_1 = \phi_0^* J \phi_1 = 0$. Analogously, for $\widetilde{\psi}_0 = (T^*)^{-1} \widetilde{\phi}_0 \in (T\mathcal{E})^\perp = (T^*)^{-1} \mathcal{E}^\perp$ and $\widetilde{\psi}_1 = (T^*)^{-1} \widetilde{\phi}_1 \in (T\mathcal{E})^\perp$, one has $(\widetilde{\psi}_0)^* J \widetilde{\psi}_1 = 0$, so that Lemma 9.3.2 implies that $T\mathcal{E}$ is J -Lagrangian. (Note that this also shows that the image of J -isotropic subspaces under a J -unitary is J -isotropic.) If $P \in \mathbb{P}(\mathcal{K}, J)$ is the range projection of $\mathcal{E}$, then the range

projection of $T\mathcal{E}$ is denoted by $T \cdot P$, namely $\cdot : \mathbb{U}(\mathcal{K}, J) \times \mathbb{P}(\mathcal{K}, J) \rightarrow \mathbb{P}(\mathcal{K}, J)$ is a group action. This action is transitive. Actually, already the subgroup $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ does so as shows the following result.

Proposition 9.3.6. *The group $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ acts continuously and transitively on $\mathbb{P}(\mathcal{K}, J)$.*

Proof. The action of $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ on $\mathbb{P}(\mathcal{K}, J)$ is simply given by $V \cdot P = VPV^*$ for $V \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ and $P \in \mathbb{P}(\mathcal{K}, J)$. One directly checks that VPV^* is in $\mathbb{P}(\mathcal{K}, J)$ and therefore the action is well defined. To show that the action is transitive, consider two J -Lagrangian projections

$$P_0 = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U_0 \\ U_0^* & \mathbf{1} \end{pmatrix} \quad \text{and} \quad P_1 = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U_1 \\ U_1^* & \mathbf{1} \end{pmatrix},$$

where $U_0, U_1 \in \mathbb{U}(\mathcal{H})$ are unitaries. One directly checks that $VP_0V^* = P_1$ for

$$V = \begin{pmatrix} U_1 & 0 \\ 0 & U_0 \end{pmatrix} \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K}),$$

finishing the proof. □

Now the action (5.7) of invertibles on projections becomes an action of $\mathbb{U}(\mathcal{K}, J)$ on $\mathbb{P}(\mathcal{K}, J)$. Recall that for $T \in \mathbb{U}(\mathcal{K}, J)$ on $P \in \mathbb{P}(\mathcal{K}, J)$, it is given by

$$T \cdot P = (TPT^*)(TPT^*)^{-2}(TPT^*).$$

The following elementary fact will be used later on.

Proposition 9.3.7. *For $T \in \mathbb{U}(\mathcal{K}, J)$ and $P \in \mathbb{P}(\mathcal{K}, J)$, one has*

$$T \cdot P = J((T^{-1})^* \cdot (\mathbf{1} - P))J.$$

Proof. The computation

$$\begin{aligned} J(T \cdot P)J &= (JTPT^*J)(JTPT^*J)^{-2}(JTPT^*J) \\ &= ((T^{-1})^*JPJT^{-1})((T^{-1})^*JPJT^{-1})^{-2}((T^{-1})^*JPJT^{-1}), \end{aligned}$$

combined with $JPJ = \mathbf{1} - P$, shows the claim. □

Under the stereographic projection, the action takes a simpler form.

Proposition 9.3.8. *The group $\mathbb{U}(\mathcal{K}, J)$ acts continuously on the Siegel disc*

$$\mathbb{D}(\mathcal{H}) = \{U \in \mathbb{B}(\mathcal{H}) : \|U\| < 1\}$$

and also on the unitary group $\mathbb{U}(\mathcal{H})$ by Möbius transformation denoted by a dot and defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot U = (AU + B)(CU + D)^{-1}, \quad U \in \mathbb{D}(\mathcal{H}).$$

The Möbius action on $\mathbb{U}(\mathcal{H})$ implements the action $\cdot : \mathbb{U}(\mathcal{K}, J) \times \mathbb{P}(\mathcal{K}, J) \rightarrow \mathbb{P}(\mathcal{K}, J)$, namely

$$T \cdot \Pi(P) = \Pi(T \cdot P).$$

Proof. One first has to show that for $U \in \mathbb{B}(\mathcal{H})$ with $\|U\| \leq 1$ and $T \in \mathbb{U}(\mathcal{K}, J)$ the inverse in the Möbius transformation $T \cdot U$ is well defined. By Proposition 9.1.3, one concludes that $(CU + D) = D(D^{-1}CU + \mathbf{1})$ is indeed invertible. Then the identities of Proposition 9.1.3 imply

$$(CU + D)^*(CU + D) - (AU + B)^*(AU + B) = \mathbf{1} - U^*U. \quad (9.8)$$

Now multiplying (9.8) from the left by $((CU + D)^*)^{-1}$ and from the right by $(CU + D)^{-1}$ and using $\mathbf{1} - U^*U > 0$ for $U \in \mathbb{D}(\mathcal{H})$ shows $(T \cdot U)^*(T \cdot U) < \mathbf{1}$ so that $T \cdot U \in \mathbb{D}(\mathcal{H})$. By the same argument, if $U \in \mathbb{U}(\mathcal{H})$, then $T \cdot U \in \mathbb{U}(\mathcal{H})$. A short algebraic calculation also shows that $(TT') \cdot U = T \cdot (T' \cdot U)$.

To prove the last formula, let us note that the range of $P = \frac{1}{2}(\begin{smallmatrix} \mathbf{1} & U \\ U^* & \mathbf{1} \end{smallmatrix})$ is

$$\text{Ran}(P) = \left\{ \begin{pmatrix} U\phi \\ \phi \end{pmatrix} : \phi \in \mathcal{H} \right\},$$

therefore

$$\begin{aligned} \text{Ran}(T \cdot P) &= \left\{ \begin{pmatrix} (AU + B)\phi \\ (CU + D)\phi \end{pmatrix} : \phi \in \mathcal{H} \right\} \\ &= \left\{ \begin{pmatrix} (AU + B)(CU + D)^{-1}\tilde{\phi} \\ \tilde{\phi} \end{pmatrix} : \tilde{\phi} \in \mathcal{H} \right\} \\ &= \text{Ran} \left(\frac{1}{2} \begin{pmatrix} \mathbf{1} & (AU + B)(CU + D)^{-1} \\ ((AU + B)(CU + D)^{-1})^* & \mathbf{1} \end{pmatrix} \right). \end{aligned}$$

Proposition 9.3.4 implies the claim. $\square$

9.4 Fredholm pairs of J -Lagrangian projections

Recall from Section 5.2 the notion of Fredholm pairs (P_0, P_1) of orthogonal projections and their index given by the difference of the finite dimensions of the subspaces $\text{Ran}(P_0) \cap \text{Ker}(P_1) = \text{Ran}(P_0) \cap \text{Ran}(P_1)^\perp$ and $\text{Ran}(P_1) \cap \text{Ker}(P_0) = \text{Ran}(P_1) \cap \text{Ran}(P_0)^\perp$. It is now natural to consider Fredholm pairs of J -Lagrangian projections and introduce the following notation:

$$\mathbb{FPF}(\mathcal{K}, J) = \{(P_0, P_1) : P_0, P_1 \in \mathbb{P}(\mathcal{K}, J) \text{ and } (P_0, P_1) \text{ Fredholm pair}\}. \quad (9.9)$$

Again $\mathcal{O}_N$ (or more precisely, $\mathcal{O}_N \times \mathcal{O}_N$) is the natural topology on this set. Noting that J -Lagrangian Fredholm pairs are always proper and recalling the notation for proper Fredholm pairs from (5.19), this can be rewritten as

$$\text{FFP}(\mathcal{K}, J) = (\mathbb{P}(\mathcal{K}, J), \mathbb{P}(\mathcal{K}, J)) \cap \text{FFP}(\mathcal{K}).$$

Let us provide a simple way to produce Fredholm pairs of J -Lagrangian projections.

Proposition 9.4.1. *Let $P \in \mathbb{P}(\mathcal{K}, J)$ and $T \in \mathbb{U}^c(\mathcal{K}, J)$. Then $(P, T \cdot P) \in \text{FFP}(\mathcal{K}, J)$.*

Proof. The hypothesis implies that $T \cdot P - P \in \mathbb{K}(\mathcal{K})$, e. g., by using the formula in Proposition 9.3.8. Therefore the Fredholm property of the pair $(P, T \cdot P)$ follows from Proposition 5.2.4. $\square$

The first aim will be to characterize the Fredholm property of pairs of J -Lagrangian projections (P_0, P_1) in terms of the associated stereographic projections. That this should be possible is plausible due to Proposition 9.3.5 which shows that the above finite-dimensional intersections can precisely be determined from the spectral theory of the stereographic projections. For the formulation of the result, which goes back at least to [113], let us recall the relevant spectral notions from Section 3.4. The discrete spectrum $\text{spec}_{\text{dis}}(A)$ of a normal operator $A \in \mathbb{B}(\mathcal{H})$ consists of all isolated eigenvalues of finite multiplicity, and then the essential spectrum is defined by $\text{spec}_{\text{ess}}(A) = \text{spec}(A) \setminus \text{spec}_{\text{dis}}(A)$. Also recall from Section 3.7 that the set of unitaries $U \in \mathbb{U}(\mathcal{H})$ such that $-1 \notin \text{spec}_{\text{ess}}(U)$ is denoted by $\mathbb{FU}(\mathcal{H})$.

Theorem 9.4.2. *Let P_0 and P_1 be two J -Lagrangian projections with stereographic projections $U_0 = \Pi(P_0)$ and $U_1 = \Pi(P_1)$. Then*

$$\begin{aligned} (P_0, P_1) \in \text{FFP}(\mathcal{K}, J) \text{ Fredholm pair} &\iff -1 \notin \text{spec}_{\text{ess}}(U_1^* U_0) \\ &\iff U_1^* U_0 \in \mathbb{FU}(\mathcal{H}) \\ &\iff U_1 U_0^* \in \mathbb{FU}(\mathcal{H}). \end{aligned}$$

Proof. As above, $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ are chiral symmetries. If (P_0, P_1) is a Fredholm pair,

$$(Q_0 + Q_1)^2 = \begin{pmatrix} (U_0 + U_1)(U_0 + U_1)^* & 0 \\ 0 & (U_0 + U_1)^*(U_0 + U_1) \end{pmatrix}$$

is Fredholm by Proposition 5.4.2 and therefore $0 \notin \text{spec}_{\text{ess}}((Q_0 + Q_1)^2)$. Multiplying out shows that

$$(U_0 + U_1)^*(U_0 + U_1) = \mathbf{2}\mathbf{1} + U_1^* U_0 + U_0^* U_1$$

and

$$(U_0 + U_1)(U_0 + U_1)^* = 2\mathbf{1} + U_1 U_0^* + U_0 U_1^*$$

are Fredholm. Let us define $U_1^* U_0 = \hat{U}$. Then

$$2\mathbf{1} + \hat{U} + \hat{U}^* = (\mathbf{1} + \hat{U})(\mathbf{1} + \hat{U})^* = (\mathbf{1} + \hat{U})^* (\mathbf{1} + \hat{U})$$

is Fredholm. Thus, by Corollary 3.4.4, $0 \notin \text{spec}_{\text{ess}}(2\mathbf{1} + \hat{U} + \hat{U}^*)$ and $\mathbf{1} + \hat{U}$ is Fredholm by Theorem 3.4.1. Again by Corollary 3.4.4, one has $-1 \notin \text{spec}_{\text{ess}}(\hat{U}) = \text{spec}_{\text{ess}}(U_0 U_1^*)$.

Conversely, if $-1 \notin \text{spec}_{\text{ess}}(U_1^* U_0)$, then $\mathbf{1} + U_1^* U_0$ is Fredholm by Corollary 3.4.4. Therefore $U_0 + U_1$ and $(U_0 + U_1)^*$ are Fredholm. Thus $Q_0 + Q_1$ is Fredholm and, by Proposition 5.4.2, (P_0, P_1) is a Fredholm pair. $\square$

Corollary 9.4.3. *Let (P_0, P_1) be a pair of J -Lagrangian projections and let furthermore $V \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Then*

$$(P_0, P_1) \in \text{FPP}(\mathcal{K}, J) \iff (V \cdot P_0, V \cdot P_1) \in \text{FPP}(\mathcal{K}, J).$$

Proof. Recall from Proposition 9.1.6 that $V = \text{diag}(V_+, V_-)$ with $V_{\pm} \in \mathbb{U}(\mathcal{H})$. By Proposition 9.3.8, one hence has $\Pi(V \cdot P_j) = V_+ \Pi(P_j) V_-^*$ so that

$$\Pi(V \cdot P_1)^* \Pi(V \cdot P_0) = V_- \Pi(P_1)^* \Pi(P_0) V_-^*. \quad (9.10)$$

Hence the claim follows from Theorem 9.4.2. $\square$

Proposition 9.4.4. *Let $P \in \mathbb{P}(\mathcal{K}, J)$ with $U = \Pi(P)$ and $T \in \mathbb{U}(\mathcal{K}, J)$. Then*

$$(P, T \cdot P) \in \text{FPP}(\mathcal{K}, J) \iff \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} \in \text{FB}(\mathcal{H}).$$

Proof. By Theorem 9.4.2, the Fredholm property of $(P, T \cdot P) \in \text{FPP}(\mathcal{K}, J)$ is equivalent to -1 not being in the essential spectrum of $U^* T \cdot U$, which is equivalent to 0 not being in the essential spectrum of the self-adjoint operator $\Re(U^* T \cdot U) + \mathbf{1}$. Now let A, B, C, D be the entries of T , e. g., as in Proposition 9.3.8. This proposition also shows that $(CU + D)^{-1}$ is invertible. Then

$$\begin{aligned} \Re(U^* T \cdot U) + \mathbf{1} &= \frac{1}{2}(U^* T \cdot U + (T \cdot U)^* U) + \mathbf{1} \\ &= \frac{1}{2}(U^* T \cdot U + \mathbf{1})^* (U^* T \cdot U + \mathbf{1}) \\ &= \frac{1}{2}((CU + D)^{-1})^* \left[\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} \right] \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} (CU + D)^{-1} \\ &= \frac{1}{2} \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} |(CU + D)^*|^{-2} \left[\begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}^* T \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix} \right]^*. \end{aligned}$$

Due to Theorem 3.4.1, the stated Fredholm property implies a lower bound on the essential spectrum of $\Re(U^*T \cdot U) + \mathbf{1}$, and, conversely, the Fredholm property is a consequence of the lower bound on the essential spectrum. $\square$

Example 9.4.5. There are $P \in \mathbb{P}(\mathcal{K}, J)$ and $T \in \mathbb{U}(\mathcal{K}, J)$ such that $(P, T \cdot P)$ is *not* a Fredholm pair. For example, take $U = \mathbf{1}$ (which corresponds to P being the reference projection P_{ref} given in (9.11) below) and $T = J$. $\diamond$

The following result provides another natural situation in which Fredholm pairs of J -Lagrangian projections arise. It merely extends Proposition 9.3.3.

Proposition 9.4.6. *Let $T \in \mathbb{U}(\mathcal{K}, J)$ satisfy $\text{spec}(T) \cap \mathbb{S}^1 = \emptyset$. Let $R^<$ and $R^>$ be the Riesz projections of T associated to the spectral subsets $\text{spec}(T) \cap B_1(0)$ and $\text{spec}(T) \setminus B_1(0)$, respectively, and let $P^<$ and $P^>$ be the orthogonal projections onto their ranges $\mathcal{E}^< = \text{Ran}(R^<)$ and $\mathcal{E}^> = \text{Ran}(R^>)$. Then $(P^<, \mathbf{1} - P^>)$ forms a Fredholm pair.*

Proof. In Proposition 9.3.3 it was already shown that $P^<$ and $P^>$ are J -Lagrangian so that also $\mathbf{1} - P^>$ is J -Lagrangian. It remains to check the conditions in Definition 5.3.2 for $P_0 = P^<$ and $P_1 = \mathbf{1} - P^>$. First of all, $\text{Ran}(P_0) + \text{Ran}(\mathbf{1} - P_1) = \mathcal{E}^< + \mathcal{E}^> = \mathcal{K}$ is closed. Secondly, $\text{Ran}(P_0) \cap \text{Ker}(P_1) = \mathcal{E}^< \cap \mathcal{E}^> = \{0\}$ is finite dimensional, and finally, $\text{Ker}(P_0)^\perp + \text{Ran}(P_1)^\perp = \text{Ran}(P_0) + \text{Ker}(P_1) = \mathcal{E}^< + \mathcal{E}^> = \mathcal{K}$ so that $\text{Ker}(P_0) \cap \text{Ran}(P_1) = \{0\}$ is also finite dimensional. $\square$

The next results states that for a Fredholm pair of J -Lagrangian projections the index as defined in Section 5.2 is of little interest (for the finite-dimensional case, see already Remark 5.2.3).

Proposition 9.4.7. *For all $(P_0, P_1) \in \text{FFP}(\mathcal{K}, J)$, one has*

$$\text{Ind}(P_0, P_1) = 0.$$

Moreover, $(\text{FFP}(\mathcal{K}, J), \mathcal{O}_N)$ is connected.

Proof. Let (P_0, P_1) be a Fredholm pair of J -Lagrangian projections. Then, by Theorem 9.4.2, -1 is not in the essential spectrum of $\Pi(P_0)\Pi(P_1)^*$. By spectral calculus with a root for which the branch cut is chosen to be on the negative real axis, the paths $s \in [0, 1] \mapsto (\Pi(P_0)\Pi(P_1)^*)^{1-s}$ lies entirely in $\text{FU}(\mathcal{H})$. For $U(s) = (\Pi(P_0)\Pi(P_1)^*)^{1-s}\Pi(P_1)$, let us define a path of J -Lagrangian projections by

$$s \in [0, 1] \mapsto P(s) = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U(s) \\ U(s)^* & \mathbf{1} \end{pmatrix}.$$

Again by Theorem 9.4.2, one checks that $s \in [0, 1] \mapsto (P(s), P_1)$ is a path of Fredholm pairs of J -Lagrangian projections. It connects (P_0, P_1) to (P_1, P_1) . Therefore by Proposition 5.2.7,

$$\text{Ind}(P_0, P_1) = \text{Ind}(P_1, P_1) = 0.$$

The second claim follows because the set $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ is connected and therefore there is a path $s \in [0, 1] \mapsto \hat{U}(s)$ of unitaries connecting $\Pi(P_1)$ to $\mathbf{1}$. Then $s \in [0, 1] \mapsto \Pi^{-1}(\hat{U}(s))$ is a path of J -Lagrangian projections connecting P_1 to the reference J -Lagrangian projection $P_{\text{ref}} = \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$. Thus $s \in [0, 1] \mapsto (\Pi^{-1}(\hat{U}(s)), \Pi^{-1}(\hat{U}(s)))$ is a path of Fredholm pairs of J -Lagrangian projections connecting (P_1, P_1) to $(P_{\text{ref}}, P_{\text{ref}})$. In conclusion, there is a path of Fredholm pairs of J -Lagrangian projections connecting (P_0, P_1) to $(P_{\text{ref}}, P_{\text{ref}})$ and therefore the set of Fredholm pairs of J -Lagrangian projections is connected. $\square$

In many applications of the Fredholm pairs of J -Lagrangian projections, one of the projections, say P_0 , is fixed and given by a reference J -Lagrangian projection which we choose to be

$$P_{\text{ref}} = \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}. \quad (9.11)$$

Thus let us introduce the Fredholm J -Lagrangian Grassmannian (with respect to P_{ref}) by

$$\mathbb{F}\mathbb{P}(\mathcal{K}, J) = \{P \in \mathbb{P}(\mathcal{K}, J) : (P_{\text{ref}}, P) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)\}.$$

As $\Pi(P_{\text{ref}}) = \mathbf{1}$, Theorem 9.4.2 implies the following

Corollary 9.4.8. *The map $\Pi : (\mathbb{F}\mathbb{P}(\mathcal{K}, J), \mathcal{O}_N) \rightarrow (\mathbb{F}\mathcal{U}(\mathcal{H}), \mathcal{O}_N)$ is a bijective isometry.*

Due to Corollary 8.1.2, this directly implies the next statement:

Corollary 9.4.9. *The homotopy groups of $(\mathbb{F}\mathbb{P}(\mathcal{K}, J), \mathcal{O}_N)$ are*

$$\pi_k(\mathbb{F}\mathbb{P}(\mathcal{K}, J)) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

The next result also allows accessing the homotopy groups of $\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$.

Proposition 9.4.10. *The space $(\mathbb{F}\mathbb{P}(\mathcal{K}, J), \mathcal{O}_N)$ is homotopy equivalent to the space $(\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J), \mathcal{O}_N)$.*

Proof. Let $(P_0, P_1) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$ be a pair. Recall that there is a unitary $U_0 \in \mathcal{U}(\mathcal{H})$ (where $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ in the grading of J) such that

$$P_0 = \frac{1}{2} \begin{pmatrix} \mathbf{1} & U_0 \\ U_0^* & \mathbf{1} \end{pmatrix}.$$

Set $V = \text{diag}(\mathbf{1}, U_0)$, which is an element in $\mathcal{U}(\mathcal{K}, J) \cap \mathcal{U}(\mathcal{K})$. Then

$$(P_0, P_1) = V^* (P_{\text{ref}}, VP_1V^*) V.$$

Due to the natural identification

$$\mathbb{F}\mathbb{P}(\mathcal{K}, J) \cong \{(P_{\text{ref}}, P) : P \in \mathbb{P}(\mathcal{K}, J) \text{ and } (P_{\text{ref}}, P) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)\},$$

one hence has $\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J) = \mathbb{F}\mathbb{P}(\mathcal{K}, J) \times \mathbb{U}(\mathcal{H})$. Therefore the claim follows from the contractibility of $\mathbb{U}(\mathcal{H})$. $\square$

Combining Corollary 9.4.9 with Proposition 9.4.10, one deduces

Theorem 9.4.11. *The homotopy groups of $(\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J), \mathcal{O}_N)$ are*

$$\pi_k(\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)) = \begin{cases} \mathbb{Z}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

9.5 Paths of Fredholm pairs of J -Lagrangian projections

Proposition 9.4.7 shows that the index of a Fredholm pair $(P_0, P_1) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$ of J -Lagrangian projections always vanishes. As already stated in Theorem 9.4.11, there is interesting topological information contained in paths in $\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$. As shown in Corollary 9.5.7 at the end of this section, this is captured by the Bott–Maslov index which will be introduced and studied in this section. For the definition, recall the characterization of Fredholm pairs of J -Lagrangian projections as given in Theorem 9.4.2.

Definition 9.5.1. Let $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$ be a path of Fredholm pairs of J -Lagrangian projections and set $U(t) = \Pi(P_0(t))^* \Pi(P_1(t)) \in \mathbb{F}\mathbb{U}(\mathcal{H})$. Then the Bott–Maslov index of the path is defined by

$$\text{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))) = \text{Sf}(t \in [0, 1] \mapsto U(t)).$$

By Proposition 9.3.5, the Bott–Maslov index counts the number of finite-dimensional intersections of $\text{Ker}(P_0(t))$ with $\text{Ran}(P_1(t))$ along the path, with the orientation of the passage through the intersection as a weight. This two-sidedness will be further discussed below, and we will also provide a crossing form formulation for the Bott–Maslov index. Let us first note a few obvious properties that the Bott–Maslov index directly inherits from the spectral flow. More precisely, the next Proposition 9.5.2 is a direct consequence of Theorem 4.5.6, and Proposition 9.5.3 further down follows from item (ii) of Theorem 4.5.5.

Proposition 9.5.2. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$ be a path of Fredholm pairs of J -Lagrangian projections. Then its Bott–Maslov index is a homotopy invariant under homotopies within the set of paths of Fredholm pairs of J -Lagrangian projections keeping the endpoints $(P_0(0), P_1(0))$ and $(P_0(1), P_1(1))$ fixed.*

In particular, the Bott–Maslov index associates to every closed path of Fredholm pairs of J -Lagrangian projections an integer invariant. As will be shown in Corollary 9.5.7 below, this characterizes the fundamental group of $\mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$.

Proposition 9.5.3. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t))$ and $t \in [0, 1] \mapsto (P'_0(t), P'_1(t))$ be two paths in $\mathbb{FPP}(\mathcal{K}, J)$ such that $P_0(1) = P'_0(0)$ and $P_1(1) = P'_1(0)$. Then their concatenation $(P_0 * P'_0, P_1 * P'_1)$, defined by*

$$P_j * P'_j(t) = \begin{cases} P_j(2t), & t \in [0, \frac{1}{2}], \\ P'_j(2t - 1), & t \in [\frac{1}{2}, 1], \end{cases}$$

has a Bott–Maslov index given by

$$\begin{aligned} \text{BM}(t \in [0, 1] \mapsto (P_0 * P'_0(t), P_1 * P'_1(t))) \\ = \text{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))) + \text{BM}(t \in [0, 1] \mapsto (P'_0(t), P'_1(t))). \end{aligned}$$

The next result also follows directly from the definition and the identity (9.10).

Proposition 9.5.4. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a path of Fredholm pairs of J -Lagrangian projections and $V \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$. Then*

$$\text{BM}(t \in [0, 1] \mapsto (V \cdot P_0(t), V \cdot P_1(t))) = \text{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))).$$

Next crossing forms for differentiable paths $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ are introduced. Let us set $U(t) = \Pi(P_0(t))^* \Pi(P_1(t)) \in \mathbb{F}\mathbb{U}(\mathcal{H})$ as in Definition 9.5.1. Then the crossing form at t as in Definition 4.5.7 is given by

$$\Gamma_t : \text{Ker}(U(t) + 1) \rightarrow \mathbb{R}, \quad \Gamma_t(\phi) = -i \langle \phi | U(t)^* \partial_t U(t) \phi \rangle.$$

A crossing is called regular if Γ_t is nondegenerate. Now Proposition 4.5.9 immediately implies the following result.

Proposition 9.5.5. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a continuously differentiable path having only regular crossings. Then*

$$\text{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))) = \frac{1}{2} \text{Sig}(\Gamma_0) + \sum_{t \in (0, 1)} \text{Sig}(\Gamma_t) + \frac{1}{2} \text{Sig}(\Gamma_1). \quad (9.12)$$

As in the finite-dimensional case (Lemma 2.1.9), it is useful to have an explicit formula for the crossing form in terms of the projections. This can be deduced from the first part of the next statement.

Lemma 9.5.6. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t))$ be a differentiable path of pairs of J -Lagrangian projections with associated $U_0(t) = \Pi(P_0(t))$ and $U_1(t) = \Pi(P_1(t))$. Then for $U(t) = U_0(t)^* U_1(t)$ one has*

$$U(t)^* \partial_t U(t) = 4 \begin{pmatrix} U_1(t) \\ 0 \end{pmatrix}^* P_0(t) \partial_t P_0(t) \begin{pmatrix} U_1(t) \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* P_1(t) \partial_t P_1(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If, moreover, $\partial_t P_0(t)$ and $\partial_t P_1(t)$ are trace class, then $U(t)^* \partial_t U(t)$ is trace class and given by

$$\mathrm{Tr}(U(t)^* \partial_t U(t)) = 2 \mathrm{Tr}(JP_0(t) \partial_t P_0(t)^*) - 2 \mathrm{Tr}(JP_1(t) \partial_t P_1(t)^*).$$

Proof. By the formulas in Proposition 9.3.4, one has for $j = 0, 1$,

$$P_j(t) \partial_t P_j(t) = \frac{1}{4} \begin{pmatrix} U_j(t) \partial_t U_j(t)^* & \partial_t U_j(t) \\ \partial_t U_j(t)^* & U_j(t)^* \partial_t U_j(t) \end{pmatrix}.$$

As

$$U(t)^* \partial_t U(t) = U_1(t)^* (U_0(t) \partial_t U_0(t)^*) U_1(t) + U_1(t)^* \partial_t U_1(t),$$

this implies the first formula. The summability in the second claim is now clear, and

$$\begin{aligned} \mathrm{Tr}(U(t)^* \partial_t U(t)) &= -\mathrm{Tr}(U(t) \partial_t U(t)^*) \\ &= 2 \mathrm{Tr}(JP_0(t) \partial_t P_0(t)^*) - 2 \mathrm{Tr}(JP_1(t) \partial_t P_1(t)^*), \end{aligned}$$

by taking the trace of the above formula for $P_j(t) \partial_t P_j(t)$ times J . $\square$

Next let us state that the Bott–Maslov index restricted to closed paths identifies the fundamental group of $\mathbb{FPP}(\mathcal{K}, J)$ given in Theorem 9.4.11.

Corollary 9.5.7. *The Bott–Maslov index defined in Definition 9.5.1 establishes an isomorphism*

$$\mathrm{BM} : \pi_1(\mathbb{FPP}(\mathcal{K}, J)) \rightarrow \mathbb{Z}.$$

For differentiable closed paths and under a trace class condition on $\partial_t U(t)$, it is now possible to plug in the formula for $\mathrm{Tr}(U(t)^* \partial_t U(t))$ given in Lemma 9.5.6 into Proposition 4.5.11.

Corollary 9.5.8. *Let $t \in [0, 1] \mapsto (P_0(t), P_1(t)) \in \mathbb{FPP}(\mathcal{K}, J)$ be a continuously differentiable closed path. Suppose that $\partial_t P_0(t)$ and $\partial_t P_1(t)$ are trace class. Then*

$$\begin{aligned} \mathrm{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))) \\ = \frac{1}{\pi i} \int_0^1 dt (\mathrm{Tr}(JP_0(t) \partial_t P_0(t)^*) - \mathrm{Tr}(JP_1(t) \partial_t P_1(t)^*)). \end{aligned}$$

Next let us note that one has an infinite-dimensional analogue of Proposition 2.2.14. Further down in Proposition 9.6.17 a link to the Conley–Zehnder index will be given.

Proposition 9.5.9. *Let $t \in [0, 1] \mapsto T_t \in \mathbb{W}^C(\mathcal{K}, J)$ be a closed path. Then the Bott–Maslov index $\mathrm{BM}(t \in [0, 1] \mapsto (P, T_t \cdot P))$ is well defined and independent of $P \in \mathbb{P}(\mathcal{K}, J)$.*

Proof. First of all, Proposition 9.4.1 indeed shows that $(P, T_t \cdot P) \in \mathbb{FP}(\mathcal{K}, J)$ so that the Bott–Maslov index of the path is well defined. As the path is closed, it is given by a winding number in the sense of Proposition 4.5.10 and thus is homotopy invariant. Now given $P_0, P_1 \in \mathbb{P}(\mathcal{K}, J)$, there exists a unitary $U \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ such that $P_1 = U \cdot P_0$ by Proposition 9.3.6. Then, because $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ is connected, one can choose a path $s \in [0, 1] \mapsto U_s \in \mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{K})$ such that $P_s = U_s \cdot P_0$ connects P_0 to P_1 in $\mathbb{P}(\mathcal{K}, J)$. Then $s \in [0, 1] \mapsto (P_s, T_t \cdot P_s)$ is a homotopy of closed loops in $\mathbb{FP}(\mathcal{K}, J)$, showing that the Bott–Maslov index of the stated path is independent of P . $\square$

Based on Proposition 9.4.4, one can also deal with other situations than that in Proposition 9.5.9 in which $t \mapsto (P, T_t \cdot P)$ has a well-defined Bott–Maslov index that has stability properties in P . For example, suppose $T_t = T_0(1 + K_t)$ for some fixed T_0 and loop $t \in [0, 1] \mapsto K_t \in \mathbb{K}(\mathcal{K})$ in the Lie algebra such that $(P, T_0 \cdot P)$ is a Fredholm pair. Then indeed $(P, T_t \cdot P)$ is a Fredholm pair (by the same argument as in Proposition 9.4.1) and, furthermore, the Fredholm property is stable along this path under small perturbations of P due to Proposition 9.4.4. By homotopy invariance of the Bott–Maslov index, one then also deduces its stability as in Proposition 9.5.9. As to explicit formulas, of course, Corollary 9.5.8 applies to the case of differentiable closed paths $(P, T_t \cdot P)$ and actually only one of the summands remains. Further formulas (such as an infinite-dimensional analogue of Proposition 2.2.14) will be given below.

As already pointed out, often one of the two projections of a pair of J -Lagrangian projections is fixed. Also Proposition 9.5.9 considers such a situation. In the following, this reference projection is again chosen to be $P_0 = P_{\text{ref}}$, and then the Bott–Maslov index of the path $t \mapsto (P_{\text{ref}}, P(t))$ is considered for $P(t) \in \mathbb{FP}(\mathcal{K}, J)$. Moreover, it will be shown below (by essentially the same argument as in the proof of Proposition 9.4.10) that one can always arrange one of the J -Lagrangian projections to be moved into the reference J -Lagrangian projection P_{ref} (or any other one). In this situation, the following is just a special case of Definition 9.5.1, simply because $\Pi(P_{\text{ref}}) = 1$.

Definition 9.5.10. For a path $t \in [0, 1] \mapsto P(t) \in \mathbb{FP}(\mathcal{K}, J)$ in the Fredholm J -Lagrangian Grassmannian, the Bott–Maslov index is defined by

$$\text{BM}(t \in [0, 1] \mapsto P(t)) = \text{Sf}(t \in [0, 1] \mapsto U(t)),$$

where $U(t) = \Pi(P(t)) \in \mathbb{F}\mathbb{U}(\mathcal{H})$.

Proposition 9.5.11. Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a differentiable closed path in $\mathbb{U}(\mathcal{K}, J)$ and $P \in \mathbb{P}(\mathcal{K}, J)$. Suppose that $(P, T_t \cdot P) \in \mathbb{FP}(\mathcal{K}, J)$ and that $\partial_t T_t J T_t^*$ is trace class. Then

$$\text{BM}(t \in [0, 1] \mapsto T_t \cdot P) = \frac{1}{i\pi} \int_0^1 dt \text{Tr}((1 - T_t \cdot P)(\partial_t T_t J T_t^*)).$$

Due to Corollary 4.5.10, the proof of Proposition 9.5.11 is completed by the following algebraic lemma which generalizes Lemma 2.2.13 dealing with the finite-dimensional case.

Lemma 9.5.12. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a differentiable path in $\mathbb{U}(\mathcal{K}, J)$ and $P \in \mathbb{P}(\mathcal{K}, J)$. Then $U_t = \Pi(P_t)$ associated to $P_t = T_t \cdot P$ satisfies*

$$U_t^* \partial_t U_t = \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}^* (\partial_t T_t J T_t^*) \begin{pmatrix} U_t \\ -\mathbf{1} \end{pmatrix}.$$

Moreover, if $\partial_t T_t J T_t^*$ is trace class, then also $\partial_t U_t$ is trace class and

$$\mathrm{Tr}(U_t^* \partial_t U_t) = 2 \mathrm{Tr}((\mathbf{1} - P_t)(\partial_t T_t J T_t^*)).$$

Proof. For sake of notational simplicity, let us suppress the index t and set $W = \Pi(P)$. First note that

$$U^* \partial U = \Pi(T \cdot P)^* \partial \Pi(T \cdot P) = (T \cdot W)^* \partial(T \cdot W),$$

because $\Pi(T \cdot P) = T \cdot \Pi(P) = T \cdot W$. Using $(T \cdot W)^* = (T \cdot W)^{-1}$ and the laws of operator differentiation, one finds

$$\begin{aligned} & (T \cdot W)^* \partial(T \cdot W) \\ &= (T \cdot W)^* (\partial(AW + B))(CW + D)^{-1} - (\partial(CW + D))(CW + D)^{-1} \\ &= ((CW + D)^{-1})^* [(AW + B)^* \partial(AW + B) - (CW + D)^* \partial(CW + D)] (CW + D)^{-1} \\ &= ((CW + D)^{-1})^* \begin{pmatrix} W \\ \mathbf{1} \end{pmatrix}^* T^* J \partial T \begin{pmatrix} W \\ \mathbf{1} \end{pmatrix} (CW + D)^{-1}. \end{aligned}$$

But

$$\begin{pmatrix} W \\ \mathbf{1} \end{pmatrix} (CW + D)^{-1} = T^{-1} \begin{pmatrix} U \\ \mathbf{1} \end{pmatrix}.$$

Now

$$(T^{-1})^* (T^* J \partial T) T^{-1} = J(\partial T J T^*) J$$

concludes the proof of the first identity. Plugging it into the trace leads to the second one. $\square$

It is always possible to recourse to the Bott–Maslov index with respect to a fixed reference plane as in Definition 9.5.10 by appealing to Proposition 9.3.6 to deform $P_0(t)$ into P_{ref} . More precisely, given a path $t \in [0, 1] \mapsto (P_0(t), P_1(t))$ of J -Lagrangian projections, set as in the proof of Proposition 9.4.10

$$V(t) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \Pi(P_0(t)) \end{pmatrix}.$$

Clearly, $t \in [0, 1] \mapsto V(t)$ is a path in $\mathbb{U}(\mathcal{K}, J) \cap \mathbb{U}(\mathcal{H})$, and one checks that

$$V(t)P_0(t)V(t)^* = P_{\text{ref}}.$$

In this manner, one obtains the path $t \in [0, 1] \mapsto (P_{\text{ref}}, V(t)P_1(t)V(t)^*)$ which consists of Fredholm pairs if $(P_0(t), P_1(t))$ are Fredholm pairs. The basis change can be suppressed in the following by setting $P(t) = V(t)P_1(t)V(t)^*$. Then $U(t) = \Pi(P(t))$ lies in $\mathbb{F}\mathbb{U}(\mathcal{H})$ by Theorem 9.4.2.

Remark 9.5.13. Another alternative to attain a situation with a fixed reference frame is a doubling procedure, e. g., [90]. Suppose given $t \mapsto (P_0(t), P_1(t)) \in \mathbb{F}\mathbb{P}\mathbb{P}(\mathcal{K}, J)$. Then one constructs a new Krein space $(\widehat{\mathcal{K}}, \widehat{J})$ by setting $\widehat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$ and $\widehat{J} = J \oplus (-J)$. Then $\widehat{P}(t) = P_0(t) \oplus (1 - P_1(t))$ is clearly $\widehat{J}$ -Lagrangian by construction. Moreover, the doubled reference frame

$$\tilde{P}_{\text{ref}} = \frac{1}{2} \begin{pmatrix} \mathbf{1}_2 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}$$

is also $\widehat{J}$ -Lagrangian. One can then check that

$$\dim(\text{Ran}(P_0(t)) \cap \text{Ker}(P_1(t))) = \dim(\text{Ran}(\widehat{P}(t)) \cap \text{Ker}(\tilde{P}_{\text{ref}}))$$

and, with $U_0(t) = \Pi(P_0(t))$ and $U_1(t) = \Pi(P_1(t))$,

$$\begin{aligned} \text{BM}(t \in [0, 1] \mapsto (P_0(t), P_1(t))) &= \text{Sf}(t \in [0, 1] \mapsto U_0(t)^* U_1(t)) \\ &= \text{Sf}\left(t \in [0, 1] \mapsto \begin{pmatrix} 0 & -U_1(t) \\ U_0(t)^* & 0 \end{pmatrix}\right) \\ &= \text{BM}(t \in [0, 1] \mapsto (\widehat{P}(t), \tilde{P}_{\text{ref}})), \end{aligned}$$

i. e., the latter expression is a Bott–Maslov index in the sense of Definition 9.5.10. This approach may be of some theoretical use, but has the disadvantage of doubling dimension and consequently only producing a special type of $\widehat{J}$ -Lagrangian subspaces, namely the diagonal ones $\widehat{P}(t)$. $\diamond$

Combining Corollaries 9.4.9 and 9.4.8 with Corollary 8.1.3 now leads to

Corollary 9.5.14. *The Bott–Maslov index induces an isomorphism*

$$\text{BM} : \pi_1(\mathbb{F}\mathbb{P}(\mathcal{K}, J)) \rightarrow \mathbb{Z}.$$

Let us also note that both Corollary 9.5.8 and Proposition 9.5.11 cover the situation of a fixed reference P_{ref} . There is, however, an even more explicit formula extending Proposition 2.2.14 of the finite-dimensional case.

Proposition 9.5.15. *Let $t \in [0, 1] \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \in \mathcal{U}(\mathcal{K}, J)$ be a closed differentiable path such that all four entries of $\partial_t T_t$ are trace class. Then*

$$\begin{aligned} \text{BM}(t \in [0, 1] \mapsto T_t \cdot P_{\text{ref}}) \\ = \frac{1}{2\pi i} \int_0^1 dt \text{Tr}((A_t + B_t)^{-1} \partial_t (A_t + B_t) - (C_t + D_t)^{-1} \partial_t (C_t + D_t)). \end{aligned}$$

Proof. Set $U_t = \Pi(T_t \cdot P_{\text{ref}})$. Then, by Lemma 9.5.12, the hypothesis implies that $U_t^* \partial_t U_t$ is trace class so that Corollary 4.5.10 can be applied to compute the Bott–Maslov index. As $U_t = T_t \cdot \Pi(P_{\text{ref}}) = T_t \cdot \mathbf{1} = (A_t + B_t)(C_t + D_t)^{-1}$, the usual derivative rule and the cyclicity of the trace then immediately lead to the claimed identity. $\square$

In the remainder of this section, let us next discuss a geometric interpretation of the Bott–Maslov index that has been put forward by Arnold. Let us now consider a path $t \in [0, 1] \mapsto P(t) \in \mathbb{FP}(\mathcal{K}, J)$. By Proposition 9.3.5, the Bott–Maslov index counts the number of intersections of $\text{Ran}(P(t))$ with the fixed subspace $\text{Ran}(JP_{\text{ref}}J)$, with their multiplicity and with an orientation as a weight. The following definition, generalizing Arnold’s definition [9], is hence natural.

Definition 9.5.16. The singular cycle of J -Lagrangian subspaces with nontrivial intersections with $JP_{\text{ref}}J$ is

$$\mathbb{SP}(\mathcal{K}, J) = \bigcup_{l \geq 1} \mathbb{SP}_l(\mathcal{K}, J),$$

where

$$\mathbb{SP}_l(\mathcal{K}, J) = \{P \in \mathbb{FP}(\mathcal{K}, J) : \dim(\text{Ran}(P) \cap \text{Ker}(P_{\text{ref}})) = l\}.$$

First of all, let us note that the Fredholm property assures that the intersection of $\text{Ran}(P)$ with $\text{Ran}(JP_{\text{ref}}J) = \text{Ker}(P_{\text{ref}})$ is always finite dimensional. Note also that Proposition 9.3.5 implies

$$\Pi(\mathbb{SP}_l(\mathcal{K}, J)) = \{U \in \mathbb{FU}(\mathcal{H}) : \dim(\text{Ker}(U + \mathbf{1})) = l\}.$$

Hence the codimension of $\mathbb{SP}_l(\mathcal{K}, J)$ increases with l and this makes $\mathbb{SP}(\mathcal{K}, J)$ into a stratified space with strata $\mathbb{SP}_l(\mathcal{K}, J)$. Finally, the singular cycle $\mathbb{SP}(\mathcal{K}, J)$ is two-sided, namely a point close to $\mathbb{SP}(\mathcal{K}, J)$ can either be on its right or its left, depending on whether the

eigenvalue of its stereographic projection has a positive or negative imaginary part. Having in mind the image of the path under the stereographic projection, all these geometric properties become self-evident.

9.6 Conley–Zehnder index

Section 2.3 analyzed the Conley–Zehnder index in finite dimensions. It turned out that the Conley–Zehnder index is nothing but the Bott–Maslov index of the graphs of J -unitaries, considered as Lagrangian subspaces in a doubled Krein space. The same algebraic setup transposes to infinite-dimensional Krein spaces, provided that suitable Fredholm conditions are imposed. This is carried out in this section. Most of the algebraic expressions and identities are identical to those in Section 2.3, but several are repeated to facilitate readability.

Associated to a Krein space $(\mathcal{K}, J)$ and a given J -unitary T is another doubled Krein space $(\mathcal{K} \oplus \mathcal{K}, (-J) \oplus J)$ on which then acts $\mathbf{1} \oplus T$ as $((-J) \oplus J)$ -unitary. The range of the operator $(\mathbf{1} \oplus T)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the graph $\mathcal{G}_T$ of T . It is hence a $((-J) \oplus J)$ -Lagrangian subspace. In order to use the stereographic projection in the form of Section 9.3, it is convenient to use the basis transformation $\hat{F}$ given in (2.24). Note that it actually is a symmetry. It then leads to a standard form for the doubled Krein space,

$$(\hat{\mathcal{K}}, \hat{J}) = (\mathcal{K} \oplus \mathcal{K}, \hat{F}((-J) \oplus J)\hat{F}),$$

with $\hat{J} = \text{diag}(\mathbf{1}, -\mathbf{1})$. The group of $\hat{J}$ -unitary operators is again denoted by $\mathcal{U}(\hat{\mathcal{K}}, \hat{J})$. A particular operator therein is

$$\hat{T} = \hat{F}(\mathbf{1} \oplus T)\hat{F} \in \mathcal{U}(\hat{\mathcal{K}}, \hat{J}),$$

and an example of a $\hat{J}$ -Lagrangian subspace is the $\hat{F}$ -transformed graph $\hat{\mathcal{G}}_T = \hat{F}\mathcal{G}_T$. The stereographic projection from the space $\mathcal{P}(\hat{\mathcal{K}}, \hat{J})$ of $\hat{J}$ -Lagrangian subspaces to $\mathcal{U}(\mathcal{K})$ defined as in Proposition 9.3.4 is denoted by $\hat{\Pi}$. As a reference $\hat{J}$ -Lagrangian projection, we will use

$$\hat{P}_{\text{ref}} = \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}. \quad (9.13)$$

It satisfies $\hat{\Pi}(\hat{P}_{\text{ref}}) = \mathbf{1}$ and $\hat{F}\hat{P}_{\text{ref}}\hat{F} = \hat{P}_{\text{ref}}$. Its range is denoted by $\hat{\mathcal{E}}_{\text{ref}} = \text{Ran}(\hat{P}_{\text{ref}})$. Moreover, it allows writing the projection on $\hat{\mathcal{G}}_T$ as $\hat{T} \cdot \hat{P}_{\text{ref}}$. The algebraic proof of the following theorem is identical to that of Theorem 2.3.1 covering the finite-dimensional case.

Theorem 9.6.1. *To a given $T \in \mathcal{U}(\mathcal{K}, J)$ let us associate a unitary $S(T)$ by*

$$S(T) = \hat{\Pi}(\hat{\mathcal{G}}_T) = \hat{\Pi}(\hat{P}_{\text{ref}})^* \hat{\Pi}(\hat{T} \cdot \hat{P}_{\text{ref}}) \in \mathcal{U}(\mathcal{K}). \quad (9.14)$$

If $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$S(T) = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix} = \begin{pmatrix} (A^*)^{-1} & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix}.$$

The map $T \in \mathbb{U}(\mathcal{K}, J) \mapsto S(T) \in \mathbb{U}(\mathcal{K})$ is a continuous embedding with image

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{U}(\mathcal{K}) : \alpha, \delta \in \text{invertible} \right\}. \quad (9.15)$$

Also the proof of the next structural result for $S(T)$ is as in the finite-dimensional case, see Proposition 2.3.2.

Proposition 9.6.2. *Given $T \in \mathbb{U}(\mathcal{K}, J)$, one has*

$$S(-T) = -JS(T)J,$$

and

$$S(T)^* = S(T)^{-1} = S(T^{-1}) = JS(T^*)J.$$

The following result justifies the above constructions. The algebraic proof is identical to the proof of Theorem 2.3.3.

Theorem 9.6.3. *Let T and $S(T)$ be as in Theorem 9.6.1. Then*

$$\text{Ker}(T - \mathbf{1}) = \text{Ker}(S(T) - \mathbf{1}), \quad \text{Ker}(T + \mathbf{1}) = J \text{Ker}(S(T) + \mathbf{1}).$$

Theorem 9.6.3, as well as the connection between eigenvectors, can easily be adapted to study other eigenvalues on the unit circle. Indeed, if $T\phi = z\phi$ for $z \in \mathbb{S}^1$, then also $(\bar{z}T)\phi = \phi$. But the operator $\bar{z}T$ is also J -unitary so that one can apply the above again to construct an associated unitary. This shows the following.

Proposition 9.6.4. *Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a J -unitary and set, for $z \in \mathbb{S}^1$,*

$$S(\bar{z}T) = \begin{pmatrix} \bar{z}(A^*)^{-1} & BD^{-1} \\ -D^{-1}C & zD^{-1} \end{pmatrix}. \quad (9.16)$$

Then the geometric multiplicity of z as eigenvalue of T is equal to the multiplicity of 1 as eigenvalue of $S(\bar{z}T)$.

Therefore, the unitaries $S(\bar{z}T)$ are a tool to study eigenvalues of T which lie on the unit circle. Let us focus again on $z = \pm 1$. Theorem 9.6.3 concerns the kernel of $S(T) \pm \mathbf{1}$. It is natural to analyze how much more spectrum $S(T)$ has close to ± 1 , or, what is equivalent, how much spectrum the self-adjoint operator $\Re e(S(T)) = \frac{1}{2}(S(T) + S(T)^*)$ has close to ± 1 .

For this purpose, it is useful to have an explicit formula for $\Re e(S(T))$. Again the algebraic proof is identical to that in the finite-dimensional cases stated in Proposition 2.3.7.

Proposition 9.6.5. *Let T be a J -unitary and $S(T)$ as above. Then*

$$\Re e(S(T)) = (\mathbf{1} + T)(\mathbf{1} + T^* T)^{-1}(\mathbf{1} + T)^* - \mathbf{1}. \quad (9.17)$$

The most robust compactness property of J -unitaries implies the following:

Proposition 9.6.6. *For $T \in \mathcal{U}^c(\mathcal{K}, J)$, one has $S(T) \in \mathcal{U}^c(\mathcal{K})$. Furthermore, the image of the map $S : T \in \mathcal{U}(\mathcal{K}, J) \mapsto S(T) \in \mathcal{U}(\mathcal{K})$ is*

$$S(\mathcal{U}^c(\mathcal{K}, J)) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{U}^c(\mathcal{K}) : \alpha, \delta \in \text{invertible} \right\}.$$

Proof. As $T = \mathbf{1} + K \in \mathcal{U}^c(\mathcal{K}, J)$, the claim directly follows from Theorem 9.6.1. $\square$

Remark 9.6.7. Proposition 9.6.6 holds irrespective of the choice of the reference projection. More precisely, if one uses some other reference plane $\tilde{P}_{\text{ref}} \in \mathbb{P}(\widehat{\mathcal{K}}, \widehat{J})$ to define $\tilde{S}(T) = \widehat{\Pi}(\tilde{P}_{\text{ref}})^* \widehat{\Pi}(\widehat{T} \cdot \tilde{P}_{\text{ref}})$, then also $\tilde{S}(T) \in \mathcal{U}^c(\mathcal{K})$ for $T \in \mathcal{U}^c(\mathcal{K}, J)$. $\diamond$

Let us now come to a Fredholm condition for the J -unitaries.

Definition 9.6.8. A J -unitary $T \in \mathcal{U}(\mathcal{K}, J)$ is called Fredholm if $S(T) \in \mathbb{F}\mathcal{U}(\mathcal{K})$. The set of all J -unitaries $T \in \mathcal{U}(\mathcal{K}, J)$ with this Fredholm property is denoted by $\mathbb{F}\mathcal{U}(\mathcal{K}, J)$.

Clearly, one has $\mathcal{U}^c(\mathcal{K}, J) \subset \mathbb{F}\mathcal{U}(\mathcal{K}, J)$. Let us now provide several characterizations of the Fredholm property of $T \in \mathcal{U}(\mathcal{K}, J)$, one of which shows that it is independent of the choice of $\widehat{P}_{\text{ref}}$ (similar as in Remark 9.6.7). Another comment is that characterization (iv) below explains that operators in $\mathbb{F}\mathcal{U}(\mathcal{K}, J)$ were called (-1) -Fredholm J -unitaries in [168] (and then the more restricted class of $\mathbb{S}^1$ -Fredholm unitaries was considered there for which $T - z\mathbf{1}$ is Fredholm for all $z \in \mathbb{S}^1$, which is a strictly larger class than the essentially $\mathbb{S}^1$ -gapped J -unitaries considered in Definition 9.2.12).

Proposition 9.6.9. *For $T \in \mathcal{U}(\mathcal{K}, J)$, the following are equivalent:*

- (i) $T \in \mathbb{F}\mathcal{U}(\mathcal{K}, J)$;
- (ii) $(\widehat{P}_{\text{ref}}, \widehat{T} \cdot \widehat{P}_{\text{ref}}) \in \mathbb{FPP}(\widehat{\mathcal{K}}, \widehat{J})$;
- (iii) $-1 \notin \text{spec}_{\text{ess}}(S(T))$;
- (iv) $T + \mathbf{1} \in \mathbb{F}\mathcal{B}(\mathcal{K})$.

Proof. (i) $\iff$ (ii). This follows directly from Theorem 9.4.2 applied to the Krein space $(\widehat{\mathcal{K}}, \widehat{J})$ and the Fredholm pair of $\widehat{J}$ -Lagrangian projections $(\widehat{P}_{\text{ref}}, \widehat{T} \cdot \widehat{P}_{\text{ref}})$.

(i) $\iff$ (iii). This follows immediately from the definition because $S(T) \in \mathbb{F}\mathcal{U}(\mathcal{K})$ is equivalent to $-1 \notin \text{spec}_{\text{ess}}(S(T))$.

(iii) $\iff$ (iv). For any unitary S , $-1 \notin \text{spec}_{\text{ess}}(S)$ is equivalent to $\min \text{spec}_{\text{ess}}(\Re(S) + \mathbf{1}) > 0$. Now $\Re(S(T)) + \mathbf{1}$ is given by Proposition 9.6.5 which can also be rewritten as

$$\Re(S(T)) + \mathbf{1} = (\mathbf{1} + T)(\mathbf{1} + T^*T)^{-1}(\mathbf{1} + T)^*.$$

Because $(\mathbf{1} + T^*T)^{-1}$ is a bounded invertible operator and therefore Fredholm, (iv) implies by Corollary 3.3.2 that $\Re(S(T)) + \mathbf{1}$ is Fredholm which, by Corollary 3.4.4, is equivalent to (iii). Conversely, if $\Re(S(T)) + \mathbf{1}$ is Fredholm also

$$J(\Re(S(T)) + \mathbf{1})J = (\mathbf{1} + T^*)(\mathbf{1} + TT^*)^{-1}(\mathbf{1} + T)$$

is Fredholm. Therefore $\dim(\text{Ker}(J(\Re(S(T)) + \mathbf{1})J)) < \infty$ and, because one moreover has $\text{Ker}(\mathbf{1} + T) \subset \text{Ker}(J(\Re(S(T)) + \mathbf{1})J)$, this implies $\dim(\text{Ker}(\mathbf{1} + T)) < \infty$. Furthermore, the range of $\Re(S(T)) + \mathbf{1}$ is closed. Thus

$$\text{Ran}(\mathbf{1} + T) = \text{Ran}(\Re(S(T)) + \mathbf{1}) \oplus (\text{Ran}(\mathbf{1} + T) \ominus \text{Ran}(\Re(S(T)) + \mathbf{1}))$$

is closed because $\text{Ran}(\mathbf{1} + T) \ominus \text{Ran}(\Re(S(T)) + \mathbf{1}) \subset \text{Ran}(\Re(S(T)) + \mathbf{1})^\perp$ is finite dimensional and therefore closed. As $\text{Ran}(\mathbf{1} + T)^\perp \subset \text{Ran}(\Re(S(T)) + \mathbf{1})^\perp$ is finite dimensional, this implies that $\mathbf{1} + T$ is Fredholm. $\square$

Combined with Theorem 9.6.1, more precisely (9.15), Proposition 9.6.9 implies the following:

Corollary 9.6.10. *The image of $\mathbb{F}\mathbb{U}(\mathcal{K}, J)$ under $S : T \in \mathbb{U}(\mathcal{K}, J) \mapsto S(T) \in \mathbb{U}(\mathcal{K})$ is*

$$S(\mathbb{F}\mathbb{U}(\mathcal{K}, J)) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{F}\mathbb{U}(\mathcal{K}) : \alpha, \delta \in \text{invertible} \right\}.$$

Corollary 9.6.10 suggests that $\mathbb{U}^c(\mathcal{K}, J)$ is a deformation retract of $\mathbb{F}\mathbb{U}(\mathcal{K}, J)$ because $\mathbb{U}^c(\mathcal{K})$ is a deformation retract of $\mathbb{F}\mathbb{U}(\mathcal{K})$ by Proposition 3.7.2. This is, however, not clear because the retract in the proof of Proposition 3.7.2 may not stay within the image $S(\mathbb{F}\mathbb{U}(\mathcal{K}, J))$ of the map S given in Corollary 9.6.10.

Now all preparations for the following definition are carried out.

Definition 9.6.11. The Conley–Zehnder index of a path $t \in [0, 1] \mapsto T_t \in \mathbb{F}\mathbb{U}(\mathcal{K}, J)$ is defined as

$$\text{CZ}(t \in [0, 1] \mapsto T_t) = \text{Sf}(t \in [0, 1] \mapsto S(T_t)).$$

Note that, indeed, Proposition 9.6.9 implies that the spectral flow of unitaries on the right-hand side is well defined (as the spectral flow through -1 in the sense of Section 4.5). As such, the Conley–Zehnder index inherits several properties of the spectral flow which are not spelled out in detail: concatenation, homotopy invariance (with fixed

endpoints), integrality, and additivity. From these properties, one directly deduces the following statement:

Proposition 9.6.12. *The Conley–Zehnder index applied to closed paths induces group homomorphisms $\text{CZ} : \pi_1(\mathbb{F}\mathbb{U}(\mathcal{K}, J)) \rightarrow \mathbb{Z}$ and $\text{CZ} : \pi_1(\mathbb{U}^{\mathbb{C}}(\mathcal{K}, J)) \rightarrow \mathbb{Z}$.*

Remark 9.6.13. Proposition 9.1.8 shows that $\pi_1(\mathbb{U}^{\mathbb{C}}(\mathcal{K}, J)) = \mathbb{Z} \oplus \mathbb{Z}$. Hence the Conley–Zehnder index extracts one of these $\mathbb{Z}$. $\diamond$

In many applications, one deals with differentiable paths $t \mapsto T_t$ of J -unitaries. Then it is useful to be able to compute the derivatives of the eigenvalues of $S(T_t)$ when they cross -1 , namely those points which can contribute to the Conley–Zehnder index. The following proposition then leads to a crossing form formulation of the Conley–Zehnder index. This is not spelled out in detail as it is essentially the same as in Section 4.3. The formulas below also allow to analyze the transversality of the path.

Proposition 9.6.14. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a differentiable path in $\mathbb{U}(\mathcal{K}, J)$. Then*

$$S(T_t)^* \partial_t S(T_t) = \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1} C_t & D_t^{-1} \end{pmatrix}^* (T_t^* J \partial_t T_t) \begin{pmatrix} \mathbf{1} & 0 \\ -D_t^{-1} C_t & D_t^{-1} \end{pmatrix}.$$

For a vector $\phi_t \in \mathcal{K}$ satisfying $T_t \phi_t = -\phi_t$, one has $S(T_t) J \phi_t = -J \phi_t$ and

$$\phi_t^* J S(T_t)^* \partial_t S(T_t) J \phi_t = \phi_t^* T_t^* J \partial_t T_t \phi_t.$$

Proof. The proof is essentially the same as that of Lemma 2.3.9. $\square$

Let us now provide an integral formula for the Conley–Zehnder index of differentiable closed paths. It is an infinite-dimensional version of Proposition 2.3.11 with an identical proof, provided supplementary trace class properties are imposed. In particular, the algebraic Lemma 2.3.10 transposes directly.

Proposition 9.6.15. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a continuously differentiable closed path in $\mathbb{U}(\mathcal{K}, J)$ such that all four entries of $\partial_t T_t$ are trace class. Then*

$$\text{CZ}(t \in [0, 1] \mapsto T_t) = \frac{1}{2\pi i} \int_0^1 dt \operatorname{Tr}((A_t)^{-1} \partial_t A_t - (D_t)^{-1} \partial_t D_t).$$

Also the statement and proof of Corollary 2.3.12 transpose to the infinite-dimensional setting. It provides a connection between the Bott–Maslov and Conley–Zehnder indices.

Corollary 9.6.16. *Let $t \mapsto T_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}$ be a continuously differentiable closed path in $\mathbb{U}^{\mathbb{C}}(\mathcal{K}, J)$ such that all four entries of $\partial_t T_t$ are trace class. Then for any $P \in \mathbb{P}(\mathcal{K}, J)$,*

$$\text{CZ}(t \in [0, 1] \mapsto T_t) = \text{BM}(t \in [0, 1] \mapsto (P, T_t \cdot P)).$$

Based on Corollary 9.6.16, as well as Propositions 9.5.9 and 9.5.15, one can now prove an infinite-dimensional version of Corollary 2.3.13, simply by realizing that the finite-dimensional arguments transpose to a trace class situation. While it is certainly possible to weaken the hypothesis, this is not further studied here.

Proposition 9.6.17. *Let $t \in [0, 1] \mapsto P_t \in \mathbb{FP}(\mathcal{K}, J)$ and $t \in [0, 1] \mapsto T_t \in \mathbb{U}^c(\mathcal{K}, J)$ be two continuously differentiable closed paths such that all four entries of $\partial_t T_t$ are trace class. Then*

$$\mathrm{BM}(t \in [0, 1] \mapsto T_t \cdot P_t) = \mathrm{BM}(t \in [0, 1] \mapsto P_t) + \mathrm{CZ}(t \in [0, 1] \mapsto T_t).$$

9.7 Oscillation theory for bound states of scattering systems

This section provides an application of the Bott–Maslov index and spectral flow in the infinite-dimensional setting as described in this chapter. It is about oscillation theory for bound states of a higher-dimensional quantum scattering system (within a single-particle framework). This basically consists of transposing the setup and results of Section 2.6 to a situation where the fibers are infinite dimensional and the locality of the scattering perturbation directly leads to the required Fredholm property. Therefore it is possible to simply refer to Section 2.6 for most of the algebraic arguments, and merely add the required functional analytic elements to the proofs. Let us also note that we are not aware of other results on oscillation theory with infinite dimensional fibers except for [101] where, however, the Fredholm property rather holds in a Breuer–Fredholm sense and the spectral flow is with respect to a semifinite trace so that it determines the density of states.

Let us begin by describing the Hamiltonian. It acts on the Hilbert space $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ over a d -dimensional lattice with N internal degrees of freedom over every site and is of the next-neighbor form

$$(H\psi)_m = \sum_{|m-k|=1} a_{m,k} \psi_k + v_m \psi_m, \quad (9.18)$$

where $\psi = (\psi_m)_{m \in \mathbb{Z}^d}$ with $\psi_m \in \mathbb{C}^N$, the sum runs over all sites neighboring n (the distance $|n - m|$ is meant in the maximum norm on $\mathbb{Z}^d$), and $a_{m,k} = a_{k,m}^*$ and v_m are $N \times N$ matrices that are invertible and self-adjoint. As in Section 2.6, we will suppose to be in a scattering situation where the coefficient matrices $a_{m,k}$ and v_m are all equal to a and v except for a finite number of sites. Let $L > 0$ be such that all these sites lie in a strip $\mathbb{Z}^{d-1} \times \{1, \dots, L\}$. Hence H is a finite-rank perturbation of a periodic Hamiltonian

$$(H_{\mathrm{per}}\psi)_m = \sum_{|m-k|=1} a \psi_k + v \psi_m.$$

By discrete Fourier transform, one can show that H_{per} has purely absolutely continuous spectrum $\text{spec}(H_{\text{per}}) = \text{spec}_{\text{ac}}(H_{\text{per}})$ consisting of at most N intervals. This section is about computing discrete eigenvalues of H not lying in $\text{spec}(H_{\text{per}})$, thus so-called bound states, by a formula similar as in Theorem 2.6.5. The dimension is throughout assumed to satisfy $d \geq 2$.

For this purpose, the Hamiltonian is rewritten as a (two-sided) infinite block Jacobi operator. The fiber Hilbert space will be $\mathcal{H} = \ell^2(\mathbb{Z}^{d-1}, \mathbb{C}^N)$. Then $\ell^2(\mathbb{Z}^d, \mathbb{C}^N) \cong \ell^2(\mathbb{Z}, \mathcal{H})$. Under this identification, the Hamiltonian (9.18) can be rewritten as

$$(H\psi)_n = A_{n+1}\psi_{n+1} + A_n\psi_{n-1} + V_n\psi_n,$$

where now $n \in \mathbb{Z}$ and $(A_n)_{n \in \mathbb{Z}}, (V_n)_{n \in \mathbb{Z}}$ are both sequences of invertible and self-adjoint operators on $\mathcal{H}$, respectively. We do not write out explicit formulas for A_n and V_n in terms of the $a_{m,k}$ and v_m , but stress that the coefficient operators are such that

$$A_n = A, \quad V_n = V, \quad n \notin \{1, \dots, L\}, \quad (9.19)$$

just as in Section 2.6. The Schrödinger equation $H\psi^E = E\psi^E$ will be considered for all sequences $\psi^E = (\psi_n^E)_{n \in \mathbb{Z}}$ of vectors $\psi_n^E \in \mathcal{H}$, and not only square-integrable states from $\ell^2(\mathbb{Z}, \mathcal{H})$. Explicitly written out, it becomes

$$A_{n+1}\psi_{n+1}^E + V_n\psi_n^E + A_n\psi_{n-1}^E = E\psi_n^E. \quad (9.20)$$

Regrouping two neighboring vectors into

$$\Psi_n^E = \begin{pmatrix} A_{n+1}\psi_{n+1}^E \\ \psi_n^E \end{pmatrix},$$

one can then rewrite (9.20) as

$$\Psi_n^E = M_n^E \Psi_{n-1}^E, \quad (9.21)$$

where the I -unitary transfer matrices M_n^E on the Krein space $(\mathcal{K}, I) = (\mathcal{H} \oplus \mathcal{H}, I)$ are defined by

$$M_n^E = \begin{pmatrix} (E\mathbf{1} - V_n)A_n^{-1} & -A_n \\ A_n^{-1} & \mathbf{0} \end{pmatrix}. \quad (9.22)$$

Let us stress that (9.21) looks exactly as the corresponding equation (2.50) in the setting with finite-dimensional fibers. Indeed, all structural algebraic facts transpose directly. In particular, we will use (9.21) also as an equation for frames $\Psi_n^E : \mathcal{H} \rightarrow \mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. If one of the Ψ_n^E spans an I -Lagrangian subspace, then all others do as well because all M_n^E are I -unitary. Let us note that due to (9.19) the M_n^E are for all $n \notin \{1, \dots, N\}$ equal to one fixed I -unitary

$$M^E = \begin{pmatrix} (E\mathbf{1} - V)A^{-1} & -A \\ A^{-1} & \mathbf{0} \end{pmatrix}.$$

The matrix entries of this transfer matrix specify H_{per} , and therefore M^E is also closely linked to the spectral properties of H_{per} . The following result extends Propositions 2.6.1 and 2.6.2.

Proposition 9.7.1. *The following statements hold:*

- (i) $E \in \sigma(H_{\text{per}}) \iff \sigma(M^E) \cap \mathbb{S}^1 \neq \emptyset$.
- (ii) For real $E \notin \sigma(H_{\text{per}})$, the subspaces $\mathcal{E}^{E,<}$ and $\mathcal{E}^{E,>}$ given by the range of the Riesz projection of M^E on $\text{spec}(M^E) \cap B_1(0)$ and $\text{spec}(M^E) \setminus B_1(0)$, respectively are I -Lagrangian.
- (iii) For real $E \notin \sigma(H_{\text{per}})$, the subspaces $\mathcal{E}^{E,<}$ and $(\mathcal{E}^{E,>})^\perp$ form a Fredholm pair of I -Lagrangian subspaces.

Proof. The first claim follows by a Weyl sequence argument, just as in the proof of Proposition 2.6.1. The second and third claims follow from Proposition 9.4.6, after a Cayley transform. $\square$

As in Section 2.6 now follows the analysis of the energy dependence of the unitaries

$$W^{E,<} = \Pi(\mathcal{C}\mathcal{E}^{E,<}), \quad W^{E,>} = \Pi(\mathcal{C}\mathcal{E}^{E,>}),$$

using the half-space restrictions of H_{per} . Let H_{per}^+ and H_{per}^- be the (Dirichlet) restrictions of H_{per} to the subspaces $\ell^2(\mathbb{N}, \mathcal{H})$ and $\ell^2(\mathbb{N}^-, \mathcal{H})$, respectively, where $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}^- = \{\dots, -1, 0\}$. In the situation of Section 2.6, the fiber Hilbert space $\mathcal{H}$ is finite dimensional, and this implies that the new spectrum $\text{spec}(H_{\text{per}}^\pm) \setminus \text{spec}(H_{\text{per}})$ only consists of a finite number of eigenvalues (bound states) of finite multiplicity. In the present situation, it is possible that $\text{spec}(H_{\text{per}}^\pm)$ acquires new essential spectrum resulting from surface states along the boundary. This spectrum is typically topologically protected. It can be studied via K -theoretic methods [152] or via transfer matrix methods along the boundary [17, 174]. We believe that the computation of the density of states of this boundary spectrum is possible by adapting Corollary 2.6.4 to a semifinite setting (either by using the Fourier decomposition along the boundary or, more generally, by transposing the techniques from [101]), but this is not carried out here. Irrespective of this, one can prove the following analogue of Proposition 2.6.3.

Proposition 9.7.2. *One has, for $E \in \mathbb{R} \setminus \text{spec}(H_{\text{per}})$,*

$$\frac{1}{i}(W^{E,<})^* \partial_E W^{E,<} < 0, \quad \frac{1}{i}(W^{E,>})^* \partial_E W^{E,>} > 0.$$

Proof. The whole setup is translation invariant with respect to shifts along the boundary. Hence it is possible to carry out a $(d-1)$ -dimensional discrete Fourier decomposition of all objects involved. In particular,

$$H_{\text{per}}^{\pm} \cong \int_{\mathbb{T}^{d-1}}^{\oplus} dk H_{\text{per}}^{\pm}(k),$$

where $k \in \mathbb{T}^{d-1} \mapsto H_{\text{per}}^{\pm}(k)$ is a real-analytic family of half-space block Jacobi matrices with a finite-dimensional fiber. Furthermore, also the transfer operators admit such a Fourier decomposition

$$M^E \cong \int_{\mathbb{T}^{d-1}}^{\oplus} dk M^E(k),$$

with finite-dimensional J -unitaries depending real analytically on $k \in \mathbb{T}^{d-1}$. Thus also $\mathcal{E}^{E,<}$ and $\mathcal{E}^{E,>}$, as well as $W^{E,<}$ and $W^{E,>}$, can be decomposed. For each $k \in \mathbb{T}^{d-1}$, one can now apply Proposition 2.6.3, and integrating over $\mathbb{T}^{d-1}$ concludes the proof. $\square$

To continue the analysis of the scattering Hamiltonian H , let us now set

$$m^E = \text{multiplicity of } E \text{ as eigenvalue of } H.$$

Each eigenstate $\psi^E \in \ell^2(\mathbb{Z}^d, \mathbb{C}^N) \cong \ell^2(\mathbb{Z}, \mathcal{H})$ decays both at $-\infty$ and $+\infty$. To construct such an eigenstate, one can again proceed as in Section 2.6. Outside of $[1, L] \cap \mathbb{Z}$, the decaying solution satisfies (9.21) with $M_n^E = M^E$. Hence neighboring sites must produce vectors lying in $\mathcal{E}^{E,>}$ on $(-\infty, 0] \cap \mathbb{Z}$ and lying in $\mathcal{E}^{E,<}$ on $[L+1, \infty) \cap \mathbb{Z}$. Matching of the solutions thus shows

$$m^E = \dim(M^E(L, 1)\mathcal{E}^{E,>} \cap \mathcal{E}^{E,<}), \quad (9.23)$$

where $M^E(L, 1) = M_L^E \cdots M_1^E$.

Proposition 9.7.3. *For $E \in \mathbb{R} \setminus \text{spec}(H_{\text{per}})$, the multiplicity m^E is finite and given by*

$$m^E = \dim(\text{Ker}(\Pi(\mathcal{C}\mathcal{E}^{E,<})^* \Pi(\mathcal{C}M^E(L, 1)\mathcal{E}^{E,>} - \mathbf{1})). \quad (9.24)$$

Proof. By Proposition 9.7.1, the right-hand side of (9.23) is an intersection between the two I -Lagrangian subspaces $M^E(L, 1)\mathcal{E}^{E,>}$ and $\mathcal{E}^{E,<}$. This intersection can thus be computed by (9.24) due to Proposition 9.3.5. It remains to show that this intersection is finite. For that purpose, let us first note that by Proposition 9.7.1 one has $\mathcal{E}^{E,<} \cap \mathcal{E}^{E,>} = \{0\}$, and therefore 1 is not in the spectrum of $\Pi(\mathcal{C}\mathcal{E}^{E,<})^* \Pi(\mathcal{C}\mathcal{E}^{E,>}) = (W^{E,<})^* W^{E,>}$, again by Proposition 9.3.5. Furthermore, let us note that $\mathcal{E}^{E,>}$ is M^E -invariant by construction. Therefore $(M^E)^L \mathcal{E}^{E,>} = \mathcal{E}^{E,>}$. Due to the assumption (9.19), $M_n^E - M^E$ is of finite rank and therefore $M_n^E (M^E)^{-1} = \mathbf{1} + F_n$ where F_n is of finite rank (and such that $\mathbf{1} + F_n$ is J -unitary). Iterating one concludes that $M^E(L, 1)(M^E)^{-L} = \mathbf{1} + F$ where F is of finite rank. Finally, by Proposition 9.3.8,

$$\begin{aligned}
 \Pi(\mathcal{C}M^E(L, 1)\mathcal{E}^{E, >}) &= \Pi(\mathcal{C}M^E(L, 1)(M^E)^{-L}\mathcal{E}^{E, >}) \\
 &= (\mathcal{C}(\mathbf{1} + F)\mathcal{C}^*) \cdot W^{E, >} \\
 &= (\mathbf{1} + \mathcal{C}F\mathcal{C}^*) \cdot W^{E, >} \\
 &= W^{E, >} + K,
 \end{aligned}$$

where K is some compact operator (such that $W^{E, >} + K$ is unitary). In conclusion, $\Pi(\mathcal{C}\mathcal{E}^{E, <})^*\Pi(\mathcal{C}M^E(L, 1)\mathcal{E}^{E, >})$ is a compact perturbation of $(W^{E, <})^*W^{E, >}$ and therefore has no essential spectrum in a neighborhood of 1. $\square$

Note that the above proof combined with Theorem 9.4.2 also shows that the subspaces $M^E(L, 1)\mathcal{E}^{E, >}$ and $(\mathcal{E}^{E, <})^\perp$ form a Fredholm pair of I -Lagrangians. As in Section 2.6, let us now set

$$\begin{aligned}
 U^E &= -\Pi(\mathcal{C}\mathcal{E}^{E, <})^*\Pi(\mathcal{C}M^E(L, 1)\mathcal{E}^{E, >}) \\
 &= -(W^{E, <})^*(\mathcal{C}M^E(L, 1)\mathcal{C}^*) \cdot W^{E, >}.
 \end{aligned}$$

Theorem 9.7.4. *One has*

$$\frac{1}{i}(U^E)^*\partial_E U^E > 0. \tag{9.25}$$

Suppose that $[E_0, E_1] \cap \text{spec}(H_{\text{per}}) = \emptyset$ and that E_0 and E_1 are not eigenvalues of H . Then the number of bound states of H in $[E_0, E_1]$ is given by

$$\#\{\text{eigenvalues of } H \text{ in } [E_0, E_1]\} = \text{Sf}(E \in [E_0, E_1] \mapsto U^E \text{ through } -1).$$

Proof. Given the preparations in Propositions 9.7.2 and 9.7.3, the proof is identical to that of Theorem 2.6.5. $\square$