

6 Unbounded Fredholm operators

This chapter offers a detailed introduction to various subsets of the unbounded Fredholm operators, with a particular focus on natural topologies thereon. This is a necessary preparation for the definition of spectral flow of unbounded self-adjoint Fredholm operators given in the next Chapter 7. First, Section 6.1 reviews various topologies on the set of closed operators. Section 6.2 recalls some fundamentals about unbounded Fredholm operators that can be found in numerous books, e. g., [80, 99, 165]. Then, following the works of Boofß-Bavnbek, Lesch, and Phillips [31], as well as Lesch [126], the set of unbounded self-adjoint Fredholm operators and its topology is studied in detail in Section 6.3. Section 6.4 considers the important subclass of self-adjoint Fredholm operators with compact resolvent and proves numerous topological results.

6.1 Topologies on closed and densely defined operators

Let us first recall that an unbounded operator is a linear map $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ where $\mathcal{D}(T)$ is a linear subspace of some Hilbert space $\mathcal{H}$, called the domain of T . It is called closed if its graph $\{(\phi, T\phi) : \phi \in \mathcal{D}(T)\}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}'$. Let us introduce a notation for the set of closed densely defined (also called regular) operators:

$$\mathbb{L}(\mathcal{H}, \mathcal{H}') = \{T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}' \text{ closed and densely defined}\}.$$

In the case $\mathcal{H}' = \mathcal{H}$, we also use the notation $\mathbb{L}(\mathcal{H}) = \mathbb{L}(\mathcal{H}, \mathcal{H})$. For any $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, the adjoint operator $T^* \in \mathbb{L}(\mathcal{H}', \mathcal{H})$ is defined by $\langle T^* \phi | \psi \rangle = \langle \phi | T\psi \rangle$ for $\psi \in \mathcal{D}(T)$ and ϕ in $\mathcal{D}(T^*) = \{\phi \in \mathcal{H}' : \psi \in \mathcal{D}(T) \mapsto \langle \phi | T\psi \rangle \text{ bounded}\}$. Then T is called symmetric if $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $T^*|_{\mathcal{D}(T)} = T$, and furthermore T is called self-adjoint whenever one has $T = T^*$ which includes $\mathcal{D}(T) = \mathcal{D}(T^*)$. As a preparation for the constructions below, some rather standard facts are needed that are included for the convenience of the reader.

Lemma 6.1.1. *Let T be a closed and densely defined operator. Then T^*T is self-adjoint with domain $\mathcal{D}(T^*T) = \{\phi \in \mathcal{H} : \phi \in \mathcal{D}(T), T\phi \in \mathcal{D}(T^*)\}$.*

Proof. (See, e. g., Korollar VII.2.13 in [204].) Clearly, T^*T is well defined and symmetric on $\mathcal{D}(T^*T)$. It remains to show that it is densely defined and self-adjoint. For that purpose, let us equip $\mathcal{D}(T)$ with the scalar product

$$\langle \phi | \psi \rangle_T = \langle T\phi | T\psi \rangle + \langle \phi | \psi \rangle.$$

Because T is closed, $(\mathcal{D}(T), \langle \cdot | \cdot \rangle_T)$ is a Hilbert space which will be denoted by $\hat{\mathcal{H}}$. Let $I \in \mathbb{B}(\hat{\mathcal{H}}, \mathcal{H})$ denote the natural embedding and $I^* \in \mathbb{B}(\mathcal{H}, \hat{\mathcal{H}})$ its adjoint. Then $I I^*$ is self-adjoint and has a trivial kernel because $\langle \phi | I I^* \psi \rangle_T = \langle I\phi | \psi \rangle = \langle \phi | \psi \rangle$ and T is densely defined. Thus $\overline{\text{Ran}(I I^*)} = \text{Ker}(I I^*)^\perp = \mathcal{H}$ and $I I^*$ has dense range. It will next be shown

that $\text{Ran}(H^*) \subset \mathcal{D}(T^*T)$. Indeed, let $\phi \in \mathcal{H}$ and $\psi = H^*\phi = I^*\phi \in \text{Ran}(H^*)$ so that $\psi \in \hat{\mathcal{H}}$ and for any $\eta \in \mathcal{D}(T)$,

$$\begin{aligned} \langle T\eta|T\psi \rangle &= \langle \eta|\psi \rangle_T - \langle \eta|\psi \rangle \\ &= \langle \eta|I^*\phi \rangle_T - \langle \eta|\psi \rangle \\ &= \langle I\eta|\phi \rangle - \langle \eta|\psi \rangle \\ &= \langle \eta|\phi - \psi \rangle. \end{aligned}$$

Hence $\eta \in \mathcal{D}(T) \mapsto \langle T\eta|T\psi \rangle$ is continuous and thus $T\psi \in \mathcal{D}(T^*)$, so that $\psi \in \mathcal{D}(T^*T)$. It follows that $\mathcal{D}(T^*T)$ is dense. As T^*T is bounded from below and symmetric, it has a self-adjoint extension with domain $\text{Ran}(H^*)$, given by the Friedrich extension (this is the only nonnegative self-adjoint extension). Hence one must have $\text{Ran}(H^*) = \mathcal{D}(T^*T)$, and T^*T is self-adjoint. $\square$

Lemma 6.1.2. *For any regular operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, the domain $\mathcal{D}(T^*T)$ is a core for T , namely T is the closure of $T|_{\mathcal{D}(T^*T)}$ which in turn is also given by the double adjoint of $T|_{\mathcal{D}(T^*T)}$. Moreover, $T(\mathbf{1} + T^*T)^{-1}$ and $T(\mathbf{1} + T^*T)^{-\frac{1}{2}}$ are bounded operators, both with norm bounded by 1.*

Proof. (See, e.g., Lemma 9.2 in [121].) Let us first show that

$$\mathbf{1} + T^*T : \mathcal{D}(\mathbf{1} + T^*T) = \mathcal{D}(T^*T) \rightarrow \mathcal{H}$$

is a bijection. For $\phi \in \mathcal{D}(T^*T)$, one has

$$\langle \phi | (\mathbf{1} + T^*T)\phi \rangle = \langle \phi | \phi \rangle + \langle T\phi | T\phi \rangle \geq \langle \phi | \phi \rangle,$$

and therefore $\|(\mathbf{1} + T^*T)\phi\| \geq \|\phi\|$. This implies that $\mathbf{1} + T^*T$ is injective. Furthermore, if $(\phi_n)_{n \geq 1}$ is a Cauchy sequence in $\text{Ran}(\mathbf{1} + T^*T)$ and $\phi_n = (\mathbf{1} + T^*T)\psi_n$, then also $(\psi_n)_{n \geq 1}$ is a Cauchy sequence converging to ψ , and then the closedness of $\mathbf{1} + T^*T$ implies that $\psi \in \mathcal{D}(\mathbf{1} + T^*T)$ and $(\mathbf{1} + T^*T)\psi = \lim \phi_n$. Thus $\text{Ran}(\mathbf{1} + T^*T)$ is closed and therefore equal to $\mathcal{H}$. Moreover, it follows that the inverse $(\mathbf{1} + T^*T)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is bounded with norm $\|(\mathbf{1} + T^*T)^{-1}\| \leq 1$ and its range is $\text{Ran}((\mathbf{1} + T^*T)^{-1}) = \mathcal{D}(T^*T)$. Let us note that, in particular, the range of the operator $(\mathbf{1} + T^*T)^{-1}$ is dense in $\mathcal{H}$. As, clearly, $(\mathbf{1} + T^*T)^{-1} > 0$, its square root $(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$ is well defined and has a dense range. Then for $\phi \in \mathcal{H}$, one has

$$\begin{aligned} \langle T(\mathbf{1} + T^*T)^{-1}\phi | T(\mathbf{1} + T^*T)^{-1}\phi \rangle &= \langle (\mathbf{1} + T^*T)^{-1}\phi | T^*T(\mathbf{1} + T^*T)^{-1}\phi \rangle \\ &\leq \langle (\mathbf{1} + T^*T)^{-1}\phi | (\mathbf{1} + T^*T)(\mathbf{1} + T^*T)^{-1}\phi \rangle \\ &= \langle (\mathbf{1} + T^*T)^{-1}\phi | \phi \rangle \\ &= \langle (\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi | (\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi \rangle, \end{aligned}$$

and therefore $\|T(\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi\| \leq \|(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi\|$. This implies that the operator $T(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) \rightarrow \mathcal{H}'$ is bounded with norm bounded by 1 and therefore has an extension from $\text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}})$ to all of $\mathcal{H}$ which is also bounded with norm bounded by 1. Next is shown that $\text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ such that this extension is given by $T(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}'$. Let $\phi \in \mathcal{H}$. As $\text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T^*T)$ is dense, there is a sequence $(\phi_n)_{n \in \mathbb{N}}$ in the range of $((\mathbf{1} + T^*T)^{-\frac{1}{2}})^2$ converging to ϕ . Then, as $(\mathbf{1} + T^*T)^{-\frac{1}{2}}$ is bounded, $\lim_{n \rightarrow \infty} (\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi_n = (\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi$. Because the operator $T(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) \rightarrow \mathcal{H}'$ is bounded, $(T(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and therefore converges to some $\psi = \lim_{n \rightarrow \infty} T(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi_n \in \mathcal{H}'$. As $\lim_{n \rightarrow \infty} ((\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi_n, T(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi_n) = ((\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi, \psi)$ in $\mathcal{H} \times \mathcal{H}'$ and T is closed, $(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi$ is in the domain of T and $T(\mathbf{1} + T^*T)^{-\frac{1}{2}}\phi = \psi$. Conversely, assume $\phi \in \mathcal{D}(T)$. Then as $(\mathbf{1} + T^*T)^{-\frac{1}{2}}T^* \subset (T(\mathbf{1} + T^*T)^{-\frac{1}{2}})^*$ is bounded, one has

$$\begin{aligned} \phi &= ((\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*T + (\mathbf{1} + T^*T)^{-1})\phi \\ &= (\mathbf{1} + T^*T)^{-\frac{1}{2}}((\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*T + (\mathbf{1} + T^*T)^{-\frac{1}{2}})\phi \in \text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}). \end{aligned}$$

This implies $\mathcal{D}(T) = \text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}})$. Thus, for $\phi \in \mathcal{D}(T)$ there is $\psi \in \mathcal{H}$ such that $\phi = (\mathbf{1} + T^*T)^{-\frac{1}{2}}\psi$. As $\text{Ran}((\mathbf{1} + T^*T)^{-1})$ is dense in $\mathcal{H}$, there is a sequence $(\theta_n)_{n \in \mathbb{N}}$ in $\text{Ran}((\mathbf{1} + T^*T)^{-1})$ such that $\lim_{n \rightarrow \infty} (\mathbf{1} + T^*T)^{-\frac{1}{2}}\theta_n = \psi$. Then

$$\lim_{n \rightarrow \infty} (\mathbf{1} + T^*T)^{-1}\theta_n = \phi$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} T(\mathbf{1} + T^*T)^{-1}\theta_n &= \lim_{n \rightarrow \infty} T(\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T)^{-\frac{1}{2}}\theta_n \\ &= T(\mathbf{1} + T^*T)^{-\frac{1}{2}}\psi \\ &= T\phi. \end{aligned}$$

One concludes that $\lim_{n \rightarrow \infty} ((\mathbf{1} + T^*T)^{-1}\theta_n, T(\mathbf{1} + T^*T)^{-1}\theta_n) = (\phi, T\phi)$ and therefore $\mathcal{D}(T^*T)$ is a core for T because $(\mathbf{1} + T^*T)^{-1}\theta_n \in \mathcal{D}(T^*T)$ for all $n \in \mathbb{N}$. $\square$

In this section two topologies on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ are studied, as well as naturally associated topologies on the image of $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ under the bounded transform that will be introduced in (6.3) below. Let us begin with the gap topology. As $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is closed, the orthogonal projection $P_T \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$ onto the graph of T is bounded. Then the *gap metric* on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is defined by

$$d_G(T_0, T_1) = \|P_{T_0} - P_{T_1}\|, \quad T_0, T_1 \in \mathbb{L}(\mathcal{H}, \mathcal{H}'). \quad (6.1)$$

The topology $\mathcal{O}_G$ on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ induced by d_G is called the gap topology. In order to get a better grip on it, let us write out the explicit form of the graph projections.

Proposition 6.1.3. *For $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$, let us set*

$$R_T = (\mathbf{1} + T^*T)^{-1}.$$

Then the orthogonal projection onto the graph of T is

$$P_T = \begin{pmatrix} R_T & T^*R_{T^*} \\ TR_T & \mathbf{1} - R_{T^*} \end{pmatrix}.$$

The gap metric is equivalent to the metric defined by

$$d'_G(T_0, T_1) = \|R_{T_0} - R_{T_1}\| + \|R_{T_0^*} - R_{T_1^*}\| + \|T_0R_{T_0} - T_1R_{T_1}\|.$$

Proof. By Lemma 6.1.2, TR_T is bounded, and thus also $T^*R_{T^*}$ is bounded. Let us first check that

$$R_{T^*}T \subset TR_T, \quad R_TT^* \subset (TR_T)^* = T^*R_{T^*}. \quad (6.2)$$

For the first equality, let $\phi \in \mathcal{D}(T)$. Then $\psi = R_T\phi \in \text{Ran}(R_T) = \mathcal{D}(T^*T) \subset \mathcal{D}(T)$. As then $(\mathbf{1} + T^*T)\psi = \phi$, one has

$$T\phi = T(\mathbf{1} + T^*T)\psi = (\mathbf{1} + TT^*)T\psi = (\mathbf{1} + TT^*)TR_T\phi,$$

and multiplying by $(\mathbf{1} + TT^*)^{-1}$ shows the first inclusion of (6.2). The second inclusion in (6.2) follows from general principles. Indeed, for $\phi \in \mathcal{D}(T^*)$, one concludes that $(TR_T)^*\phi = R_TT^*\phi = T^*R_{T^*}\phi$ where the last equality follows from the first inclusion of (6.2). As $\mathcal{D}(T^*)$ is dense, this implies the last equality in (6.2). Using (6.2), an algebraic computation shows that P_T is an orthogonal projection. Moreover, one readily verifies

$$P_T \begin{pmatrix} \phi \\ T\phi \end{pmatrix} = \begin{pmatrix} (\mathbf{1} + T^*T)^{-1}\phi + T^*(\mathbf{1} + TT^*)^{-1}T\phi \\ T(\mathbf{1} + T^*T)^{-1}\phi + (\mathbf{1} - (\mathbf{1} + TT^*)^{-1})T\phi \end{pmatrix} = \begin{pmatrix} \phi \\ T\phi \end{pmatrix}$$

for all $\phi \in \mathcal{D}(T)$, due to $(\mathbf{1} + TT^*)^{-1}T = R_{T^*}T \subset TR_T = T(\mathbf{1} + T^*T)^{-1}$. Note that the set $\left\{ \begin{pmatrix} T^*\psi \\ -\psi \end{pmatrix} : \psi \in \mathcal{D}(T^*) \right\}$ is the orthogonal complement of the graph of T in $\mathcal{H} \oplus \mathcal{H}'$. One checks that for $\psi \in \mathcal{D}(T^*)$,

$$P_T \begin{pmatrix} T^*\psi \\ -\psi \end{pmatrix} = \begin{pmatrix} (\mathbf{1} + T^*T)^{-1}T^*\psi - T^*(\mathbf{1} + TT^*)^{-1}\psi \\ T(\mathbf{1} + T^*T)^{-1}T^*\psi - (\mathbf{1} - (\mathbf{1} + TT^*)^{-1})\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $(\mathbf{1} + T^*T)^{-1}T^* = R_{T^*}T^* \subset T^*R_{T^*} = T^*(\mathbf{1} + TT^*)^{-1}$ was used. Hence P_T is the orthogonal projection onto the graph of T . Replacing the formula for P_T twice in definition (6.1), one readily deduces the equivalence of d_G and d'_G . $\square$

The key element for the definition of the Riesz topology on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is the bounded transform (sometimes also called Riesz transform due to the work of Riesz and Lorch, on which it is elaborated in the textbook [158]; note though that there is no square root in these works)

$$\mathcal{F}(T) = T(\mathbf{1} + T^*T)^{-\frac{1}{2}} \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \quad (6.3)$$

of $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$. By Lemma 6.1.2, the operator $\mathcal{F}(T)$ is well defined and bounded so that also the map $\mathcal{F} : \mathbb{L}(\mathcal{H}, \mathcal{H}') \rightarrow \mathbb{B}(\mathcal{H}, \mathcal{H}')$ given by (6.3) is well defined. To analyze its mapping properties, let us introduce the ball of bounded operators of radius $a > 0$,

$$\mathbb{B}_a(\mathcal{H}, \mathcal{H}') = \{F \in \mathbb{B}(\mathcal{H}, \mathcal{H}') : \|F\| \leq a\},$$

as well as the following subset of the unit ball:

$$\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}') = \{F \in \mathbb{B}(\mathcal{H}, \mathcal{H}') : \|F\| \leq 1, \text{Ker}(\mathbf{1} - F^*F) = \{0\}\}.$$

This notation fits with that of Section 4.6, namely the lower index 1 indicates that the norm is bounded by 1 and the upper index 0 denotes that 1 is not a singular value of F .

Theorem 6.1.4. *The bounded transform establishes a bijection*

$$\mathcal{F} : \mathbb{L}(\mathcal{H}, \mathcal{H}') \rightarrow \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}').$$

Moreover, $\mathcal{F}(T)^* = \mathcal{F}(T^*)$.

Proof. (See, e.g., Theorem 10.4 in [121].) In the proof of Lemma 6.1.2, it was shown that $(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$ is well defined and bounded with norm bounded by 1. Moreover, $\text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ and $\mathcal{F}(T) : \mathcal{H} \rightarrow \mathcal{H}$ is well defined and bounded with norm $\|\mathcal{F}(T)\| \leq 1$, see the proof of Lemma 6.1.2.

Clearly,

$$(\mathbf{1} + T^*T)^{-\frac{1}{2}} T^* \subset \mathcal{F}(T)^*, \quad (6.4)$$

and therefore one has for $\phi \in \mathcal{H}$,

$$\begin{aligned} \mathcal{F}(T)^* \mathcal{F}(T) (\mathbf{1} + T^*T)^{-\frac{1}{2}} \phi &= (\mathbf{1} + T^*T)^{-\frac{1}{2}} T^* T (\mathbf{1} + T^*T)^{-1} \phi \\ &= (\mathbf{1} + T^*T)^{-\frac{1}{2}} (\mathbf{1} + T^*T - \mathbf{1}) (\mathbf{1} + T^*T)^{-1} \phi \\ &= (\mathbf{1} - (\mathbf{1} + T^*T)^{-1}) (\mathbf{1} + T^*T)^{-\frac{1}{2}} \phi. \end{aligned}$$

As $\text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \mathcal{D}(T)$ is dense in $\mathcal{H}$, this implies

$$\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T) = (\mathbf{1} + T^*T)^{-1}. \quad (6.5)$$

Thus as $\text{Ker}((\mathbf{1} + T^*T)^{-\frac{1}{2}}) = \text{Ker}((\mathbf{1} + T^*T)^{-1}) = \{0\}$, the kernel of $\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T)$ is trivial. This shows that the map $\mathcal{F}$ is well defined.

Let us next show that the map $\mathcal{F}$ is surjective. Let $F \in \mathbb{B}(\mathcal{H}, \mathcal{H}')$ be such that $\|F\| \leq 1$ and $\text{Ker}(\mathbf{1} - F^*F) = \{0\}$. As $\text{Ker}(\mathbf{1} - F^*F)$ is trivial, it follows that $\mathcal{D} = \text{Ran}((\mathbf{1} - F^*F)^{\frac{1}{2}}) \subset \mathcal{H}$ is dense. Then an unbounded operator $T : \mathcal{D} \rightarrow \mathcal{H}'$ is defined by $T(\mathbf{1} - F^*F)^{\frac{1}{2}}\phi = F\phi$ for $\phi \in \mathcal{H}$. As $\mathbf{1} - F^*F$ is injective, this is well defined and

$$T = F(\mathbf{1} - F^*F)^{-\frac{1}{2}}. \quad (6.6)$$

Clearly, T is densely defined and it remains to show that it is closed and $\mathcal{F}(T) = F$. We next show that the kernel of $\mathbf{1} - FF^*$ is trivial. Suppose to the contrary, namely that there is $\phi \in \mathcal{H}'$ with $\|\phi\| = 1$ such that $FF^*\phi = \phi$. This implies that $FF^*FF^*\phi = \phi$ and therefore

$$1 = \langle \phi | FF^*FF^*\phi \rangle = \langle F^*\phi | F^*F(F^*\phi) \rangle.$$

As $\|F^*\phi\| \leq 1$, this implies by the Cauchy–Schwarz inequality that $\|F^*\phi\| = 1$ and that $F^*F(F^*\phi) = F^*\phi$, which is a contradiction to $\text{Ker}(\mathbf{1} - F^*F) = \{0\}$. Thus $\text{Ker}(\mathbf{1} - FF^*) = \{0\}$ and the range $\mathcal{D}^* = \text{Ran}((\mathbf{1} - FF^*)^{\frac{1}{2}}) \subset \mathcal{H}'$ is dense. Then $S : \mathcal{D}^* \rightarrow \mathcal{H}$, defined by $S(\mathbf{1} - FF^*)^{\frac{1}{2}}\phi = F^*\phi$ for $\phi \in \mathcal{H}'$, is well defined and $S = F^*(\mathbf{1} - FF^*)^{-\frac{1}{2}}$. Next setting $\phi = (\mathbf{1} - F^*F)^{\frac{1}{2}}\phi' \in \mathcal{D}$ and $\psi = (\mathbf{1} - FF^*)^{\frac{1}{2}}\psi' \in \mathcal{D}^*$, one has

$$\begin{aligned} \langle T\phi | \psi \rangle &= \langle F\phi' | (\mathbf{1} - FF^*)^{\frac{1}{2}}\psi' \rangle \\ &= \langle (\mathbf{1} - FF^*)^{\frac{1}{2}}F\phi' | \psi' \rangle \\ &= \langle F(\mathbf{1} - F^*F)^{\frac{1}{2}}\phi' | \psi' \rangle \\ &= \langle (\mathbf{1} - F^*F)^{\frac{1}{2}}\phi' | F^*\psi' \rangle \\ &= \langle \phi | S\psi \rangle. \end{aligned}$$

This implies $S \subset T^*$ and, in particular, T^* is densely defined (and T is closable). One directly checks that

$$P = \begin{pmatrix} \mathbf{1} - F^*F & (\mathbf{1} - F^*F)^{\frac{1}{2}}F^* \\ F(\mathbf{1} - F^*F)^{\frac{1}{2}} & FF^* \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$$

is an orthogonal projection. An explicit computation shows that the graph of T is $\text{Ran}(P)$ and therefore T is closed. Moreover, $\{(-S\psi, \psi) : \psi \in \mathcal{D}^*\} = \text{Ker}(P)$ and, because one has $\{(\phi, T\phi) : \phi \in \mathcal{D}\}^\perp = \{(-T^*\psi, \psi) : \psi \in \mathcal{D}(T^*)\}$, this implies $\mathcal{D}^* = \mathcal{D}(T^*)$ and $S = T^*$. Next let us verify that $F = \mathcal{F}(T)$. By Lemma 6.1.2, $\mathcal{D}(T) \subset \text{Ran}((\mathbf{1} + T^*T)^{-\frac{1}{2}})$ and therefore

$$\mathcal{F}(T)(\mathbf{1} - \mathcal{F}(T)^*\mathcal{F}(T))^{-\frac{1}{2}} = T(\mathbf{1} + T^*T)^{-\frac{1}{2}}(\mathbf{1} + T^*T)^{\frac{1}{2}} = T.$$

This implies that T is given by (6.6) for $F = \mathcal{F}(T)$ and the map $\mathcal{F}$ is injective. Let T be as in (6.6), then $T^* = F^*(\mathbf{1} - FF^*)^{-\frac{1}{2}}$ and

$$\begin{aligned}\mathcal{F}(T) &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}(\mathbf{1} + F^*(\mathbf{1} - FF^*)^{-\frac{1}{2}}F(\mathbf{1} - F^*F)^{-\frac{1}{2}})^{-\frac{1}{2}} \\ &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}(\mathbf{1} + F^*F(\mathbf{1} - F^*F)^{-1})^{-\frac{1}{2}} \\ &= F(\mathbf{1} - F^*F)^{-\frac{1}{2}}((\mathbf{1} - F^*F + F^*F)(\mathbf{1} - F^*F)^{-1})^{-\frac{1}{2}} \\ &= F.\end{aligned}$$

Therefore $\mathcal{F}$ is bijective and $\mathcal{F}(T)^* = \mathcal{F}(T^*)$. □

The so-called *Riesz metric* on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is defined by

$$d_R(T_0, T_1) = \|\mathcal{F}(T_0) - \mathcal{F}(T_1)\|, \quad T_0, T_1 \in \mathbb{L}(\mathcal{H}, \mathcal{H}').$$

Using Theorem 6.1.4, one checks the nondegeneracy assumption for d_R . The triangle inequality and symmetry are obvious. The topology $\mathcal{O}_R$ on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ induced by the Riesz metric is also called the Riesz topology. Henceforth we use both notations $(\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_R)$ and $(\mathbb{L}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R)$ depending on whether we want to stress the metric structure when discussing the continuity of maps on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. Similarly, we will proceed with other spaces below.

As d_R is naturally associated to the bounded transform, the following holds:

Proposition 6.1.5. *The bounded transform*

$$\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_R) \rightarrow (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_N)$$

is a homeomorphism. As above, $d_N(T_0, T_1) = \|T_0 - T_1\|$ is here the norm distance.

Proof. By Theorem 6.1.4, $\mathcal{F} : \mathbb{L}(\mathcal{H}, \mathcal{H}') \rightarrow \mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$ is bijective and, by the very definition of the Riesz metric, it is a homeomorphism. □

Proposition 6.1.6. *An operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is bounded if and only if its bounded transform has norm less than 1, namely $\|\mathcal{F}(T)\| < 1$.*

Proof. Let us first suppose that $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is bounded. Then it is sufficient to show that $\|\mathcal{F}(T)^*\mathcal{F}(T)\| = \|\mathcal{F}(T)\|^2 < 1$. As

$$\mathcal{F}(T)^*\mathcal{F}(T) = (\mathbf{1} + T^*T)^{-\frac{1}{2}}T^*T(\mathbf{1} + T^*T)^{-\frac{1}{2}} = T^*T(\mathbf{1} + T^*T)^{-1},$$

by the spectral radius theorem one has

$$\begin{aligned}\|\mathcal{F}(T)^*\mathcal{F}(T)\| &= \sup(\text{spec}(\mathcal{F}(T)^*\mathcal{F}(T))) \\ &= \sup\{\lambda(1 + \lambda)^{-1} : \lambda \in \text{spec}(T^*T)\} < 1,\end{aligned}$$

where the spectral mapping theorem was used. Conversely, assume that $\|\mathcal{F}(T)\| < 1$, then $\|\mathcal{F}(T)^* \mathcal{F}(T)\| < 1$ and therefore $\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T)$ is invertible with bounded inverse. This implies that

$$T = \mathcal{F}(T)(\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T))^{-\frac{1}{2}}$$

is bounded. □

Next let us introduce a pseudometric on the unit ball $\mathbb{B}_1(\mathcal{H}, \mathcal{H}')$ by setting

$$\begin{aligned} d_E(F_0, F_1) \\ = \max \left\{ \|F_0^* F_0 - F_1^* F_1\|, \|F_0 F_0^* - F_1 F_1^*\|, \|F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| \right\}. \end{aligned}$$

Clearly, d_E satisfies the triangle inequality and is symmetric. Note that this is an extension of the pseudometric introduced in Lemma 4.6.3 to operators which are not self-adjoint any more. As discussed after Lemma 4.6.3, it goes back to [108] and is called the extended gap metric, and the topology is then called the extended gap topology. The next result justifies this terminology, namely the extended gap metric is just the push-forward of the gap metric under the bounded transform.

Proposition 6.1.7. *The bounded transform*

$$\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_G) \rightarrow (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_E)$$

is a bi-Lipshitz-continuous homeomorphism. In particular, d_E defines a metric on $\mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$.

Proof. In the proof of Theorem 6.1.4, it is shown that for $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$,

$$P_T = \begin{pmatrix} \mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T) & (\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T))^{\frac{1}{2}} \mathcal{F}(T)^* \\ \mathcal{F}(T)(\mathbf{1} - \mathcal{F}(T)^* \mathcal{F}(T))^{\frac{1}{2}} & \mathcal{F}(T) \mathcal{F}(T)^* \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$$

is the projection onto the graph of T . Comparing this to the definition of d_E leads to

$$d_E(\mathcal{F}(T_0), \mathcal{F}(T_1)) \leq d_G(T_0, T_1) \leq \sqrt{2} d_E(\mathcal{F}(T_0), \mathcal{F}(T_1)).$$

This implies all statements. □

The next result extends the applicability of Lemma 4.6.3.

Lemma 6.1.8. *The extended gap topology on $\mathbb{B}_1(\mathcal{H}, \mathcal{H}')$ is weaker than the norm topology. More precisely,*

$$d_E(F_0, F_1) \leq 2\sqrt{2} d_N(F_0, F_1)^{\frac{1}{2}}, \quad F_0, F_1 \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}'). \quad (6.7)$$

Moreover, for $a < 1$,

$$d_N(F_0, F_1) \leq \frac{1 + \sqrt{2}}{1 - a^2} d_E(F_0, F_1)^{\frac{1}{2}}, \quad F_0, F_1 \in \mathbb{B}_a(\mathcal{H}, \mathcal{H}').$$

Proof. For the third term in d_E , let us start with

$$\begin{aligned} & \|F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| \\ & \leq \|F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_0(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| + \|F_0(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| \\ & \leq \|(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - (\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| + \|F_0 - F_1\|. \end{aligned}$$

For the first summand, recall the fact (Proposition A.2.2) that for two nonnegative operators $A \geq 0, B \geq 0$ and $\alpha \in (0, 1)$, one has $\|A^\alpha - B^\alpha\| \leq \|A - B\|^\alpha$. Hence

$$\|F_0(\mathbf{1} - F_0^* F_0)^{\frac{1}{2}} - F_1(\mathbf{1} - F_1^* F_1)^{\frac{1}{2}}\| \leq \|F_0^* F_0 - F_1^* F_1\|^{\frac{1}{2}} + \|F_0 - F_1\|.$$

Now

$$\|F_0^* F_0 - F_1^* F_1\| \leq \|(F_0 - F_1)^* F_0\| + \|F_1^* (F_0 - F_1)\| \leq 2\|F_0 - F_1\|,$$

and similarly

$$\|F_0 F_0^* - F_1 F_1^*\| \leq 2\|F_0 - F_1\|.$$

Therefore

$$d_E(F_0, F_1) \leq \sqrt{2}\|F_0 - F_1\|^{\frac{1}{2}} + \|F_0 - F_1\|,$$

so that

$$d_E(F_0, F_1) \leq 2\sqrt{2}d_N(F_0, F_1)^{\frac{1}{2}},$$

because $d_N(F_0, F_1) = \|F_0 - F_1\| \leq 2$ for $F_0, F_1 \in \mathbb{B}_1(\mathcal{H})$. The proof of the other bound (6.7) is as in Lemma 4.6.3, upon replacing H^2 by $F^* F$. $\square$

Next comes an extension of a result of Nicolaescu [139] showing that the gap topology is weaker than the Riesz topology.

Proposition 6.1.9. *The gap topology on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ is strictly weaker than the Riesz topology.*

Proof. The fact that the gap topology is weaker than the Riesz topology on $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ directly follows from the first part of Lemma 6.1.8 combined with Propositions 6.1.5 and 6.1.7.

To show that the Riesz topology is different from the gap topology, we choose an orthonormal basis $\{\phi_k : k \in \mathbb{N}\}$ of $\mathcal{H}$ and define the linear operator

$$H : \mathcal{D}(H) \rightarrow \mathcal{H}, \quad \sum_{k \in \mathbb{N}} a_k \phi_k \mapsto \sum_{k \in \mathbb{N}} k a_k \phi_k$$

with domain $\mathcal{D}(H) = \{\sum_{k \in \mathbb{N}} a_k \phi_k : \sum_{k \in \mathbb{N}} k^2 |a_k|^2 < \infty\}$. Clearly, H is self-adjoint and therefore in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. For $n \in \mathbb{N}$, let us define

$$H_n : \mathcal{D}(H_n) \rightarrow \mathcal{H}, \quad \sum_{k \in \mathbb{N}} a_k \phi_k \mapsto \sum_{k \in \mathbb{N}} k a_k \phi_k - 2n a_n \phi_n$$

with domain $\mathcal{D}(H_n) = \mathcal{D}(H)$. Then H_n is self-adjoint and therefore in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$. As $H_n^2 = H^2$ for all n and thus $R_{H_n} = R_H$ and

$$\lim_{n \rightarrow \infty} \|H_n R_{H_n} - H R_H\| = \lim_{n \rightarrow \infty} \|H_n R_{H_n} \phi_n - H R_H \phi_n\| = \lim_{n \rightarrow \infty} \|2n(1 + n^2)^{-1}\| = 0,$$

the sequence $(H_n)_{n \in \mathbb{N}}$ converges to H with respect to the gap topology. For the Riesz topology, one has

$$\lim_{n \rightarrow \infty} \|\mathcal{F}(H_n) \phi_n - \mathcal{F}(H) \phi_n\| = \lim_{n \rightarrow \infty} \left\| -\frac{n}{\sqrt{1 + n^2}} \phi_n - \frac{n}{\sqrt{1 + n^2}} \phi_n \right\| = \lim_{n \rightarrow \infty} \frac{2n}{\sqrt{1 + n^2}} = 2.$$

Therefore $(H_n)_{n \in \mathbb{N}}$ does not converge to H with respect to the Riesz topology and the gap topology is strictly weaker than the Riesz topology. $\square$

Proposition 6.1.9 directly implies that the bounded transform $\mathcal{F}$ is *not* continuous as a map $\mathcal{F} : (\mathbb{L}(\mathcal{H}, \mathcal{H}'), d_G) \rightarrow (\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_N)$. In other words, there are not enough open sets in the gap topology to assure continuity of $\mathcal{F}$ in this sense.

The following is due to Cordes and Labrousse, see the addendum to [66]. However, the proof presented here is considerably simpler.

Theorem 6.1.10. *On the space of bounded operators $\mathbb{B}(\mathcal{H}, \mathcal{H}')$, the topologies induced by d_G and d_R coincide with the norm topology. Moreover, with respect to both the gap and Riesz topologies, $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open and dense in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$.*

Proof. Let us introduce the set

$$\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}') = \{F \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}') : \|F\| < 1\}.$$

Then $\mathcal{F}(\mathbb{B}(\mathcal{H}, \mathcal{H}')) = \mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$ by Proposition 6.1.6 and, furthermore, by the definition of the bounded transform,

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_N) \rightarrow (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_N)$$

is a homeomorphism. On the other hand, the two maps,

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_G) \rightarrow (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_E)$$

and

$$\mathcal{F} : (\mathbb{B}(\mathcal{H}, \mathcal{H}'), d_R) \rightarrow (\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}'), d_N),$$

are also homeomorphisms by Propositions 6.1.7 and 6.1.5, respectively.

But Lemma 6.1.8 implies that the metrics d_E and d_N induce the same topologies on $\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$, showing the first claim.

By Proposition 6.1.6, the image of $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ under the bounded transform is dense and open in $\mathcal{F}(\mathbb{L}(\mathcal{H}, \mathcal{H}'))$ with respect to the norm topology. By Proposition 6.1.5, this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is dense and open in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the Riesz topology. As the gap topology is weaker than the Riesz topology by Proposition 6.1.9, this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is also dense in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the gap topology. Furthermore, $\mathbb{B}_{<1}(\mathcal{H}, \mathcal{H}')$ is open in $(\mathbb{B}_1^0(\mathcal{H}, \mathcal{H}'), d_E)$. Combined with Proposition 6.1.7 this implies that $\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is open in $\mathbb{L}(\mathcal{H}, \mathcal{H}')$ with respect to the gap topology. $\square$

6.2 Basic properties of unbounded Fredholm operators

This section introduces unbounded Fredholm operators. As for bounded Fredholm operators, we recall several basic facts about them which can also be found in the literature, e. g., [99, 165]. Most of the results presented here are similar to the properties of bounded Fredholm operators studied in Section 3.2. However, as several modifications are necessary, the proofs are provided with full details, even though this leads to some repetitions.

Let us first recall that the quotient $\mathcal{H}/\mathcal{E}$ of $\mathcal{H}$ with respect to a subspace $\mathcal{E} \subset \mathcal{H}$ is the set of equivalence classes of the relation $\phi \sim \psi \iff \phi - \psi \in \mathcal{E}$.

Definition 6.2.1. A linear operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is a Fredholm operator if and only if

- (i) T is regular,
- (ii) $\dim(\text{Ker}(T)) < \infty$,
- (iii) $\dim(\mathcal{H}'/\text{Ran}(T)) < \infty$.

The set of Fredholm operators is denoted by $\mathbb{F}(\mathcal{H}, \mathcal{H}')$ and simply by $\mathbb{F}(\mathcal{H}) = \mathbb{F}(\mathcal{H}, \mathcal{H})$ whenever $\mathcal{H}' = \mathcal{H}$.

For a closed operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$, the linear space $\mathcal{D}(T)$ equipped with the T -norm $\|\phi\|_T = (\|\phi\|_{\mathcal{H}}^2 + \|T\phi\|_{\mathcal{H}'}^2)^{\frac{1}{2}}$ is a Hilbert space. Associated with T there is a bounded operator $\tilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}'$ defined by $\tilde{T}\phi = T\phi$.

Proposition 6.2.2. A closed operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is Fredholm if and only if the associated bounded operator $\tilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}'$ is Fredholm.

Proof. As $\text{Ker}(T) = \text{Ker}(\tilde{T})$ and $\text{Ran}(T) = \text{Ran}(\tilde{T})$, the claim directly follows from item (ii) of Theorem 3.2.2. $\square$

As for bounded Fredholm operators, one has the following characterization.

Lemma 6.2.3. *A linear operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is Fredholm if and only if*

- (i) *T is regular,*
- (ii) $\dim(\text{Ker}(T)) < \infty$,
- (iii) $\dim(\text{Ker}(T^*)) < \infty$,
- (iv) *$\text{Ran}(T)$ is closed in $\mathcal{H}'$.*

Proof. Let us first assume that T is Fredholm. Then, by Proposition 6.2.2, the associated bounded operator $\tilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}'$ is Fredholm and $\text{Ran}(\tilde{T}) = \text{Ran}(T)$ is closed. Therefore

$$\dim(\mathcal{H}' / \text{Ran}(T)) = \dim(\text{Ran}(T)^\perp) = \dim(\text{Ker}(T^*))$$

is finite. Conversely, if $\text{Ran}(T)$ is closed then $\mathcal{H}' / \text{Ran}(T)$ is known to be a Hilbert space of dimension $\dim(\mathcal{H}' / \text{Ran}(T)) = \dim(\text{Ker}(T^*))$. Thus the equivalence is shown. $\square$

The following extends Theorem 3.2.2 to unbounded operators.

Theorem 6.2.4. *For a regular operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$, the following are equivalent:*

- (i) *T is a Fredholm operator.*
- (ii) *There exists a unique $S_0 \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ such that*

$$\text{Ker}(S_0) = \text{Ran}(T)^\perp, \quad \text{Ker}(S_0^*) = \text{Ker}(T),$$

and such that $S_0 T$ can be continuously extended to the orthogonal projection onto $\text{Ker}(T)^\perp$. Moreover, TS_0 is the orthogonal projection onto $\text{Ran}(T)$ and

$$\dim(\text{Ran}(\mathbf{1} - S_0 T)) < \infty, \quad \dim(\text{Ran}(\mathbf{1} - TS_0)) < \infty.$$

- (iii) *There exists a so-called pseudoinverse $S \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ such that $TS - \mathbf{1}$ and $ST - \mathbf{1}$ can be extended to compact operators on $\mathcal{H}$ and $\mathcal{H}'$, respectively.*

Proof. (i) $\implies$ (ii). First note that $T|_{\text{Ker}(T)^\perp} : \mathcal{D}(T) \cap \text{Ker}(T)^\perp \rightarrow \text{Ran}(T)$ is bijective and the graph of its inverse $\{(T\phi, \phi) : \phi \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp\}$ is closed as T is closed. Now, as $\text{Ran}(T)$ is closed and therefore a Hilbert space, the closed graph theorem shows that the inverse $S_0 : \text{Ran}(T) \rightarrow \text{Ker}(T)^\perp$ is bounded. It can be extended to all of $\mathcal{H}'$ by $S_0\psi = 0$ for $\psi \in \text{Ran}(T)^\perp$. Then by construction TS_0 is the projection in $\mathcal{H}'$ onto $\text{Ran}(T)$ and $S_0 T$ is bounded and can be extended to the projection in $\mathcal{H}$ onto $\text{Ker}(T)^\perp$. This implies all the stated properties. Uniqueness is obvious.

(ii) $\implies$ (iii). This is obvious.

(iii) $\implies$ (i). Suppose that $(\psi_n)_{n \geq 1}$ is an infinite orthonormal basis of $\text{Ker}(T)$. As these vectors are all eigenvectors of the compact operator $K = ST - \mathbf{1}$ for the eigenvalue 1, this is a contradiction to Riesz' theorem (Theorem 3.1.6). Suppose that $(\phi_n)_{n \geq 1}$ is an infinite orthonormal basis of $\text{Ran}(T)^\perp$. Consequently, one has $\|(TS - \mathbf{1})\phi_n\| = \|TS\phi_n - \phi_n\| \geq 1$ as $TS\phi_n \perp \phi_n$, a contradiction to the compactness of $TS - \mathbf{1}$. It remains to show that $\text{Ran}(T)$ is closed. Let K be the compact extension of $ST - \mathbf{1}$. Choose $L \in \mathcal{K}(\mathcal{H})$ with a finite-dimensional range and such that

$$\|K - L\| \leq \frac{1}{2}.$$

Then for all $\phi \in \text{Ker}(L) \cap \mathcal{D}(T)$:

$$\begin{aligned} \|S\| \|T\phi\| &\geq \|ST\phi\| \\ &= \|(\mathbf{1} + K)\phi\| \\ &\geq \|\phi\| - \|K\phi\| \\ &\geq \|\phi\| - \|(K - L)\phi\| - \|L\phi\| \\ &\geq \frac{1}{2} \|\phi\|. \end{aligned}$$

Thus $\|\phi\| \leq 2\|S\| \|T\phi\|$ for all $\phi \in \text{Ker}(L) \cap \mathcal{D}(T)$. This implies that $T(\text{Ker}(L) \cap \mathcal{D}(T))$ is closed. Indeed, given a convergent sequence $(T\phi_n)_{n \geq 1}$ with $\phi_n \in \text{Ker}(L) \cap \mathcal{D}(T)$, one can set $\psi = \lim_n T\phi_n$. Then

$$\|\phi_n - \phi_m\| \leq 2\|S\| \|T\phi_n - T\phi_m\|.$$

Thus $(\phi_n)_{n \geq 1}$ is a Cauchy sequence and hence has a limit point $\phi = \lim \phi_n \in \mathcal{H}$. As T is closed, one has $\psi = T\phi \in T(\text{Ker}(L) \cap \mathcal{D}(T))$. On the other hand,

$$T(\text{Ker}(L)^\perp \cap \mathcal{D}(T)) = T(\text{Ran}(L^*) \cap \mathcal{D}(T)).$$

As L^* also has a finite-dimensional image, it follows that $T(\text{Ker}(L)^\perp \cap \mathcal{D}(T))$ is finite dimensional. Thus $\text{Ran}(T) = T(\text{Ker}(L) \cap \mathcal{D}(T)) + T(\text{Ker}(L)^\perp \cap \mathcal{D}(T))$ is closed. $\square$

The following two propositions present criteria for regular operators to be Fredholm. They are the analogues of Lemma 3.4.2 and Proposition 3.2.6 for bounded operators.

Proposition 6.2.5. *For a regular operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$, the following are equivalent:*

- (i) $\dim(\text{Ker}(T)) < \infty$ and $\text{Ran}(T)$ is closed.
- (ii) $\dim(\text{Ker}(T)) < \infty$ and there is a constant $c > 0$ such that $\|T\phi\| > c\|\phi\|$ for all vectors $\phi \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$.

(iii) If $(\phi_n)_{n \geq 1}$ is a bounded sequence in $\mathcal{D}(T)$ such that $(T\phi_n)_{n \geq 1}$ is convergent, then there is a convergent subsequence of $(\phi_n)_{n \geq 1}$.

Proof. (i) $\implies$ (ii). The restriction T_r of T is a bijection from $\mathcal{D}(T) \cap \text{Ker}(T)^\perp$ to $\text{Ran}(T)$. The graph of its inverse T_r^{-1} is $\{ (T\phi, \phi) : \phi \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp \}$. As T is a closed operator, also its restriction T_r can be seen to be closed, so that the graph of $T_r^{-1} : \text{Ran}(T) \rightarrow \mathcal{H}$ is closed. As $\text{Ran}(T_r)$ is closed and therefore a Hilbert space, the closed graph theorem shows that T_r^{-1} is bounded. Therefore $\|\phi\| = \|T_r^{-1}T\phi\| \leq \|T_r^{-1}\| \|T\phi\|$ holds for all $\phi \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$.

(ii) $\implies$ (i). Let $(\psi_n)_{n \geq 1}$ be a sequence in $\text{Ran}(T)$ converging to $\psi \in \mathcal{H}'$. Then there are $\phi_n \in \text{Ker}(T)^\perp \cap \mathcal{D}(T)$ with $T\phi_n = \psi_n$. By (ii), one has $\|\phi_n - \phi_m\| < \frac{1}{c} \|\psi_n - \psi_m\|$ so that $(\phi_n)_{n \geq 1}$ is Cauchy and thus converges to some $\phi \in \mathcal{H}$. As $(\phi_n, T\phi_n)$ converges to (ϕ, ψ) and T is closed, one has $T\psi = \phi$ so that $\phi \in \text{Ran}(T)$ and $\text{Ran}(T)$ is closed.

(ii) $\implies$ (iii). Let $(\phi_n)_{n \geq 1}$ be a bounded sequence in $\mathcal{D}(T)$ such that $(T\phi_n)_{n \geq 1}$ is convergent. One has $\phi_n = \theta_n + \psi_n$ with $\theta_n \in \text{Ker}(T)$ and $\psi_n \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$. Because $\|\psi_n - \psi_m\| < \frac{1}{c} \|T\phi_n - T\phi_m\|$ by (ii), $(\psi_n)_{n \geq 1}$ is Cauchy and therefore convergent. As $(\phi_n)_{n \geq 1}$ and $(\psi_n)_{n \geq 1}$ are bounded, also $(\theta_n)_{n \geq 1}$ is bounded. Because the dimension of the kernel of T is finite, $(\theta_n)_{n \geq 1}$ and therefore $(\phi_n)_{n \geq 1}$ has a convergent subsequence.

(iii) $\implies$ (ii). Suppose that the kernel of T is infinite dimensional and that $(\phi_n)_{n \in \mathbb{N}}$ is an orthonormal basis of it. Then $(\phi_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{H}$ such that $T\phi_n$ is constant (equal to 0) and therefore convergent. As there is no convergent subsequence of $(\phi_n)_{n \in \mathbb{N}}$, this is a contradiction to (ii). Thus $\text{Ker}(T)$ is finite dimensional. Moreover, there is a constant $c > 0$ such that $\|\phi\| \leq c \|T\phi\|$ for all $\phi \in \text{Ker}(T)^\perp \cap \mathcal{D}(T)$, because otherwise there is a sequence $(\phi_n)_{n \in \mathbb{N}}$ in $\text{Ker}(T)^\perp \cap \mathcal{D}(T)$ such that $\|\phi_n\| = 1$ for all $n \in \mathbb{N}$ and $\|T\phi_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. As $(T\phi_n)_{n \in \mathbb{N}}$ is convergent, by assumption there is a subsequence $(\phi_{n_k})_{k \in \mathbb{N}}$ converging to some vector $\phi \in \text{Ker}(T)^\perp$ with $\|\phi\| = 1$. As $(\phi_{n_k}, T\phi_{n_k})$ converges to $(\phi, 0)$ and T is closed, one has $\phi \in \mathcal{D}(T)$ and $T\phi = 0$. This is a contradiction to $\phi \in \text{Ker}(T)^\perp$. $\square$

Proposition 6.2.6. Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ be a regular operator. If there is a compact operator $K \in \mathbb{K}(\mathcal{H}, \mathcal{H}'')$ and a constant $c > 0$ such that

$$\|\phi\| \leq c(\|T\phi\| + \|K\phi\|)$$

for all $\phi \in \mathcal{D}(T)$, then T has a closed range and a finite-dimensional kernel.

Proof. Let $(\phi_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathcal{D}(T)$ such that $T\phi_n$ is convergent, namely there is a $\psi \in \mathcal{H}'$ such that $\lim_{n \rightarrow \infty} T\phi_n = \psi$. As K is compact, there is a subsequence $(\phi_{n_k})_{k \in \mathbb{N}}$ such that $K\phi_{n_k}$ is convergent. Then $(K\phi_{n_k})_{k \in \mathbb{N}}$ is a Cauchy sequence and as $\lim_{k \rightarrow \infty} T\phi_{n_k} = \psi$, also $(T\phi_{n_k})_{k \in \mathbb{N}}$ is a Cauchy sequence. Therefore for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\max\{\|T\phi_{n_k} - T\phi_{n_m}\|, \|K\phi_{n_k} - K\phi_{n_m}\|\} < \frac{\epsilon}{2c}$ for all $k, m > N$. Thus

$$\|\phi_{n_k} - \phi_{n_m}\| \leq c(\|T\phi_{n_k} - T\phi_{n_m}\| + \|K\phi_{n_k} - K\phi_{n_m}\|) < \epsilon$$

or all $k, m > N$, which shows that $(\phi_{n_k})_{k \in \mathbb{N}}$ is a Cauchy sequence and therefore convergent. Finally, item (iii) of Proposition 6.2.5 shows the assertion. $\square$

Definition 6.2.7. The index of a Fredholm operator $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ is

$$\text{Ind}(T) = \dim(\text{Ker}(T)) - \dim(\mathcal{H}' / \text{Ran}(T)).$$

Next let us generalize Corollary 3.3.2 to unbounded Fredholm operators.

Corollary 6.2.8. (i) For $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, $T' \in \mathbb{F}(\mathcal{H}'', \mathcal{H})$, also $TT' \in \mathbb{F}(\mathcal{H}'', \mathcal{H}')$.

(ii) If $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, then $T^* \in \mathbb{F}(\mathcal{H}', \mathcal{H})$. Moreover,

$$\text{Ind}(T) = \dim(\text{Ker}(T)) - \dim(\text{Ker}(T^*))$$

and

$$\text{Ind}(T^*) = -\text{Ind}(T).$$

(iii) If $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$, then

$$\text{Ind}(T) = \dim(\text{Ker}(T^*T)) - \dim(\text{Ker}(TT^*)).$$

(iv) For $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ and $T' \in \mathbb{F}(\mathcal{H}'', \mathcal{H}''')$, one has $T \oplus T' \in \mathbb{F}(\mathcal{H} \oplus \mathcal{H}'', \mathcal{H}' \oplus \mathcal{H}''')$ and

$$\text{Ind}(T \oplus T') = \text{Ind}(T) + \text{Ind}(T').$$

Proof. For the proof of (i), let us first show that TT' is densely defined, namely that $\mathcal{D}(TT') = \{\phi \in \mathcal{D}(T') : T'\phi \in \mathcal{D}(T)\}$ is dense in $\mathcal{H}''$. First, it is checked that $\mathcal{D}(T) \cap \text{Ran}(T')$ is dense in $\text{Ran}(T')$. As T' is Fredholm, $\text{Ran}(T')^\perp$ is finite dimensional. Let $\{\psi_1, \dots, \psi_n\}$ be an orthonormal basis of $\text{Ran}(T')^\perp$. Let $\epsilon_1 > 0$. Because $\mathcal{D}(T)$ is dense in $\mathcal{H}$, there are $\theta_i \in \mathcal{D}(T)$ for $i = 1, \dots, n$ such that $\|\phi_i - \theta_i\| < \epsilon_1$. Then $\mathcal{E} = \text{span}(\{\theta_1, \dots, \theta_n\})$ is a subspace of $\mathcal{D}(T)$ and, for ϵ_1 sufficiently small, $\text{Ran}(T') \cap \mathcal{E} = \{0\}$ and $\text{Ran}(T') \oplus \mathcal{E} = \mathcal{H}$. By Proposition 5.1.6, there is a projection $P \in \mathbb{B}(\mathcal{H})$ with $\text{Ran}(P) = \text{Ran}(T')$ and $\text{Ker}(P) = \mathcal{E}$. Because $\mathcal{D}(T) \subset \mathcal{H}$ is dense, for any vector $\psi \in \text{Ran}(T')$ and $\epsilon_2 > 0$ there is $\psi' \in \mathcal{D}(T)$ such that $\|\psi - \psi'\| < \epsilon_2$. Then $P\psi' = \psi' - (1-P)\psi' \in \mathcal{D}(T) \cap \text{Ran}(T')$ and

$$\begin{aligned} \|\psi - P\psi'\| &\leq \|\psi - \psi'\| + \|(1-P)\psi'\| \\ &< \epsilon_2 + \|(1-P)(\psi' - \psi)\| \\ &\leq \epsilon_2(1 + \|1-P\|), \end{aligned}$$

where the second step follows as $\psi \in \text{Ran}(T') = \text{Ker}(1-P)$. This shows that $\mathcal{D}(T) \cap \text{Ran}(T')$ is dense in $\text{Ran}(T')$. To show that $\mathcal{D}(TT')$ is dense in $\mathcal{H}''$, it is sufficient to show that for $\epsilon > 0$ and $\phi \in \mathcal{D}(T')$ there is $\tilde{\phi} \in \mathcal{D}(TT')$ such that $\|\phi - \tilde{\phi}\| < \epsilon$ (because $\mathcal{D}(T') \subset \mathcal{H}''$ is dense). For $\phi \in \mathcal{D}(T')$, there is $\phi' \in \mathcal{D}(T') \cap \text{Ker}(T')^\perp$ such that $T'\phi' = T'\phi$ and

thus $\phi - \phi' \in \text{Ker}(T) \subset \mathcal{D}(TT')$. By the above, there is $\psi \in \text{Ran}(T') \cap \mathcal{D}(T)$ such that $\|\psi - T'\phi\| < \epsilon c$ for $c > 0$ as in item (ii) of Proposition 6.2.5 applied to T' . Then there is $\theta \in \mathcal{D}(T') \cap \text{Ker}(T')^\perp$ such that $\psi = T'\theta$ and therefore $\theta \in \mathcal{D}(TT')$. Thus one concludes that $\phi' - \theta \in \mathcal{D}(T') \cap \text{Ker}(T')^\perp$ and, by Proposition 6.2.5,

$$\|\phi' - \theta\| < \frac{1}{c} \|T'(\phi' - \theta)\| = \frac{1}{c} \|T'\phi - \psi\| < \epsilon.$$

By construction, $\tilde{\phi} = \phi - \phi' + \theta \in \mathcal{D}(TT')$ fulfills

$$\|\phi - \tilde{\phi}\| = \|\phi' - \theta\| < \epsilon.$$

This shows that TT' is densely defined.

To show that TT' is closed, let us choose a sequence $(\phi_n)_{n \geq 1}$ in $\mathcal{D}(TT')$ such that $(\phi_n, TT'\phi_n)$ converges to (ϕ, θ) . For $\psi_n = T'\phi_n$, there are $\psi'_n \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$ and $\psi''_n \in \text{Ker}(T)$ such that $\psi_n = \psi'_n + \psi''_n$. Then $(\psi'_n)_{n \geq 1}$ is Cauchy, as, by Proposition 6.2.5, there is a constant $c > 0$ such that

$$\|\psi'_n - \psi'_m\| < c \|T\psi'_n - T\psi'_m\| = c \|TT'\phi_n - TT'\phi_m\|$$

and $(TT'\phi_n)_{n \geq 1}$ is Cauchy by assumption. Therefore $(\psi'_n)_{n \geq 1}$ is convergent, and one can define $\psi = \lim_{n \rightarrow \infty} \psi'_n \in \mathcal{H}$. As $(\psi'_n, T\psi'_n) = (\psi'_n, TT'\phi_n)$ converges to (ψ, θ) and T is closed, one has $\psi \in \mathcal{D}(T)$ and $T\psi = \theta$. We show that $(\psi''_n)_{n \geq 1}$ is bounded. Suppose that $(\psi''_n)_{n \geq 1}$ is unbounded, then there is a subsequence, again denoted by $(\psi''_n)_{n \geq 1}$, such that $\lim_{n \rightarrow \infty} \|\psi''_n\| = \infty$. Then $(\frac{\psi''_n}{\|\psi''_n\|})_{n \geq 1}$ is a bounded sequence in the finite-dimensional kernel of T . Again by choosing a subsequence, without loss of generality one can assume that $(\frac{\psi''_n}{\|\psi''_n\|})_{n \geq 1}$ converges to $\tilde{\psi} \in \text{Ker}(T)$ with $\|\tilde{\psi}\| = 1$. As $(\phi_n)_{n \geq 1}$ is bounded, one has $\lim_{n \rightarrow \infty} \frac{\phi_n}{\|\psi''_n\|} = 0$ and $T' \frac{\phi_n}{\|\psi''_n\|} = \frac{\psi'_n + \psi''_n}{\|\psi''_n\|}$ converges to $\tilde{\psi}$. As T' is closed, this implies $\tilde{\psi} = 0$, which is a contradiction. Therefore the sequence $(\psi''_n)_{n \geq 1}$ is bounded. As the dimension of $\text{Ker}(T)$ is finite, there is a convergent subsequence $(\psi''_{n_j})_{j \geq 1}$ of $(\psi''_n)_{n \geq 1}$. Setting $\psi'' = \lim_{j \rightarrow \infty} \psi''_{n_j} \in \text{Ker}(T)$, one has $\lim_{j \rightarrow \infty} \psi_{n_j} = \psi + \psi''$. As $(\phi_{n_j})_{j \geq 1}$ converges to ϕ and $(T\phi_{n_j})_{j \geq 1} = (\psi_{n_j})_{j \geq 1}$ converges to $\psi + \psi''$, one has $\phi \in \mathcal{D}(T')$ and $T'\phi = \psi + \psi''$. As $T\psi = \theta$ and $\psi'' \in \text{Ker}(T)$, one has $\theta = T(\psi + \psi'') = TT'\phi$. In conclusion, (ϕ, θ) is an element of the graph of TT' and therefore TT' is closed.

We next use Proposition 6.2.5 to show that $\text{Ran}(TT')$ is closed and that the dimension of the kernel of TT' is finite. Let $(\phi_n)_{n \geq 1}$ in $\mathcal{D}(TT')$ be a bounded sequence such that $(TT'\phi_n)_{n \geq 1}$ is convergent. For $\psi_n = T'\phi_n$, there are $\psi'_n \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$ and $\psi''_n \in \text{Ker}(T)$ such that $\psi_n = \psi'_n + \psi''_n$. Then $(\psi'_n)_{n \geq 1}$ is Cauchy, as, by Proposition 6.2.5, there is a constant $c > 0$ such that

$$\|\psi'_n - \psi'_m\| < c \|T\psi'_n - T\psi'_m\| = c \|TT'\phi_n - TT'\phi_m\|$$

and $(TT'\phi_n)_{n \geq 1}$ is Cauchy by assumption. Therefore $(\psi'_n)_{n \geq 1}$ is convergent and one can set $\psi = \lim_{n \rightarrow \infty} \psi'_n \in \mathcal{H}$. As above one can show that $(\psi''_n)_{n \geq 1}$ is bounded. As the dimension of $\text{Ker}(T)$ is finite, there is a convergent subsequence $(\psi''_{n_j})_{j \geq 1}$ of $(\psi''_n)_{n \geq 1}$. Next setting $\psi'' = \lim_{j \rightarrow \infty} \psi''_{n_j} \in \text{Ker}(T)$, one has $\lim_{j \rightarrow \infty} \psi_{n_j} = \psi + \psi''$. Thus $(T'\phi_{n_j})_{j \geq 1} = (\psi_{n_j})_{j \geq 1}$ converges to $\psi + \psi''$. By item (iii) of Proposition 6.2.5 applied to the Fredholm operator T' , there is a convergent subsequence of $(\phi_{n_j})_{j \geq 1}$. Thus $\text{Ran}(TT')$ is closed and $\dim(\text{Ker}(TT'))$ is finite. To show that $\text{Ran}(TT')^\perp$ is finite dimensional, note that $\dim(\text{Ran}(T')^\perp)$ is finite and thus the dimension of $T(\text{Ran}(T')^\perp)$ is finite. As

$$\text{Ran}(T) = T(\text{Ran}(T')) + T(\text{Ran}(T')^\perp),$$

one has

$$\text{Ran}(T)^\perp = (T(\text{Ran}(T'))^\perp \cap (T(\text{Ran}(T')^\perp)^\perp)^\perp.$$

As $\text{Ran}(T)^\perp$ and $T(\text{Ran}(T')^\perp)$ are finite dimensional, this implies that the dimension of $(T(\text{Ran}(T'))^\perp)^\perp = \text{Ran}(TT')^\perp$ is finite.

In order to show (ii), let us note that T^* is regular and $\text{Ker}(T^*) = \text{Ran}(T)^\perp$ and $\text{Ran}(T^*)^\perp = \text{Ker}(T)$ are finite dimensional. It remains to show that $\text{Ran}(T^*)$ is closed. This follows from Proposition 6.2.5, because $\text{Ker}(T^*)^\perp = \text{Ran}(T)$ as $\text{Ran}(T)$ is closed. Therefore for $\theta \in \mathcal{D}(T^*) \cap \text{Ker}(T^*)^\perp$ there is $\phi \in \mathcal{D}(T) \cap \text{Ker}(T)^\perp$ such that $T\phi = \theta$. Then

$$\|\phi\| \|T^* \theta\| \geq \langle T^* \theta | \phi \rangle = \langle T\phi | T\phi \rangle = \|T\phi\|^2 \geq c \|\phi\| \|T\phi\| = c \|\phi\| \|\theta\|$$

for a constant $c > 0$ by Proposition 6.2.5. Thus $\|T^* \theta\| \geq c \|\theta\|$ for all $\theta \in \mathcal{D}(T^*) \cap \text{Ker}(T^*)^\perp$ and $\text{Ran}(T^*)$ is closed, again by Proposition 6.2.5. The claim about the index of T follows directly from Definition 6.2.7.

As $\text{Ker}(T) = \text{Ker}(T^* T)$ and $\text{Ker}(T^*) = \text{Ker}(TT^*)$, item (iii) is a direct consequence of (ii).

The last claim follows from the obvious identities $\text{Ker}(T \oplus T') = \text{Ker}(T) \oplus \text{Ker}(T')$ and $\text{Ran}(T \oplus T') = \text{Ran}(T) \oplus \text{Ran}(T')$. $\square$

Proposition 6.2.9. *If $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ and $T' \in \mathbb{F}(\mathcal{H}'', \mathcal{H})$, then the index of the Fredholm operator $TT' \in \mathbb{F}(\mathcal{H}'', \mathcal{H}')$ is given by*

$$\text{Ind}(TT') = \text{Ind}(T) + \text{Ind}(T').$$

Proof. Recall that TT' is Fredholm by Corollary 6.2.8. One has

$$\dim(\text{Ker}(TT')) = \dim(\text{Ker}(T')) + \dim(\text{Ker}(T) \cap \text{Ran}(T')).$$

Setting $\mathcal{N}_1 = \text{Ker}(T) \cap \text{Ran}(T')$, there is a finite-dimensional subspace $\mathcal{N}_2 \subset \mathcal{H}$ such that

$$\text{Ker}(T) = \mathcal{N}_1 \oplus \mathcal{N}_2.$$

Note that $\text{Ran}(T') \cap \mathcal{N}_2 = \{0\}$ and $\text{Ran}(T') \oplus \mathcal{N}_2$ is closed. Next it is shown that there is a finite-dimensional subspace $\mathcal{N}_3 \subset \mathcal{D}(T)$ such that

$$\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 = \mathcal{H}.$$

Because $(\text{Ran}(T') \oplus \mathcal{N}_2)^\perp$ is a subspace of the finite-dimensional space $\text{Ran}(T')^\perp$, it is finite dimensional. If $\dim((\text{Ran}(T') \oplus \mathcal{N}_2)^\perp) = 0$, the claim holds for $\mathcal{N}_3 = \{0\}$. Therefore, without loss of generality, one can assume $\dim((\text{Ran}(T') \oplus \mathcal{N}_2)^\perp) = l \in \mathbb{N}$. Next since $\text{Ran}(T') \oplus \mathcal{N}_2$ is closed and T is densely defined, there is a vector $\phi_1 \in \mathcal{D}(T) \setminus (\text{Ran}(T') \oplus \mathcal{N}_2)$. Then $\mathcal{H}_1 = \text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\})$ is closed and $\dim((\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\}))^\perp) = l-1$. If $l \geq 2$, there is a vector $\phi_2 \in \mathcal{D}(T) \setminus (\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1\}))$. Repeating this procedure l times, one finds vectors $\phi_1, \dots, \phi_l \in \mathcal{D}(T)$ such that $\text{Ran}(T') \oplus \mathcal{N}_2 \oplus \text{span}(\{\phi_1, \dots, \phi_l\}) = \mathcal{H}$. Then the claim holds for $\mathcal{N}_3 = \text{span}(\{\phi_1, \dots, \phi_l\})$.

The restriction $T|_{\mathcal{N}_3}$ is injective and

$$\text{Ran}(T) = \text{Ran}(TT') \oplus T\mathcal{N}_3.$$

The last claim holds as $\text{Ran}(T) = \text{Ran}(TT') + T\mathcal{N}_3$ by construction and since, for vectors $\phi \in \text{Ran}(T')$ and $\psi \in \mathcal{N}_3$ such that $T\phi = T\psi \in \text{Ran}(TT')$, one has $\psi \in \text{Ran}(T') + \text{Ker}(T)$ and therefore $\psi = 0$ by definition of $\mathcal{N}_3$. Thus

$$\dim(\text{Ran}(TT')^\perp) = \dim(\text{Ran}(T)^\perp) + \dim(\mathcal{N}_3).$$

One can conclude that

$$\begin{aligned} \text{Ind}(TT') &= \dim(\text{Ker}(TT')) - \dim(\mathcal{H}' / \text{Ran}(TT')) \\ &= \dim(\text{Ker}(T')) + \dim(\mathcal{N}_1) - \dim(\text{Ran}(T)^\perp) - \dim(\mathcal{N}_3) \\ &= \dim(\text{Ker}(T')) + \dim(\mathcal{N}_1) + \dim(\mathcal{N}_2) \\ &\quad - \dim(\text{Ran}(T)^\perp) - \dim(\mathcal{N}_3) - \dim(\mathcal{N}_2) \\ &= \dim(\text{Ker}(T')) + \dim(\text{Ker}(T)) - \dim(\text{Ran}(T)^\perp) - \dim(\text{Ran}(T')^\perp) \\ &= \text{Ind}(T) + \text{Ind}(T'), \end{aligned}$$

by definition of $\mathcal{N}_1, \mathcal{N}_2$, and $\mathcal{N}_3$. □

The next aim is to show that the Fredholm property and that the index is invariant under small or compact perturbations. Therefore we introduce the notion of relatively bounded and relatively compact operators.

Definition 6.2.10. Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ be a closed linear operator. Another operator $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}'$ with $\mathcal{D}(T) \subset \mathcal{D}(S)$ is called relatively bounded with respect to T (or T -bounded) if the restriction $S|_{\mathcal{D}(T)}$ is bounded as operator $S : \mathcal{D}(T) \rightarrow \mathcal{H}'$ where $\mathcal{D}(T)$ is equipped with the T -norm $\|\cdot\|_T$. Analogously, S is called relatively compact with

respect to T (or T -compact) if the restriction $S|_{\mathcal{D}(T)} : \mathcal{D}(T) \rightarrow \mathcal{H}'$ is compact, where again $\mathcal{D}(T)$ is equipped with the T -norm.

Note that $\phi \mapsto \|T\phi\| + \|\phi\|$ defines a norm on $\mathcal{D}(T)$ that is equivalent to the T -norm. Therefore an operator $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is relatively bounded with respect to T if and only if there are constants $c_1, c_2 > 0$ such that

$$\|S\phi\| \leq c_1\|T\phi\| + c_2\|\phi\| \quad (6.8)$$

for all $\phi \in \mathcal{D}(T)$. In particular, every bounded operator $S : \mathcal{H} \rightarrow \mathcal{H}'$ is T -bounded and every compact operator $S : \mathcal{H} \rightarrow \mathcal{H}'$ is T -compact.

Lemma 6.2.11. *If $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is a closed operator and $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is relatively bounded with respect to T and the relative bound c_1 in (6.8) is less than 1, then $T + S : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is a closed operator.*

Proof. Equation (6.8) with $c_1, c_2 > 0$ implies

$$\|(T + S)\phi\| \leq (1 + c_1)\|T\phi\| + c_2\|\phi\| \quad (6.9)$$

and

$$\|(T + S)\phi\| \geq \|T\phi\| - \|S\phi\| \geq (1 - c_1)\|T\phi\| - c_2\|\phi\|.$$

As $c_1 < 1$, the last inequality is equivalent to

$$\|T\phi\| \leq \frac{1}{1 - c_1} (\|(T + S)\phi\| + c_2\|\phi\|). \quad (6.10)$$

Let $(\phi_n)_{n \geq 1}$ be a sequence in $\mathcal{D}(T)$ such that $(\phi_n, (T + S)\phi_n)$ converges to (ϕ, θ) . By (6.10),

$$\|T\phi_n - T\phi_m\| \leq \frac{1}{1 - c_1} (\|(T + S)(\phi_n - \phi_m)\| + c_2\|\phi_n - \phi_m\|)$$

and therefore $(T\phi_n)_{n \geq 1}$ is Cauchy and thus convergent. Setting $\psi = \lim_{n \rightarrow \infty} T\phi_n$, this implies that $(\phi_n, T\phi_n)_{n \geq 1}$ converges to (ϕ, ψ) . As T is closed, ϕ is in $\mathcal{D}(T) = \mathcal{D}(T + S)$ and $T\phi = \psi$. Moreover, by (6.9),

$$\|(T + S)(\phi - \phi_n)\| \leq (1 + c_1)\|T(\phi - \phi_n)\| + c_2\|\phi - \phi_n\|$$

converges to 0. Therefore $(T + S)\phi = \lim_{n \rightarrow \infty} (T + S)\phi_n = \theta$ and the graph of $T + S$ is closed. $\square$

A similar result holds for relatively compact operators.

Lemma 6.2.12. *If $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is a Fredholm operator and $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is relatively compact with respect to T , then $T + S : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ is a closed operator.*

Proof. Let $\tilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}$ be the bounded operator associated with T . Similarly, define $\tilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}$ by $\tilde{S}\phi = S\phi$. Then by Proposition 6.2.2, $\tilde{T}$ and therefore $\tilde{T} + \tilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}$ are bounded Fredholm operators. Let us define the embedding $I : \mathcal{D}(T) \subset (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \rightarrow (\mathcal{D}(T), \|\cdot\|_T)$ by $I\phi = \phi$. Then I is invertible and I^{-1} is bounded and therefore closed. Thus also I is closed and as $\text{Ker}(I) = \{0\}$ and $\text{Ran}(I) = \mathcal{D}(T)$, I is a Fredholm operator. Therefore, by item (i) of Corollary 6.2.8, $T + S = (\tilde{T} + \tilde{S})I$ is Fredholm and, in particular, closed. $\square$

After these preparations, we can now show that the Fredholm property is invariant under small or compact perturbations.

Proposition 6.2.13. *Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}'$ be a Fredholm operator and furthermore let $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}'$ be relatively compact with respect to T or relatively bounded with respect to T such that the constants c_1, c_2 in (6.8) are sufficiently small, then the operator $T + S : \mathcal{D}(T) \rightarrow \mathcal{H}'$ is Fredholm and*

$$\text{Ind}(T + S) = \text{Ind}(T).$$

Proof. By the above lemmata, where $c_1 < 1$ is assumed, operator $T + S$ is closed. Let $\tilde{T} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}$ be the operator associated with T and again let $\tilde{S} : (\mathcal{D}(T), \|\cdot\|_T) \rightarrow \mathcal{H}$ be given by $\tilde{S}\phi = S\phi$. Then by Proposition 6.2.2, $T + S$ is Fredholm if and only if $\tilde{T} + \tilde{S}$ is Fredholm. If S is relatively compact with respect to T , $\tilde{T} + \tilde{S}$ and therefore $T + S$ are Fredholm by Theorem 3.3.4. Moreover, $\text{Ind}(T + S) = \text{Ind}(\tilde{T} + \tilde{S}) = \text{Ind}(\tilde{T}) = \text{Ind}(T)$ again by Theorem 3.3.4. If $\tilde{T}$ is Fredholm, as the set of bounded Fredholm operators is open, see Theorem 3.3.5, there is a constant $c > 0$ such that $\tilde{T} + A$ is Fredholm for all $A \in \mathcal{B}((\mathcal{D}(T), \|\cdot\|_T), \mathcal{H}')$ such that $\|A\| < c$. If S is relatively bounded with respect to T , then $\tilde{S} \in \mathcal{B}((\mathcal{D}(T), \|\cdot\|_T), \mathcal{H}')$ has norm less than c provided the constants c_1 and c_2 in (6.8) are sufficiently small. Then, by the above, $T + S$ is Fredholm with an index satisfying $\text{Ind}(T + S) = \text{Ind}(\tilde{T} + \tilde{S}) = \text{Ind}(\tilde{T}) = \text{Ind}(T)$. $\square$

As in the bounded case for self-adjoint operators, there is another characterization using the notion of essential spectrum. The essential spectrum of a self-adjoint operator $H : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$ is defined as in Section 3.4 for bounded self-adjoint operators, namely $\text{spec}_{\text{ess}}(H) = \text{spec}(H) \setminus \text{spec}_{\text{dis}}(H)$ where the discrete spectrum $\text{spec}_{\text{dis}}(H)$ consists of all isolated eigenvalues of H of finite multiplicity.

Theorem 6.2.14. *A self-adjoint operator $H = H^* \in \mathbb{L}(\mathcal{H})$ is Fredholm if and only if one has $0 \notin \text{spec}_{\text{ess}}(H)$.*

Proof. Let us first assume that H is Fredholm. As $\text{Ran}(H) = \text{Ker}(H)^\perp$, then either H is invertible with a bounded inverse, by the Hellinger–Toeplitz theorem, or 0 is an eigenvalue of finite multiplicity. It remains to show that there exists $\epsilon > 0$ such that one has $\text{spec}(H) \cap (-\epsilon, \epsilon) \setminus \{0\} = \emptyset$. The restriction H' of H to $\mathcal{D}(H) \cap \text{Ker}(H)^\perp$ is a bijection onto its range, which is a Hilbert space. Its graph $\{(\phi, H\phi) : \phi \in \mathcal{D}(H) \cap \text{Ker}(H)^\perp\}$ is closed

because H is a closed operator. Therefore the closed graph theorem shows that $(H')^{-1}$ is bounded and therefore 0 lies in the resolvent set of H' . Thus there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap \text{spec}(H') = \emptyset$. Furthermore, $H + \delta \mathbf{1} : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a Fredholm operator of all $\delta \in (-\epsilon, \epsilon)$. Then $\text{Ran}(H - \delta \mathbf{1}) = \text{Ker}(H - \delta \mathbf{1})^\perp$ and therefore δ is an eigenvalue of H or $H - \delta \mathbf{1}$ is invertible with bounded inverse so that $\delta \notin \text{spec}(H)$. If δ is an eigenvalue of H , there is $\phi = \phi_1 + \phi_2 \in \mathcal{D}(H)$ with $\phi_1 \in \text{Ker}(H)$ and $\phi_2 \in \mathcal{D}(H) \cap \text{Ker}(H)^\perp$ such that $H\phi = H\phi_2 = \delta\phi = \delta\phi_1 + \delta\phi_2$. Therefore $\delta\phi_1 = (H - \delta)\phi_2$ and, as $\phi_1 \in \text{Ker}(H)$ and $(H - \delta)\phi_2 \in \text{Ker}(H)^\perp$, this implies $\phi_1 = 0$. Therefore $H'\phi_2 = H\phi_2 = \delta\phi_2$, which is a contradiction.

Conversely assume that $0 \notin \text{spec}_{\text{ess}}(H)$. Then $\dim(\text{Ker}(H)) < \infty$ and $\|H\phi\| \geq c\|\phi\|$ for some $c > 0$ for all $\phi \in \mathcal{D}(H) \cap \text{Ker}(H)^\perp$, which is, by Proposition 6.2.5, equivalent to the Fredholm property of H . $\square$

The following generalizes Theorem 3.4.1 to unbounded operators.

Theorem 6.2.15. *A regular operator $T \in \mathbb{L}(\mathcal{H}, \mathcal{H}')$ is Fredholm if and only if one has $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and $0 \notin \text{spec}_{\text{ess}}(TT^*)$.*

Proof. Let us first suppose that T is Fredholm. Then by Corollary 6.2.8, T^* is Fredholm and therefore T^*T and TT^* are Fredholm. As T^*T and also TT^* are self-adjoint by Lemma 6.1.1 (note also that $(T^*)^* = \bar{T} = T$), this implies $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and furthermore $0 \notin \text{spec}_{\text{ess}}(TT^*)$ by Theorem 6.2.14.

Conversely assume that $0 \notin \text{spec}_{\text{ess}}(T^*T)$ and $0 \notin \text{spec}_{\text{ess}}(TT^*)$. Then by Theorem 6.2.14 and Lemma 6.1.1, T^*T and TT^* are Fredholm. Therefore the dimensions of $\text{Ker}(T) = \text{Ker}(T^*T)$ and $\text{Ker}(T^*) = \text{Ker}(TT^*)$ are finite. Moreover, Lemma 5.3.3 implies that $\text{Ran}(T) = \text{Ran}(TT^*) \oplus (\text{Ran}(T) \cap \text{Ran}(TT^*)^\perp)$ is closed. This implies by Lemma 6.2.3 that T is a Fredholm operator. $\square$

As for bounded Fredholm operators, there is another characterization of the index of a Fredholm operator $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ using the operator $L : \mathcal{D}(T) \oplus \mathcal{D}(T^*) \rightarrow \mathcal{H} \oplus \mathcal{H}'$ defined by

$$L = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}. \quad (6.11)$$

Note that the square L^2 commutes with $J = \text{diag}(\mathbf{1}, -\mathbf{1})$ and therefore $\text{Ker}(L) = \text{Ker}(L^2)$ is invariant under J . Now $\text{Ind}(T)$ can be calculated as follows.

Proposition 6.2.16. *Let $T \in \mathbb{F}(\mathcal{H}, \mathcal{H}')$ be a Fredholm operator. Then the operator L defined by (6.11) is self-adjoint. Moreover, the index of T is equal to the signature of the operator $J = \mathbf{1} \oplus -\mathbf{1} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}')$ restricted to the kernel of L , namely*

$$\text{Ind}(T) = \text{Sig}(J|_{\text{Ker}(L)}).$$

Proof. One directly checks that L is symmetric. Therefore it is sufficient to show that $\mathcal{D}(L^*) \subset \mathcal{D}(L)$. As $\text{Ker}(L) = \text{Ker}(T) \oplus \text{Ker}(T^*)$ and

$$\text{Ran}(L) = \text{Ran}(T^*) \oplus \text{Ran}(T) = \text{Ker}(T)^\perp \oplus \text{Ker}(T^*)^\perp = \text{Ker}(L)^\perp,$$

one has $\text{Ran}(L^*) \subset \text{Ker}(L)^\perp = \text{Ran}(L)$ and $\text{Ker}(L^*) = \text{Ran}(L)^\perp = \text{Ker}(L)$. Now let be given $\psi \in \mathcal{D}(L^*) \setminus \mathcal{D}(L)$. Then $L^*\psi \in \text{Ran}(L^*) \subset \text{Ran}(L)$ so that there is a $\phi \in \mathcal{D}(L)$ with $L^*\psi = L\phi = L^*\phi$. Hence $0 = L^*(\psi - \phi) = L(\psi - \phi)$ as $\text{Ker}(L) = \text{Ker}(L^*)$, and one concludes $\psi \in \mathcal{D}(L)$, in contradiction to the assumption. Hence $\mathcal{D}(L^*) \subset \mathcal{D}(L)$ and L is self-adjoint. As $\text{Ker}(L) = \text{Ker}(T) \oplus \text{Ker}(T^*)$, one concludes that

$$\text{Sig}(J|_{\text{Ker}(L)}) = \dim(\text{Ker}(T)) - \dim(\text{Ker}(T^*)) = \text{Ind}(T),$$

completing the proof. $\square$

As the final topic of this section, let us examine the image of Fredholm operators under the bounded transform $\mathcal{F}$, namely let us restrict the bounded transform $\mathcal{F}$ to the subset $\mathbb{F}(\mathcal{H}, \mathcal{H}') \subset \mathbb{L}(\mathcal{H}, \mathcal{H}')$. Combining Theorems 6.1.4 and 6.2.15 and using the identity $\mathcal{F}(T)^*\mathcal{F}(T) = T^*T(\mathbf{1} + T^*T)^{-1}$ where $(\mathbf{1} + T^*T)^{-1} : \mathcal{H} \rightarrow \mathcal{D}(T^*T)$ is a bijection so that $\mathcal{F}(T)^*\mathcal{F}(T)$ is Fredholm if and only if T^*T is Fredholm, one obtains

$$\begin{aligned} \mathcal{F}(\mathbb{F}(\mathcal{H}, \mathcal{H}')) \\ = \{F \in \mathbb{B}_1(\mathcal{H}, \mathcal{H}') : \text{Ker}(\mathbf{1} - F^*F) = \{0\}, 0 \notin \text{spec}_{\text{ess}}(F^*F) \cup \text{spec}_{\text{ess}}(FF^*)\}, \end{aligned}$$

so that, by Theorem 6.2.15,

$$\mathcal{F}(\mathbb{F}(\mathcal{H}, \mathcal{H}')) = \mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \quad (6.12)$$

where $\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}') = \mathbb{FB}(\mathcal{H}, \mathcal{H}') \cap \mathbb{B}_1^0(\mathcal{H}, \mathcal{H}')$. Moreover, one has

$$\text{Ind}(\mathcal{F}(T)) = \text{Ind}(T), \quad T \in \mathbb{F}(\mathcal{H}, \mathcal{H}').$$

Now Propositions 6.1.5 and 6.1.7 immediately imply the following

Proposition 6.2.17. *The bounded transform provides two homeomorphisms:*

$$\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R) \rightarrow (\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N)$$

and

$$\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_G) \rightarrow (\mathbb{FB}_1^0(\mathcal{H}, \mathcal{H}'), \mathcal{O}_E).$$

Proposition 6.2.17 leads to the following result that will be used in Section 8.2 for the computation of the homotopy groups of $(\mathbb{F}(\mathcal{H}), \mathcal{O}_R)$.

Proposition 6.2.18. *The inclusion $i : (\mathbb{FB}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N) \rightarrow (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R)$ is a homotopy equivalence with homotopy inverse $\mathcal{F} : (\mathbb{F}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_R) \rightarrow (\mathbb{FB}(\mathcal{H}, \mathcal{H}'), \mathcal{O}_N)$.*

Proof. (Modification of the proof of Theorem 5.10 in [126].) Let us first show that the composition $\mathcal{F} \circ i : \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}') \rightarrow \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}')$ is a homotopic to the identity. Consider the norm-continuous homotopy $h : \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}') \times [0, \frac{1}{2}] \rightarrow \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}')$ defined by

$$h(T, t) = T(\mathbf{1} + T^*T)^{-t}.$$

Then, clearly, $h(T, 0) = T$ and $h(T, \frac{1}{2}) = (\mathcal{F} \circ i)(T)$ for all $T \in \mathbb{F}\mathbb{B}(\mathcal{H}, \mathcal{H}')$. By Proposition 6.2.17, this implies that $i \circ \mathcal{F} = \mathcal{F}^{-1} \circ (\mathcal{F} \circ i) \circ \mathcal{F}$ is also homotopic to the identity. Putting these facts together, one concludes that i is a homotopy equivalence. $\square$

6.3 Unbounded self-adjoint Fredholm operators

This section analyzes the set $\mathbb{F}_{\text{sa}}(\mathcal{H})$ of unbounded self-adjoint Fredholm operators on $\mathcal{H}$. As a subset of the set $\mathbb{L}(\mathcal{H}) = \mathbb{L}(\mathcal{H}, \mathcal{H})$ of closed densely defined operators, it inherits two natural metrics, namely the Riesz metric d_R and the gap metric d_G . The induced topologies will still be called Riesz and gap topologies, respectively. Let us begin by analyzing the image of $\mathbb{F}_{\text{sa}}(\mathcal{H})$ under the bounded transform. Recall from Section 4.6 the notations

$$\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}) = \{H \in \mathbb{B}_{\text{sa}}(\mathcal{H}) : \|H\| \leq 1, \text{Ker}(H^2 - \mathbf{1}) = \{0\}\}$$

and

$$\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}) = \mathbb{B}_{1,\text{sa}}^0(\mathcal{H}) \cap \mathbb{F}\mathbb{B}(\mathcal{H}).$$

Proposition 6.3.1. *The bounded transform $\mathcal{F}$ maps $\mathbb{L}_{\text{sa}}(\mathcal{H})$ and $\mathbb{F}_{\text{sa}}(\mathcal{H})$ bijectively onto $\mathbb{B}_{1,\text{sa}}^0(\mathcal{H})$ and $\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H})$, respectively.*

Proof. By Theorem 6.1.4, one has $\mathcal{F}(T^*) = \mathcal{F}(T)^*$ for all $T \in \mathbb{L}(\mathcal{H})$. Therefore T is self-adjoint if and only if $\mathcal{F}(T)$ is self-adjoint. Moreover, as $(\mathbf{1} + T^*T)^{-\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{D}(T)$ is bijective, $\text{Ran}(T) = \text{Ran}(\mathcal{F}(T))$ and $\dim(\text{Ker}(T)) = \dim(\text{Ker}(\mathcal{F}(T)))$. This implies that T is Fredholm if and only if $\mathcal{F}(T)$ is Fredholm. Theorem 6.1.4 implies the claim. $\square$

Even though it is not the main focus of this section, let us begin by studying the Riesz metric. As it is obtained (by definition) via the bounded transform from the norm on the bounded linear operators on $\mathcal{H}$, the following is natural and actually directly follows by combining Propositions 6.3.1 and 6.1.5.

Corollary 6.3.2. *The bounded transform*

$$\mathcal{F} : (\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R) \rightarrow (\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_N)$$

is a homeomorphism.

Corollary 6.3.2 allows deducing the next result which later on allows us to determine the homotopy groups of $(\mathbb{F}_{\text{sa}}(\mathcal{H}), d_R)$, see Theorem 8.6.1. By repeating the proof of Proposition 6.2.18 for self-adjoint operators, one obtains

Proposition 6.3.3. *The inclusion $i : (\mathbb{F}\mathbb{B}_{\text{sa}}(\mathcal{H}), \mathcal{O}_N) \rightarrow (\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R)$ is a homotopy equivalence with homotopy inverse $\mathcal{F} : (\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_R) \rightarrow (\mathbb{F}\mathbb{B}_{\text{sa}}(\mathcal{H}), \mathcal{O}_N)$.*

The remainder of this section concerns the gap topology. First, let us combine Proposition 6.3.1 with Proposition 6.1.7 which concerns the continuity properties of $\mathcal{F}$ when the gap metric d_G and the extended gap metric d_E are used. One immediately deduces

Corollary 6.3.4. *The bounded transform $\mathcal{F}$ provides two bi-Lipshitz-continuous homeomorphisms*

$$\mathcal{F} : (\mathbb{L}_{\text{sa}}(\mathcal{H}), d_G) \rightarrow (\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), d_E), \quad \mathcal{F} : (\mathbb{F}_{\text{sa}}(\mathcal{H}), d_G) \rightarrow (\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), d_E).$$

The metric space $(\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), d_E)$ was already analyzed in Section 4.6. In particular, Corollary 4.6.10 showed that $\mathcal{G} : (\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_E) \rightarrow (\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$ with $\mathcal{G}$ defined by (4.22) and

$$\mathbb{U}^0(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : \text{Ker}(U - \mathbf{1}) = \{0\}\} \quad (6.13)$$

is a homeomorphism. Moreover, Corollary 4.6.12 already stated that also the map $\mathcal{G} : (\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_E) \rightarrow (\mathbb{F}\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$ is a homeomorphism. Combining this with Corollary 6.3.4, one immediately obtains a central result of this section.

Theorem 6.3.5. *The maps*

$$\mathcal{G} \circ \mathcal{F} : (\mathbb{L}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G) \rightarrow (\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$$

and

$$\mathcal{G} \circ \mathcal{F} : (\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G) \rightarrow (\mathbb{F}\mathbb{U}^0(\mathcal{H}), \mathcal{O}_N)$$

are homeomorphisms.

Based on Theorem 6.3.5, one can then define the spectral flow of gap-continuous paths in $\mathbb{F}_{\text{sa}}(\mathcal{H})$ as the spectral flow of essentially gapped unitaries introduced in Section 4.5. This will be carried out in detail in Section 7.1 below.

Let us next compute the map $\mathcal{G} \circ \mathcal{F}$. Using spectral calculus of the self-adjoint operator $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$, one has

$$\begin{aligned} \mathcal{G} \circ \mathcal{F}(H) &= 2H^2(\mathbf{1} + H^2)^{-1} - \mathbf{1} - 2iH(\mathbf{1} + H^2)^{-\frac{1}{2}}(\mathbf{1} - H^2(\mathbf{1} + H^2)^{-1})^{\frac{1}{2}} \\ &= 2H^2(\mathbf{1} + H^2)^{-1} - \mathbf{1} - 2iH(\mathbf{1} + H^2)^{-1} \\ &= 2H(H - i\mathbf{1})(\mathbf{1} + H^2)^{-1} - \mathbf{1} \end{aligned}$$

$$= (H - i\mathbf{1})(H + i\mathbf{1})^{-1}.$$

This shows that $\mathcal{G} \circ \mathcal{F} = \mathcal{C}$, where the Cayley transform is defined by

$$\mathcal{C} : \mathbb{R} \rightarrow \mathbb{S}^1 \setminus \{1\}, \quad x \mapsto \frac{x - i}{x + i}. \quad (6.14)$$

Then the operator

$$\mathcal{C}(H) = (H - i\mathbf{1})(H + i\mathbf{1})^{-1} = \mathbf{1} - 2i(H + i\mathbf{1})^{-1} \quad (6.15)$$

is called the *Cayley transform* of H . It is a unitary operator $\mathcal{C}(H) \in \mathcal{U}(\mathcal{H})$ by the spectral theorem (this will also be proved more directly below). The mapping properties in the first formula for $\mathcal{C}(H)$ in (6.15) are given by $(H + i\mathbf{1})^{-1} : \mathcal{H} \rightarrow \text{Ran}((H + i\mathbf{1})^{-1}) = \mathcal{D}(H)$ and afterwards $H - i\mathbf{1} : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$.

Theorem 6.3.5 was deduced from the results of $\mathcal{G}$ as given in Section 4.6 combined with those on $\mathcal{F}$ given in Section 6.1. While this is clearly sufficient to go on to the definition of the spectral flow in Section 7.1, we will provide also a direct proof of Theorem 6.3.5 along the works [31, 126]. This also provides several useful metrics that are equivalent to the gap metric d_G . Moreover, these direct arguments are useful in other contexts, e. g., [38]. Let us start by analyzing the mapping properties of the Cayley transform and its inverse.

Proposition 6.3.6. *If $U \in \mathcal{U}(\mathcal{H})$ and $U - \mathbf{1}$ is injective, then $H = i(\mathbf{1} + U)(\mathbf{1} - U)^{-1}$ is self-adjoint on $\mathcal{D}(H) = \text{Ran}(\mathbf{1} - U)$. Moreover, $H = i(\mathbf{1} - U)^{-1}(\mathbf{1} + U)$.*

Proof. Since U is normal, $\text{Ker}(\mathbf{1} - U^*) = \text{Ker}(\mathbf{1} - U)$ and thus

$$\overline{\text{Ran}(\mathbf{1} - U)} = \text{Ker}(\mathbf{1} - U^*)^\perp = \text{Ker}(\mathbf{1} - U)^\perp = \mathcal{H},$$

as $\mathbf{1} - U$ is injective. Consequently, $\mathcal{D}(H) = \text{Ran}(\mathbf{1} - U)$ is dense in $\mathcal{H}$. From

$$(\mathbf{1} - U)(\mathbf{1} + U) = \mathbf{1} - U^2 = (\mathbf{1} + U)(\mathbf{1} - U), \quad (6.16)$$

it follows that

$$\begin{aligned} (\mathbf{1} + U)(\mathbf{1} - U)^{-1} &= (\mathbf{1} - U)^{-1}(\mathbf{1} - U)(\mathbf{1} + U)(\mathbf{1} - U)^{-1} \\ &= (\mathbf{1} - U)^{-1}(\mathbf{1} + U)|_{\text{Ran}(\mathbf{1} - U)}. \end{aligned} \quad (6.17)$$

On the other hand, if $\psi \in \mathcal{D}((\mathbf{1} - U)^{-1}(\mathbf{1} + U))$, then $(\mathbf{1} + U)\psi \in \mathcal{D}((\mathbf{1} - U)^{-1}) = \text{Ran}(\mathbf{1} - U)$ and accordingly there exists $\phi \in \mathcal{H}$ such that $(\mathbf{1} + U)\psi = (\mathbf{1} - U)\phi$. Thus $\psi = (\mathbf{1} - U)\phi + (\mathbf{1} - U)\psi - \psi$ and hence

$$\psi = \frac{1}{2}(\mathbf{1} - U)(\psi + \phi) \in \mathcal{D}((\mathbf{1} + U)(\mathbf{1} - U)^{-1}).$$

It follows from (6.17) that

$$H = \iota(\mathbf{1} + U)(\mathbf{1} - U)^{-1} = \iota(\mathbf{1} - U)^{-1}(\mathbf{1} + U).$$

Next, let us show that H is symmetric. If $\psi, \phi \in \mathcal{D}(H) = \text{Ran}(\mathbf{1} - U)$, then there exist $\phi', \psi' \in \mathcal{H}$ such that $\psi = \psi' - U\psi'$ and $\phi = \phi' - U\phi'$ and therefore $H\psi = \iota(\psi' + U\psi')$ and $H\phi = \iota(\phi' + U\phi')$. One gets

$$\begin{aligned} \langle \phi | H\psi \rangle &= \iota \langle \phi' - U\phi' | \psi' + U\psi' \rangle \\ &= \iota (\langle \phi' | \psi' \rangle - \langle U\phi' | \psi' \rangle + \langle \phi' | U\psi' \rangle - \langle U\phi' | U\psi' \rangle) \\ &= -\iota \langle U\phi' | \psi' \rangle + \iota \langle \phi' | U\psi' \rangle \\ &= \langle \iota(\phi' + U\phi') | \psi' - U\psi' \rangle \\ &= \langle H\phi | \psi \rangle. \end{aligned}$$

Hence H is symmetric and

$$H \subset H^* = -\iota(\mathbf{1} - U^*)^{-1}(\mathbf{1} + U^*). \quad (6.18)$$

As $U^* - \mathbf{1}$ is injective, arguing as above one gets

$$H^* = -\iota(\mathbf{1} - U^*)^{-1}(\mathbf{1} + U^*) = -\iota(\mathbf{1} + U^*)(\mathbf{1} - U^*)^{-1},$$

thus exchanging U and U^* shows that H^* is symmetric. Hence

$$H^* \subset H^{**} = \iota(\mathbf{1} - U)^{-1}(\mathbf{1} + U) = H$$

and it follows from (6.18) that $H = H^*$. □

Corollary 6.3.7. *If U and H are as in Proposition 6.3.6, then $\mathcal{C}(H) = U$. Moreover, the Cayley transform $\mathcal{C} : \mathbb{L}_{\text{sa}}(\mathcal{H}) \rightarrow \mathbb{W}^0(\mathcal{H})$ is a bijection.*

Proof. By Proposition 6.3.6, $H = \iota(\mathbf{1} + U)(\mathbf{1} - U)^{-1}$. Hence

$$H + \iota\mathbf{1} = \iota(\mathbf{1} + U)(\mathbf{1} - U)^{-1} + \iota(\mathbf{1} - U)(\mathbf{1} - U)^{-1} = 2\iota(\mathbf{1} - U)^{-1},$$

and thus

$$(H + \iota\mathbf{1})^{-1} = \frac{1}{2\iota}(\mathbf{1} - U).$$

Analogously,

$$H - \iota\mathbf{1} = \iota(\mathbf{1} + U)(\mathbf{1} - U)^{-1} - \iota(\mathbf{1} - U)(\mathbf{1} - U)^{-1} = 2\iota U(\mathbf{1} - U)^{-1},$$

and one obtains

$$\mathcal{C}(H) = (H - i\mathbf{1})(H + i\mathbf{1})^{-1} = U(\mathbf{1} - U)^{-1}(\mathbf{1} - U) = U.$$

To prove the second claim, one only has to show that $U = \mathcal{C}(H)$ is unitary and $\mathbf{1} - \mathcal{C}(H)$ injective for all $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$. It is clear that U is surjective. For $\phi \in \mathcal{D}(H)$,

$$\begin{aligned} \|H\phi + i\phi\|^2 &= \langle H\phi + i\phi | H\phi + i\phi \rangle \\ &= \|H\phi\|^2 + \|\phi\|^2 - i\langle \phi | H\phi \rangle + i\langle H\phi | \phi \rangle \\ &= \|H\phi\|^2 + \|\phi\|^2 \\ &= \|H\phi - i\phi\|^2 \end{aligned}$$

and, since $U(H\phi + i\phi) = H\phi - i\phi$, it follows that $\|U\psi\| = \|\psi\|$ for all $\psi \in \mathcal{H}$. Hence U is a surjective isometry defined on all of $\mathcal{H}$, and consequently it is a unitary operator. Now let us assume that $\psi \in \mathcal{H}$ is such that $\mathcal{C}(H)\psi = \psi$. Then one obtains from (6.15)

$$\psi = \mathcal{C}(H)\psi = \psi - 2i(H + i\mathbf{1})^{-1}\psi,$$

and hence $(H + i\mathbf{1})^{-1}\psi = 0$ which implies that $\psi = 0$. $\square$

The following connection of the spectrum of $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ to the spectrum of its image $\mathcal{C}(H) \in \mathbb{U}(\mathcal{H})$ follows from the spectral mapping theorem, but again a direct proof is provided due to its importance for the definition of the spectral flow of paths of unbounded self-adjoint Fredholm operators in Section 7.1.

Corollary 6.3.8. *If $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ and $\lambda \in \mathbb{R}$, then*

- (i) $\text{Ker}(\lambda\mathbf{1} - H) = \text{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$;
- (ii) $\text{Ran}(\lambda\mathbf{1} - H) = \text{Ran}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$;
- (iii) $\lambda \in \text{spec}(H) \iff \mathcal{C}(\lambda) \in \text{spec}(\mathcal{C}(H))$;
- (iv) $\lambda \in \text{spec}_p(H) \iff \mathcal{C}(\lambda) \in \text{spec}_p(\mathcal{C}(H))$;
- (v) $\lambda \in \text{spec}_{\text{ess}}(H) \iff \mathcal{C}(\lambda) \in \text{spec}_{\text{ess}}(\mathcal{C}(H))$.

The proof is based on the following lemma.

Lemma 6.3.9. *For $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ and $\lambda \in \mathbb{R}$, one has*

$$\lambda\mathbf{1} - H = (\lambda + i)(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1}.$$

Proof. The equality

$$\begin{aligned} \lambda\mathbf{1} - H &= \lambda\mathbf{1} - i(\mathbf{1} + \mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1} \\ &= (\lambda(\mathbf{1} - \mathcal{C}(H)) - i(\mathbf{1} + \mathcal{C}(H)))(\mathbf{1} - \mathcal{C}(H))^{-1} \\ &= (\lambda\mathbf{1} - \lambda\mathcal{C}(H) - i\mathbf{1} - i\mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1} \\ &= ((\lambda - i)\mathbf{1} - (\lambda + i)\mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1} \end{aligned}$$

$$\begin{aligned}
&= (\lambda + \iota)((\lambda - \iota)(\lambda + \iota)^{-1}\mathbf{1} - \mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1} \\
&= (\lambda + \iota)(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))(\mathbf{1} - \mathcal{C}(H))^{-1}
\end{aligned}$$

implies the claim. $\square$

Proof of Corollary 6.3.8. First of all, let us note that $(\mathbf{1} - \mathcal{C}(H))^{-1}$ maps $\mathcal{D}(H)$ bijectively onto $\mathcal{H}$ as $\mathbf{1} - \mathcal{C}(H) = 2\iota(H + \iota\mathbf{1})^{-1}$. Thus by the previous Lemma 6.3.9,

$$\begin{aligned}
\text{Ker}(\lambda\mathbf{1} - H) &= (\mathbf{1} - \mathcal{C}(H))(\text{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))) \\
&= \text{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H)),
\end{aligned}$$

where the second equality follows from the fact that $\text{Ker}(\mathcal{C}(\lambda\mathbf{1}) - \mathcal{C}(H))$ is invariant under $\mathcal{C}(H)$. This implies the assertion (i). As $(\mathbf{1} - \mathcal{C}(H))^{-1} : \mathcal{D}(H) \rightarrow \mathcal{H}$ is a bijection, Lemma 6.3.9 directly implies (ii). All other claims are immediate consequences of (i) and (ii). $\square$

Let us recall that for an operator $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ that is bounded, the spectrum of its image $\mathcal{C}(H) \in \mathbb{U}(\mathcal{H})$ does not contain 1. This is made more precise in the following statement.

Lemma 6.3.10. *For $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$, one has*

- (i) $1 \notin \text{spec}(\mathcal{C}(H)) \iff \mathcal{D}(H) = \mathcal{H}$, and this is true if and only if H is bounded.
- (ii) $1 \in \text{spec}_{\text{ess}}(\mathcal{C}(H)) \iff \mathcal{D}(H) \neq \mathcal{H}$, and this is true if and only if H is unbounded.

Proof. The assertions regarding the boundedness and unboundedness of H follow as any self-adjoint operator $H : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$ is bounded if and only if $\mathcal{D}(H) = \mathcal{H}$. By (6.15), one has

$$\mathbf{1} - \mathcal{C}(H) = 2\iota(H + \iota\mathbf{1})^{-1} \in \mathbb{B}(\mathcal{H})$$

mapping $\mathcal{H}$ bijectively onto $\mathcal{D}(H)$. Accordingly, if 1 is in the resolvent set of $\mathcal{C}(H)$, one infers $\mathcal{H} = \text{Ran}(\mathbf{1} - \mathcal{C}(H)) = \mathcal{D}(H)$. Conversely, if $\mathcal{D}(H) = \mathcal{H}$, then $\mathbf{1} - \mathcal{C}(H)$ maps $\mathcal{H}$ bijectively onto $\mathcal{H}$, showing that 1 is in the resolvent set of $\mathcal{C}(H)$. Hence assertion (i) is proved.

In order to show (ii), we note at first that by (i), $1 \in \text{spec}(\mathcal{C}(H))$ if and only if $\mathcal{D}(H) \neq \mathcal{H}$. Now it remains to show that if $1 \in \text{spec}(\mathcal{C}(H))$, then we actually have $1 \in \text{spec}_{\text{ess}}(\mathcal{C}(H))$. But, if $\mathcal{D}(H) \neq \mathcal{H}$, we see that $\text{Ran}(\mathbf{1} - \mathcal{C}(H)) = \mathcal{D}(H)$ is a proper dense subspace of $\mathcal{H}$ and hence in particular not closed. Accordingly, $\mathbf{1} - \mathcal{C}(H)$ is not a Fredholm operator and, by Corollary 3.4.4, $1 \in \text{spec}_{\text{ess}}(\mathcal{C}(H))$. $\square$

Corollary 6.3.8 implies:

Corollary 6.3.11. *If $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$, then*

- (i) $\mathcal{C}(\text{spec}(H)) = \text{spec}(\mathcal{C}(H))$ if H is bounded.
- (ii) $\mathcal{C}(\text{spec}(H)) \cup \{1\} = \text{spec}(\mathcal{C}(H))$ if H is unbounded.

Now all is prepared to state and prove the result that is essentially already contained in Theorem 6.3.5. However, as already stressed above, the result also feature a metric d_G'' on $\mathbb{L}_{\text{sa}}(\mathcal{H})$ defined by

$$d_G''(H_0, H_1) = \|\mathcal{C}(H_0) - \mathcal{C}(H_1)\|, \quad H_0, H_1 \in \mathbb{L}_{\text{sa}}(\mathcal{H}).$$

Due to (6.15), one then has

$$d_G''(H_0, H_1) = 2\|(H_0 + i\mathbf{1})^{-1} - (H_1 + i\mathbf{1})^{-1}\|, \quad H_0, H_1 \in \mathbb{L}_{\text{sa}}(\mathcal{H}). \quad (6.19)$$

Hence the following theorem shows that the gap topology can be obtained from the Cayley transform, similarly as the Riesz topology is obtained from the bounded transform in Proposition 6.1.5.

Theorem 6.3.12. *On $\mathbb{L}_{\text{sa}}(\mathcal{H})$ the gap metric d_G is equivalent to the metric d_G'' . The Cayley transform*

$$\mathcal{C} : (\mathbb{L}_{\text{sa}}(\mathcal{H}), d_G) \rightarrow (\mathbb{U}^0(\mathcal{H}), d_N)$$

is a Lipschitz-continuous homeomorphism.

Proof. Recall from Proposition 6.1.3 that d_G is equivalent to

$$d_G'(H_0, H_1) = 2\|R_{H_0} - R_{H_1}\| + \|H_0 R_{H_0} - H_1 R_{H_1}\|, \quad H_0, H_1 \in \mathbb{L}_{\text{sa}}(\mathcal{H}).$$

The identities

$$\begin{aligned} (H - i\mathbf{1})^{-1} &= (H + i\mathbf{1})(H^2 + \mathbf{1})^{-1} = HR_H + iR_H, \\ (H + i\mathbf{1})^{-1} &= (H - i\mathbf{1})(H^2 + \mathbf{1})^{-1} = HR_H - iR_H \end{aligned}$$

imply

$$\begin{aligned} R_H &= \frac{1}{2i}((H - i\mathbf{1})^{-1} - (H + i\mathbf{1})^{-1}), \\ HR_H &= \frac{1}{2}((H - i\mathbf{1})^{-1} + (H + i\mathbf{1})^{-1}). \end{aligned}$$

Therefore the metric d_G' is equivalent to the metric d_G'' as, for $H_0, H_1 \in \mathbb{L}_{\text{sa}}(\mathcal{H})$,

$$d_G''(H_0, H_1) = \|(H_0 + i\mathbf{1})^{-1} - (H_1 + i\mathbf{1})^{-1}\| + \|(H_0 - i\mathbf{1})^{-1} - (H_1 - i\mathbf{1})^{-1}\|,$$

where it was used that $\|A\| = \|A^*\|$ for all $A \in \mathbb{B}(\mathcal{H})$. Now all claims follow from Corollary 6.3.7 and (6.19). $\square$

Theorem 6.3.13. *With respect to the gap metric, the set $\mathbb{B}_{\text{sa}}(\mathcal{H})$ is dense in $\mathbb{L}_{\text{sa}}(\mathcal{H})$.*

Proof. For $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$, let the spectral resolution of H be denoted by $(E_\lambda)_{\lambda \in \mathbb{R}}$. For $n \in \mathbb{N}$, let us define the bounded self-adjoint operator

$$H_n = \int_{[-n, n]} \lambda dE_\lambda + \int_{|\lambda| > n} n \operatorname{sgn}(\lambda) dE_\lambda.$$

Then using the metric d_G'' as in (6.19), one has

$$\begin{aligned} d_G''(H, H_n) &= 2\|(H + \imath \mathbf{1})^{-1} - (H_n + \imath \mathbf{1})^{-1}\| \\ &= 2\left\| \int_{|\lambda| > n} (\lambda + \imath)^{-1} - (n \operatorname{sgn}(\lambda) + \imath)^{-1} dE_\lambda \right\| \\ &\leq \frac{4}{n}. \end{aligned}$$

Hence H_n converges to H with respect to the metric d_G'' and, by Theorem 6.3.12, also with respect to the gap metric. $\square$

Next let us focus on the set

$$\mathbb{F}_{\text{sa}}(\mathcal{H}) = \{H \in \mathbb{F}(\mathcal{H}) : H = H^*\}$$

of self-adjoint (unbounded) Fredholm operators on $\mathcal{H}$. By Corollary 6.3.8, the Cayley transform maps $\mathbb{F}_{\text{sa}}(\mathcal{H})$ bijectively onto $\mathbb{F}\mathbb{U}^0(\mathcal{H}) = \mathbb{F}\mathbb{U}(\mathcal{H}) \cap \mathbb{U}^0(\mathcal{H})$. Hence Theorem 6.3.12 also implies the second statement of Theorem 6.3.5, namely

Theorem 6.3.14. *The Cayley transform*

$$\mathcal{C} : (\mathbb{F}_{\text{sa}}(\mathcal{H}), d_G) \rightarrow (\mathbb{F}\mathbb{U}^0(\mathcal{H}), d_N)$$

is a Lipschitz-continuous homeomorphism.

Theorem 6.3.14 directly implies the following because $\mathbb{F}\mathbb{U}^0(\mathcal{H}) \subset \mathbb{U}^0(\mathcal{H})$ is open.

Corollary 6.3.15. *With respect to the gap metric, the set $\mathbb{F}_{\text{sa}}(\mathcal{H})$ is open in $\mathbb{L}_{\text{sa}}(\mathcal{H})$.*

In contrast to the set of bounded self-adjoint Fredholm operators consisting of three connected components as studied in Section 3.6, $\mathbb{F}_{\text{sa}}(\mathcal{H})$ is connected when equipped with the gap metric. Following [31], this is now proved directly by a spectral-theoretic argument. Let us note that an alternative proof, actually leading to a stronger statement, is given in Section 8.6.

Theorem 6.3.16. *With respect to the gap metric, the set $\mathbb{F}_{\text{sa}}(\mathcal{H})$ is connected.*

Proof. We show that $\mathbb{F}\mathbb{U}^0(\mathcal{H})$ is connected with respect to $\mathcal{O}_N$, which by Theorem 6.3.14 implies the claim. For $U \in \mathbb{F}\mathbb{U}^0(\mathcal{H})$, we show that there is a norm-continuous path within $\mathbb{F}\mathbb{U}^0(\mathcal{H})$ connecting U to $\imath \mathbf{1}$. First, we decompose $\mathcal{H}$ into the spectral subspaces $\mathcal{H}_\pm$ of U

corresponding to $\{e^{i\varphi} : \varphi \in [0, \pi)\}$ and $\{e^{i\varphi} : \varphi \in [\pi, 2\pi]\}$. Respectively, we decompose $U = U_+ \oplus U_-$. There is no intersection of the spectral subspaces as if $-1 \in \text{spec}(U)$ it is an isolated eigenvalue and hence belongs to $\text{spec}(U_-)$. And if $1 \in \text{spec}(U)$, it does not contribute to the decomposition of U as it is not an eigenvalue. Then by spectral deformation we contract U_+ to $i\mathbf{1}_+$ and U_- to $-i\mathbf{1}_-$ where $\mathbf{1}_\pm$ denotes the identity on $\mathcal{H}_\pm$. During this contraction, 1 does not become an eigenvalue and -1 does not become an element of the essential spectrum. Thus we have connected U to $i\mathbf{1}_+ \oplus -i\mathbf{1}_-$ within $\text{FU}^0(\mathcal{H})$.

If $\mathcal{H}_-$ is finite dimensional, we rotate $-i\mathbf{1}_-$ through -1 into $i\mathbf{1}_-$. Otherwise, we identify $\mathcal{H}_-$ with $L^2([0, 1])$. Now the multiplication operator by $-i$ on $L^2([0, 1])$ can be connected to the multiplication by the function $f : [0, 1] \rightarrow \mathbb{S}^1$, $f(t) = e^{i(\frac{3}{2}\pi + t - \frac{1}{2})}$ within the unitaries in such a way that one does not introduce spectrum at ± 1 . Then $s \in [0, \pi] \mapsto e^{is}f$ connects f to $g : [0, 1] \rightarrow \mathbb{S}^1$, $g(t) = e^{i(\frac{1}{2}\pi + t - \frac{1}{2})}$ such that -1 is not in the spectrum and 1 does not become an eigenvalue. Finally, g can be contracted to the multiplication by i . Thus, in both cases U can be connected to $i\mathbf{1}$ within $\text{FU}^0(\mathcal{H})$ completing the argument. $\square$

The following result is due to Nicolaescu [139], see also [126] and Proposition 6.3.3.

Proposition 6.3.17. *The Riesz topology on $\mathbb{F}_{\text{sa}}(\mathcal{H})$ is strictly finer than the gap topology.*

Proof. By Proposition 6.1.9, the topology induced by the Riesz metric on $\mathbb{F}_{\text{sa}}(\mathcal{H})$ is finer than the topology induced by the gap metric. In the proof of Proposition 6.1.9, a sequence $(H_n)_{n \in \mathbb{N}}$ of operators in $\mathbb{F}_{\text{sa}}(\mathcal{H})$ converging to $H \in \mathbb{F}_{\text{sa}}(\mathcal{H})$ with respect to the gap topology, but not converging with respect to the Riesz topology was constructed. This implies the claim. $\square$

Note that Proposition 6.3.17 implies, in particular, that every path in $\mathbb{F}_{\text{sa}}(\mathcal{H})$ which is continuous with respect to the Riesz metric is also continuous with respect to the gap metric. Next let us transfer the theorem of Cordes and Labrousse (see Theorem 6.1.10) to the subset of self-adjoint operators. One immediately deduces the following result (also discussed in [126]).

Corollary 6.3.18. *With respect to the gap metric, the set $\mathbb{FB}_{\text{sa}}(\mathcal{H})$ is open in $\mathbb{F}_{\text{sa}}(\mathcal{H})$. On $\mathbb{FB}_{\text{sa}}(\mathcal{H})$ the topologies induced by d_N , d_R , and d_G coincide.*

Finally, the next result is a direct consequence of Theorem 6.3.13 and Corollary 6.3.15.

Corollary 6.3.19. *With respect to the gap metric, the set $\mathbb{FB}_{\text{sa}}(\mathcal{H})$ is dense in $\mathbb{F}_{\text{sa}}(\mathcal{H})$.*

6.4 Self-adjoint Fredholm operators with compact resolvent

This section analyzes the set $\mathbb{F}_{\text{sa}}^{\text{C}}(\mathcal{H})$ of self-adjoint Fredholm operators with compact resolvent

$$\mathbb{F}_{\text{sa}}^{\text{C}}(\mathcal{H}) = \{H \in \mathbb{F}_{\text{sa}}(\mathcal{H}) : (H - i\mathbf{1})^{-1} \in \mathbb{K}(\mathcal{H})\}.$$

By the resolvent identity, the compactness of the resolvent $(H - z\mathbf{1})^{-1}$ at some other point $z \in \mathbb{C} \setminus \text{spec}(H)$ is equivalent to the compactness of $(H - i\mathbf{1})^{-1}$. Further recall that the compactness of the resolvent directly implies the Fredholm property:

Proposition 6.4.1. *Let $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ have a compact resolvent $(H - i\mathbf{1})^{-1} \in \mathbb{K}(\mathcal{H})$. Then $H \in \mathbb{F}_{\text{sa}}(\mathcal{H})$ is a Fredholm operator so that $H \in \mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H})$.*

Proof. If $H \in \mathbb{L}_{\text{sa}}(\mathcal{H})$ has a compact resolvent, then $0 \notin \text{spec}_{\text{ess}}(H)$, which, by Theorem 6.2.14, directly implies that H is a Fredholm operator. $\square$

Operators from $\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H})$ play a central role in index theory and noncommutative geometry [63] where they appear as unbounded Fredholm modules, which are also a special case of unbounded Kasparov modules (namely those representing elements from $KK(\mathbb{B}(\mathcal{H}), \mathbb{C})$ or $KK(\mathbb{C}, \mathbb{B}(\mathcal{H}))$). While both Riesz and gap topologies can be used on $\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H})$, the focus will here be on the gap topology. One of the main final results of this section is the following:

Theorem 6.4.2. *Space $(\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H}), \mathcal{O}_G)$ is homotopy equivalent to $(\mathbb{F}_{\text{sa}}(\mathcal{H}), \mathcal{O}_G)$.*

The proof of this result is surprisingly intricate and will make up a large part of the remainder of the section. While it will mainly pend on the use of the bounded transform of the set $\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H})$, let us start the analysis of the Cayley transform of $(\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H}), \mathcal{O}_G)$. Recall from Section 3.7 that $\mathbb{U}^{\mathbb{C}}(\mathcal{H})$ is the set of unitaries U with $U - \mathbf{1} \in \mathbb{K}(\mathcal{H})$ and furthermore from (6.13) that $\mathbb{U}^0(\mathcal{H})$ is the set of unitaries U with $\text{Ker}(U - \mathbf{1}) = \{0\}$. Here the intersection of these sets will appear naturally

$$\mathbb{U}^{\mathbb{C},0}(\mathcal{H}) = \{U \in \mathbb{U}(\mathcal{H}) : U - \mathbf{1} \in \mathbb{K}(\mathcal{H}), \text{Ker}(U - \mathbf{1}) = \{0\}\}.$$

Theorem 6.4.3. *The Cayley transform*

$$\mathcal{C} : (\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H}), d_G) \rightarrow (\mathbb{U}^{\mathbb{C},0}(\mathcal{H}), d_N)$$

is a Lipschitz-continuous homeomorphism.

Proof. By (6.15), the compactness of $\mathcal{C}(H) - \mathbf{1}$ and that of the resolvent are equivalent. Therefore the claim directly follows from Theorem 6.3.12 (or equivalently, Theorem 6.3.5). $\square$

Next let us consider the bounded transform of the set $\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H})$. For this purpose, let us introduce the set

$$\mathbb{FB}_{1,\text{sa}}^{\mathbb{C},0}(\mathcal{H}) = \{H \in \mathbb{FB}_{1,\text{sa}}(\mathcal{H}) : \mathbf{1} - H^2 \in \mathbb{K}(\mathcal{H}), \text{Ker}(\mathbf{1} - H^2) = \{0\}\}.$$

Note that this is a subset of $\mathbb{FB}_{1,\text{sa}}^{\mathbb{C}}(\mathcal{H})$ studied in Proposition 3.6.3, specified by the supplementary condition $\text{Ker}(\mathbf{1} - H^2) = \{0\}$.

Proposition 6.4.4. *The bounded transform $\mathcal{F}$ provides a bi-Lipshitz-continuous homeomorphism*

$$\mathcal{F} : (\mathbb{F}_{\text{sa}}^{\mathbb{C}}(\mathcal{H}), d_G) \rightarrow (\mathbb{FB}_{1,\text{sa}}^{\mathbb{C},0}(\mathcal{H}), d_E).$$

Proof. This follows from Corollary 6.3.4 by implementing the compactness condition $\mathbf{1} - H^2 \in \mathbb{K}(\mathcal{H})$. Indeed, the identity

$$H = \mathcal{F}(H)(\mathbf{1} - \mathcal{F}(H)^2)^{-\frac{1}{2}}$$

following from (6.6) implies

$$(H - i\mathbf{1})^{-1} = (\mathbf{1} - \mathcal{F}(H)^2)^{\frac{1}{2}} (\mathcal{F}(H) - i(\mathbf{1} - \mathcal{F}(H)^2)^{\frac{1}{2}})^{-1},$$

which shows that the compactness of the resolvent of H is equivalent to the compactness of $\mathbf{1} - \mathcal{F}(H)^2$ because $(\mathcal{F}(H) - i(\mathbf{1} - \mathcal{F}(H)^2)^{\frac{1}{2}})^{-1}$ is unitary and hence bounded. $\square$

For the following it is necessary to use yet another topology on $\mathbb{FB}_{1,\text{sa}}^{\mathbb{C},0}(\mathcal{H})$ and some of its supersets. The so-called strong extended gap topology on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ is defined by

$$\mathcal{O}_{SE} = \mathcal{O}(\mathcal{O}_E, \mathcal{O}_S),$$

where on the right-hand side $\mathcal{O}_E$ denotes the extended gap topology generated by d_E , $\mathcal{O}_S$ is the strong operator topology, and the remaining $\mathcal{O}$ denotes the generated topology. In other words, $\mathcal{O}_{SE}$ is the weakest (or smallest) topology on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ containing both $\mathcal{O}_E$ and $\mathcal{O}_S$. The topology $\mathcal{O}_{SE}$ was introduced in [108] under the name strict extended gap topology, but in the Hilbert space framework the strict and strong topologies coincide. The strong topology is not metrizable on the set of all bounded operators, but on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ it is metrizable. This leads to the following statement which, in particular, implies that sequential compactness and compactness are equivalent in $(\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Lemma 6.4.5. *The topology $\mathcal{O}_{SE}$ on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ is metrizable.*

Proof. It will first be shown that $\mathcal{O}_S$ is metrizable on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$. For an orthonormal basis $(\phi_n)_{n \geq 1}$ of $\mathcal{H}$, consider the metric

$$d_S(H_1, H_0) = \sum_{n=1}^{\infty} 2^{-n} \|(H_1 - H_0)\phi_n\|,$$

and let $B_\varepsilon(H_0)$ be a ball of radius $\varepsilon > 0$ in $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ with respect to d_S . Let $N \in \mathbb{N}$ be sufficiently large such that $\sum_{n=N}^{\infty} 2^{-n+1} < \frac{\varepsilon}{2}$. With $H \in \mathbb{B}_{1,\text{sa}}(\mathcal{H})$, $\psi \in \mathcal{H}$, and $\eta > 0$, the sets

$$\mathcal{U}_\eta(H, \psi) = \{H' \in \mathbb{B}_{1,\text{sa}}(\mathcal{H}) : \|H'\psi - H\psi\| < \eta\}$$

form a subbase of $\mathcal{O}_S$ and thus

$$\mathcal{V} = \bigcap_{n=1}^N \mathcal{U}_{\frac{\varepsilon}{2}}(H_0, \phi_n) \in \mathcal{O}_S.$$

It follows for $H_1 \in \mathcal{V}$ that

$$\begin{aligned} d_S(H_1, H_0) &= \sum_{n=1}^N 2^{-n} \|(H_1 - H_0)\phi_n\| + \sum_{n=N+1}^{\infty} 2^{-n} \|(H_1 - H_0)\phi_n\| \\ &\leq \frac{\varepsilon}{2} \sum_{n=1}^N 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n+1} < \varepsilon. \end{aligned}$$

Thus $\mathcal{V} \subset B_\varepsilon(H_0)$, and it is shown that every open neighborhood of H_0 in the metric topology contains an open neighborhood of H_0 in $\mathcal{O}_S$. This implies that $\mathcal{O}_S$ is finer than the metric topology.

For the converse inclusion, let us first note that $\mathcal{O}_S$ is already generated by the sets $\mathcal{U}_\eta(H, \psi)$ with ψ only taken from any dense subset of $\mathcal{H}$. Moreover, the set of all finite linear combinations of elements of $(\phi_n)_{n \geq 1}$ is dense in $\mathcal{H}$. Now let $\psi = \sum_{n=1}^N a_n \phi_n$. Then

$$\begin{aligned} \|(H_1 - H_0)\psi\| &\leq \sum_{n=1}^N |a_n| \|(H_1 - H_0)\phi_n\| \\ &\leq 2^N \max_{n=1, \dots, N} |a_n| \sum_{n=1}^{\infty} 2^{-n} \|(H_1 - H_0)\phi_n\|. \end{aligned}$$

Thus if $d_S(H_1, H_0) < \frac{\varepsilon}{2^N \max\{|a_n|\}}$, then $H_1 \in \mathcal{U}_\varepsilon(H_0, \psi)$. As these sets are a subbase of $\mathcal{O}_S$, it follows that the metric topology is finer than $\mathcal{O}_S$.

Finally, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,sa}(\mathcal{H})$. $\square$

Proposition 6.4.6. *The following pairs of topological spaces are identical:*

- (i) $(\mathbb{B}_{1,sa}^0(\mathcal{H}), \mathcal{O}_{SE})$ and $(\mathbb{B}_{1,sa}^0(\mathcal{H}), \mathcal{O}_E)$;
- (ii) $(\mathbb{F}\mathbb{B}_{1,sa}^0(\mathcal{H}), \mathcal{O}_{SE})$ and $(\mathbb{F}\mathbb{B}_{1,sa}^0(\mathcal{H}), \mathcal{O}_E)$;
- (iii) $(\mathbb{F}\mathbb{B}_{1,sa}^{C,0}(\mathcal{H}), \mathcal{O}_{SE})$ and $(\mathbb{F}\mathbb{B}_{1,sa}^{C,0}(\mathcal{H}), \mathcal{O}_E)$.

Proof. Let $(H_j)_{j \geq 1}$ be a sequence in $\mathbb{B}_{1,sa}^0(\mathcal{H})$ converging to $H \in \mathbb{B}_{1,sa}^0(\mathcal{H})$ with respect to d_E , namely $\|H_j^2 - H^2\| \rightarrow 0$ and $\|H_j(\mathbf{1} - H_j^2)^{\frac{1}{2}} - H(\mathbf{1} - H^2)^{\frac{1}{2}}\| \rightarrow 0$. One needs to show that for any $\phi \in \mathcal{H}$, one has $\|(H_j - H)\phi\| \rightarrow 0$ so that the sequence also converges strongly. As $H \in \mathbb{B}_{1,sa}^0(\mathcal{H})$, one has $\text{Ker}(\mathbf{1} - H^2) = \{0\}$ and therefore the range of $(\mathbf{1} - H^2)^{\frac{1}{2}}$ is dense in $\mathcal{H}$. Hence, for a given $\varepsilon > 0$ there exists $\psi \in \mathcal{H}$ with $\|\phi - (\mathbf{1} - H^2)^{\frac{1}{2}}\psi\| \leq \varepsilon$. Then there is a j_0 such that for $j \geq j_0$

$$\begin{aligned} \|(H_j - H)\phi\| &\leq \|(H_j(\mathbf{1} - H^2)^{\frac{1}{2}} - H(\mathbf{1} - H^2)^{\frac{1}{2}})\psi\| + 2\varepsilon \\ &\leq \|(H_j(\mathbf{1} - H^2)^{\frac{1}{2}} - H_j(\mathbf{1} - H_j^2)^{\frac{1}{2}})\psi\| + 3\varepsilon \end{aligned}$$

$$\begin{aligned}
&\leq \|((1 - H^2)^{\frac{1}{2}} - (1 - H_j^2)^{\frac{1}{2}})\psi\| + 3\epsilon \\
&\leq \|H^2 - H_j^2\|^{\frac{1}{2}} \|\psi\| + 3\epsilon,
\end{aligned}$$

where the last step follows from Proposition A.2.2. Choosing j_0 possibly even larger, this shows that $\|(H_j - H)\phi\| \leq 4\epsilon$ for all $j \geq j_0$. As ϵ was arbitrary, this shows the first claim, which directly implies the second and third. $\square$

Proposition 6.4.7. *One has the following deformation retracts:*

- (i) $(\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_{SE})$ is a deformation retract of $(\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$;
- (ii) $(\mathbb{FB}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_{SE})$ is a deformation retract of $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$;
- (iii) $(\mathbb{FB}_{1,\text{sa}}^{\mathcal{C},0}(\mathcal{H}), \mathcal{O}_{SE})$ is a deformation retract of $(\mathbb{FB}_{1,\text{sa}}^{\mathcal{C}}(\mathcal{H}), \mathcal{O}_{SE})$.

Proof. (Inspired by Proposition 2.13 in [108].) Let us focus on the proof of (ii) and later on explain that the argument also covers the cases (i) and (iii). Let $K \in \mathbb{K}(\mathcal{H})$ be a nonnegative compact operator with norm less than or equal to $\frac{1}{2}$. To construct such an operator, recall that $\mathcal{H}$ is separable and thus has a countable orthonormal basis $(\phi_n)_{n \geq 1}$. Then $K = \sum_{n \geq 1} \frac{1}{n+1} |\phi_n\rangle\langle\phi_n|$ has all the desired properties. Then define

$$f : \mathbb{FB}_{1,\text{sa}}(\mathcal{H}) \rightarrow \mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \quad f(H) = (1 - K)H(1 - K).$$

Note that $f(H)$ is indeed self-adjoint and Fredholm by the compact stability of the Fredholm operators, and that it has norm less than or equal to 1 because $\|H\| \leq 1$ and $\|1 - K\| \leq 1$. Now let ϕ be a normalized vector. Then, using again $\|H\| \leq 1$ and $\|1 - K\| \leq 1$, the Cauchy–Schwarz inequality implies

$$\begin{aligned}
\langle\phi|f(H)^2\phi\rangle^2 &= \langle(1 - K)^{\frac{1}{2}}\phi|(1 - K)^{\frac{1}{2}}H(1 - K)^2H(1 - K)\phi\rangle^2 \\
&\leq \langle\phi|(1 - K)\phi\rangle\langle\phi|(1 - K)^{\frac{1}{2}}H(1 - K)^2H(1 - K)\phi\rangle^2 \\
&\leq \langle\phi|(1 - K)\phi\rangle \\
&= 1 - \langle\phi|K\phi\rangle.
\end{aligned}$$

Therefore

$$\langle\phi|(1 - f(H)^2)\phi\rangle = 1 - \langle\phi|f(H)^2\phi\rangle \geq 1 - \sqrt{1 - \langle\phi|K\phi\rangle} > 0,$$

because K has a trivial kernel. Hence $\text{Ker}(1 - f(H)^2) = \{0\}$ so that $f(H)$ indeed lies in $\mathbb{FB}_{1,\text{sa}}^0(\mathcal{H})$.

Let us now show that f is continuous with respect to the topology $\mathcal{O}_{SE}$. Hence let $(H_j)_{j \geq 1}$ be a sequence converging to H in $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. It has to be shown that then also $(f(H_j))_{j \geq 1}$ converges to $f(H)$ in $(\mathbb{FB}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$. Clearly, $(f(H_j))_{j \geq 1}$ converges strongly to $f(H)$. For the convergence with respect to d_E , let us begin by estimating

$$\|f(H_j)^2 - f(H)^2\|$$

$$\begin{aligned}
&= \|(1-K)H_j(1-K)^2H_j(1-K) - (1-K)H(1-K)^2H(1-K)\| \\
&\leq \|(1-K)(H_j^2 - H^2)(1-K)\| \\
&\quad + \|(1-K)H_jK(21-K)H_j(1-K) - (1-K)HK(21-K)H(1-K)\| \\
&\leq \|H_j^2 - H^2\| + \|H_jK(21-K)H_j - HK(21-K)H\| \\
&\leq \|H_j^2 - H^2\| + \|(H_j - H)K(21-K)H_j\| + \|HK(21-K)(H - H_j)\| \\
&\leq \|H_j^2 - H^2\| + 4\|(H_j - H)K\|.
\end{aligned}$$

Now K can be approximated in the operator norm by a finite-dimensional matrix, namely for all $\epsilon > 0$ one can find some finite-rank operator M with $\|K - M\| < \epsilon$ (this can readily be written out explicitly from K as given above). Due to the strong convergence $s\text{-}\lim_{j \rightarrow \infty} H_j = H$, one can then find a j_0 such that $\|(H_j - H)M\| \leq \epsilon$ for all $j \geq j_0$ (this is just the standard argument showing that a strongly converging sequence of compact operators is norm convergent). Choosing j_0 possibly even larger so that also $\|H_j^2 - H^2\| < \epsilon$ for $j \geq j_0$, one then finds

$$\|f(H_j)^2 - f(H)^2\| < \epsilon + 8\epsilon + 4\|(H_j - H)M\| < 13\epsilon$$

for all $j \geq j_0$. Hence $\lim_{j \rightarrow \infty} \|f(H_j)^2 - f(H)^2\| = 0$. By a similar argument, one also checks that the second norm difference in the definition of d_E vanishes in the limit so that $\lim_{j \rightarrow \infty} d_E(f(H_j), f(H)) = 0$. In conclusion, f is a continuous map on $(\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$.

Next it will be shown that the map f is actually a homotopy inverse to the inclusion $i : \mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}) \rightarrow \mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H})$, namely both of the maps $i \circ f : \mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}) \rightarrow \mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ and $f \circ i : \mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}) \rightarrow \mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H})$ are homotopic to the identity on $(\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ and $(\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_{SE})$, respectively. One can use the homotopy $h_s(H) = (1-sK)H(1-sK)$ which is indeed continuous by similar arguments as above, and it also satisfies the inclusion $h_s(\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H})) \subset \mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H})$ so that the case of $f \circ i$ is also dealt with.

The argument directly covers item (i) and also (iii), the latter because indeed one has $f(H) \in \mathbb{F}\mathbb{B}_{1,\text{sa}}^C(\mathcal{H})$ for $H \in \mathbb{F}\mathbb{B}_{1,\text{sa}}^C(\mathcal{H})$. $\square$

Remark 6.4.8. The essence of the above proof is that the perturbation can be chosen such that it eliminates the point spectrum of H^2 at 1 for *all* H . The above proof also shows that $(\mathbb{F}\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$, and also that $(\mathbb{B}_{1,\text{sa}}^0(\mathcal{H}), \mathcal{O}_N)$ is a deformation retract of $(\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$. $\diamond$

Remark 6.4.9. In Proposition 6.4.7, on $\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ the strong essential gap topology $\mathcal{O}_{SE}$ appears. It is strictly weaker than the norm topology $\mathcal{O}_N$ on $\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H})$. This can be seen by analyzing the bounded transform of the sequence $(H_n)_{n \geq 1}$ studied in the proof of Proposition 6.1.9 and realizing that $\mathcal{F}(H_n) \rightarrow \mathcal{F}(H)$ in the strong topology. Another manifestation is that $(\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_N)$ has 3 components, while $(\mathbb{F}\mathbb{B}_{1,\text{sa}}(\mathcal{H}), \mathcal{O}_{SE})$ has one component by Theorem 6.3.16 combined with Corollary 6.3.4. $\diamond$

Example 6.4.10. Proposition 6.4.6 showed that the extended gap topology $\mathcal{O}_E$ and the strong extended gap topology $\mathcal{O}_{SE}$ coincide on $\mathbb{FB}_{1,sa}^0(\mathcal{H})$. In view of Proposition 6.4.7, one might wonder whether the same holds true for the supersets $\mathbb{FB}_{1,sa}(\mathcal{H})$ and $\mathbb{B}_{1,sa}(\mathcal{H})$. In fact, this is *not* true as shows the following example. Consider the sequence $(H_j)_{j \geq 1}$ in $\mathbb{FB}_{1,sa}(\mathcal{H})$ given by $H_j = -(1 - \frac{1}{j})\mathbf{1}$. It converges to $H = \mathbf{1}$ with respect to d_E because

$$\begin{aligned} \|(H_j)^2 - H^2\| &= \left(1 - \frac{1}{j}\right)^2 - 1 \rightarrow 0, \\ \|H_j(\mathbf{1} - H_j^2)^{\frac{1}{2}} - H(\mathbf{1} - H^2)^{\frac{1}{2}}\| &= \left(1 - \frac{1}{j}\right) \left(1 - \left(1 - \frac{1}{j}\right)^2\right)^{\frac{1}{2}} \rightarrow 0. \end{aligned}$$

However, the sequence $(H_j)_{j \geq 1}$ does not converge strongly to H as $H_j\phi \rightarrow -\phi \neq \phi = H\phi$ for all $\phi \in \mathcal{H} \setminus \{0\}$. Hence $\mathcal{O}_{SE}$ is strictly stronger than $\mathcal{O}_E$ on $\mathbb{FB}_{1,sa}(\mathcal{H})$.

Working with the same sequence, one can show that the map f defined in the proof of Proposition 6.4.7 is *not* continuous with respect to $\mathcal{O}_E$ on $\mathbb{FB}_{1,sa}(\mathcal{H})$. Indeed,

$$f(H_j) = -\left(1 - \frac{1}{j}\right)(\mathbf{1} - K)^2, \quad f(H) = f(\mathbf{1}) = (\mathbf{1} - K)^2,$$

so that

$$\|f(H_j)(\mathbf{1} - f(H_j)^2)^{\frac{1}{2}} - f(\mathbf{1})(\mathbf{1} - f(\mathbf{1})^2)^{\frac{1}{2}}\| \rightarrow \|-2(\mathbf{1} - K)^2(\mathbf{1} - (\mathbf{1} - K)^4)^{\frac{1}{2}}\|.$$

Hence $d_E(f(H_j), f(\mathbf{1}))$ does not converge to zero. The problem is that f moves all the spectrum away from ± 1 to the inside which is a discontinuous procedure at $\mathbf{1}$ in the topology $\mathcal{O}_E$ (but the sequence $(H_j)_{j \geq 1}$ does not converge to $\mathbf{1}$ with respect to $\mathcal{O}_{SE}$ and hence does not disprove continuity with respect to $\mathcal{O}_{SE}$). $\diamond$

Example 6.4.11. This example shows that the quotient topologies $\mathcal{O}_E^\sim$ and $\mathcal{O}_{SE}^\sim$ on $\mathbb{FB}_{1,sa}^{\mathcal{C},\sim}(\mathcal{H}) = \mathbb{FB}_{1,sa}^{\mathcal{C}}(\mathcal{H})/\sim$ do not coincide. Let us consider $\mathcal{H} = \ell^2(\mathbb{N})$ and the following sequence of operators from $\mathbb{FB}_{1,sa}^{\mathcal{C}}(\mathcal{H})$:

$$H_n = \sum_{k \neq 1, n} \left(1 - \frac{1}{kn}\right) |k\rangle\langle k| + \left(1 - \frac{1}{n}\right) (|1\rangle, |n\rangle)^T \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\langle 1|, \langle n|).$$

Then $\|H_n^2 - \mathbf{1}\| \rightarrow 0$ and hence $d_E(H_n, \mathbf{1}) \rightarrow 0$. Thus $\{H_n : n \geq 2\}$ is not closed with respect to $\mathcal{O}_E$. As each class $[H_n]$ with respect to $\sim$ has only one representative, it follows that also $\{[H_n] : n \geq 2\}$ is not closed with respect to $\mathcal{O}_E^\sim$. On the other hand, it will be shown that the set $\{[H_n] : n \geq 2\}$ is closed with respect to $\mathcal{O}_{SE}^\sim$. Indeed, as

$$\|\langle H_n - H_m | 1 \rangle\| = \frac{1}{\sqrt{2}} \left\| \left(-\frac{1}{n} + \frac{1}{m}\right) |1\rangle + \left(1 - \frac{1}{n}\right) |n\rangle - \left(1 - \frac{1}{m}\right) |m\rangle \right\| \geq \frac{1}{2\sqrt{2}},$$

the sequence $(H_n)_{n \geq 2}$ has no strongly convergent subsequence so that the subspace topology on $\{[H_n] : n \geq 2\}$ induced by $\mathcal{O}_{SE}$ is the discrete topology. $\diamond$

The next step will be to realize that $\mathbb{FB}_{1,sa}^C(\mathcal{H})$ is a deformation retract of $\mathbb{FB}_{1,sa}(\mathcal{H})$ by spectral analysis. This was already proved in Proposition 3.6.3, see also Remark 3.6.4, however, for the norm topology. It turns out that one can prove that this retraction is also continuous with respect to $\mathcal{O}_{SE}$. A generalization of this fact is proved in [108]. Here we provide an elementary proof.

Proposition 6.4.12. *The space $(\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE})$ is a deformation retract of the space $(\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_{SE})$.*

Proof. It will be checked that the maps in the proof of Proposition 3.6.3 are continuous with respect to $\mathcal{O}_{SE}$ so that they provide the desired retraction. For $H \in \mathbb{FB}_{1,sa}(\mathcal{H})$, let us define $\delta(H) = \min\{1, \min(\text{spec}_{\text{ess}}(H^2))^{\frac{1}{2}}\} > 0$. Then by the spectral radius theorem in the Calkin algebra, it follows that $H \mapsto \delta(H)$ is continuous with respect to $\mathcal{O}_{SE}$. For $\delta \in (0, 1]$, let now $f_\delta : [-1, 1] \rightarrow \mathbb{R}$ be the monotone continuous function defined by

$$f_\delta(x) = \chi_{[\delta, 1]}(x) - \chi_{[-1, -\delta]}(x) + \frac{x}{\delta} \chi_{(-\delta, \delta)}(x).$$

Then set $\tilde{f} : \mathbb{FB}_{1,sa}(\mathcal{H}) \rightarrow \mathbb{FB}_{1,sa}^C(\mathcal{H})$ defined by $\tilde{f}(H) = f_{\delta(H)}(H)$ and consider the linear homotopy

$$h : \mathbb{FB}_{1,sa}(\mathcal{H}) \times [0, 1] \rightarrow \mathbb{FB}_{1,sa}(\mathcal{H}), \quad h(H, t) = (1 - t)H + t\tilde{f}(H).$$

To show that this homotopy is continuous, let us first note that if a sequence $(H_n)_{n \geq 1}$ in $\mathbb{FB}_{1,sa}(\mathcal{H})$ converges to H with respect to $\mathcal{O}_{SE}$ and f is a continuous function, then also $(f(H_n))_{n \geq 1}$ converges strongly to $f(H)$. Indeed, for all even polynomials p , this follows from the convergence of $(H_n^2)_{n \geq 1}$ to H^2 in norm, while odd polynomials can be written as $H_n p(H_n)$ for an even polynomial p so that the strong convergence of H_n to H implies that $s\text{-}\lim_{n \rightarrow \infty} H_n p(H_n) = H p(H)$. Then the strong continuity for any function follows from the Weierstrass approximation theorem which can be applied since $\|H_n^2 - H^2\| \rightarrow 0$ and therefore the sequence $(H_n)_{n \geq 1}$ is bounded.

To show that the homotopy h is continuous, it is shown that for any sequence $(H_n, t_n)_{n \geq 1}$ in $\mathbb{FB}_{1,sa} \times [0, 1]$ converging to $(H, t) \in \mathbb{FB}_{1,sa} \times [0, 1]$ with respect to $\mathcal{O}_{SE} \times |\cdot|$, the sequence $h(H_n, t_n)$ converges to $h(H, t)$ with respect to $\mathcal{O}_{SE}$. By Lemma 6.4.5, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,sa}(\mathcal{H})$ where

$$d_S(H'_0, H'_1) = \sum_{n=1}^{\infty} 2^{-n} \|(H'_1 - H'_0)\phi_n\|, \quad H'_0, H'_1 \in \mathbb{B}_{1,sa}(\mathcal{H}),$$

for a fixed orthonormal basis $(\phi_n)_{n \geq 1}$ of $\mathcal{H}$ as in the proof of Lemma 6.4.5. Thus it is sufficient to show that

$$\lim_{n \rightarrow \infty} (d_E(h(H_n, t_n), h(H, t)) + d_S(h(H_n, t_n), h(H, t))) = 0. \quad (6.20)$$

The second summand is bounded by

$$\begin{aligned} d_S(h(H_n, t_n), h(H, t)) &= d_S((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H + t\tilde{f}(H)) \\ &\leq d_S((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n)) \\ &\quad + d_S((1 - t)H_n + t\tilde{f}(H_n), (1 - t)H + t\tilde{f}(H)). \end{aligned}$$

The second summand in this expression converges to 0 because $f_{\delta(H_n)}$ converges to $f_{\delta(H)}$ with respect to $\|\cdot\|_{L^\infty}$ and therefore by the first part of the above argument $\text{s-lim}_{n \rightarrow \infty} h(H_n, t) = h(H, t)$. The first summand is bounded by

$$\begin{aligned} &d_S((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n)) \\ &= \sum_{m=1}^{\infty} 2^{-m} \|(t - t_n)H_n\phi_n + (t_n - t)\tilde{f}(H_n)\phi_m\| \\ &\leq \sum_{m=1}^{\infty} 2^{-m} |t_n - t| (\|H_n\| + \|\tilde{f}(H_n)\|) \\ &\leq 2 \sum_{m=1}^{\infty} 2^{-m} |t_n - t|. \end{aligned}$$

Thus $d_S((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n))$ converges to 0 uniformly in H_n , and one concludes that $\lim_{n \rightarrow \infty} d_S(h(H_n, t_n), h(H, t)) = 0$. The first summand in (6.20) is bounded by

$$\begin{aligned} d_E(h(H_n, t_n), h(H, t)) &= d_E((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H + t\tilde{f}(H)) \\ &\leq d_E((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n)) \\ &\quad + d_E((1 - t)H_n + t\tilde{f}(H_n), (1 - t)H + t\tilde{f}(H)). \end{aligned}$$

By Lemma 6.1.8,

$$\begin{aligned} &d_E((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n)) \\ &\leq 2\sqrt{2} \|(1 - t_n)H_n + t_n\tilde{f}(H_n) - (1 - t)H_n + t\tilde{f}(H_n)\|^{\frac{1}{2}} \\ &\leq 2\sqrt{2} (|t - t_n| (\|H_n\| + \|\tilde{f}(H_n)\|))^{\frac{1}{2}} \\ &\leq 2\sqrt{2} (2|t - t_n|)^{\frac{1}{2}}. \end{aligned}$$

Thus $d_E((1 - t_n)H_n + t_n\tilde{f}(H_n), (1 - t)H_n + t\tilde{f}(H_n))$ converges to 0 uniformly in H_n . It remains to show $\lim_{n \rightarrow \infty} d_E((1 - t)H_n + t\tilde{f}(H_n), (1 - t)H + t\tilde{f}(H)) = 0$. It is therefore sufficient to show that $h_t : \mathbb{FB}_{1, \text{sa}}(\mathcal{H}) \rightarrow \mathbb{FB}_{1, \text{sa}}(\mathcal{H})$ defined by $h_t(H) = h(t, H)$ is continuous with respect to $\mathcal{O}_E$. Because h_t is a class map with respect to $\sim$, this can be checked using

the last claim of Lemma 4.6.6, namely it is sufficient to prove the continuity of the map $h_t : \mathbb{FB}_{1,sa}^{\sim}(\mathcal{H}) \rightarrow \mathbb{FB}_{1,sa}^{\sim}(\mathcal{H})$ with respect to $\mathcal{O}_E$. By Corollary 4.6.11, this is equivalent to the continuity of $\mathcal{G} \circ h_t \circ \mathcal{G}^{-1}$ on $(\mathbb{FU}(\mathcal{H}), \mathcal{O}_N)$. This is, in turn, guaranteed by the continuity of the map

$$(e^{i\varphi}, \delta) \in \mathbb{S}^1 \times (0, 1] \mapsto \mathcal{G} \circ h_{t,\delta} \circ \mathcal{G}^{-1}(e^{i\varphi}), \quad (6.21)$$

where $\varphi \in (0, 2\pi]$ and $h_{t,\delta} : [-1, 1] \rightarrow [-1, 1]$ is defined by

$$h_{t,\delta}(x) = (1-t)x + tf_{\delta}(x).$$

As

$$\mathcal{G} \circ h_{t,\delta} \circ \mathcal{G}^{-1}(e^{i\varphi}) = \mathcal{G}\left(-(1-t)\cos\left(\frac{\varphi}{2}\right) + tf_{\delta}\left(-\cos\left(\frac{\varphi}{2}\right)\right)\right) \in \mathbb{S}^1,$$

for $(e^{i\varphi}, \delta) \in \mathbb{S}^1 \times (0, 1]$ the continuity of (6.21) can readily be checked. $\square$

Corollary 6.4.13. *The space $(\mathbb{FB}_{1,sa}^{C,0}(\mathcal{H}), \mathcal{O}_{SE})$ is homotopy equivalent to the space $(\mathbb{FB}_{1,sa}^0(\mathcal{H}), \mathcal{O}_{SE})$.*

Proof. Proposition 6.4.7(iii) implies that $(\mathbb{FB}_{1,sa}^{C,0}(\mathcal{H}), \mathcal{O}_{SE})$ is homotopy equivalent to $(\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE})$, which, by Proposition 6.4.12, is homotopy equivalent to the space $(\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_{SE})$. But Proposition 6.4.7(ii) shows that the latter is homotopy equivalent to $(\mathbb{FB}_{1,sa}^0(\mathcal{H}), \mathcal{O}_{SE})$. $\square$

Proof of Theorem 6.4.2. By Proposition 6.4.6, the topologies $\mathcal{O}_E$ and $\mathcal{O}_{SE}$ coincide on both $\mathbb{FB}_{1,sa}^{C,0}(\mathcal{H})$ and $\mathbb{FB}_{1,sa}^0(\mathcal{H})$. Due to Corollary 6.4.13, one concludes that $(\mathbb{FB}_{1,sa}^{C,0}(\mathcal{H}), \mathcal{O}_E)$ and $(\mathbb{FB}_{1,sa}^0(\mathcal{H}), \mathcal{O}_E)$ are homotopy equivalent. The claim now follows from Proposition 6.4.4 and Corollary 6.3.4. $\square$

In order to further complete the analysis of the strong extended gap topology $\mathcal{O}_{SE}$ on $\mathbb{FB}_{1,sa}^C(\mathcal{H})$, let us prove that it is equivalent to the Kasparov topology as introduced by Bunke, Joachim, and Stolz [44].

Definition 6.4.14. The Kasparov topology $\mathcal{O}_K$ on $\mathbb{FB}_{1,sa}^C(\mathcal{H})$ is the weakest topology containing the strong topology $\mathcal{O}_S$ and such that the map

$$H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_K) \mapsto \mathbf{1} - H^2 \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$$

is continuous.

Proposition 6.4.15. *The strong extended gap topology $\mathcal{O}_{SE}$ on $\mathbb{FB}_{1,sa}^C(\mathcal{H})$ is identical to the Kasparov topology $\mathcal{O}_K$.*

Proof. (Following Proposition 3.3 in [108].) The extended gap topology on $\mathbb{FB}_{1,sa}^C(\mathcal{H})$ is the weakest topology such that $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H^2 \in (\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ and

$H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H(1 - H^2)^{\frac{1}{2}} \in (\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ are continuous. Clearly, the continuity of the two maps $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE}) \mapsto 1 - H^2 \in (\mathbb{FB}_{1,sa}(\mathcal{H}), \mathcal{O}_N)$ and $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE}) \mapsto H^2 \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ are equivalent. As both $\mathcal{O}_{SE}$ and $\mathcal{O}_K$ contain $\mathcal{O}_S$, it follows that $\mathcal{O}_{SE}$ is finer than $\mathcal{O}_K$.

Next let us come to the the converse. It will be used that the continuity of the map $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_K) \mapsto 1 - H^2 \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ implies, by Proposition A.2.2, also the continuity of $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_K) \mapsto (1 - H^2)^{\frac{1}{2}} \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$. Because a strongly continuous map of compact operators is norm-continuous, it follows that also the map $H \in (\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_K) \mapsto H(1 - H^2)^{\frac{1}{2}} \in (\mathbb{K}(\mathcal{H}), \mathcal{O}_N)$ is continuous, showing that $\mathcal{O}_K$ is also finer than $\mathcal{O}_{SE}$. $\square$

Next let us provide an application of the Kasparov topology. In the set $\mathbb{FB}_{1,sa}^C(\mathcal{H})$, there are two subsets with opposite properties: one is $\mathbb{FB}_{1,sa}^{C,0}(\mathcal{H})$ in which neither -1 nor 1 is an eigenvalue, the other has both as eigenvalues with infinite multiplicity,

$$\mathbb{FB}_{1,sa}^{C,\infty}(\mathcal{H}) = \{H \in \mathbb{FB}_{1,sa}^C(\mathcal{H}) : \dim(\text{Ker}(H \pm 1)) = \infty\}.$$

The analogue of Proposition 6.4.7 is the following result (that is not used for the proof of Theorem 6.4.2):

Proposition 6.4.16. *The space $(\mathbb{FB}_{1,sa}^{C,\infty}(\mathcal{H}), \mathcal{O}_{SE})$ is homotopy equivalent to the space $(\mathbb{FB}_{1,sa}^C(\mathcal{H}), \mathcal{O}_{SE})$.*

Proof. (Inspired by Lemma 2.5 of [44].) Let us denote $L^2 = L^2([0, 1]) \otimes \mathbb{C}^2$ and choose a unitary

$$U : \mathcal{H} \rightarrow L^2.$$

Further let $Q_0 = 1 \otimes \text{diag}(1, -1)$ be a proper symmetry on L^2 . Next let us introduce the unitary $W = (W_0, W_1) : L^2 \rightarrow L^2 \oplus L^2$ by

$$(W_0\psi)(x) = 2^{-\frac{1}{2}}\psi\left(\frac{x}{2}\right), \quad (W_1\psi)(x) = 2^{-\frac{1}{2}}\psi\left(\frac{x+1}{2}\right),$$

where $x \in [0, 1]$ and the 2×2 matrix component is the identity and suppressed in the notation. Then set

$$\tilde{H} = U^* W^* (U H U^* \oplus Q_0) W U$$

for $H \in \mathbb{FB}_{1,sa}^C(\mathcal{H})$. By construction, one has $\tilde{H} \in \mathbb{FB}_{1,sa}^{C,\infty}(\mathcal{H})$. It remains to construct a homotopy $h : \mathbb{FB}_{1,sa}^C(\mathcal{H}) \times [\frac{1}{2}, 1] \rightarrow \mathbb{FB}_{1,sa}^C(\mathcal{H})$ from $h_1(H) = H$ to $h_{\frac{1}{2}}(H) = \tilde{H}$, continuous with respect to $\mathcal{O}_{SE}$. For this purpose, one can now proceed using a family of partial isometries $V_t : L^2 \rightarrow L^2$ first introduced by Dixmier and Douady [73]. Set

$$(V_t\psi)(x) = \begin{cases} t^{-\frac{1}{2}}\psi\left(\frac{x}{t}\right), & x \in [0, t], \\ 0, & x \in (t, 1]. \end{cases}$$

Note that $V_t^* V_t = \mathbf{1}$ and $V_t V_t^* = \chi_{[0,t]}$ is the projection onto $L^2([0, t])$ (again tensorized with the identity on $\mathbb{C}^2$), so that, in particular, V_1 is unitary. Also $[V_t, Q_0] = 0$. Moreover, by a standard approximation argument with smooth functions, one can check that both $t \in [\frac{1}{2}, 1] \mapsto V_t$ and $t \in [\frac{1}{2}, 1] \mapsto V_t^*$ are strongly continuous. Then set

$$h_t(H) = U^*(V_t U H U^* V_t^* + (\mathbf{1} - V_t V_t^*) Q_0) U.$$

Due to $V_t^*(\mathbf{1} - V_t V_t^*) = 0$, $(\mathbf{1} - V_t V_t^*)^2 = \mathbf{1} - V_t V_t^*$ and $Q_0^2 = \mathbf{1}$, one has

$$\begin{aligned} \mathbf{1} - h_t(H)^2 &= U^*(\mathbf{1} - V_t U H U^* V_t^* V_t U H U^* V_t^* - (\mathbf{1} - V_t V_t^*)^2 Q_0^2) U \\ &= U^*(V_t V_t^* - V_t U H^2 U^* V_t^*) U \\ &= U^* V_t U (\mathbf{1} - H^2) U^* V_t^* U, \end{aligned}$$

which is compact so that indeed $h_t(H) \in \mathbb{F}\mathbb{B}_{1,\text{sa}}^{\mathbb{C}}(\mathcal{H})$. Next let us verify that h is continuous and therefore a homotopy on $(\mathbb{F}\mathbb{B}_{1,\text{sa}}^{\mathbb{C}}(\mathcal{H}), \mathcal{O}_{SE})$. For this purpose, it is shown that, for any sequence $(H_n, t_n)_{n \geq 1}$ in $\mathbb{F}\mathbb{B}_{1,\text{sa}}^{\mathbb{C}} \times [\frac{1}{2}, 1]$ converging to $(H, t) \in \mathbb{F}\mathbb{B}_{1,\text{sa}}^{\mathbb{C}} \times [\frac{1}{2}, 1]$ with respect to $\mathcal{O}_{SE} \times |\cdot|$, the sequence $h_{t_n}(H_n)$ converges to $h_t(H)$ with respect to $\mathcal{O}_{SE}$. By Lemma 6.4.5, $\mathcal{O}_{SE}$ is the topology induced by the metric $d = d_E + d_S$ on $\mathbb{B}_{1,\text{sa}}(\mathcal{H})$ where

$$d_S(H'_0, H'_1) = \sum_{n=1}^{\infty} 2^{-n} \|(H'_1 - H'_0) \phi_n\|, \quad H'_0, H'_1 \in \mathbb{B}_{1,\text{sa}}(\mathcal{H}),$$

for a fixed orthonormal basis $(\phi_n)_{n \geq 1}$ of $\mathcal{H}$ as in the proof of Lemma 6.4.5. Thus it is sufficient to show that

$$\lim_{n \rightarrow \infty} (d_E(h_{t_n}(H_n), h_t(H)) + d_S(h_{t_n}(H_n), h_t(H))) = 0. \quad (6.22)$$

The second summand is bounded by

$$d_S(h_{t_n}(H_n), h_t(H)) \leq d_S(h_{t_n}(H_n), h_t(H_n)) + d_S(h_t(H_n), h_t(H)). \quad (6.23)$$

Then

$$\lim_{n \rightarrow \infty} d_S(h_{t_n}(H_n), h_t(H)) = \lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} 2^{-m} \|U^* V_t U (H_n - H) U^* V_t^* U \phi_m\| = 0$$

because $s\text{-}\lim_{n \rightarrow \infty} H_n = H$ by assumption and $\|U^* V_t U (H_n - H) U^* V_t^* U\| \leq 2$ for all $(n, t) \in \mathbb{N} \times [\frac{1}{2}, 1]$. The first summand in (6.23) is bounded by

$$\begin{aligned} &d_S(h_{t_n}(H_n), h_t(H_n)) \\ &= \sum_{m=1}^{\infty} 2^{-m} \|U^* (V_{t_n} U H_n U^* V_{t_n}^* - V_t U H_n U^* V_t^* + Q_0 (V_t V_t^* - V_{t_n} V_{t_n}^*)) U \phi_m\| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m=1}^{\infty} 2^{-m} \|U^*(V_{t_n} UH_n U^* V_{t_n}^* - V_t UH_n U^* V_t^*) U\phi_m\| \\
&\quad + \sum_{m=1}^{\infty} 2^{-m} \|(V_t V_t^* - V_{t_n} V_{t_n}^*) U\phi_m\|.
\end{aligned}$$

The second summand converges to 0 and the first summand is bounded by

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{-m} \|U^*(V_{t_n} UH_n U^* V_{t_n}^* - V_t UH_n U^* V_t^*) U\phi_m\| \\
&\leq \sum_{m=1}^{\infty} 2^{-m} (\|V_{t_n} UH_n U^* (V_{t_n}^* - V_t^*) U\phi_m\| \\
&\quad + \|(V_{t_n} - V_t) UH_n U^* V_t^* U\phi_m\|) \\
&\leq \sum_{m=1}^{\infty} 2^{-m} (\|V_{t_n} UH_n U^* (V_{t_n}^* - V_t^*) U\phi_m\| \\
&\quad + \|(V_{t_n} - V_t) U(H_n - H) U^* V_t^* U\phi_m\| \\
&\quad + \|(V_{t_n} - V_t) UH U^* V_t^* U\phi_m\|) \\
&\leq \sum_{m=1}^{\infty} 2^{-m} (2\|(V_{t_n}^* - V_t^*) U\phi_m\| + 2\|(H_n - H) U^* V_t^* U\phi_m\| \\
&\quad + \|(V_{t_n} - V_t) UH U^* V_t^* U\phi_m\|)
\end{aligned}$$

and all three summands converge to 0 by the same argument as above using that $\text{s-lim}_{n \rightarrow \infty} V_{t_n}^* = V_t^*$ and $\text{s-lim}_{n \rightarrow \infty} H_n = H$. Finally, it remains to verify that $\lim_{n \rightarrow \infty} d_E(h_{t_n}(H_n), h_t(H)) = 0$. As $\mathcal{O}_{SE} = \mathcal{O}_K$ on $\mathbb{FB}_{1, \text{sa}}^C(\mathcal{H})$ by Proposition 6.4.15 and $\text{s-lim}_{n \rightarrow \infty} h_{t_n}(H_n) = h_t(H)$ by the above, it is sufficient to show

$$\lim_{n \rightarrow \infty} \|h_{t_n}(H_n)^2 - h_t(H)^2\| = 0.$$

This follows from

$$\begin{aligned}
\|h_{t_n}(H_n)^2 - h_t(H)^2\| &= \|V_{t_n} U(1 - H_n^2) U^* V_{t_n}^* - V_t U(1 - H^2) U^* V_t^*\| \\
&\leq \|(V_{t_n} - V_t) U(1 - H_n^2) U^* V_{t_n}^*\| \\
&\quad + \|V_t U((1 - H_n^2) - (1 - H^2)) U^* V_{t_n}^*\| \\
&\quad + \|V_t U(1 - H^2) U^* (V_{t_n}^* - V_t^*)\| \\
&\leq \|(V_{t_n} - V_t) U(1 - H_n^2)\| \\
&\quad + \|H_n^2 - H^2\| \\
&\quad + \|(1 - H^2) U^* (V_{t_n}^* - V_t^*)\| \\
&\leq \|(V_{t_n} - V_t) U(1 - H^2)\| \\
&\quad + \|(V_{t_n} - V_t) U(H^2 - H_n^2)\|
\end{aligned}$$

$$\begin{aligned}
& + \|H_n^2 - H^2\| \\
& + \|(\mathbf{1} - H^2)U^*(V_{t_n}^* - V_t^*)\|.
\end{aligned}$$

The first summand converges to 0 because $s\text{-}\lim_{n \rightarrow \infty} V_{t_n} = V_t$ and $\mathbf{1} - H^2 \in \mathbb{K}(\mathcal{H})$ is compact. Because $((V_{t_n} - V_t)U(\mathbf{1} - H^2))^* = (\mathbf{1} - H^2)U^*(V_{t_n}^* - V_t^*)$, this implies that also the last summand converges to 0. By assumption $\lim_{n \rightarrow \infty} d_E(H_n, H) = 0$ and by definition of the extended gap metric, this implies that the third summand converges to 0 and therefore also the second summand converges to 0. One concludes that (6.22) holds and therefore h is continuous.

It only remains to note that indeed $h_1(H) = H$ and $h_1(H) = \tilde{H}$, and furthermore $h_t(H) \in \mathbb{FB}_{1,\text{sa}}^{C,\infty}(\mathcal{H})$ for all $H \in \mathbb{FB}_{1,\text{sa}}^{C,\infty}(\mathcal{H})$ and all $t \in [\frac{1}{2}, 1]$. Therefore also the map $f : \mathbb{FB}_{1,\text{sa}}^C(\mathcal{H}) \rightarrow \mathbb{FB}_{1,\text{sa}}^{C,\infty}(\mathcal{H})$ defined by $f(H) = \tilde{H}$ is a homotopy equivalence with respect to $\mathcal{O}_{SE}$ with homotopy inverse given by the embedding $i : \mathbb{FB}_{1,\text{sa}}^{C,\infty}(\mathcal{H}) \rightarrow \mathbb{FB}_{1,\text{sa}}^C(\mathcal{H})$. $\square$

The following is a direct consequence of Theorems 6.4.2 and 6.3.16.

Theorem 6.4.17. *With respect to the gap metric, the set $\mathbb{F}_{\text{sa}}^C(\mathcal{H})$ is connected.*

Because this result may seem surprising at first sight, a direct proof is provided.

Proof. It is shown that $\mathbb{U}^{C,0}(\mathcal{H})$ is connected, which, by Theorem 6.4.3, implies the claim. For $U \in \mathbb{U}^{C,0}(\mathcal{H})$, a norm-continuous path within $\mathbb{U}^{C,0}(\mathcal{H})$ connecting U to

$$U_{\text{ref}} = \sum_{n \geq 1} e^{\frac{i}{n+1}} |\phi_n\rangle \langle \phi_n|,$$

where $(\phi_n)_{n \geq 1}$ is an orthonormal basis of $\mathcal{H}$, is constructed. Note that $U_{\text{ref}} = e^{iK_{\text{ref}}}$ for the self-adjoint and compact operator $K_{\text{ref}} = \sum_{n \geq 1} \frac{1}{n+1} |\phi_n\rangle \langle \phi_n|$.

First, let us decompose $\mathcal{H}$ into the spectral subspaces $\mathcal{H}_{\pm}$ of U corresponding to $\{e^{i\varphi} : \varphi \in [0, \pi]\}$ and $\{e^{i\varphi} : \varphi \in (\pi, 2\pi]\}$. Respectively, we decompose $U = U_+ \oplus U_-$. There is no intersection of the spectral subspaces as, if $-1 \in \text{spec}(U)$, it is an isolated eigenvalue and hence belongs to $\text{spec}(U_+)$. And if $1 \in \text{spec}(U)$, it does not contribute to the decomposition of U as it is not an eigenvalue.

If $\mathcal{H}_-$ is finite dimensional, we rotate U_- through -1 into $U'_- = -U_-$. More precisely, the path $t \in [0, 1] \mapsto e^{-i\pi t} U_-$ lies entirely in $\mathbb{U}^{C,0}(\mathcal{H}_-)$ and connects U_- to U'_- where $\text{spec}(U'_-) \subset \{e^{i\varphi} : \varphi \in (0, \pi)\}$. Otherwise, we identify $\mathcal{H}_-$ with $L^2([0, 1])$. Then U_- is of the form $U_- = e^{iK_-}$ for some self-adjoint injective compact operator $K_- \in \mathbb{K}(L^2([0, 1]))$ with $\text{spec}(K_-) \subset (-\pi, 0]$. For $t \in [0, 2]$, let $M_t \in \mathbb{B}(L^2([0, 1]))$ denote the multiplication operator given by multiplication with the function $f_t : [0, 1] \rightarrow [0, 1]$ defined by

$$f_t(x) = (-1 + 2tx)\chi_{[0,1]}(t) + (2(1-x)(t-2) + 1)\chi_{(1,2]}(t)$$

for $t \in [0, 2]$ and $x \in [0, 1]$. Then

$$t \in [0, 2] \mapsto K_t = -|K_-|^{\frac{1}{2}} M_t |K_-|^{\frac{1}{2}}$$

is a continuous path of injective compact operators connecting K_- to $-K_-$ such that $\|K_t\| \leq \|K_-\| < \pi$ for all $t \in [0, 2]$. Therefore the path $t \in [0, 2] \mapsto e^{iK_t}$ lies in $\mathcal{U}^{C,0}(\mathcal{H})$ and connects U_- to $U'_- = e^{-iK_-}$.

In both cases taking the pointwise direct sum of the constructed path and the constant path $t \mapsto U_+$ gives a path in $\mathcal{U}^{C,0}(\mathcal{H})$ connecting U to $U_+ \oplus U'_-$ with spectrum satisfying $\text{spec}(U_+ \oplus U'_-) \subset \{e^{i\varphi} : \varphi \in [0, \pi]\}$. Then there is an injective compact operator $K \in \mathcal{K}(\mathcal{H})$ with $\text{spec}(K) \subset [0, \pi]$ such that $U_+ \oplus U'_- = e^{iK}$. The linear path $t \in [0, 1] \mapsto K'_t = (1-t)K + tK_{\text{ref}}$ connecting K to K_{ref} is within the injective compact operators with spectrum $\text{spec}(K_t) \subset [0, \pi]$. Therefore the path $t \in [0, 1] \mapsto e^{iK'_t}$ is within $\mathcal{U}^{C,0}(\mathcal{H})$ and connects $U_+ \oplus U'_-$ to U_{ref} . Thus U can be connected to U_{ref} within $\mathcal{U}^{C,0}(\mathcal{H})$, which implies the claim. $\square$