5 Fredholm pairs and their index

This chapter is about Fredholm pairs of projections and their index, a concept introduced by Kato [112], and independently also by Brown, Douglas, and Fillmore [42] where the index is called essential codimension. Section 5.2 gives different characterizations of Fredholm pairs of projections and collects basic facts about them, to a large extend following the influential work by Avron, Seiler, and Simon [18]. It avoids to use the orthogonality of the projections, and supplementary aspects linked to self-adjointness are then regrouped in Section 5.3. Section 5.4 then accesses the same Fredholm concept from the point of view of symmetry operators which provides yet another formula for the index which readily allows connecting it to the spectral flow later on. Section 5.5 focusses on a special type of Fredholm pairs where one projection is unitary conjugate to the other. Sections 5.6, 5.7, and 5.8 provide several formulas connecting the spectral flow to the index of a Fredholm pair of projections. In particular, the spectral flow of a path of self-adjoint Fredholm operators is expressed as the sum of indices of pairs of projections. The chapter concludes by introducing the relative Morse index in Section 5.9 and giving a formula for the spectral flow as sum of relative Morse indices, as in the work of Fitzpatrick, Pejsachowicz, and Recht [84].

5.1 Projections and orthogonal projections

This short section merely reviews some well-known basic definitions and facts about projections, frames, and the action of invertible operators thereon.

Definition 5.1.1. Let $P \in \mathbb{B}(\mathcal{H})$.
(i) P is called a projection if $P^2 = P$.
(ii) A projection P is called orthogonal if, moreover, $P = P^*$.
(iii) A projection P is called finite or finite dimensional if $\dim(\mathrm{Ran}(P)) < \infty$.
(iv) A projection P is called proper if $\dim(\mathrm{Ker}(P)) = \dim(\mathrm{Ran}(P)) = \infty$.
(v) The complementary projection of a projection P is $\mathbf{1}-P$ and it is denoted by $P^\perp = \mathbf{1}-P$.

The set of all proper orthogonal projections on $\mathcal{H}$ is denoted by $\mathbb{P}(\mathcal{H})$.

In a large part but not nearly all of the literature, projections are called idempotent (as all powers are the same) and orthogonal projections are called projections. We hope that the reader can get accustomed to Definition 5.1.1. From $P = P^2$ one gets $\|P\| \leq \|P\|^2$ so that $\|P\| \geq 1$ for every projection $P \neq 0$. However, nonvanishing orthogonal projections always have norm 1.

There is a tight connection between closed subspaces $\mathcal{E} \subset \mathcal{H}$ of $\mathcal{H}$ and orthogonal projections. In fact, for any $P \in \mathbb{P}(\mathcal{H})$ the range $\mathrm{Ran}(P) = \mathrm{Ker}(\mathbf{1} - P)$ is a closed subspace, and given a closed subspace, there is always an associated orthogonal projection. For this reason, $\mathbb{P}(\mathcal{H})$ is also called the (closed proper) Grassmannian of $\mathcal{H}$. Furthermore,

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given a projection P (not necessarily orthogonal), one can always construct two naturally associated orthogonal projections: the range projection P_{R} onto $\mathrm{Ran}(P) = \mathrm{Ker}(\mathbf{1}-P)$ and the kernel projection P_{K} onto $\mathrm{Ker}(P) = \mathrm{Ran}(\mathbf{1}-P)$.

Proposition 5.1.2. *The range and kernel projection associated to a projection P satisfy*

$$\mathrm{Ran}(P_{\mathrm{K}}) \cap \mathrm{Ran}(P_{\mathrm{R}}) = \{0\}, \quad \mathrm{Ran}(P_{\mathrm{K}}) + \mathrm{Ran}(P_{\mathrm{R}}) = \mathcal{H}, \tag{5.1}$$

and are given by

$$P_{\mathrm{R}} = P(P^*P)^{-1}P^*, \quad P_{\mathrm{K}} = P^{\perp}((P^{\perp})^*P^{\perp})^{-1}(P^{\perp})^*. \tag{5.2}$$

Then one has

$$P = P_{\mathrm{R}}(P_{\mathrm{K}}^{\perp}P_{\mathrm{R}})^{-1}P_{\mathrm{K}}^{\perp}. \tag{5.3}$$

Inversely, given two orthogonal projections P_{R} and P_{K} satisfying (5.1)*, formula* (5.3) *defines a projection with range projection P_{R} and kernel projection P_{K}.*

Proof. Both claims in (5.1) follow from the well-known fact that each vector $\phi \in \mathcal{H}$ can be uniquely decomposed into $\phi = \phi_{\mathrm{R}}+\phi_{\mathrm{K}}$ with $P\phi_{\mathrm{R}} = \phi_{\mathrm{R}}$ and $P\phi_{\mathrm{K}} = 0$. In the first formula of (5.2), note that P^*P is not an invertible operator, however, it maps $\mathrm{Ker}(P)^{\perp} = \mathrm{Ran}(P^*)$ bijectively onto $\mathrm{Ran}(P^*)$. Hence $P^*P : \mathrm{Ran}(P^*) \to \mathrm{Ran}(P^*)$ is an invertible operator by the inverse mapping theorem. Thus $P(P^*P)^{-1}P^*$ is well defined, and one readily sees that it is indeed an orthogonal projection, with range given by $\mathrm{Ran}(P)$. The formula for P_{K} can be verified in the same manner. To check (5.3), one notes that

$$P_{\mathrm{K}}^{\perp}P_{\mathrm{R}} : \mathrm{Ker}(P_{\mathrm{R}})^{\perp} = \mathrm{Ran}(P_{\mathrm{R}}) \to \mathrm{Ran}(P_{\mathrm{K}}^{\perp}) = \mathrm{Ker}(P_{\mathrm{K}})$$

is a bijection. Indeed, if $\phi \in \mathrm{Ran}(P_{\mathrm{R}})$, then $P_{\mathrm{R}}\phi = \phi$ so that $0 = P_{\mathrm{K}}^{\perp}P_{\mathrm{R}}\phi = P_{\mathrm{K}}^{\perp}\phi$ implies $\phi \in \mathrm{Ker}(P_{\mathrm{K}}^{\perp}) = \mathrm{Ran}(P_{\mathrm{K}})$, and hence $\phi = 0$ by (5.1); moreover, if $\psi = P_{\mathrm{K}}^{\perp}\psi \in \mathrm{Ran}(P_{\mathrm{K}}^{\perp})$, then by (5.1) one can decompose uniquely $\psi = \psi^{\perp} + P_{\mathrm{R}}\phi$ with $\psi^{\perp} \in \mathrm{Ran}(P_{\mathrm{K}}) = \mathrm{Ker}(P_{\mathrm{K}}^{\perp})$ and some $\phi = P_{\mathrm{R}}\phi \in \mathrm{Ran}(P_{\mathrm{R}})$, so that $\psi = P_{\mathrm{K}}^{\perp}(\psi^{\perp}+P_{\mathrm{R}}\phi) = P_{\mathrm{K}}^{\perp}P_{\mathrm{R}}\phi$. Again the inverse mapping theorem implies that $(P_{\mathrm{K}}^{\perp}P_{\mathrm{R}})^{-1} : \mathrm{Ran}(P_{\mathrm{K}}^{\perp}) \to \mathrm{Ran}(P_{\mathrm{R}})$ is well defined, and then one can check that (5.3) holds. The last claim follows from the above argument. □

Remark 5.1.3. There is an alternative way to write out the range projection, namely it will be checked that

$$P_{\mathrm{R}} = P(\mathbf{1} - (P - P^*)^2)^{-1}P^*.$$

Note that $-(P - P^*)^2 = (P - P^*)^*(P - P^*) \geq 0$, which implies that the inverse exists. Moreover, an explicit computation shows that P commutes with $\mathbf{1} - (P - P^*)^2$ and thus so does P^*. Furthermore, $PP^*P = P(\mathbf{1} - (P - P^*)^2)$. Now let P_K' denote the right-hand side

$P(\mathbf{1} - (P - P^*)^2)^{-1}P^*$. Combining the above facts allows to check $(P'_{\mathrm{R}})^2 = P'_{\mathrm{R}}$ and, clearly, also $(P'_{\mathrm{R}})^* = P'_{\mathrm{R}}$. As $\mathrm{Ran}(P'_{\mathrm{R}}) = \mathrm{Ran}(P)$, this implies that $P'_{\mathrm{R}} = P_{\mathrm{R}}$. Similarly,

$$P_{\mathrm{K}} = (\mathbf{1} - P)\big(\mathbf{1} - (P - P^*)^2\big)^{-1}(\mathbf{1} - P)^*,$$

which follows from the above applied to $\mathbf{1}-P$, or can be checked in the same manner. ◇

Corollary 5.1.4. *Every projection can be connected to its range projection within the set of projections.*

Proof. Note that $P_{\mathrm{R}} = P(P^*P)^{-1}P^*$ satisfies $P_{\mathrm{R}}P = P$ and $PP_{\mathrm{R}} = P_{\mathrm{R}}$. Therefore one readily checks that

$$t \in [0,1] \mapsto P_t = (1 - t)P + tP_{\mathrm{R}}$$

is indeed a path of projections connecting P to P_{R}. □

Next let us introduce the concept of a frame. While this was already used in Chapter 2, let us here give a precise definition for the case of infinite-dimensional Hilbert spaces.

Definition 5.1.5. A frame is a bounded injective linear map $\Phi : \mathfrak{h} \to \mathcal{H}$ with closed range, from an auxiliary Hilbert space $\mathfrak{h}$ into $\mathcal{H}$. The frame is called normalized if $\Phi^*\Phi = \mathbf{1}_{\mathfrak{h}}$. Furthermore, $\Phi^\perp : \mathfrak{h}' \to \mathcal{H}$ denotes a frame with $\mathrm{Ran}(\Phi^\perp) = \mathrm{Ran}(\Phi)^\perp$.

Given a frame Φ, one can always associate an orthogonal projection onto its range by

$$P = \Phi\big(\Phi^*\Phi\big)^{-1}\Phi^*. \tag{5.4}$$

Note that this is well defined because $\Phi^*\Phi : \mathfrak{h} \to \mathfrak{h}$ is invertible. Let us then also say that Φ is a frame for P. If, moreover, Φ is normalized, the formula reduces to $P = \Phi\Phi^*$. One particular frame for P is always given by choosing $\mathfrak{h} = \mathrm{Ran}(P)$ and Φ the embedding. Another standard way to construct normalized frames, say for an infinite-dimensional projection P, is to choose an orthonormal basis $(\phi_n)_{n\geq1}$ of $\mathrm{Ran}(P)$ and then set $\mathfrak{h} = \ell^2(\mathbb{N})$ and

$$\Phi = \sum_{n\geq1} |\phi_n\rangle\langle n|.$$

Note, however, that there are many frames for a given P. Indeed, given a frame Φ for P and any invertible map $a \in \mathbb{B}(\mathfrak{h})$, also Φa is a frame for P. Furthermore, if Φ is normalized and $u \in \mathbb{B}(\mathfrak{h})$ is unitary, also Φu is normalized. Let us also note that, clearly, $\Phi^*\Phi^\perp = 0$. Finally, $(\Phi, \Phi^\perp) : \mathfrak{h} \oplus \mathfrak{h}' \to \mathcal{H}$ is an isomorphism which is unitary if both Φ

and $\Phi^\perp$ are normalized. Now one can also use frames to write out an arbitrary (not necessarily orthogonal) projection, analogous to Proposition 5.1.2. The proof is essentially the same and therefore skipped.

Proposition 5.1.6. *Let P be a projection and Φ_{R} and Φ_{K} be frames for P_{R} and P_{K}. Then*

$$P = \Phi_{\mathrm{R}}\big((\Phi_{\mathrm{K}}^\perp)^*\Phi_{\mathrm{R}}\big)^{-1}(\Phi_{\mathrm{K}}^\perp)^*. \tag{5.5}$$

Inversely, given two frames Φ_{R} and Φ_{K} satisfying

$$\mathrm{Ran}(\Phi_{\mathrm{K}}) \cap \mathrm{Ran}(\Phi_{\mathrm{R}}) = \{0\}, \quad \mathrm{Ran}(\Phi_{\mathrm{K}}) + \mathrm{Ran}(\Phi_{\mathrm{R}}) = \mathcal{H}, \tag{5.6}$$

formula (5.5) *defines a projection with range and kernel projection given as in* (5.4).

To illustrate the use of frames, let us prove a result that will be used several times later on.

Proposition 5.1.7. *If P_0 and P_1 are proper orthogonal projections, then there exists a unitary U such that $P_1 = U^*P_0U$.*

Proof. Let Φ_0 and Φ_1 be normalized frames for P_0 and P_1, respectively. Then

$$V = \Phi_1\Phi_0^*$$

is a partial isometry from $\mathrm{Ran}(P_0)$ to $\mathrm{Ran}(P_1)$, namely $V^*V = P_0$ and $VV^* = P_1$. Similarly, let W be a partial isometry satisfying $W^*W = \mathbf{1} - P_0$ and $WW^* = \mathbf{1} - P_1$. Multiplying two of these identities shows $VV^*WW^* = 0$ so that $V^*W = 0$ and $V^*VW^*W = 0$ so that $VW^* = 0$. Hence $U = V^* + W^*$ is a unitary because $UU^* = V^*V + W^*W = P_0 + \mathbf{1} - P_0 = \mathbf{1}$ and $U^*U = \mathbf{1}$. By construction, $P_1 = U^*P_0U$. □

In the remainder of this section, the action of an invertible operator $T \in \mathbb{G}(\mathcal{H})$ on projections will be introduced and studied. Let us first begin with the action on an orthogonal projection P. Then the formula

$$T \cdot P = (TPT^*)(TPT^*)^{-2}(TPT^*) \tag{5.7}$$

is well defined because $TPT^* : \mathrm{Ran}(TP) \to \mathrm{Ran}(TP)$ is invertible (even though TPT^* is not invertible as an operator on all $\mathcal{H}$). Clearly, $T \cdot P$ is the orthogonal projection onto

$$T\,\mathrm{Ran}(P) = \mathrm{Ran}(T \cdot P),$$

and one has

$$\begin{aligned}\mathrm{Ker}(T \cdot P) &= \{\phi \in \mathcal{H} : PT^*\phi = 0\}\\ &= (T^*)^{-1}\{T^*\phi \in \mathcal{H} : PT^*\phi = 0\}\end{aligned}$$

$$= (T^*)^{-1}\,\mathrm{Ker}(P).$$

Moreover, (5.7) defines a group action of the group $\mathbb{G}(\mathcal{H})$ on the set of orthogonal projections, namely one has $S\cdot(T\cdot P) = (ST)\cdot P$ for $S, T \in \mathbb{G}(\mathcal{H})$. Let us also note that for the subgroup $\mathbb{U}(\mathcal{H}) \subset \mathbb{G}(\mathcal{H})$ of unitary operators, the action reduces to $U \cdot P = UPU^* = UPU^{-1}$. Another property worth mentioning is that

$$(T\cdot P)^\perp = (T^*)^{-1}\cdot P^\perp. \tag{5.8}$$

Indeed, both sides are orthogonal projections, and one has

$$\mathrm{Ran}\big((T\cdot P)^\perp\big) = \mathrm{Ker}(T\cdot P) = (T^*)^{-1}\,\mathrm{Ker}(P) = (T^*)^{-1}\,\mathrm{Ran}(P^\perp).$$

Furthermore, if $P = \Phi(\Phi^*\Phi)^{-1}\Phi^*$ is given in terms of a frame as in (5.4), then $T\Phi$ is a frame for $T\cdot P$ and therefore

$$T\cdot P = T\Phi\big(\Phi^* T^* T\Phi\big)^{-1}\Phi^* T^*. \tag{5.9}$$

Based on this, there is an alternative way to verify (5.8) by checking that $T\cdot P$ is orthogonal to $(T^*)^{-1}\cdot P^\perp$.

While it is not possible to extend the action (5.7) to projections that are not orthogonal, one can define another group action of $\mathbb{G}(\mathcal{H})$ by $(T, P) \mapsto TPT^{-1}$. One readily checks that this is indeed well defined and is a group action on all projections. When restricted to the unitary group $\mathbb{U}(\mathcal{H}) \subset \mathbb{G}(\mathcal{H})$, this action coincides with (5.7). In general, however, it does not conserve the orthogonality of projections. This second action will be used at several instances below, e. g., Proposition 5.2.9.

5.2 Characterization of Fredholm pairs of projections

The definition of Fredholm pairs of projections and many of the results of this section and the next sections are due to Kato [112, Chapter IV.4] and Avron, Seiler, and Simon [18], see also [3].

Definition 5.2.1. Let (P_0, P_1) be a pair of projections and consider the operator

$$A : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$$

defined by

$$A\phi = P_1P_0\phi, \quad \phi \in \mathrm{Ran}(P_0).$$

Then (P_0, P_1) is a Fredholm pair of projections if and only if A is a Fredholm operator. The index of a Fredholm pair (P_0, P_1) of projections is defined by

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Ind}(A).$$

For the case of two orthogonal projections, it will be shown in Proposition 5.3.2 below that for a Fredholm pair the projections P_0 and $\mathbf{1}-P_1$ are complementary up to finite-dimensional defects, in the sense that $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1)$ has finite codimension and $\mathrm{Ran}(P_0) \cap \mathrm{Ran}(\mathbf{1} - P_1)$ is finite dimensional. Of course, in interesting cases both $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ are infinite dimensional. If they are both finite dimensional, then the index is simply given by the difference of the dimensions of the ranges, as shown next.

Proposition 5.2.2. *Let P_0 and P_1 be two finite-dimensional projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair of projections with index*

$$\mathrm{Ind}(P_0, P_1) = \dim(\mathrm{Ran}(P_0)) - \dim(\mathrm{Ran}(P_1)).$$

Proof. Consider the linear operator $A = P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$. By the rank theorem,

$$\dim(\mathrm{Ran}(P_0)) = \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ran}(A)).$$

Moreover,

$$\dim(\mathrm{Ran}(A)) + \dim(\mathrm{Ran}(A)^\perp) = \dim(\mathrm{Ran}(P_1)),$$

where the orthogonal complement is taken in the Hilbert space $\mathrm{Ran}(P_1)$. Hence from the definition of the index,

$$\begin{aligned}
\mathrm{Ind}(A) &= \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*)) \\
&= \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ran}(A)^\perp) \\
&= \dim(\mathrm{Ker}(A)) + \dim(\mathrm{Ran}(A)) - (\dim(\mathrm{Ran}(A)^\perp) + \dim(\mathrm{Ran}(A))) \\
&= \dim(\mathrm{Ker}(A)) - (\dim(\mathrm{Ran}(P_1)) - \dim(\mathrm{Ran}(A))) \\
&= \dim(\mathrm{Ran}(P_0)) - \dim(\mathrm{Ran}(P_1)),
\end{aligned}$$

concluding the proof. □

Remark 5.2.3. Let us suppose, just for this remark, that $\mathcal{H} = \mathbb{C}^{2N}$ is finite dimensional with Krein quadratic form $J = \mathrm{diag}(\mathbf{1}_N, -\mathbf{1}_N)$ and that P_0 and P_1 project on two J-Lagrangian subspaces, as defined in Chapter 2. Then

$$\dim(\mathrm{Ran}(P_0)) = N = \dim(\mathrm{Ran}(P_1))$$

and hence $\mathrm{Ind}(P_0, P_1) = 0$ by Proposition 5.2.2. This remains true in the infinite-dimensional setting, see Proposition 9.4.7. ◇

The most elementary example of a Fredholm pair arises as follows:

Proposition 5.2.4. *Let P_0 and P_1 be two projections such that $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$ is compact. Then (P_0, P_1) and (P_1, P_0) are both Fredholm pairs of projections.*

Proof. Set $A_0 = P_0P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_0)$ and $A_1 = P_1P_0P_1 : \mathrm{Ran}(P_1) \to \mathrm{Ran}(P_1)$. Then

$$A_0 = P_0 + P_0(P_1 - P_0)P_0 = \mathbf{1}_{\mathrm{Ran}(P_0)} + P_0(P_1 - P_0)P_0$$

is a compact perturbation of the identity on $\mathrm{Ran}(P_0)$ and hence a Fredholm operator (with vanishing index). Hence $\mathrm{Ker}(A) \subset \mathrm{Ker}(A_0)$ is finite dimensional. Similarly, A_1 is a Fredholm operator so that also $\mathrm{Ran}(A) \supset \mathrm{Ran}(A_1)$ has finite codimension. Hence (P_0, P_1) is a Fredholm pair. For (P_1, P_0), one argues in the same way, namely exchanges P_0 and P_1 in the above. □

Remark 5.2.5. In general, it is not true that the Fredholm property of (P_0, P_1) implies that also (P_1, P_0) is a Fredholm pair. Let us illustrate this with an example on an infinite-dimensional Hilbert space of the form $\mathcal{H} \oplus \mathcal{H}$. Two projections are given by

$$P_0 = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix}, \quad P_1 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}\Bigg|_{\mathrm{Ran}(P_0)}$$

is surjective, namely $\mathrm{Ran}(A) = \mathrm{Ran}(P_1)$. As $\mathrm{Ker}(A) = \{0\}$, A is Fredholm and therefore (P_0, P_1) is a Fredholm pair (with vanishing index). On the other hand, $P_0P_1 = 0$ and therefore (P_1, P_0) is *not* a Fredholm pair. ◇

Next let us come to some basic properties of Fredholm pairs of projections. First of all, Fredholm pairs have a natural transformation property under invertible linear maps, namely if (P_0, P_1) is a Fredholm pair of projections and $T \in \mathbb{G}(\mathcal{H})$ an invertible operator, then (TP_0T^{-1}, TP_1T^{-1}) is a Fredholm pair of projections and

$$\mathrm{Ind}\big(TP_0T^{-1}, TP_1T^{-1}\big) = \mathrm{Ind}(P_0, P_1). \tag{5.10}$$

Secondly, one has the following concatenation formula for the index of Fredholm pairs.

Proposition 5.2.6. *Suppose given three projections P_0, P_1 and P_2 such that (P_0, P_1) is a Fredholm pair and $P_2 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact (or vice versa). Then also (P_1, P_2) and (P_0, P_2) are Fredholm pairs and*

$$\mathrm{Ind}(P_0, P_2) = \mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2). \tag{5.11}$$

Proof. By Proposition 5.2.4, (P_1, P_2) is a Fredholm pair. Now consider the equality

$$P_2P_0 = (P_2)^2P_0 = P_2P_1P_0 + P_2(P_2 - P_1)P_0 = (P_2P_1)(P_1P_0) + P_2(P_2 - P_1)P_0,$$

as operators from $\mathrm{Ran}(P_0)$ to $\mathrm{Ran}(P_2)$. Then $(P_2P_1)(P_1P_0) : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is a concatenation of two Fredholm operators with index given by $\mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2)$. As

$P_2(P_2 - P_1)P_0$ is compact by hypothesis, also $P_2P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is a Fredholm operator with the same index by Theorem 3.3.4. □

Next let us show a stability result for the index of Fredholm pairs.

Proposition 5.2.7. *Let $t \in [0,1] \mapsto P_0(t)$ and $t \in [0,1] \mapsto P_1(t)$ be norm-continuous paths of projections such that $(P_0(t), P_1(t))$ is a Fredholm pair for every $t \in [0,1]$. Then $t \in [0,1] \mapsto \mathrm{Ind}(P_0(t), P_1(t))$ is constant.*

Proof. It is clearly sufficient to prove local constancy of the index. Hence let us fix some $t_0 \in [0,1]$ and consider the paths

$$B_j(t) = \mathbf{1} - P_j(t) + P_j(t_0), \quad t \in [0,1], \quad j = 0,1.$$

As the set of invertibles is open in $\mathbb{B}(\mathcal{H})$, there exists a neighborhood N of t_0 such that $B_j(t)$ is invertible for $t \in N$. Consequently, the restrictions $C_j(t) = B_j(t)|_{\mathrm{Ran}(P_j(t))}$ map $\mathrm{Ran}(P_j(t))$ bijectively onto $\mathrm{Ran}(P_j(t_0))$. Thus

$$C_1(t) \circ P_1(t) \circ C_0(t)^{-1} : \mathrm{Ran}\big(P_0(t_0)\big) \to \mathrm{Ran}\big(P_1(t_0)\big),$$

are Fredholm operators with index

$$\mathrm{Ind}\big(C_1(t) \circ P_1(t) \circ C_0(t)^{-1}\big) = \mathrm{Ind}\big(P_1(t)|_{\mathrm{Ran}(P_0(t))}\big) = \mathrm{Ind}\big(P_0(t), P_1(t)\big),$$

where the last step is the definition. On the left-hand side, one has the index of a path of Fredholm operators on the same Hilbert space, which is constant by Theorem 3.3.4. Thus also the index on the right-hand side is constant in t. □

If two projections with compact difference are sufficiently close to each other, then one can actually construct a path of Fredholm pairs connecting the pair to a trivial pair.

Proposition 5.2.8. *Let $P_0, P_1 \in \mathbb{B}(\mathcal{H})$ be projections with $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ satisfying*

$$\|P_0 - P_1\| < \|\mathbf{1} - 2P_1\|^{-1}.$$

Then there is a path $t \in [0,1] \mapsto (P_0, P_1(t))$ of Fredholm pairs with $P_1(1) = P_1$ and $P_1(0) = P_0$. Along this path the index vanishes.

Proof. The path is constructed just as in Proposition 4.3.2 in [23]. Let us set

$$M = \frac{1}{2}(\mathbf{1} - 2P_0)(\mathbf{1} - 2P_1) + \frac{1}{2}\mathbf{1}.$$

Then $\mathbf{1} - M = (\mathbf{1} - 2P_0)(P_1 - P_0) = (P_0 - P_1)(\mathbf{1} - 2P_1)$ is a compact operator. Moreover, by hypothesis this operator satisfies $\|\mathbf{1} - M\| < 1$. Therefore $M = \mathbf{1} - (\mathbf{1} - M)$ is invertible with inverse given by the Neumann series. Furthermore, one has $P_0 M = P_0 P_1 = M P_1$ so that $P_0 = M P_1 M^{-1}$. Now set

$$M_t = (1-t)M + t\mathbf{1} = \mathbf{1} - (1-t)(\mathbf{1} - M), \quad t \in [0,1],$$

and $P_1(t) = M_t P_1 M_t^{-1}$. This is a path of projections connecting $P_1(1) = P_1$ to $P_1(0) = P_0$, and one has that $P_0 - P_1(t) = (P_0 - P_1) - (P_1(t) - P_1) \in \mathbb{K}(\mathcal{H})$ because $\mathbf{1} - M_t \in \mathbb{K}(\mathcal{H})$. By Proposition 5.2.4, one concludes that indeed $(P_0, P_1(t))$ is a Fredholm pair. The last claim follows from Proposition 5.2.7. □

The construction in the proof of Proposition 5.2.8 leads to another important result on the lifting of paths of idempotents that is at the root of numerous arguments later on. It does not pend on Fredholm properties.

Proposition 5.2.9. *Let $t \in [0,1] \mapsto P_t$ be a path of projections. Then there exists a path $t \in [0,1] \mapsto M_t$ of invertibles such that*

$$P_t = M_t P_0 M_t^{-1}.$$

Proof. Let us begin by setting

$$M_t = \frac{1}{2}(\mathbf{1} - 2P_0)(\mathbf{1} - 2P_t) + \frac{1}{2}\mathbf{1}.$$

As above, $\mathbf{1} - M_t = (P_0 - P_t)(\mathbf{1} - 2P_t)$ so that $\|\mathbf{1} - M_t\| \le \|P_0 - P_t\| \|\mathbf{1} - 2P_t\|$. As $t \mapsto P_t$ is norm continuous, this implies that M_t is invertible for t sufficiently small, say $t \in [0, t_1]$. Therefore $t \in [0, t_1] \mapsto M_t$ and $t \in [0, t_1] \mapsto M_t^{-1}$ are both continuous and, as in the proof of Proposition 5.2.8, one has $P_t = M_t P_0 M_t^{-1}$. Also note that $M_0 = \mathbf{1}$. Next one can start out with the path $t \in [t_1, 1] \mapsto P_t$ and construct in the same manner a $t_2 > t_1$ and a path $t \in [t_1, t_2] \mapsto M'_t$ such that $P_t = M'_t P_{t_1} (M'_t)^{-1}$ and $M'_{t_1} = \mathbf{1}$. By replacing, one gets $P_t = M'_t M_{t_1} P_0 (M'_t M_{t_1})^{-1}$. Thus setting $M_t = M'_t M_{t_1}$ for $t \in [t_1, t_2]$ completes the construction on the interval $[0, t_2]$. Iterating the procedure a final number of times completes the proof. □

Next let us turn to formulas for the index of a Fredholm pair of projections. The index of a Fredholm operator can be computed by the Calderon–Fedosov formula given in Theorem 3.3.7, provided that some trace class conditions hold. The following statement spells this out for a Fredholm pair of projections.

Proposition 5.2.10. *Let $P_0, P_1 \in \mathbb{B}(\mathcal{H})$ be projections and $n \in \mathbb{N}$ such that*

$$P_0 - P_0 P_1 P_0 \in \mathcal{L}^n(\mathrm{Ran}(P_0)), \quad P_1 - P_1 P_0 P_1 \in \mathcal{L}^n(\mathrm{Ran}(P_1)).$$

Then (P_0, P_1) is a Fredholm pair of projections, and for all $m \ge n$ one has

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}\big((P_0 - P_0 P_1 P_0)^m\big) - \mathrm{Tr}\big((P_1 - P_1 P_0 P_1)^m\big).$$

Proof. Let us apply Theorem 3.3.7 to the operator $A = P_1P_0|_{\mathrm{Ran}(P_0)} : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ with pseudoinverse $B = P_0P_1|_{\mathrm{Ran}(P_1)} : \mathrm{Ran}(P_1) \to \mathrm{Ran}(P_0)$. Due to the hypothesis, Theorem 3.3.7 then implies that A is Fredholm with index

$$\mathrm{Ind}(A) = \mathrm{Tr}\big((P_0 - P_0P_1P_0)^m\big) - \mathrm{Tr}\big((P_1 - P_1P_0P_1)^m\big)$$

for all $m \geq n$. By Definition 5.2.1, this implies that (P_0, P_1) is a Fredholm pair with the same index. □

One way to reformulate Proposition 5.2.2 is to state that, for finite-rank projections P_0 and P_1,

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}(P_0) - \mathrm{Tr}(P_1) = \mathrm{Tr}(P_0 - P_1).$$

The right-hand side not only makes sense if P_0 and P_1 are finite dimensional, but also if $P_0 - P_1$ is a trace class operator. The following result shows that then $\mathrm{Tr}(P_0 - P_1)$ is indeed equal to the index, actually under the even weaker assumption that some power of $P_0 - P_1$ is trace class. This provides yet another formula for the index of a Fredholm pair of projections.

Theorem 5.2.11. *Let (P_0, P_1) be a Fredholm pair of projections. If the operator $(P_0 - P_1)^{2n+1}$ is trace class for some integer $n \geq 0$, then for all $k \geq n$,*

$$\mathrm{Ind}(P_0, P_1) = \mathrm{Tr}\big((P_0 - P_1)^{2k+1}\big).$$

Proof. First let us note the following algebraic identities:

$$P_0 - P_0P_1P_0 = P_0(P_0 - P_1)P_0 = P_0(P_0 - P_1)^2P_0 = P_0(P_0 - P_1)^2 = (P_0 - P_1)^2P_0.$$

Therefore,

$$\begin{aligned}
(P_0 - P_0P_1P_0)^{k+1} &= (P_0 - P_0P_1P_0)^k(P_0 - P_0P_1P_0) \\
&= \big(P_0(P_0 - P_1)^2P_0\big)^k\big(P_0(P_0 - P_1)P_0\big) \\
&= P_0(P_0 - P_1)^{2k}(P_0 - P_1)P_0 \\
&= P_0(P_0 - P_1)^{2k+1}P_0.
\end{aligned}$$

In particular, the trace class condition on $(P_0 - P_1)^{2k+1}$ implies that $(P_0 - P_0P_1P_0)^{k+1}$ is trace class. This holds for all $k \geq n$. Similarly, one can deduce

$$(P_1 - P_1P_0P_1)^{k+1} = P_1(P_1 - P_0)^{2k+1}P_1,$$

and verify the trace class property of $(P_1 - P_1P_0P_1)^{k+1}$. Now by Proposition 5.2.10 and the cyclicity of the trace, one has

$$\begin{aligned}\operatorname{Ind}(P_0,P_1) &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1}P_0\big) - \operatorname{Tr}\big(P_1(P_1-P_0)^{2k+1}P_1\big)\\ &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1} - P_1(P_1-P_0)^{2k+1}\big)\\ &= \operatorname{Tr}\big(P_0(P_0-P_1)^{2k+1} + P_1(P_0-P_1)^{2k+1}\big)\\ &= \operatorname{Tr}\big((P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k}\big).\end{aligned}$$

It remains to show

$$\operatorname{Tr}\big((P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k}\big) = \operatorname{Tr}\big((P_0-P_1)^{2k+1}\big). \tag{5.12}$$

Note that

$$\begin{aligned}&(P_0+P_1)(P_0-P_1)(P_0-P_1)^{2k} - (P_0-P_1)^{2k+1}\\ &\quad= (P_1P_0 - P_0P_1)(P_0-P_1)^{2k}\\ &\quad= \big(P_1(P_0-P_1) - (P_0-P_1)P_1\big)(P_0-P_1)^{2k}\\ &\quad= P_1(P_0-P_1)^{2k+1} - (P_0-P_1)P_1(P_0-P_1)^{2k}.\end{aligned}$$

As in the last line both summands are trace class, (5.12) now follows from the cyclicity of the trace. □

Based on Theorem 5.2.11, one can derive integral formulas for the index of a pair of projections which is due to Phillips [148]. They directly lead to formulas for the spectral flow in Section 5.6.

Theorem 5.2.12. *Let (P_0,P_1) be a Fredholm pair of projections on $\mathcal{H}$ such that $(P_0-P_1)^{2n+1}$ is trace class for some integer $n \geq 0$. For $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ consider the linear path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$. Then for all integers $k \geq n$,*

$$\operatorname{Ind}(P_0,P_1) = \frac{1}{C_k}\int_0^1 dt\, \operatorname{Tr}\big((\partial_t Q)_t(1-Q_t^2)^k\big),$$

where, with $(2k+1)!! = (2k+1)(2k-1)\cdots 3\cdot 1$,

$$C_k = \int_{-1}^{1} dt(1-t^2)^k = \frac{k!\,2^{k+1}}{(2k+1)!!}. \tag{5.13}$$

Proof. One directly checks

$$(\partial_t Q)_t = 2(P_0 - P_1)$$

and

$$\mathbf{1} - Q_t^2 = t(1-t)(Q_0-Q_1)^2 = 4t(1-t)(P_0-P_1)^2.$$

Hence $(\partial_t Q)_t(\mathbf{1} - Q_t^2)^k$ is trace class by assumption, and thus

$$\begin{aligned}\int_0^1 dt\,\mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1} - Q_t^2)^k\big) &= \int_0^1 dt\,\mathrm{Tr}\big(2(P_0 - P_1)((4(t - t^2))(P_0 - P_1)^2)^k\big) \\ &= 2 \cdot 4^k \int_0^1 dt(t - t^2)^k\,\mathrm{Ind}(P_0, P_1) \\ &= \int_{-1}^1 ds(1 - s^2)^k\,\mathrm{Ind}(P_0, P_1),\end{aligned}$$

where the last step follows after the change of variables $s = 2t - 1$. The value of the integral can be computed and gives the constant C_k. □

5.3 Fredholm pairs of orthogonal projections

In this section, unless otherwise stated, all projections are supposed to be orthogonal, namely to be self-adjoint idempotents. Let us begin by proving two results that reformulate the definition and give a geometric interpretation of the index of a Fredholm pair of orthogonal projections.

Proposition 5.3.1. *Let P_0 and P_1 be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*

$$P_0P_1P_0 + \mathbf{1} - P_0 \quad \textit{and} \quad P_1P_0P_1 + \mathbf{1} - P_1$$

are Fredholm operators on $\mathcal{H}$. If (P_0, P_1) is a Fredholm pair, then

$$\mathrm{Ind}(P_0, P_1) = \dim\big(\mathrm{Ker}(P_0P_1P_0 + \mathbf{1} - P_0)\big) - \dim\big(\mathrm{Ker}(P_1P_0P_1 + \mathbf{1} - P_1)\big).$$

Proof. This follows directly from Theorem 3.4.1 applied to the operator A of Definition 5.2.1, after complementing A^*A and AA^* to operators on all of $\mathcal{H}$. □

Proposition 5.3.2. *Let P_0 and P_1 be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*

(i) *the linear span* $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1) = \mathrm{Ran}(P_0) + \mathrm{Ker}(P_1)$ *is a closed subspace;*
(ii) $\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$ *is finite dimensional;*
(iii) $\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)$ *is finite dimensional.*

The index $\mathrm{Ind}(P_0, P_1)$ *of the Fredholm pair is then given by*

$$\mathrm{Ind}(P_0, P_1) = \dim\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big) - \dim\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big).$$

Proof. First of all, let us note that $\mathrm{Ran}(P_0)$ and $\mathrm{Ran}(P_1)$ are closed subspaces and thus Hilbert spaces. Now by Definition 3.2.1, the operator $A = P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ is a Fredholm operator if and only if

$$\mathrm{Ker}(A) = \mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$$

is finite dimensional,

$$\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp = \mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)$$

is finite dimensional, and $\mathrm{Ran}(A) = \mathrm{Ran}(P_1P_0)$ is closed. Now

$$\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1) = \mathrm{Ran}(\mathbf{1} - P_1) \oplus \mathrm{Ran}(P_1P_0),$$

where $\oplus$ denotes the orthogonal sum. Thus $\mathrm{Ran}(P_0) + \mathrm{Ran}(\mathbf{1} - P_1)$ is closed if and only if $\mathrm{Ran}(A) = \mathrm{Ran}(P_1P_0)$ is closed by Lemma 5.3.3 below. Therefore A is indeed a Fredholm operator if and only if (i), (ii), and (iii) hold. Furthermore, by definition, the index $\mathrm{Ind}(P_0, P_1) = \mathrm{Ind}(A) = \dim(\mathrm{Ker}(A)) - \dim(\mathrm{Ker}(A^*))$ is given by the formula claimed. □

Lemma 5.3.3. *Let P be a projection (not necessarily orthogonal) and $\mathcal{E} \subset \mathrm{Ker}(P)$ as well as $\mathcal{F} \subset \mathrm{Ran}(P)$ subspaces. Then $\mathcal{E} + \mathcal{F}$ is closed if and only if $\mathcal{E}$ and $\mathcal{F}$ are closed.*

Proof. Suppose that $\mathcal{E} + \mathcal{F}$ is closed. Let $(\phi_n)_{n\geq 1}$ be a convergent sequence in $\mathcal{E}$ with limit $\phi \in \mathcal{H}$. It is then also convergent in $\mathcal{E} + \mathcal{F}$ and therefore $\phi \in \mathcal{E} + \mathcal{F}$ as $\mathcal{E} + \mathcal{F}$ is closed. But $P\phi = \lim P\phi_n = 0$ so that $\phi \in \mathrm{Ker}(P)$ and thus $\phi \in \mathcal{E}$. Similarly, one checks that $\mathcal{F}$ is closed. For the converse, let $(\phi_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{E} + \mathcal{F}$. Then $(P\phi_n)_{n\in\mathbb{N}}$ and $((\mathbf{1} - P)\phi_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{F}$ and $\mathcal{E}$, respectively. As $\mathcal{F}$ and $\mathcal{E}$ are closed, $(P\phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}$ and $((\mathbf{1} - P)\phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{E}$ and hence also the sequence $\phi_n = P\phi_n + (\mathbf{1} - P)\phi_n$ converges in $\mathcal{E} + \mathcal{F}$. □

It follows directly from Definition 5.2.1 that for a Fredholm pair (P_0, P_1) of orthogonal projections also the pair (P_1, P_0) is Fredholm (because then the corresponding Fredholm operators are A and its adjoint A^*, respectively) and that one has

$$\mathrm{Ind}(P_0, P_1) = -\,\mathrm{Ind}(P_1, P_0).$$

Moreover, by Proposition 5.3.2, $(\mathbf{1}-P_0, \mathbf{1}-P_1)$ is Fredholm if and only if (P_0, P_1) is Fredholm and then

$$\mathrm{Ind}(\mathbf{1} - P_0, \mathbf{1} - P_1) = \mathrm{Ind}(P_1, P_0).$$

Finally, it follows from Proposition 5.3.2, or alternatively from (5.10), that for every Fredholm pair (P_0, P_1) of orthogonal projections and any unitary operator U, also (UP_0U^*, UP_1U^*) is a Fredholm pair of orthogonal projections with index

$$\mathrm{Ind}\big(UP_0U^*, UP_1U^*\big) = \mathrm{Ind}(P_0, P_1). \tag{5.14}$$

Generalizing the unitary conjugation, one can also consider the natural action (5.7) of invertibles on orthogonal projections.

Proposition 5.3.4. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections and furthermore let $T \in \mathbb{G}(\mathcal{H})$ be invertible. Then $(T\cdot P_0, (T^{-1})^* \cdot P_1)$ is a Fredholm pair of orthogonal projections with the same index. Moreover,*

$$\dim\big(\mathrm{Ran}(T\cdot P_0) \cap \mathrm{Ker}\big((T^{-1})^* \cdot P_1\big)\big) = \dim\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big)$$

and

$$\dim\big(\mathrm{Ran}\big((T^{-1})^* \cdot P_1\big) \cap \mathrm{Ker}(T\cdot P_0)\big) = \dim\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big).$$

Proof. For any orthogonal projection P, one deduces from the definition of $T\cdot P$ that

$$\mathrm{Ran}(T\cdot P) = T\,\mathrm{Ran}(P), \quad \mathrm{Ker}(T\cdot P) = \big(T^{-1}\big)^*\,\mathrm{Ker}(P),$$

see the argument after equation (5.7). The Fredholm property of $(T\cdot P_0, (T^{-1})^* \cdot P_1)$ is checked by verifying the three conditions (i)–(iii) stated in Proposition 5.3.2. One has

$$\begin{aligned}
\mathrm{Ran}(T\cdot P_0) + \mathrm{Ran}\big(\mathbf{1} - \big(T^{-1}\big)^* \cdot P_1\big) &= \mathrm{Ran}(T\cdot P_0) + \mathrm{Ker}\big(\big(T^{-1}\big)^* \cdot P_1\big) \\
&= T\,\mathrm{Ran}(P_0) + T\,\mathrm{Ker}(P_1) \\
&= T\big(\mathrm{Ran}(P_0) + \mathrm{Ker}(P_1)\big),
\end{aligned}$$

showing that this is a closed subspace because T is invertible. Moreover,

$$\begin{aligned}
\mathrm{Ran}(T\cdot P_0) \cap \mathrm{Ker}\big(\big(T^{-1}\big)^* \cdot P_1\big) &= \big(T\,\mathrm{Ran}(P_0)\big) \cap \big(T\,\mathrm{Ker}(P_1)\big) \\
&= T\big(\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)\big),
\end{aligned}$$

which has the same finite dimension as $\mathrm{Ran}(P_0) \cap \mathrm{Ker}(P_1)$. In the same way,

$$\mathrm{Ran}\big(\big(T^{-1}\big)^* \cdot P_1\big) \cap \mathrm{Ker}(T\cdot P_0) = \big(T^{-1}\big)^*\big(\mathrm{Ran}(P_1) \cap \mathrm{Ker}(P_0)\big),$$

implying all remaining claims. □

One may wonder if for a Fredholm pair (P_0, P_1) of orthogonal projections and an invertible $T \in \mathbb{G}(\mathcal{H})$ the pair $(T\cdot P_0, T\cdot P_1)$ is Fredholm. In general, however, this is not true as is shown by the next example.

Example 5.3.5. For a fixed grading $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$ where $\mathcal{H}'$ is an infinite-dimensional separable Hilbert space, let us set

$$P_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_1 = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \quad T = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathrm{Ker}(P_1P_0) = \mathrm{Ker}(P_0)$ and $\mathrm{Ran}(P_1P_0) = \mathrm{Ran}(P_1)$, and therefore (P_0, P_1) is a Fredholm pair. Moreover,

$$\mathrm{Ran}(T \cdot P_0) = \mathrm{span}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}, \quad \mathrm{Ran}(T \cdot P_1) = \mathrm{span}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$

This shows $T \cdot P_1 = \mathbf{1} - T \cdot P_0$ and therefore $(T \cdot P_0, T \cdot P_1)$ is not a Fredholm pair. ◇

The following aim is to give a spectral theoretic approach to Fredholm pairs of orthogonal projections. As a preparation for the proofs, let us present a set of algebraic relations satisfied by two projections (which need not be orthogonal). They can be traced back to Kato [112], see also [49] and [18].

Lemma 5.3.6. *Let P_0 and P_1 be projections. Set*

$$R_0 = \mathbf{1} - P_0 - P_1, \quad R_1 = P_0 - P_1.$$

Then the following identities hold:

$$R_0^2 + R_1^2 = \mathbf{1}, \quad R_0R_1 = -R_1R_0. \tag{5.15}$$

Moreover,

$$\begin{aligned} R_0P_0 &= P_1R_0, & R_0P_1 &= P_0R_0, \\ R_1(\mathbf{1} - P_0) &= P_1R_1, & R_1(\mathbf{1} - P_1) &= P_0R_1, \\ R_0^2P_0 &= P_0R_0^2, & R_0R_1(\mathbf{1} - P_0) &= P_0R_0R_1. \end{aligned}$$

Proof. Multiplying out, one finds

$$R_0^2 = \mathbf{1} - P_0 - P_1 + P_0P_1 + P_1P_0,$$

and similarly

$$R_1^2 = P_0 + P_1 - P_0P_1 - P_1P_0.$$

Adding this up, leads to the first identity. The others are also verified by straightforward algebraic computations. □

The identities of Lemma 5.3.6 lead to interesting spectral information of $P_1 - P_0$ and $P_1 + P_0$, stated in terms of R_0 and R_1.

Proposition 5.3.7. *Let P_0 and P_1 be orthogonal projections and R_0 and R_1 as in Lemma 5.3.6. Then for $j = 0, 1$, the spectrum $\mathrm{spec}(R_j)$ of R_j lies in $[-1, 1]$ and satisfies*

$$\mathrm{spec}(R_j) \setminus \{-1, 1\} = -(\mathrm{spec}(R_j) \setminus \{-1, 1\}).$$

Moreover, for any $\lambda \notin \{-1, 1\}$,

$$\dim(\mathrm{Ker}(R_j - \lambda\mathbf{1})) = \dim(\mathrm{Ker}(R_j + \lambda\mathbf{1})).$$

Proof. Let us focus on $j = 1$. The proof for the case $j = 0$ is the same as only the relations (5.15) are used and they are symmetric in the indices. The inclusion $\mathrm{spec}(R_1) \subset [-1, 1]$ follows from $R_1^2 = \mathbf{1} - R_0^2 \leq \mathbf{1}$. The symmetry of the spectrum can be shown using Weyl sequences, namely if $(R_1 - \lambda\mathbf{1})\phi_n \to 0$ for $\lambda \in (-1, 1)$ and a sequence of unit vectors $(\phi_n)_{n\geq 1}$, then, by the identity $R_0R_1 = -R_1R_0$ of Lemma 5.3.6, one has $(R_1 + \lambda\mathbf{1})R_0\phi_n \to 0$. As $R_0^2\phi_n = (1-\lambda^2)\phi_n - (R_1^2 - \lambda^2\mathbf{1})\phi_n$ by the first relation of (5.15) and $(R_1^2 - \lambda^2\mathbf{1})\phi_n \to 0$, it follows from $|\lambda| < 1$ that $\|R_0\phi_n\| \geq c$ for some constant $c > 0$ and n sufficiently large. Hence $(R_1 + \lambda\mathbf{1})\frac{R_0\phi_n}{\|R_0\phi_n\|} \to 0$ and $(\frac{R_0\phi_n}{\|R_0\phi_n\|})_{n\geq 1}$ is a Weyl sequence for $-\lambda$. Finally, set $\mathcal{H}_\lambda = \mathrm{Ker}(R_1 - \lambda\mathbf{1})$. Then by the same identity $R_0(\mathcal{H}_\lambda) \subset \mathcal{H}_{-\lambda}$ and $R_0(\mathcal{H}_{-\lambda}) \subset \mathcal{H}_\lambda$. As $R_0^2|_{\mathcal{H}_\lambda} = (1 - \lambda^2)\mathbf{1}|_{\mathcal{H}_\lambda}$ by the identity $R_0^2 = \mathbf{1} - R_1^2$ in (5.15), it follows that R_0 is an isomorphism from $\mathcal{H}_\lambda$ to $\mathcal{H}_{-\lambda}$ for any value $\lambda \notin \{-1, 1\}$. $\square$

Now the main spectral theoretic result for the index of a Fredholm pair of orthogonal projections can be stated and proved.

Theorem 5.3.8. *Two orthogonal projections P_0 and P_1 form a Fredholm pair if and only if ± 1 are not in the essential spectrum of the operator $P_0 - P_1$. Then*

$$\mathrm{Ind}(P_0, P_1) = \dim(\mathrm{Ker}(P_0 - P_1 - \mathbf{1})) - \dim(\mathrm{Ker}(P_0 - P_1 + \mathbf{1})). \tag{5.16}$$

Proof. Recall that (P_0, P_1) is a Fredholm pair if and only if $A : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_1)$ defined by $A\phi = P_1P_0\phi$ for $\phi \in \mathrm{Ran}(P_0)$ is a Fredholm operator.

Let 1 be in the essential spectrum of $P_0 - P_1$. By Proposition 3.4.7, there is a singular Weyl sequence $(\phi_n)_{n\geq 1}$ such that $(P_0 - P_1 - \mathbf{1})\phi_n \to 0$. Then $\langle\phi_n|(P_0 - P_1)\phi_n\rangle \to 1$, thus $\|P_0\phi_n\| \to 1$ and $\|P_1\phi_n\| \to 0$. Therefore $\psi_n = \frac{P_0\phi_n}{\|P_0\phi_n\|}$ has norm 1, converges weakly to 0, and $P_0P_1P_0\psi_n \to 0$, which shows that $0 \in \mathrm{spec}_{\mathrm{ess}}(A^*A)$ by Proposition 3.4.7. By Theorem 3.4.1, this is a contradiction to the Fredholm property of A. Therefore (P_0, P_1) is no Fredholm pair. Similarly, $-1 \in \mathrm{spec}_{\mathrm{ess}}(P_0 - P_1)$ implies $0 \in \mathrm{spec}_{\mathrm{ess}}(AA^*)$ and, again by Theorem 3.4.1, this is a contradiction to the Fredholm property of A, thus (P_0, P_1) is no Fredholm pair.

Conversely, let ± 1 be not in the essential spectrum of the operator $P_0 - P_1$. By the spectral radius theorem, one has $P_0 - P_1 = B + F$ where F is of finite rank and, moreover, $(\epsilon - 1)\mathbf{1} \leq B \leq (1 - \epsilon)\mathbf{1}$ for some $\epsilon > 0$. As

$$P_0P_1P_0 = P_0(\mathbf{1} - (P_0 - P_1))P_0$$

$$
\begin{aligned}
&= -P_0FP_0 + P_0(\mathbf{1} - B)P_0 \\
&\geq -P_0FP_0 + \epsilon P_0,
\end{aligned}
$$

this implies $0 \notin \operatorname{spec}_{\mathrm{ess}}(A^*A)$. Analogously, one has $P_1P_0P_1 \geq P_1FP_1 + \epsilon P_1$ and consequently $0 \notin \operatorname{spec}_{\mathrm{ess}}(AA^*)$. By Theorem 3.4.1, this implies that A is Fredholm and (P_0, P_1) is a Fredholm pair.

It remains to show (5.16) if (P_0, P_1) is a Fredholm pair. The kernel of a sum of two nonnegative operators is the intersection of their kernels. Therefore

$$
\begin{aligned}
\operatorname{Ker}(P_0 - P_1 - \mathbf{1}) &= \operatorname{Ker}\big(P_1 + (\mathbf{1} - P_0)\big) \\
&= \operatorname{Ker}(P_1) \cap \operatorname{Ker}(\mathbf{1} - P_0) \\
&= \operatorname{Ker}(P_1) \cap \operatorname{Ran}(P_0).
\end{aligned}
$$

Similarly, $\operatorname{Ker}(P_0 - P_1 + \mathbf{1}) = \operatorname{Ker}(P_0) \cap \operatorname{Ran}(P_1)$, and this implies the claimed identity due to Proposition 5.3.2. □

Remark 5.3.9. Proposition 5.3.7 and Theorem 5.3.8 allow giving an alternative proof of Theorem 5.2.11 for orthogonal projections. Under the hypothesis that $(P_0-P_1)^{2k+1}$ is trace-class, the spectrum of $(P_0 - P_1)^{2k+1} = R_1^{2k+1}$ consists of eigenvalues accumulating only at 0. By Proposition 5.3.7 and because $2k + 1$ is odd, this spectrum is symmetric and the eigenspaces $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$ have the same dimension for $\lambda \notin \{-1, 0, 1\}$ which, moreover, is finite. Thus by Lidskii's theorem,

$$
\begin{aligned}
\operatorname{Tr}\big((P_0 - P_1)^{2k+1}\big) &= \sum_{\lambda \in \operatorname{spec}(P_0 - P_1)} \lambda^{2k+1} \dim(\mathcal{H}_\lambda) \\
&= \sum_{\substack{\lambda \in \operatorname{spec}(P_0 - P_1), \\ \lambda > 0}} \lambda^{2k+1}\big(\dim(\mathcal{H}_\lambda) - \dim(\mathcal{H}_{-\lambda})\big) \\
&= \dim(\mathcal{H}_1) - \dim(\mathcal{H}_{-1}) \\
&= \operatorname{Ind}(P_0, P_1),
\end{aligned}
$$

where the last equality follows from Theorem 5.3.8. ◇

Let us also provide a slight generalization of Theorem 5.2.11 going back to [56].

Proposition 5.3.10. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections such that $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact and let $f : [-1, 1] \to \mathbb{R}$ be a continuous odd function such that $f(1) = 1$ and such that $f(P_0 - P_1)$ is trace class, then*

$$
\operatorname{Ind}(P_0, P_1) = \operatorname{Tr}\big(f(P_0 - P_1)\big).
$$

Proof. Recall from Proposition 5.3.7 that

$$
\operatorname{spec}(P_0 - P_1) \setminus \{-1, 1\} = -\big(\operatorname{spec}(P_0 - P_1) \setminus \{-1, 1\}\big)
$$

and

$$\dim\big(\mathrm{Ker}(P_0 - P_1 - \lambda\mathbf{1})\big) = \dim\big(\mathrm{Ker}(P_0 - P_1 + \lambda\mathbf{1})\big)$$

for any $\lambda \notin \{-1, 1\}$. Therefore, by the same argument as in Remark 5.3.9,

$$\begin{aligned}\mathrm{Tr}\big(f(P_0 - P_1)\big) &= \big(\dim\big(\mathrm{Ker}(P_0 - P_1 - \mathbf{1})\big) - \dim\big(\mathrm{Ker}(P_0 - P_1 + \mathbf{1})\big)\big)\\ &= \mathrm{Ind}(P_0, P_1),\end{aligned}$$

where the last step follows from Theorem 5.3.8. □

Based on Proposition 5.3.10, one can also express the index of a pair of projections as an integral similar as in Theorem 5.2.12, but under weaker hypothesis. Combined with the results of Section 5.6, this leads to integral formulas for the spectral flow of paths between Fredholm pairs of symmetries. In the following, functions $f : [0, \infty) \to [0, \infty)$ of the form $f(x) = x^{-r} e^{-x^{-\sigma}}$ for $r \geq 0$ and $\sigma \geq 1$ are considered. These functions are defined to be 0 at $x = 0$.

Proposition 5.3.11. *Let $P_0, P_1 \in \mathbb{P}(\mathcal{H})$ be orthogonal projections such that the operator $\exp(-((P_0 - P_1)^2)^{-\frac{1}{q}})$ is trace class for some $0 < q \leq 1$. Then (P_0, P_1) is a Fredholm pair of projections and for $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ the linear path*

$$t \in [0, 1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$$

is within the Fredholm operators. Moreover,

$$\mathrm{Ind}(P_0, P_1) = \frac{1}{C_{r,q}} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t \big(\mathbf{1} - Q_t^2\big)^{-r} e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}}\big)$$

for $r \geq 0$, where

$$C_{r,q} = \int_{-1}^{1} du \big(1 - u^2\big)^{-r} e^{-(1-u^2)^{-\frac{1}{q}}}. \tag{5.17}$$

Proof. First of all, let us note that $e^{-((P_0 - P_1)^2)^{-\frac{1}{q}}}$ is trace class and, in particular, compact so that $P_0 - P_1 \in \mathbb{K}(\mathcal{H})$ is compact. Thus (P_0, P_1) is a Fredholm pair and Q_t is Fredholm for all $t \in [0, 1]$. Now recall from the proof of Theorem 5.2.12 that $(\partial_t Q)_t = 2(P_0 - P_1)$ and $\mathbf{1} - Q_t^2 = 4t(1 - t)(P_0 - P_1)^2$. Thus

$$e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}} = \big(e^{-((P_0 - P_1)^2)^{-\frac{1}{q}}}\big)^{(4t(1-t))^{-\frac{1}{q}}}$$

is trace class as $(4t(1 - t))^{-\frac{1}{q}} \geq 1$ for $t \in (0, 1)$ while it trivially is trace class for $t \in \{0, 1\}$. One obtains

$$\int_0^1 dt\,\mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1}-Q_t^2)^{-r}e^{-(\mathbf{1}-Q_t^2)^{-\frac{1}{q}}}\big)$$
$$= \int_0^1 dt\,\mathrm{Tr}\big(2(P_0-P_1)(4(t-t^2)(P_0-P_1)^2)^{-r}e^{-(4(t-t^2)(P_0-P_1)^2)^{-\frac{1}{q}}}\big).$$

For $t \in (0,1)$, the function $f_t : \mathbb{R} \to \mathbb{R}$ defined by

$$f_t(x) = 2x\big(4(t-t^2)x^2\big)^{-r}e^{-(4(t-t^2)x^2)^{-\frac{1}{q}}}$$

is odd and $f_t(P_0 - P_1)$ is trace class. Thus by Proposition 5.3.10,

$$\mathrm{Tr}\big(f_t(P_0-P_1)\big) = f_t(1)\,\mathrm{Ind}(P_0,P_1)$$

and therefore

$$\int_0^1 dt\,\mathrm{Tr}\big((\partial_t Q)_t(\mathbf{1}-Q_t^2)^{-r}e^{-(\mathbf{1}-Q_t^2)^{-\frac{1}{q}}}\big)$$
$$= \int_0^1 dt\,f_t(1)\,\mathrm{Ind}(P_0,P_1)$$
$$= \mathrm{Ind}(P_0,P_1)\int_0^1 dt\,2\big(4(t-t^2)\big)^{-r}e^{-(4(t-t^2))^{-\frac{1}{q}}}$$
$$= \mathrm{Ind}(P_0,P_1)\int_{-1}^1 du(1-u^2)^{-r}e^{-(1-u^2)^{-\frac{1}{q}}},$$

where the last step follows from the change of variables $u = 2t-1$. Dividing by $C_{r,q}$ shows the claim. □

Theorem 5.3.8 has several other consequences. The first gives an important criterion for a pair of projections to be a Fredholm pair with vanishing index.

Proposition 5.3.12. *Let P_0 and P_1 be a pair of orthogonal projections on $\mathcal{H}$. If*

$$\|P_0 - P_1\| < 1,$$

then (P_0, P_1) is a Fredholm pair and $\mathrm{Ind}(P_0, P_1) = 0$.

Proof. This follows immediately from Theorem 5.3.8 because the hypothesis implies that ± 1 are not in the spectrum of $P_0 - P_1$. □

One can go beyond Proposition 5.3.12 and show that $\|P_0 - P_1\| < 1$ implies that there exists a unitary V such that $VP_0V^* = P_1$ and $VP_1V^* = P_0$, see Proposition 5.5.6. The next consequence is a characterization of the Fredholmness of a pair of orthogonal projections that is often used as the definition of a Fredholm pair.

Corollary 5.3.13. *Two orthogonal projections P_0 and P_1 form a Fredholm pair if and only if the norm of their difference in the Calkin algebra is less than* 1,

$$\|\pi(P_1 - P_0)\|_{\mathbb{Q}} < 1.$$

The following characterization of the Fredholmness of a pair of orthogonal projections is another direct consequence of Theorem 5.3.8.

Corollary 5.3.14. *A pair of orthogonal projections (P_0, P_1) is a Fredholm pair if and only if*

$$P_0 - P_1 = B + F,$$

where B, F are self-adjoint operators on $\mathcal{H}$, $\|B\| < 1$ and F is of finite rank.

Proof. If $P_0 - P_1 = B + F$, then (P_0, P_1) is a Fredholm pair by Theorem 5.3.8.

For the converse, set $P_\pm = \chi_{\{\pm 1\}}(P_0 - P_1)$. Then define $F = P_+ - P_-$ which is of finite rank and $B = (\mathbf{1} - P_+ - P_-)(P_0 - P_1)(\mathbf{1} - P_+ - P_-)$ for which $\|B\| < 1$. One directly checks that $P_0 - P_1 = B + F$. □

Next let us strengthen Proposition 5.2.6 on the concatenation of Fredholm pairs.

Proposition 5.3.15. *Suppose given three orthogonal projections P_0, P_1 and P_2 such that $\|\pi(P_0) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_2)\|_{\mathbb{Q}} < 1$. Then (P_0, P_1), (P_1, P_2) and (P_0, P_2) are all Fredholm pairs and*

$$\mathrm{Ind}(P_0, P_2) = \mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2). \tag{5.18}$$

Proof. By Corollary 5.3.13, (P_0, P_1), (P_1, P_2), and (P_0, P_2) are Fredholm pairs. Therefore, by definition $P_iP_j : \mathrm{Ran}(P_j) \to \mathrm{Ran}(P_i)$ is Fredholm for $i, j \in \{0, 1, 2\}$ with $i > j$ and $\mathrm{Ind}(P_j, P_i) = \mathrm{Ind}(P_iP_j)$. Thus, by item (iii) of Theorem 3.3.4,

$$\mathrm{Ind}(P_0, P_1) + \mathrm{Ind}(P_1, P_2) = \mathrm{Ind}(P_2P_1P_0),$$

where $P_2P_1P_0 : \mathrm{Ran}(P_0) \to \mathrm{Ran}(P_2)$ is Fredholm by item (i) of Corollary 3.3.2. Then (5.18) is equivalent to

$$\mathrm{Ind}(P_2P_1P_0) = \mathrm{Ind}(P_2P_0),$$

which is, again by Corollary 3.3.2, equivalent to

$$\operatorname{Ind}(P_0P_2P_1P_0) = \operatorname{Ind}\big((P_2P_0)^*(P_2P_1P_0)\big) = 0.$$

As

$$\begin{aligned}\|\pi(P_0P_2P_1P_0) - \pi(P_0)\|_{\mathbb{Q}} &\le \|\pi(P_2P_1) - \pi(P_0)\|_{\mathbb{Q}} \\ &\le \|\pi(P_2P_1) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_0)\|_{\mathbb{Q}} \\ &\le \|\pi(P_0) - \pi(P_1)\|_{\mathbb{Q}} + \|\pi(P_1) - \pi(P_2)\|_{\mathbb{Q}} < 1,\end{aligned}$$

there is a compact operator $K : \operatorname{Ran}(P_0) \to \operatorname{Ran}(P_0)$ such that

$$\|P_0P_2P_1P_0 + K - P_0\| < 1.$$

This implies that $P_0P_2P_1P_0 + K - P_0 + P_0 : \operatorname{Ran}(P_0) \to \operatorname{Ran}(P_0)$ is invertible and therefore $\operatorname{Ind}(P_0P_2P_1P_0) = \operatorname{Ind}(P_0P_2P_1P_0 + K) = 0$. □

Remark 5.3.16. It is not sufficient to suppose that (P_0, P_1) and (P_1, P_2) are Fredholm pairs, because then (P_0, P_2) is not necessarily a Fredholm pair. Indeed, let us set

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \mathbf{1} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \mathbf{1} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

acting on $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. One directly checks $\|P_0 - P_1\| = \|P_1 - P_2\| = \frac{1}{\sqrt{2}} < 1$, thus (P_0, P_1) and (P_1, P_2) are Fredholm pairs by Corollary 5.3.13. But $\|\pi(P_0 - P_2)\|_{\mathbb{Q}} = 1$ and therefore (P_0, P_2) does not form a Fredholm pair, again by Corollary 5.3.13. ◇

Even though a Fredholm pair (P_0, P_1) with compact difference $P_1 - P_0$ is only a special case, it nevertheless appears often, as in the following situation:

Proposition 5.3.17. *Let $H_0, H_1 \in \mathbb{FB}_{sa}(\mathcal{H})$ be two self-adjoint bounded Fredholm operators such that the difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ is compact. Then the spectral projections $P_0 = \chi(H_0 < 0)$ and $P_1 = \chi(H_1 < 0)$ form a Fredholm pair with compact difference $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$.*

Proof. Because H_0 and H_1 are Fredholm, and therefore $0 \notin \operatorname{spec}_{\mathrm{ess}}(H_0) \cup \operatorname{spec}_{\mathrm{ess}}(H_1)$ by Corollary 3.4.4, 0 is not an accumulation point of $\Sigma = \operatorname{spec}(H_0) \cup \operatorname{spec}(H_1)$. Therefore $\chi_{(-\infty,0)}|_\Sigma : \Sigma \to \{0, 1\}$ is a continuous function on the compact domain Σ and one has $P_1 - P_0 = \chi_{(-\infty,0)}|_\Sigma(H_1) - \chi_{(-\infty,0)}|_\Sigma(H_0)$. As

$$H_1^n - H_0^n = H_1(H_1^{n-1} - H_0^{n-1}) + (H_1 - H_0)H_0^{n-1}, \quad n \ge 2,$$

$p(H_1) - p(H_0)$ is compact for any polynomial $p : \mathbb{C} \to \mathbb{C}$. As the set of compact operators $\mathbb{K}(\mathcal{H})$ is a closed subset of the set of bounded operators $\mathbb{B}(\mathcal{H})$ and the polynomials are dense in set of continuous functions on compact domains, we see that $h(H_1) - h(H_0)$ is compact for every continuous function $h : \Sigma \to \mathbb{C}$. In conclusion, $P_1 - P_0 \in \mathbb{K}(\mathcal{H})$ is compact and therefore (P_0, P_1) is a Fredholm pair by Corollary 5.3.13. □

In Remark 5.3.16, the orthogonal projection P_1 is obtained from P_0 by a rotation of less than a right angle. The following result states that, inversely, one can always rotate one of the orthogonal projections of a Fredholm pair to attain a Fredholm pair with compact difference.

Proposition 5.3.18. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections. Then there exists a path $t \in [0,1] \mapsto P_1(t)$ of orthogonal projections such that $(P_0, P_1(t))$ is a Fredholm pair for all $t \in [0,1]$, $P_1(1) = P_1$ and $P_0 - P_1(0)$ is compact.*

Note that by Proposition 5.2.7, $t \in [0,1] \mapsto \mathrm{Ind}(P_0, P_1(t))$ is constant along this path. The proof of Proposition 5.3.18 starts out with a special case.

Proposition 5.3.19. *Let (P_0, P_1) be a pair of orthogonal projections satisfying the bound $\|P_0 - P_1\| < 1$. Then there exists a path $t \in [0,1] \mapsto P_t$ of orthogonal projections connecting P_0 with P_1 such that (P_0, P_t) is a Fredholm pair for all $t \in [0,1]$.*

Proof. (This uses the construction after Proposition 4.6.6 in [23].) Let $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ be the two associated symmetries and then set

$$R = Q_0 Q_1 + Q_1 Q_0 = 2\mathbf{1} - 4(P_0 - P_1)^2.$$

Then one has $[R, Q_0] = 0 = [R, Q_1]$. Let $a = \|P_0 - P_1\| < 1$. Then $-2\mathbf{1} < (2 - 4a^2)\mathbf{1} \le R \le 2\mathbf{1}$ so that $\mathbf{1} + \lambda R > 0$ uniformly in $\lambda \in [0, \frac{1}{2}]$. Therefore one can set

$$Q_t = \left(\mathbf{1} + R\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}} \left(Q_0 \cos\left(\frac{\pi}{2}t\right) + Q_1 \sin\left(\frac{\pi}{2}t\right)\right), \quad t \in [0,1].$$

Clearly, $Q_t^* = Q_t$ and computing the square shows $Q_t^2 = \mathbf{1}$, so this is a path of symmetries which, indeed, connects Q_0 and Q_1. Set $P_t = \frac{1}{2}(\mathbf{1} - Q_t)$. To verify the Fredholm property along this path, let us compute

$$(P_t - P_0)^2 = \frac{1}{2}\mathbf{1} - \frac{1}{4}\left(\mathbf{1} + R\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}} \left(2\cos\left(\frac{\pi}{2}t\right)\mathbf{1} + R\sin\left(\frac{\pi}{2}t\right)\right).$$

The right-hand side is merely a function of the self-adjoint operator R. Hence the norm is bounded by the maximum of the function

$$f(t,r) = \frac{1}{2} - \frac{1}{4}\left(1 + r\cos\left(\frac{\pi}{2}t\right)\sin\left(\frac{\pi}{2}t\right)\right)^{-\frac{1}{2}} \left(2\cos\left(\frac{\pi}{2}t\right) + r\sin\left(\frac{\pi}{2}t\right)\right)$$

on the rectangle $[0,1] \times [2 - 4a^2, 2]$. One finds

$$\sup_{t \in [0,1]} f(t,r) = f(1,r) = \frac{1}{2} - \frac{r}{4} \le a^2,$$

so that $\|P_t - P_0\| \le a$ uniformly in $t \in [0,1]$. By Proposition 5.3.12, this implies that (P_0, P_t) is a Fredholm pair for all $t \in [0,1]$. □

Proof of Proposition 5.3.18. Let us set $K_0 = \chi_{\{1\}}(P_0 - P_1)$ and $K_1 = \chi_{\{-1\}}(P_0 - P_1)$ which are finite-dimensional orthogonal projections satisfying $K_0K_1 = 0$. For $\phi \in \mathrm{Ran}(K_0)$, one has $P_0\phi = \phi$ and $P_1\phi = 0$ so that $\mathrm{Ran}(K_0)$ is left invariant by both P_0 and P_1. The same holds for $\mathrm{Ran}(K_1)$. Then consider $\mathcal{H}' = \mathcal{H} \ominus (\mathrm{Ran}(K_0) \oplus \mathrm{Ran}(K_1))$ and the restrictions $P_0' = P_0|_{\mathcal{H}'}$ and $P_1' = P_1|_{\mathcal{H}'}$. By construction, P_0' and P_1' are orthogonal projections on $\mathcal{H}'$ satisfying $\|P_0' - P_1'\| < 1$. Let $P_1'(t)$ be the path of orthogonal projections on $\mathcal{H}'$ given by Proposition 5.3.19. Finally, set $P_1(t) = P_1'(t) \oplus K_1$ which is an orthogonal projection on $\mathcal{H}$. The pair $(P_0, P_1(t))$ is Fredholm and satisfies the claim. □

The next aim is to lift the path of Proposition 5.3.19 by generalizing Proposition 5.2.9 in the following manner.

Proposition 5.3.20. *Let $t \in [0,1] \mapsto P_t$ be a path of orthogonal projections. Then there exists a path $t \in [0,1] \mapsto U_t$ of unitaries such that*

$$P_t = U_t^* P_0 U_t.$$

Proof. The operator M_t used in Proposition 5.2.9 satisfies $M_tP_0 = P_tM_t$ so that also $P_0M_t^* = M_t^*P_t$. Therefore $P_t = M_tP_0(M_t)^{-1}$ and $P_0 = M_t^*P_t(M_t^*)^{-1}$ so that upon replacing also

$$P_t = \left(M_tM_t^*\right)^{-1} P_t(M_tM_t^*).$$

This implies $P_t = (M_tM_t^*)^{-\frac{1}{2}} P_t (M_tM_t^*)^{\frac{1}{2}}$. Now set

$$U_t = M_t^* \left(M_tM_t^*\right)^{-\frac{1}{2}}.$$

This is indeed unitary and satisfies the claim. □

Remark 5.3.21. If the path $t \mapsto P_t$ is differentiable, then there is another standard way to obtain the path $t \mapsto U_t$ as the solution to Kato's adiabatic time-evolution:

$$\imath\partial_t U_t = U_t \imath[P_t, \partial_t P_t], \quad U_0 = \mathbf{1}.$$

Note that $\imath[P_t, \partial_t P_t]$ is self-adjoint so that indeed U_t is unitary. Furthermore, one has

$$\begin{aligned}\partial_t\left(U_tP_tU_t^*\right) &= (\partial_tU_t)P_tU_t^* + U_t(\partial_tP_t)U_t^* - U_tP_tU_t^*(\partial_tU_t)U_t^* \\ &= U_t[P_t, \partial_tP_t]P_tU_t^* + U_t(\partial_tP_t)U_t^* - U_tP_t[P_t, \partial_tP_t]U_t^* \\ &= 0,\end{aligned}$$

the latter because $\partial_tP_t = \partial_tP_t^2 = \partial_tP_tP_t + P_t\partial_tP_t$ and $P_t\partial_tP_tP_t = 0$ for any differentiable path of projections. Hence the initial condition implies indeed that $P_t = U_t^*P_0U_t$. This argument can be modified to show that there are many possible choices for the path $t \mapsto U_t$. More precisely, one can modify the adiabatic equation to

$$\imath \partial_t U_t = U_t\big(\imath[P_t, \partial_t P_t] - H_t\big), \quad U_0 = \mathbf{1},$$

where $t \mapsto H_t$ is an arbitrary path of self-adjoints satisfying $[H_t, P_t] = 0$, without spoiling the conjugacy relation $P_t = U_t^* P_0 U_t$. ◇

Combining Proposition 5.3.20 with Proposition 5.3.19 one obtains the following:

Corollary 5.3.22. *Let P_0 and P_1 be a pair of orthogonal projections satisfying the bound $\|P_0 - P_1\| < 1$. Then exists a path $t \in [0,1] \mapsto U_t$ of unitaries such that for $P_1(t) = U_t^* P_0 U_t$ one has $P_1(1) = P_1$ and $P_1(0) = P_0$, and $(P_0, P_1(t))$ is a Fredholm pair for all $t \in [0,1]$.*

As an application of Proposition 5.3.18 let us prove a statement on the connected components of Fredholm pairs of proper orthogonal projections:

$$\mathbb{FPP}(\mathcal{H}) = \big\{(P_0, P_1) \text{ Fredholm pair} : \dim(P_j) = \dim(\mathbf{1} - P_j) = \infty\big\}. \tag{5.19}$$

The result is the equivalent of Theorem 3.3.5 for Fredholm operators.

Proposition 5.3.23. *With respect to the norm topology on $\mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H})$, the set*

$$\mathbb{F}_n\mathbb{PP}(\mathcal{H}) = \big\{(P_0, P_1) \textit{ Fredholm} : \mathrm{Ind}(P_0, P_1) = n,\ \dim(P_j) = \dim(\mathbf{1} - P_j) = \infty\big\}$$

is connected.

Proof. Let $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}}) \in \mathbb{F}_n\mathbb{PP}(\mathcal{H})$ be a fixed Fredholm pair with index n such that $P_{0,\mathrm{ref}} \geq P_{1,\mathrm{ref}}$ if $n > 0$, $P_{0,\mathrm{ref}} \leq P_{1,\mathrm{ref}}$ if $n < 0$ and $P_{0,\mathrm{ref}} = P_{1,\mathrm{ref}}$ if $n = 0$. It will be shown that for $(P_0, P_1) \in \mathbb{F}_n\mathbb{PP}(\mathcal{H})$ there is a norm-continuous path of Fredholm pairs connecting (P_0, P_1) to $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}})$. First recall from the proof of Proposition 5.3.18 that there is a norm-continuous path of Fredholm pairs connecting (P_0, P_1) to (P_0, P_1') where with respect to the grading

$$\mathcal{H} = \mathrm{Ran}\big(\chi_{\{1\}}(P_0 - P_1)\big) \oplus \mathrm{Ran}\big(\chi_{\{-1\}}(P_0 - P_1)\big) \oplus \mathrm{Ran}\big(\chi_{\{-1,1\}}(P_0 - P_1)\big)^{\perp}$$

one has $P_1' = 0 \oplus \mathbf{1} \oplus \widetilde{P}_0$ and $\widetilde{P}_0 = P_0 \chi_{(-1,1)}(P_0 - P_1)$. In this grading, P_0 is of the form $P_0 = \mathbf{1} \oplus 0 \oplus \widetilde{P}_0$. Moreover, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ acting nontrivially only on $\mathrm{Ran}(\chi_{\{1\}}(P_0 - P_1)) \oplus \mathrm{Ran}(\chi_{\{-1\}}(P_0 - P_1))$ such that $P_1'' = U P_1' U^*$ fulfils $P_0 \geq P_1''$ if $n > 0$, $P_0 \leq P_1''$ if $n < 0$ and $P_0 = P_1''$ if $n = 0$. As $\mathbf{1} - U \in \mathbb{K}(\mathcal{H})$ is a compact operator, $t \in [0,1] \mapsto (P_0, U^t P_1' (U^t)^*)$ is a continuous path of Fredholm pairs connecting (P_0, P_1') to (P_0, P_1''). Finally, there is a unitary $V \in \mathbb{U}(\mathcal{H})$ such that $V P_0 V^* = P_{0,\mathrm{ref}}$ and $V P_1'' V^* = P_{1,\mathrm{ref}}$. Indeed, say for $n \geq 0$, one can first rotate P_0 to $P_{0,\mathrm{ref}}$ via a unitary $\hat{V}$, namely $\hat{V} P_0 \hat{V}^* = P_{0,\mathrm{ref}}$; then $\hat{V} P_1'' \hat{V}^* \leq P_{0,\mathrm{ref}}$; thus one can choose $\tilde{V}$ commuting with $P_{0,\mathrm{ref}}$ so that $\tilde{V} \hat{V} P_1'' \hat{V}^* \tilde{V}^* = P_{1,\mathrm{ref}}$; finally, set $V = \tilde{V} \hat{V}$. Then $t \in [0,1] \mapsto (V^t P_0 (V^t)^*, V^t P_1'' (V^t)^*)$ is a norm-continuous path of Fredholm pairs connecting (P_0, P_1'') to $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}})$. Concatenation of these paths leads to a path of Fredholm pairs connecting (P_0, P_1) to $(P_{0,\mathrm{ref}}, P_{1,\mathrm{ref}})$ (as the index of Fredholm pairs is locally constant by Proposition 5.2.7 the path lies in $\mathbb{F}_n\mathbb{PP}(\mathcal{H})$). As the Fredholm pair $(P_0, P_1) \in \mathbb{F}_n\mathbb{PP}(\mathcal{H})$ was arbitrary, this shows that $\mathbb{F}_n\mathbb{PP}(\mathcal{H})$ is connected. □

Corollary 5.3.24. *The path-connected components of* $\mathbb{FPP}(\mathcal{H})$ *are labeled by the index map* $\mathrm{Ind} : \mathbb{FPP}(\mathcal{H}) \to \mathbb{Z}$.

Remark 5.3.25. Given an arbitrary pair (P_0, P_1) of orthogonal projections, it is always possible to find a path $t \in [0, 1] \mapsto P_t$ connecting them. Indeed, there always exists a unitary U such that $P_1 = UP_0U^*$ (see Proposition 5.1.7) and then one can simply set $P_t = U^tP_0(U^t)^*$ where U^t is the $\frac{1}{t}$th root of U defined by spectral calculus. However, along this path, the Fredholm property is in general violated. ◇

5.4 Fredholm pairs of symmetries

Associated to an orthogonal projection P is always a symmetry, that is, a self-adjoint unitary, by the formula

$$Q = \mathbf{1} - 2P.$$

Definition 5.2.1 therefore naturally leads to the following:

Definition 5.4.1. Two symmetries Q_0 and Q_1 form a Fredholm pair of symmetries if and only if $P_0 = \frac{1}{2}(\mathbf{1} - Q_0)$ and $P_1 = \frac{1}{2}(\mathbf{1} - Q_1)$ are a Fredholm pair of orthogonal projections. Then the index of the Fredholm pair of symmetries is given by

$$\mathrm{Ind}(Q_0, Q_1) = \mathrm{Ind}(P_0, P_1).$$

Of course, Fredholm pairs of symmetries are merely a reformulation of Fredholm pairs of orthogonal projections, but in some instances below this leads to nicer formulas. The first result shows that a pair of symmetries is Fredholm if and only if the sum of this symmetries is Fredholm.

Proposition 5.4.2. *A pair* (P_0, P_1) *of orthogonal projections is Fredholm if and only the operator if* $Q_0 + Q_1$ *is Fredholm, where* $Q_0 = \mathbf{1} - 2P_0$ *and* $Q_1 = \mathbf{1} - 2P_1$.

Proof. As

$$Q_0 + Q_1 = 2(\mathbf{1} - P_0 - P_1),$$

$Q_0 + Q_1$ is Fredholm if and only if $\mathbf{1} - P_0 - P_1$ is Fredholm. Moreover,

$$(\mathbf{1} - P_0 - P_1)^2 = \mathbf{1} - P_0 - P_1 + P_0P_1 + P_1P_0 = (\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0).$$

By Theorem 5.3.8, (P_0, P_1) is a Fredholm pair if and only if $\mathbf{1} - P_0 + P_1$ and $\mathbf{1} - P_1 + P_0$ are Fredholm. Therefore $Q_0 + Q_1$ is Fredholm if (P_0, P_1) is a Fredholm pair. Conversely, if $Q_0 + Q_1$ is Fredholm, $(\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0) = (\mathbf{1} - P_1 + P_0)(\mathbf{1} - P_0 + P_1)$ is Fredholm by the above. We show that $(\mathbf{1} - P_0 + P_1)$ and $(\mathbf{1} - P_1 + P_0)$ are Fredholm. First

$$\operatorname{Ker}(\mathbf{1} - P_0 + P_1) \subset \operatorname{Ker}\big((\mathbf{1} - P_1 + P_0)(\mathbf{1} - P_0 + P_1)\big)$$

is finite dimensional. Analogously, $\operatorname{Ker}(\mathbf{1} - P_1 + P_0)$ is finite dimensional. The range of $\mathbf{1} - P_0 + P_1$ can be decomposed into a direct sum of two subspaces

$$\operatorname{Ran}(\mathbf{1} - P_0 + P_1) = \operatorname{Ran}\big((\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0)\big) \oplus \operatorname{Ran}\big((\mathbf{1} - P_0 + P_1)|_{\operatorname{Ker}(\mathbf{1} - P_1 + P_0)}\big).$$

The first summand is closed by the Fredholm property of $(\mathbf{1} - P_0 + P_1)(\mathbf{1} - P_1 + P_0)$, the second is finite dimensional. Thus $\operatorname{Ran}(\mathbf{1} - P_0 + P_1)$ is closed, and one concludes that $(\mathbf{1} - P_0 + P_1)$ is Fredholm. Analogously, $(\mathbf{1} - P_1 + P_0)$ is Fredholm. Theorem 5.3.8 allows concluding that (P_0, P_1) is a Fredholm pair. □

Lemma 5.4.3. *For symmetries Q_0 and Q_1, one has*

$$\operatorname{Ker}(Q_0 + Q_1) = \big(\operatorname{Ker}(Q_0 - \mathbf{1}) \cap \operatorname{Ker}(Q_1 + \mathbf{1})\big) \oplus \big(\operatorname{Ker}(Q_0 + \mathbf{1}) \cap \operatorname{Ker}(Q_1 - \mathbf{1})\big).$$

Proof. If Q_0 and Q_1 are expressed in terms of orthogonal projections P_0 and P_1, then

$$\operatorname{Ker}(Q_0 + Q_1) = \operatorname{Ker}(\mathbf{1} - P_0 - P_1).$$

For some vector $\phi = \phi_1 + \phi_2$ in this kernel such that $P_0\phi_1 = \phi_1$ and $P_0\phi_2 = 0$, one has

$$\begin{aligned} (\mathbf{1} - P_0 - P_1)\phi = 0 \quad &\Longleftrightarrow \quad \phi_2 - P_1\phi_1 - P_1\phi_2 = 0 \\ &\Longleftrightarrow \quad (\mathbf{1} - P_1)\phi_2 = P_1\phi_1. \end{aligned}$$

Hence $(\mathbf{1} - P_1)\phi_2 = 0 = P_1\phi_1$ and therefore

$$\operatorname{Ker}(Q_0 + Q_1) \subset \Big(\big(\operatorname{Ker}(Q_0 - \mathbf{1}) \cap \operatorname{Ker}(Q_1 + \mathbf{1})\big) \oplus \big(\operatorname{Ker}(Q_0 + \mathbf{1}) \cap \operatorname{Ker}(Q_1 - \mathbf{1})\big)\Big)$$

As the reverse inclusion is obvious, this implies the claim. □

If Q_0 and Q_1 are expressed in terms of P_0 and P_1, then the operators R_0 and R_1 defined in Lemma 5.3.6 are given by

$$R_0 = \mathbf{1} - (P_0 + P_1) = \frac{1}{2}(Q_0 + Q_1), \quad R_1 = P_0 - P_1 = \frac{1}{2}(Q_1 - Q_0).$$

Then the second set of identities of Lemma 5.3.6 becomes

$$\begin{aligned} R_0 Q_0 &= Q_1 R_0, & R_1 Q_0 &= -Q_1 R_1, & R_0^2 Q_0 &= Q_0 R_0^2, \\ R_0 Q_1 &= Q_0 R_0, & R_1 Q_1 &= -Q_0 R_1, & R_0 R_1 Q_0 &= -Q_0 R_0 R_1. \end{aligned}$$

Replacing R_1 in the formula in Theorem 5.3.8, one finds for a Fredholm pair (Q_0, Q_1) of symmetries

$$\mathrm{Ind}(Q_0, Q_1) = \dim(\mathrm{Ker}(R_1 - \mathbf{1})) - \dim(\mathrm{Ker}(R_1 + \mathbf{1})). \tag{5.20}$$

This leads to the following further formula for the index of Fredholm pairs.

Proposition 5.4.4. *For a Fredholm pair (Q_0, Q_1) of symmetries,*

$$\mathrm{Ind}(Q_0, Q_1) = \mathrm{Sig}\big((Q_1 - Q_0)|_{\mathrm{Ker}(Q_0+Q_1)}\big).$$

Proof. First of all, let us note that $\mathrm{Ker}(R_0)$ is an invariant subspace for R_1. Indeed, if $\phi \in \mathrm{Ker}(R_0)$, then, exploring the second identity in (5.15), one finds $R_0R_1\phi = -R_1R_0\phi = 0$. Moreover, the first identity $R_0^2 + R_1^2 = \mathbf{1}$ implies that $R_1|_{\mathrm{Ker}(R_0)}$ is nondegenerate. Hence the signature of this finite-dimensional operator is well defined. More precisely, one has

$$(R_1|_{\mathrm{Ker}(R_0)})^2 = \mathbf{1}|_{\mathrm{Ker}(R_0)},$$

namely $R_1|_{\mathrm{Ker}(R_0)}$ is a symmetry on $\mathrm{Ker}(R_0)$. Using again that on the spectral subspaces $\chi_{\{\pm1\}}(R_1)$ of R_1 the projections P_0 and P_1 are either the identity or the zero map, one obtains

$$\mathrm{Ker}(R_1 - \mathbf{1}) = \big(\mathrm{Ker}(Q_1 - \mathbf{1}) \cap \mathrm{Ker}(Q_0 + \mathbf{1})\big)$$

and

$$\mathrm{Ker}(R_1 + \mathbf{1}) = \big(\mathrm{Ker}(Q_1 + \mathbf{1}) \cap \mathrm{Ker}(Q_0 - \mathbf{1})\big).$$

By Lemma 5.4.3,

$$\mathrm{Ker}(R_0) = \mathrm{Ker}(R_1 - \mathbf{1}) \oplus \mathrm{Ker}(R_1 + \mathbf{1}).$$

Thus $\mathrm{Sig}((Q_1 - Q_0)|_{\mathrm{Ker}(Q_0+Q_1)})$ is given by the difference of dimension on the right-hand side of (5.20). □

5.5 Fredholm pairs of unitary conjugate projections

In many applications Fredholm pairs are explicitly given by pairs of unitary conjugate orthogonal projections, namely given in the form $(P_0, P_1) = (P, U^*PU)$ with a unitary operator U. Conversely, if P_0 and P_1 are both proper, namely have infinite-dimensional range and kernel, then they are always unitarily equivalent, as Proposition 5.1.7 shows. Hence many of the results of the last two sections transfer to this case, but sometimes take a slightly different form worth noting, in particular, for the context of applications.

Let us begin rewriting the Fredholm condition in this situation, which follows directly from Proposition 5.3.1.

Corollary 5.5.1. *Let P_0 and $P_1 = U^* P_0 U$ be orthogonal projections on $\mathcal{H}$. Then (P_0, P_1) is a Fredholm pair if and only if*

$$P_0 U^* P_0 U P_0 + \mathbf{1} - P_0 \quad \text{and} \quad P_0 U P_0 U^* P_0 + \mathbf{1} - P_0$$

are Fredholm operators on $\mathcal{H}$. If (P_0, P_1) is a Fredholm pair, then

$$\operatorname{Ind}(P_0, P_1) = \dim(\operatorname{Ker}(P_0 U^* P_0 U P_0 + \mathbf{1} - P_0)) - \dim(\operatorname{Ker}(P_0 U P_0 U^* P_0 + \mathbf{1} - P_0)).$$

Note that the Fredholm property of $P_0 U^* P_0 U P_0 + \mathbf{1} - P_0$ is not sufficient for $(P_0, U^* P_0 U)$ to be a Fredholm pair. This can be shown by considering $\mathcal{H} = \mathcal{H}' \otimes \mathbb{C}^3$ for an infinite-dimensional separable Hilbert space $\mathcal{H}'$ and setting

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by Proposition 5.1.7, there is a unitary $U \in \mathbb{U}(\mathcal{H})$ such that

$$P_1 = U^* P_0 U = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One directly checks that $P_0 U^* P_0 U P_0 + \mathbf{1} - P_0 = \mathbf{1}$ is Fredholm but $P_1 P_0 P_1 + \mathbf{1} - P_1 = P_0 + \mathbf{1} - P_1$ is not Fredholm. Therefore by Proposition 5.3.1, $(P_0, U^* P_0 U)$ is not a Fredholm pair.

In many situations one has the property that $[P, U]$ is compact. This does, however, not necessary hold for every Fredholm pair $(P, U^* P U)$, as shows the following remark.

Remark 5.5.2. This elaborates on Remark 5.3.16. Let

$$P_0 = \mathbf{1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \mathbf{1} \otimes \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

act on $\ell^2(\mathbb{N}) \otimes \mathbb{C}^2$. Then $P_1 = U^* P_0 U$ for the unitary operator

$$U = \mathbf{1} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and, by Remark 5.3.16, (P_0, P_1) is a Fredholm pair. On the other hand, neither the commutator $[U, P_0]$ nor $P_0 - P_1$ is compact. This generalizes as follows: Let P_0 and P_1 be two orthogonal projections such that $P_1 = U^* P_0 U$ for a unitary U. As

$$P_0 - P_1 = P_0 - U^* P_0 U = U^* [U, P_0],$$

the difference $P_0 - P_1$ is compact if and only if $[U, P_0]$ is compact. ◇

Proposition 5.5.3. *Let P be an orthogonal projection and $U \in \mathbb{U}(\mathcal{H})$ unitary. Then (P, U^*PU) is a Fredholm pair if and only if $PUP + \mathbf{1} - P$ is Fredholm and in this case*

$$\operatorname{Ind}(P, U^*PU) = \operatorname{Ind}(PUP + \mathbf{1} - P).$$

Proof. Let us set $T = PUP + \mathbf{1} - P$. Then, by Corollary 5.5.1, (P, U^*PU) is a Fredholm pair if and only if $PU^*PUP + \mathbf{1} - P = T^*T$ and $PUPU^*P + \mathbf{1} - P = TT^*$ are Fredholm, which is, by Theorem 3.4.1, equivalent to the Fredholm property of T. Moreover, by the expression for the index given in by Corollary 5.5.1,

$$\begin{aligned}
&\operatorname{Ind}(P, U^*PU) \\
&\quad = \dim(\operatorname{Ker}(PU^*PUP + \mathbf{1} - P)) - \dim(\operatorname{Ker}(PUPU^*P + \mathbf{1} - P)) \\
&\quad = \dim(\operatorname{Ker}(T^*T)) - \dim(\operatorname{Ker}(TT^*)).
\end{aligned}$$

As $\operatorname{Ind}(T) = \dim(\operatorname{Ker}(T^*T)) - \dim(\operatorname{Ker}(TT^*))$, this implies the claim. □

The following corollary is a consequence of Proposition 5.3.15.

Corollary 5.5.4. (i) *Suppose given an orthogonal projection P and furthermore two unitaries $U_1, U_2 \in \mathbb{U}(\mathcal{H})$ such that $\|\pi([U_1, P])\|_{\mathbb{Q}} + \|\pi([U_2, U_1^*PU_1])\|_{\mathbb{Q}} < 1$ holds. Then $(P, U_1^*PU_1)$, $(U_1^*PU_1, (U_1U_2)^*PU_1U_2)$, and $(P, (U_1U_2)^*PU_1U_2)$ are all Fredholm pairs and*

$$\operatorname{Ind}(P, (U_1U_2)^*PU_1U_2) = \operatorname{Ind}(P, U_1^*PU_1) + \operatorname{Ind}(U_1^*PU_1, (U_1U_2)^*PU_1U_2).$$

(ii) *Let $P \in \mathbb{B}(\mathcal{H})$ be an orthogonal projection and $U \in \mathbb{U}(\mathcal{H})$ unitary such that for some $n \in \mathbb{N}$ and all $k \in \{0, \dots, n-1\}$ one has $\|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, P])\|_{\mathbb{Q}} < 1$. Then (P, U^*PU), $(P, (U^*)^nPU^n)$, and $(P, U^nP(U^*)^n)$ are Fredholm pairs with index*

$$\operatorname{Ind}(P, (U^*)^nPU^n) = n \operatorname{Ind}(P, U^*PU) = -\operatorname{Ind}(P, U^nP(U^*)^n).$$

Proof. To show (i), note that $P - U^*PU = U^*[U, P]$ and therefore

$$\|\pi(P - U^*PU)\|_{\mathbb{Q}} = \|\pi([U, P])\|_{\mathbb{Q}}$$

for any orthogonal projection $P \in \mathbb{B}(\mathcal{H})$ and unitary $U \in \mathbb{U}(\mathcal{H})$. Hence

$$\|\pi(P - U_1^*PU_1)\|_{\mathbb{Q}} + \|\pi(U_1^*PU_1 - (U_1U_2)^*PU_1U_2)\|_{\mathbb{Q}} < 1,$$

and the claim follows from Proposition 5.3.15.

Because $\|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, U^*PU])\|_{\mathbb{Q}} = \|\pi([U, P])\|_{\mathbb{Q}} + \|\pi([U^k, P])\|_{\mathbb{Q}} < 1$, the first part of this corollary implies that (P, U^*PU), $(U^*PU, (U^*)^{k+1}PU^{k+1})$, and $(P, (U^*)^{k+1}PU^{k+1})$ are Fredholm pairs and

$$\operatorname{Ind}(P, (U^*)^{k+1}PU^{k+1}) = \operatorname{Ind}(P, U^*PU) + \operatorname{Ind}(U^*PU, (U^*)^{k+1}PU^{k+1}).$$

As $\mathrm{Ind}(U^*PU, (U^*)^{k+1}PU^{k+1}) = \mathrm{Ind}(P, (U^*)^k PU^k)$, it follows iteratively that

$$\mathrm{Ind}(P, (U^*)^{k+1} PU^{k+1}) = (k+1)\,\mathrm{Ind}(P, U^*PU),$$

which implies the first claim. The claim on $\mathrm{Ind}(P, U^n P(U^*)^n)$ follows by exchanging the roles of U and U^*. □

The following is merely a reformulation of Proposition 5.2.10 and Theorem 5.2.11.

Proposition 5.5.5. *Let $P \in \mathbb{B}(\mathcal{H})$ be an orthogonal projection, $U \in \mathbb{U}(\mathcal{H})$ unitary, and $n \in \mathbb{N}$ be such that*

$$P - PU^*PUP \in \mathcal{L}^n(\mathrm{Ran}(P)), \quad P - PUPU^*P \in \mathcal{L}^n(\mathrm{Ran}(P)).$$

Then (P, U^*PU) is a Fredholm pair of orthogonal projections, and for all $m \geq n$ one has

$$\mathrm{Ind}(P, U^*PU) = \mathrm{Tr}\big((P - PU^*PUP)^m\big) - \mathrm{Tr}\big((P - PUPU^*P)^m\big).$$

If (P, U^*PU) is a Fredholm pair of orthogonal projections and $(P - U^*PU)^{2n'+1}$ is trace class for some integer $n' \geq 0$, then for all $m' \geq n'$,

$$\mathrm{Ind}(P, U^*PU) = \mathrm{Tr}\big((P - U^*PU)^{2m'+1}\big).$$

Proof. As the property $P - PUPU^*P \in \mathcal{L}^n(\mathrm{Ran}(P))$ is equivalent to $U^*PU - U^*PUPU^*PU$ lying in $\mathcal{L}^n(\mathrm{Ran}(U^*PU))$ and

$$\mathrm{Tr}\big((P - PUPU^*P)^m\big) = \mathrm{Tr}\big((U^*PU - U^*PUPU^*PU)^m\big),$$

Proposition 5.2.10 implies the first claim. The second claim directly follows from Theorem 5.2.11. □

From the formula in Theorem 5.3.8, one can directly deduce the next result (taken from [18]).

Proposition 5.5.6. *Let (P_0, P_1) be a Fredholm pair of orthogonal projections. There exists a unitary $V \in \mathbb{U}(\mathcal{H})$ such that*

$$VP_0V^* = P_1 \quad \textit{and} \quad VP_1V^* = P_0$$

if and only if $\mathrm{Ind}(P_0, P_1) = 0$.

Proof. If such a V exists, then $V(P_0 - P_1)V^* = P_1 - P_0$ and thus, by Theorem 5.3.8, $\mathrm{Ind}(P_0, P_1) = 0$.

Conversely, let (P_0, P_1) be a Fredholm pair with vanishing index. As above, define $P_\pm = \chi_{\{\pm 1\}}(P_0 - P_1)$. As $\mathrm{Ind}(P_0, P_1) = 0$, $\mathrm{Ran}(P_+)$ and $\mathrm{Ran}(P_-)$ have the same dimensions by Theorem 5.3.8, there is a unitary operator $U_0 : \mathrm{Ran}(P_+) \to \mathrm{Ran}(P_-)$. Then the operator

$V_0 : \mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-) \to \mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-)$ defined by $V_0(\phi_+ + \phi_-) = U_0^* \phi_- + U_0 \phi_+$ for $\phi_+ \in \mathrm{Ran}(P_+)$ and $\phi_- \in \mathrm{Ran}(P_-)$ is unitary. Let $\tilde{V}$ denote the partial isometry in the polar decomposition $\mathbf{1} - P_0 - P_1 = \tilde{V}|\mathbf{1} - P_0 - P_1|$ of the operator $\mathbf{1} - P_0 - P_1$. Then also the restriction $V_1 = \tilde{V}|_{\mathrm{Ran}(\mathbf{1}-P_0-P_1)} : \mathrm{Ran}(\mathbf{1} - P_0 - P_1) \to \mathrm{Ran}(\mathbf{1} - P_0 - P_1)$ is unitary. Note that, by Lemma 5.4.3, $\mathrm{Ran}(\mathbf{1} - P_0 - P_1) = \mathcal{H} \ominus (\mathrm{Ran}(P_+) \oplus \mathrm{Ran}(P_-))$. As $\mathbf{1} - P_0 - P_1$ and $P_0 - P_1$ anticommute,

$$\tilde{V}(P_0 - P_1) = (P_1 - P_0)\tilde{V} \quad \text{and} \quad \tilde{V}(P_0 + P_1) = (P_0 + P_1)\tilde{V},$$

thus

$$\tilde{V}P_0 = P_1\tilde{V} \quad \text{and} \quad \tilde{V}P_1 = P_0\tilde{V}.$$

One directly checks that

$$VP_0V^* = P_1 \quad \text{and} \quad VP_1V^* = P_0$$

hold for $V = V_0 \oplus \tilde{V}$. □

5.6 Spectral flow of linear paths between Fredholm pairs

This section collects several formulas connecting the index of Fredholm pairs of orthogonal projections to a spectral flow. Let us begin with an expression of the spectral flow of the linear path connecting two symmetries that form a Fredholm pair by the index of this Fredholm pair.

Theorem 5.6.1. *For any Fredholm pair of symmetries Q_0, Q_1 on $\mathcal{H}$, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big) = \mathrm{Ind}(Q_0, Q_1). \tag{5.21}$$

Proof. The operators $H_t = (1-t)Q_0 + tQ_1$ are Fredholm. Indeed, $H_t = Q_0 + t(Q_1 - Q_0)$ is for $t \in [0, \frac{1}{2}]$ a perturbation of an operator Q_0 with spectrum $\{-1, 1\}$. As the Fredholm condition of the pair (Q_0, Q_1) is equivalent to $\|\pi(Q_0 - Q_1)\|_{\mathcal{Q}} < 2$ by Corollary 5.3.13, it follows that H_t has its essential spectrum bounded away from 0 for $t \in [0, \frac{1}{2}]$. Furthermore, for $t \in [\frac{1}{2}, 1]$ one can write $H_t = Q_1 + (1-t)(Q_0 - Q_1)$ so that the same argument applies. Moreover, H_t is invertible except possibly at $t = \frac{1}{2}$. The derivative at this point is

$$\partial_t H_t|_{t=\frac{1}{2}} = Q_1 - Q_0.$$

Hence the crossing form at $t = \frac{1}{2}$ is

$$\Gamma_{\frac{1}{2}} : \mathrm{Ker}(H_{\frac{1}{2}}) \to \mathrm{Ker}(H_{\frac{1}{2}}), \quad \Gamma_{\frac{1}{2}}(\phi) = \big\langle \phi \big| (Q_1 - Q_0)\phi \big\rangle,$$

and its signature is equal to the spectral flow by Proposition 4.3.6 which applies because the crossing was shown to be regular in the proof of Proposition 5.4.4. But, by Proposition 5.4.4, this signature is precisely the index of the Fredholm pair (Q_0, Q_1) of symmetries. □

Let us stress that in the earlier works [207, 148, 84], the equality (5.21) was only shown under the hypothesis that $Q_0 - Q_1$ is compact (or equivalently that the associated orthogonal projections have a compact difference $P_0 - P_1$). Next recall from Proposition 5.1.7 that, given two proper symmetries Q_0 and Q_1, it is always possible to find a unitary $U \in \mathbb{U}(\mathcal{H})$ such that $Q_1 = U^* Q_0 U$. For this situation, one thus obtains from Proposition 5.5.3:

Corollary 5.6.2. *For a Fredholm pair of symmetries $Q_0 = \mathbf{1} - 2P_0$ and $Q_1 = \mathbf{1} - 2P_1$ on $\mathcal{H}$ and a unitary U such that $Q_1 = U^* Q_0 U$, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big) = \mathrm{Ind}(P_0 U P_0 + \mathbf{1} - P_0).$$

Proof. As $P_1 = U^* P_0 U$, the claim directly follows from Theorem 5.6.1 and Proposition 5.5.3. □

Combined with Theorem 5.2.12, one also deduces the following formula for the spectral flow which is similar in spirit to Proposition 4.3.12.

Corollary 5.6.3. *Let (Q_0, Q_1) be a Fredholm pair of symmetries such that $(Q_0 - Q_1)^{2n+1}$ is trace class for some integer $n \geq 0$. Then for the linear path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$ and any $k \geq n$,*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) = \frac{1}{C_k} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (1 - Q_t^2)^k\big), \tag{5.22}$$

with C_k given in (5.13).

Similarly, also Proposition 5.3.11 leads to a formula for the spectral flow, see [56].

Corollary 5.6.4. *Let (Q_0, Q_1) be a Fredholm pair of symmetries on $\mathcal{H}$ such that the operator* $\exp(-((Q_0 - Q_1)^2)^{-\frac{1}{q}})$ *is trace class for some* $0 < q \leq 1$. *Then the path*

$$t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$$

satisfies

$$\mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) = \frac{1}{C_{r,q}} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (\mathbf{1} - Q_t^2)^{-r} e^{-(\mathbf{1} - Q_t^2)^{-\frac{1}{q}}}\big)$$

for $r \geq 0$ and where $C_{r,q}$ is given in (5.17).

5.7 Spectral flow formulas for paths with compact difference

Section 4.3 already presented a quite diverse selection of formulas for the spectral flow. Here further formulas are provided, all based on the results of the last Section 5.6. First let us generalize Theorem 5.6.1 to linear paths connecting two invertible self-adjoint operators (instead of symmetries) with compact difference.

Corollary 5.7.1. *For self-adjoint invertible operators $H_0, H_1 \in \mathbb{B}(\mathcal{H})$ such that the difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ is compact, one has*

$$\begin{aligned}\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big) &= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big)\\ &= \mathrm{Ind}(Q_0, Q_1),\end{aligned}$$

where $Q_i = H_i|H_i|^{-1}$ is the unitary phase of H_i for $i = 0, 1$.

Proof. (Some elements are similar to the proof of Proposition 5.3.17.) As $H_1 - H_0$ is compact, $(1-t)H_0 + tH_1 = H_0 + t(H_1 - H_0)$ is Fredholm for all $t \in [0,1]$ by Corollary 3.2.3. Moreover, $h(H_1) - h(H_0)$ is compact for every continuous function $h : \Sigma \to \mathbb{C}$ where $\Sigma = \mathrm{spec}(H_0) \cup \mathrm{spec}(H_1)$. As

$$H_1^n - H_0^n = H_1(H_1^{n-1} - H_0^{n-1}) + (H_1 - H_0)H_0^{n-1}, \quad n \geq 2,$$

$p(H_1) - p(H_0)$ is compact for any polynomial $p : \mathbb{C} \to \mathbb{C}$. As the set of compact operators $\mathbb{K}(\mathcal{H})$ is a closed subset of the set of bounded operators $\mathbb{B}(\mathcal{H})$ and the polynomials are dense in the set of continuous functions on compact domains, we see that $h(H_1) - h(H_0)$ is compact for every continuous function $h : \mathrm{spec}(H_0) \cup \mathrm{spec}(H_1) \to \mathbb{C}$. Therefore $(t,s) \in [0,1] \times [0,1] \mapsto (1-t)H_0|H_0|^{-s} + tH_1|H_1|^{-s}$ is a continuous homotopy of Fredholm operators. By Theorem 4.2.2,

$$\begin{aligned}&\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big)\\ &\quad = \mathrm{Sf}\big(s \in [0,1] \mapsto H_0|H_0|^{-s}\big)\\ &\qquad + \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0|H_0|^{-1} + tH_1|H_1|^{-1}\big)\\ &\qquad - \mathrm{Sf}\big(s \in [0,1] \mapsto H_1|H_1|^{-s}\big).\end{aligned}$$

As

$$s \in [0,1] \mapsto H_0|H_0|^{-s} \quad \text{and} \quad s \in [0,1] \mapsto H_1|H_1|^{-s}$$

are paths of invertibles and therefore the spectral flow along these paths vanishes, and $H_0|H_0|^{-1} = Q_0$ and $H_1|H_1|^{-1} = Q_1$, one has

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tH_1\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)Q_0 + tQ_1\big).$$

The remaining claim follows from Theorem 5.6.1. □

Similar as Corollary 5.6.2 is a special case of Theorem 5.6.1, one can now state Corollary 5.7.1 for the special case of paths with unitary equivalent endpoints.

Corollary 5.7.2. *For a self-adjoint invertible operator $H \in \mathbb{B}(\mathcal{H})$ and a unitary $U \in \mathbb{U}(\mathcal{H})$ such that the commutator $[H, U] \in \mathbb{K}(\mathcal{H})$ is compact, one has*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H + tU^*HU\big) = \mathrm{Ind}(PUP + \mathbf{1} - P),$$

where $P = \chi(H \leq 0)$ is the orthogonal projection onto the negative spectrum of H.

Proof. As $H - U^*HU = [H, U^*]U$ is compact,

$$\begin{aligned}&\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H + tU^*HU\big)\\ &\quad = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)(\mathbf{1}-2P) + tU^*(\mathbf{1}-2P)U\big),\end{aligned}$$

by Corollary 5.7.1. Now the claim follows from Corollary 5.6.2. □

The next result is the starting point for many applications, e. g., all of Chapter 10. It also considers a situation similar to Corollary 5.7.2, namely paths with unitary conjugate endpoints, but does not require the paths to be linear. The result goes back to the work of Phillips [148] with precursors like Wojciechowski [207], see also [70].

Theorem 5.7.3. *Let $t \in [0,1] \mapsto H_t$ be a norm-continuous path of self-adjoint operators with invertible endpoints H_0 and H_1 such that $H_t - H_0$ is compact for all $t \in [0,1]$ and $H_1 = U^*H_0U$. If $P = \chi(H_0 \leq 0)$, then*

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Ind}(PUP + \mathbf{1} - P).$$

In particular, one has for the linear path connecting $\mathbf{1} - 2P$ and $U^*(\mathbf{1} - 2P)U$,

$$\mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)(\mathbf{1}-2P) + tU^*(\mathbf{1}-2P)U\big) = \mathrm{Ind}(PUP + \mathbf{1} - P).$$

Proof. First $H_1 - H_0 = U^*H_0U - H_0 = U^*[H_0, U] \in \mathbb{K}(\mathcal{H})$ is compact by assumption. Thus $[H_0, U] \in \mathbb{K}(\mathcal{H})$ is compact and, by Corollary 5.7.2,

$$\mathrm{Ind}(PUP + \mathbf{1} - P) = \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tU^*H_0U\big). \tag{5.23}$$

The homotopy $h : [0,1] \times [0,1] \to \mathbb{B}(\mathcal{H})$,

$$h(t,s) = (1-s)H_t + s\big((1-t)H_0 + tH_1\big),$$

is within the Fredholm operators as $h(t,s) = H_0 + (1-s)(H_t - H_0) + st(H_1 - H_0)$ and $(1-s)(H_t - H_0) + st(H_1 - H_0)$ is compact for all $(t,s) \in [0,1] \times [0,1]$. As $h(0,s) = H_0$ and $h(1,s) = H_1$ for all $s \in [0,1]$, Theorem 4.2.2 implies

$$\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto h(t,0)\big)$$

$$
\begin{aligned}
&= \mathrm{Sf}\big(t \in [0,1] \mapsto h(t,1)\big) \\
&= \mathrm{Sf}\big(t \in [0,1] \mapsto (1-t)H_0 + tU^* H_0 U\big),
\end{aligned}
$$

which by (5.23) implies the claim. □

Let us give an elementary example illustrating some of the above facts.

Example 5.7.4. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with orthonormal basis $|n\rangle$, $n \in \mathbb{Z}$. Introduce the symmetry

$$
Q_0 = \sum_{n\geq 0} |n\rangle\langle n| - \sum_{n<0} |n\rangle\langle n|.
$$

Furthermore, let S be the left-shift on $\mathcal{H}$ given by $S|n\rangle = |n-1\rangle$. For $k \in \mathbb{N}$, choose $U = (S^k)^*$ and set $Q_k = (S^k)^* Q_0 S^k$. Now, roughly stated, Q_k has k less positive eigenvalues than Q_0. This difference between infinities is taken into account by the spectral flow. Calculating the spectrum on the straight line path $H_t = (1-t)Q_0 + tQ_k \in \mathbb{F}\mathbb{B}^*_{\mathrm{sa}}(\mathcal{H})$ explicitly shows

$$
\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = -k.
$$

Alternatively, $(\partial_t H)_t = Q_k - Q_0 = -2P_k$ where P_k is the finite-dimensional orthogonal projection on the span of $|n\rangle$, $n = 0, \dots, k-1$. Then by Proposition 4.3.12,

$$
\mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big) = \frac{1}{2}\int_0^1 dt\, \mathrm{Tr}\big(g'(H_t)(\partial_t H)_t\big) = -k,
$$

where g is a smooth nonnegative function of integral 1 which is supported in the gap of the essential spectrum of H_t for all $t \in [0,1]$. ◇

The Fredholm operators in Theorem 5.7.3 often appear as the result of an even index pairing between a K_0-class specified by P (e. g., of the C^*-algebra generated by H_0) and a graded Dirac operator, see Section 10.1 for a detailed description. Section 10.1 also describes odd index pairings and the following result can be interpreted as an odd (or dual) analogue to Theorem 5.7.3, namely an index formula for paths of unitaries with conjugate endpoints by a self-adjoint conjugation operator. It will use the notion of spectral flow of paths of normal operators as given in (4.14).

Theorem 5.7.5. *Let $t \in [0,1] \mapsto U_t$ be a path of unitaries such that $U_t - U_0$ is compact. Suppose that there is a self-adjoint unitary G such that $U_1 = GU_0G$. If $E = \chi(G \geq 0)$, then $EU_0E + \mathbf{1} - E$ is a Fredholm operator with index given by*

$$
\mathrm{Ind}(EU_0E + \mathbf{1} - E) = \mathrm{Sf}\big(t \in [0,1] \mapsto \mathbb{R}e(W_t)\big), \tag{5.24}
$$

where $W_t = GU_tU_0^*$ is unitary.

Proof. To show that the spectral flow is well defined, first note that $U_t U_0^* - \mathbf{1} = (U_t - U_0)U_0^*$ is compact by hypothesis, so that $W_t - G$ is compact and so is $\mathbb{R}e(W_t) - G$. Moreover, $W_0 = G = \mathbb{R}e(W_0)$ and $W_1 = U_0 G U_0^* = \mathbb{R}e(W_1)$ are both self-adjoint, and one has $\mathbb{R}e(W_1) = U_0 \mathbb{R}e(W_0) U_0^*$. Thus Theorem 5.7.3 can be applied to $t \in [0,1] \mapsto H_t = \mathbb{R}e(W_t)$. There are now two sign changes in the index pairing involved, one because E is the spectral projection onto the positive spectrum of G and one because U_0 is on the left-hand side in $\mathbb{R}e(W_1) = U_0 \mathbb{R}e(W_0) U_0^*$ (while P is the negative spectral projection of H_0 and $H_1 = U^* H_0 U$ in Theorem 5.7.3). This concludes the proof. □

The spectral flow of unitaries appearing on the right-hand side of (5.24) inherits natural homotopy invariance properties. For example, choosing $U_0 U_t^* G$ instead of W_t is another natural choice giving a different path connecting G and $U_0 G U_0^*$. The choices $G U_t^* U_0$ and $U_0^* U_t G$ reverse the path and thus the sign of the spectral flow. A standard choice of $t \in [0,1] \mapsto U_t$ leading to a path $t \in [0,1] \mapsto G U_t U_0^*$ from G to $U_0 G U_0^*$, expressed merely in terms of U_0 and G, is given by

$$U_t = U_0 \exp\left(\frac{\imath\pi}{2} (G - \mathbf{1} + t U_0^* [G, U_0]) \right) G.$$

As explained in [70], this path leads to a K-theoretic interpretation of Theorem 5.7.5.

The next set of results generalizes the formula given in Corollary 5.6.3. The proofs are based on Singer's idea to use closed 1-forms [183] which in this context was further developed in the work of Getzler [96] and more thoroughly in the works of Carey and Phillips [55, 56]. The latter two papers contain more general versions of the next results. More precisely, these works require fewer summability assumptions and also deal with the case of semifinite spectral flow discussed in Chapter 11.

Proposition 5.7.6. *Let (Q_0, Q_1) be a Fredholm pair of symmetries on $\mathcal{H}$ connected by a path $t \in [0,1] \mapsto Q_t \in \mathbb{B}_{sa}(\mathcal{H})$ such that $(Q_t - Q_0)^{2n+1}$ is trace class for some integer $n \geq 0$ and the path is continuously $\mathcal{L}^{2n+1}$-differentiable. Then one has for any $k \geq n$,*

$$\mathrm{Sf}(t \in [0,1] \mapsto Q_t) = \frac{1}{C_k} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t Q)_t (\mathbf{1} - Q_t^2)^k\big), \tag{5.25}$$

with C_k given in (5.13).

Proof. Let $\mathcal{L}^{2n+1}_{sa}(\mathcal{H})$ denote the set of self-adjoint operators in the $(2n+1)$th Schatten ideal $\mathcal{L}^{2n+1}(\mathcal{H})$. Then consider the set $\mathcal{M} = Q_0 + \mathcal{L}^{2n+1}_{sa}(\mathcal{H})$ as a manifold with tangent space $T\mathcal{M} = \mathcal{L}^{2n+1}_{sa}(\mathcal{H})$. By assumption the path $t \in [0,1] \mapsto Q_t$ lies in $\mathcal{M}$ and is differentiable with derivatives $(\partial_t Q)_t$ lying in $\mathcal{L}^{2n+1}_{sa}(\mathcal{H})$. Let us introduce a 1-form α_k on $\mathcal{M}$ by setting

$$\alpha_{k,Q}(X) = \frac{1}{C_k} \mathrm{Tr}\big(X(\mathbf{1} - Q^2)^k\big), \quad X \in T\mathcal{M},\ Q \in \mathcal{M}. \tag{5.26}$$

Note that $\alpha_{k,Q}(X)$ is real because it is given by the trace of a product of two self-adjoint operators. The integral on the right-hand side of (5.25) is by definition the integral of α_k over the path $t \in [0,1] \mapsto Q_t$. To show the claim (5.25), it will be verified that this integral of α_k over a path is invariant under changes of the path inside $\mathcal{M}$ with fixed endpoints, or alternatively that it vanishes on closed curves. This will follow by adapting a standard argument. Let us first show that the form α_k is closed, namely that one has

$$\partial_s|_{s=0}\alpha_{k,Q+sY}(X) = \partial_s|_{s=0}\alpha_{k,Q+sX}(Y), \quad X, Y \in T\mathcal{M}.$$

This follows from a computation based on the Leibniz rule, $\partial_s|_{s=0}(Q + sY)^2 = QY + YQ$ and the cyclicity of the trace:

$$\begin{aligned}
C_k\partial_s|_{s=0}\alpha_{k,Q+sY}(X) &= \partial_s|_{s=0} \operatorname{Tr}\big(X(\mathbf{1} - (Q + sY)^2)^k\big) \\
&= -\sum_{l=0}^{k-1} \operatorname{Tr}\big(X(\mathbf{1} - Q^2)^l(QY + YQ)(\mathbf{1} - Q^2)^{k-1-l}\big) \\
&= -\sum_{l=0}^{k-1} \operatorname{Tr}\big(Y(\mathbf{1} - Q^2)^{k-1-l}(QX + XQ)(\mathbf{1} - Q^2)^l\big) \\
&= C_k\partial_s|_{s=0}\alpha_{k,Q+sX}(Y).
\end{aligned}$$

Now given that α_k is closed, one can deduce that the integral of α_k over a closed curve vanishes. This can first be shown for rectangles lying in a two-dimensional plane spanned by two vectors $X, Y \in T\mathcal{M}$, by transposing Pirkheimer's proof of the Goursat lemma. Then one can deduce it by the usual approximation arguments for an arbitrary differentiable curve in $\mathcal{M}$. Let us stress that the argument only requires that the derivatives of α_k exist (and neither their continuity nor the exactness of the form which in the present situation is given, but not when the above argument is applied in the proof of Theorem 7.2.2 later on). Therefore the right-hand side of (5.25) equals the integral of α_k over the linear path connecting Q_0 to Q_1 which, by Corollary 5.6.3, equals the spectral flow of the linear path connecting Q_0 to Q_1. As $Q_0 - Q_t$ is compact for all $t \in [0,1]$, the claim follows from the homotopy invariance of the spectral flow. □

Remark 5.7.7. The 1-form defined in (5.26) satisfies $\alpha_k = d\beta_{k,F'}$ where, for an arbitrary fixed point $F' \in \mathcal{M} = Q_0 + \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$, the 0-form $\beta_{k,F'} : \mathcal{M} \to \mathbb{C}$ is defined by

$$\beta_{k,F'}(F) = \frac{1}{C_k}\int_0^1 dt\, \operatorname{Tr}\big((\partial_t F)_t(\mathbf{1} - F_t^2)^k\big),$$

where $F_t = F' + t(F - F')$ is the linear path between F' and F. This is merely the Poincaré lemma for the 1-form α_k which holds globally in all of $\mathcal{M}$. This can be verified by an explicit computation as the one in the proof of Proposition 5.7.6 which we provide for the convenience of the reader. The claim $\alpha_k = d\beta_{k,F'}$ explicitly means

$$\partial_s|_{s=0}\beta_{k,F'}(F+sX) = \alpha_{k,F}(X), \quad X \in T\mathcal{M}.$$

To verify this, let us set

$$F_r(s) = (1-r)F' + r(F+sX),$$

so that $\partial_s F_r(s) = rX$ and $\partial_r F_r(s) = F + sX - F'$, as well as $F_r = F_r(0)$. Then

$$\begin{aligned} C_k \partial_s|_{s=0}\beta_{k,F'}(F+sX) &= \partial_s|_{s=0} \int_0^1 dr\, \mathrm{Tr}\big((F+sX-F')(1-F_r(s)^2)^k\big) \\ &= \int_0^1 dr\big[\mathrm{Tr}\big(X(1-F_r^2)^k\big) + \partial_s|_{s=0}\, \mathrm{Tr}\big((F-F')(1-F_r(s)^2)^k\big)\big]. \end{aligned}$$

The derivative is computed as above, using the Leibniz rule and the cyclicity of the trace,

$$\begin{aligned} &\partial_s|_{s=0}\, \mathrm{Tr}\big((F-F')(1-F_r(s)^2)^k\big) \\ &= -\sum_{l=0}^{k-1} \mathrm{Tr}\big((F-F')(1-F_r^2)^l \partial_s|_{s=0}F_r(s)^2(1-F_r^2)^{k-1-l}\big) \\ &= -r\sum_{l=0}^{k-1} \mathrm{Tr}\big((F-F')(1-F_r^2)^l (XF_r + F_rX)(1-F_r^2)^{k-1-l}\big) \\ &= -r\sum_{l=0}^{k-1} \mathrm{Tr}\big(X(1-F_r^2)^{k-1-l}(F_r(F-F') + (F-F')F_r)(1-F_r^2)^l\big) \\ &= r\partial_r\, \mathrm{Tr}\big(X(1-F_r^2)^k\big). \end{aligned}$$

By replacing, one finds

$$\begin{aligned} C_k\partial_s|_{s=0}\beta_{k,F'}(F+sX) &= \int_0^1 dr\big[\mathrm{Tr}\big(X(1-F_r^2)^k\big) + r\partial_r\, \mathrm{Tr}\big(X(1-F_r^2)^k\big)\big] \\ &= \int_0^1 dr \partial_r\big[r\, \mathrm{Tr}\big(X(1-F_r^2)^k\big)\big] \\ &= \mathrm{Tr}\big(X(1-F_1^2)^k\big) \\ &= \mathrm{Tr}\big(X(1-F^2)^k\big) = C_k\alpha_{k,F}(X), \end{aligned}$$

which shows the claim. ◇

In the following, Proposition 5.7.6 will be further generalized to paths for which the endpoints are not necessarily symmetries. The following object is needed.

Definition 5.7.8. Let $F_0 \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ be a base point satisfying $F_0^2 - \mathbf{1} \in \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$ for some $n \in \mathbb{N}$. Then for an invertible $F \in \mathcal{M} = F_0 + \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$ with phase $Q = \mathrm{sgn}(F)$ set

$$\beta_k(F) = \frac{1}{C_k} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t F)_t (\mathbf{1} - F_t^2)^k\big),$$

where $t \mapsto F_t = F + t(Q - F)$ is the linear path from F to Q and $k \geq n$.

Theorem 5.7.9. *Let $t \in [0,1] \mapsto F_t \in \mathbb{B}_{\mathrm{sa}}(\mathcal{H})$ be such that*
(i) $F_0^2 - \mathbf{1} \in \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$,
(ii) $F_t - F_0 \in \mathcal{L}^{2n+1}_{\mathrm{sa}}(\mathcal{H})$,
(iii) *the path $t \mapsto F_t$ is continuously $\mathcal{L}^{2n+1}$-differentiable.*

Then one has for any $k \geq n$,

$$\mathrm{Sf}\big(t \in [0,1] \mapsto F_t\big) = \beta_k(F_1) - \beta_k(F_0) + \frac{1}{C_k} \int_0^1 dt\, \mathrm{Tr}\big((\partial_t F)_t (\mathbf{1} - F_t^2)^k\big).$$

Proof. First of all, note that

$$\beta_k(F_0) = \int_{[F_0,Q_0]} \alpha_k, \quad \beta_k(F_1) = \int_{[F_1,Q_1]} \alpha_k,$$

where $[F_j, Q_j]$ denotes the straight-line path from F_j to $Q_j = \mathrm{sgn}(F_j)$ for $j = 0,1$ (these paths lie in $\mathcal{M}$). Moreover, let $[Q_0, Q_1]$ denote the path $t \in [0,1] \mapsto Q_t = Q_0 + t(Q_1 - Q_0)$ from Q_0 to Q_1 (attention: Q_t is *not* equal to $\mathrm{sgn}(F_t)$). Then the path $t \in [0,1] \mapsto F_t$ is homotopic to $[F_0, Q_0] * [Q_0, Q_1] * (-[F_1, Q_1])$ where $-[F_1, Q_1]$ denotes the reversed path of $[F_1, Q_1]$. As the paths $[F_0, Q_0]$ and $[F_1, Q_1]$ lie in the invertibles, there is no spectral flow along them. Hence by the homotopy invariance of the spectral flow and Proposition 5.7.6,

$$\begin{aligned}
\mathrm{Sf}\big(t \in [0,1] \mapsto F_t\big) &= \mathrm{Sf}\big(t \in [0,1] \mapsto Q_t\big) \\
&= \int_{[Q_0,Q_1]} \alpha_k \\
&= \int_{[Q_0,F_0]} \alpha_k + \int_{[t \in [0,1] \mapsto F_t]} \alpha_k + \int_{[F_1,Q_1]} \alpha_k,
\end{aligned}$$

where in the last step the closedness of the 1-form α_k was used in order to deform the integration path. The middle term is precisely the integral in the statement, which is hence verified. □

Remark 5.7.10. The essential ingredient of the proof of Theorem 5.7.9 is Corollary 5.6.3. It is possible to carry out a similar reasoning based on Corollary 5.6.4. This is carried out in the work of Carey and Phillips [56]. If the statement is then applied to the bounded transform of paths of self-adjoint Fredholm operators with compact resolvent, one obtains, after the change of variables connected to the bounded transform, a proof of Theorem 7.2.2, namely the equivalents of the boundary terms $\beta_k(F)$ become the η-invariants. ◇

5.8 Spectral flow as sum of indices of Fredholm pairs

In this section, it is shown how the spectral flow of an arbitrary norm-continuous path $t \in [0,1] \mapsto H_t$ of bounded self-adjoint Fredholm operators can be expressed as a sum of indices of Fredholm pairs of orthogonal projections. The outcome is Proposition 5.8.2 below. As a preparation for the statement, the following lemma is needed.

Lemma 5.8.1. *For every $H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ there is $a > 0$ and a neighborhood $\mathcal{N}'_{H,a} \subset \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous finite-rank projection-valued function on $\mathcal{N}'_{H,a}$, $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair, where $P^{\geq}(A) = \chi_{[0,\infty)}(A)$ for every self-adjoint Fredholm operator A, and*

$$\mathrm{Ind}\big(P^{\geq}(S), P^{\geq}(H)\big) = \mathrm{Ind}\big(\chi_{[0,a]}(S), \chi_{[0,a]}(H)\big)$$

for all $S \in \mathcal{N}'_{H,a}$.

Proof. By Lemma 4.1.1, there is a neighborhood $\mathcal{N}$ of H and $a > 0$ such that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous finite-rank projection-valued function on $\mathcal{N}$. By construction (see the proof of Lemma 4.1.1), the function $S \mapsto \chi_{(a,\infty)}(S)$ is norm-continuous on $\mathcal{N}$. Define $\tilde{\mathcal{N}}'_{H,a}$ as

$$\tilde{\mathcal{N}}'_{H,a} = \{S \in \mathcal{N} : \|\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S)\| < 1\}.$$

As $S \mapsto \chi_{(a,\infty)}(S)$ is norm-continuous on $\mathcal{N}$, this is a neighborhood of H. Then define $\mathcal{N}'_{H,a}$ as the connected component of $\tilde{\mathcal{N}}_{H,a}$ containing H. It remains to show that $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair with index $\mathrm{Ind}(\chi_{[0,a]}(S), \chi_{[0,a]}(H))$ for all $S \in \mathcal{N}'_{H,a}$. As $P^{\geq}(H) = \chi_{[0,a]}(H) + \chi_{(a,\infty)}(H)$ and, similarly, $P^{\geq}(S) = \chi_{[0,a]}(S) + \chi_{(a,\infty)}(S)$, where $\chi_{[0,a]}(H)$ and $\chi_{[0,a]}(S)$ are compact,

$$\begin{aligned}\|\pi(P^{\geq}(H) - P^{\geq}(S))\|_{\mathcal{Q}} &= \|\pi(\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S))\|_{\mathcal{Q}} \\ &\leq \|\chi_{(a,\infty)}(H) - \chi_{(a,\infty)}(S)\| < 1.\end{aligned}$$

Therefore, by Corollary 5.3.13, $(P^{\geq}(S), P^{\geq}(H))$ is a Fredholm pair. Its index equals the index of $P^{\geq}(H)P^{\geq}(S)|_{\mathrm{Ran}(P^{\geq}(S))} : \mathrm{Ran}(P^{\geq}(S)) \to \mathrm{Ran}(P^{\geq}(H))$ by Definition 5.2.1. Thus

$$\begin{aligned}\mathrm{Ind}\big(P^{\geq}(S), P^{\geq}(H)\big) &= \mathrm{Ind}\big((\chi_{[0,a]}(H) + \chi_{(a,\infty)}(H))(\chi_{[0,a]}(S) + \chi_{(a,\infty)}(S))\big) \\ &= \mathrm{Ind}\big(\chi_{[0,a]}(H)\chi_{[0,a]}(S) + \chi_{[0,a]}(H)\chi_{(a,\infty)}(S)\end{aligned}$$

$$+\chi_{(a,\infty)}(H)\chi_{[0,a]}(S)+\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big).$$

As the second and third summands in the last expression are compact, Theorem 3.3.4 implies

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(H)\chi_{[0,a]}(S)+\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big).$$

By Corollary 3.3.2, this implies

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(H)\chi_{[0,a]}(S)\big)+\operatorname{Ind}\big(\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S)\big),$$

where $\chi_{[0,a]}(H)\chi_{[0,a]}(S) : \operatorname{Ran}(\chi_{[0,a]}(S)) \to \operatorname{Ran}(\chi_{[0,a]}(H))$ and $\chi_{(a,\infty)}(H)\chi_{(a,\infty)}(S) : \operatorname{Ran}(\chi_{(a,\infty)}(S)) \to \operatorname{Ran}(\chi_{(a,\infty)}(H))$ are Fredholm operators. Again by Definition 5.2.1,

$$\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}\big(\chi_{[0,a]}(S),\chi_{[0,a]}(H)\big)+\operatorname{Ind}\big(\chi_{(a,\infty)}(S),\chi_{(a,\infty)}(H)\big)$$

follows. By definition of $\mathcal{N}'_{H,a}$, $\|\chi_{(a,\infty)}(H)-\chi_{(a,\infty)}(S)\| < 1$ which, by Proposition 5.3.12, implies

$$\operatorname{Ind}\big(\chi_{(a,\infty)}(S),\chi_{(a,\infty)}(H)\big)=0.$$

Therefore $\operatorname{Ind}(P^{\geq}(S),P^{\geq}(H))=\operatorname{Ind}(\chi_{[0,a]}(S),\chi_{[0,a]}(H))$, finishing the proof. □

By compactness, it is possible to choose a finite partition

$$0=t_0<t_1<\cdots<t_{M-1}<t_M=1, \tag{5.27}$$

of $[0,1]$ and $a_m>0$, $m=1,\dots,M$, such that

$$t\in[t_{m-1},t_m]\mapsto H_t$$

lies entirely in the neighborhood $\mathcal{N}'_{H_{t_m},a_m}$ of H_{t_m} defined in Lemma 5.8.1.

Proposition 5.8.2. *For a partition* $0=t_0<t_1<\cdots<t_{M-1}<t_M=1$ *as above, one has*

$$\begin{aligned}&\operatorname{Sf}(t\in[0,1]\mapsto H_t)\\&\quad=\frac{1}{2}\dim(\operatorname{Ker}(H_0))+\sum_{m=1}^{M}\operatorname{Ind}(P^{\geq}(H_{t_m}),P^{\geq}(H_{t_{m-1}}))-\frac{1}{2}\dim(\operatorname{Ker}(H_1)).\end{aligned}$$

Proof. By Lemma 5.8.1,

$$\operatorname{Ind}(P^{\geq}(H_{t_m}),P^{\geq}(H_{t_{m-1}}))=\operatorname{Ind}(P^{\geq}_{a_m,t_m},P^{\geq}_{a_m,t_{m-1}})$$

for all $m=1,2,\dots,M$ where $P^{\geq}_{a_m,t}=\chi_{[0,a_m]}(H_t)$. As $P^{\geq}_{a_m,t_m}$ and $P^{\geq}_{a_m,t_{m-1}}$ are finite-dimensional,

$$\begin{aligned}\operatorname{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) &= \dim(\operatorname{Ran}(P^{\geq}_{a_m,t_m})) - \dim(\operatorname{Ran}(P^{\geq}_{a_m,t_{m-1}}))\\ &= \operatorname{Tr}(P^{\geq}_{a_m,t_m} - P^{\geq}_{a_m,t_{m-1}})\\ &= \operatorname{Tr}(P^{>}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}}) + \dim(\operatorname{Ker}(H_{t_m})) - \dim(\operatorname{Ker}(H_{t_{m-1}}))\end{aligned} \tag{5.28}$$

by Proposition 5.2.2. By Definition 4.1.2,

$$\operatorname{Sf}(t \in [0,1] \mapsto H_t) = \frac{1}{2}\sum_{m=1}^{M} \operatorname{Tr}(P^{>}_{a_m,t_m} - P^{<}_{a_m,t_m} - P^{>}_{a_m,t_{m-1}} + P^{<}_{a_m,t_{m-1}}).$$

As $t \in [t_{m-1}, t_m] \mapsto H_t$ lies entirely in the neighborhood $\mathcal{N}'_{H_{t_m},a_m}$ of H_{t_m}, one concludes that the path $t \in [t_{m-1}, t_m] \mapsto \operatorname{Tr}(\chi_{[-a_m,a_m]}(H_t))$ is constant. Therefore

$$\begin{aligned}&\operatorname{Tr}(P^{<}_{a_m,t_{m-1}}) - \operatorname{Tr}(P^{<}_{a_m,t_m})\\ &\quad = \operatorname{Tr}(P^{>}_{a_m,t_m}) + \dim(\operatorname{Ker}(H_{t_m})) - \operatorname{Tr}(P^{>}_{a_m,t_{m-1}}) - \dim(\operatorname{Ker}(H_{t_{m-1}}))\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Sf}(t \in [0,1] \mapsto H_t)\\ &= \frac{1}{2}\sum_{m=1}^{M}(\operatorname{Tr}(2P^{>}_{a_m,t_m} - 2P^{>}_{a_m,t_{m-1}}) + \dim(\operatorname{Ker}(H_{t_m})) - \dim(\operatorname{Ker}(H_{t_{m-1}})))\\ &= \sum_{m=1}^{M} \operatorname{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) - \dim(\operatorname{Ker}(H_{t_m})) + \dim(\operatorname{Ker}(H_{t_{m-1}}))\\ &\quad + \frac{1}{2}\dim(\operatorname{Ker}(H_{t_m})) - \frac{1}{2}\dim(\operatorname{Ker}(H_{t_{m-1}}))\\ &= \sum_{m=1}^{M} \operatorname{Ind}(P^{\geq}_{a_m,t_m}, P^{\geq}_{a_m,t_{m-1}}) - \frac{1}{2}\dim(\operatorname{Ker}(H_{t_m})) + \frac{1}{2}\dim(\operatorname{Ker}(H_{t_{m-1}}))\\ &= \frac{1}{2}\dim(\operatorname{Ker}(H_0)) + \sum_{m=1}^{M} \operatorname{Ind}(P^{\geq}(H_{t_m}), P^{\geq}(H_{t_{m-1}})) - \frac{1}{2}\dim(\operatorname{Ker}(H_1)),\end{aligned}$$

where the second step follows from (5.28). □

5.9 Relative Morse indices and spectral flow

The Morse index of an invertible self-adjoint matrix is defined as the number of negative eigenvalues. It is a standard object in Morse and stability theories as it is used to determine the qualitative behavior of flow lines of gradient flows on Riemannian manifolds close to rest points. It is possible to define the Morse index for self-adjoint Fredholm operators $H \in \mathbb{FB}^{+}_{\mathrm{sa}}(\mathcal{H})$ with positive essential spectrum as the same object. However, for a self-adjoint Fredholm operator $H \in \mathbb{FB}^{*}_{\mathrm{sa}}(\mathcal{H})$ having both positive and negative essential spectrum, there is no interesting definition of the Morse index itself. It is, however, possible to define a relative Morse index for a pair $H_0, H_1 \in \mathbb{FB}^*_{sa}(\mathcal{H})$ with compact difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$ (namely, H_0 and H_1 are Calkin equivalent). Indeed, due to Proposition 5.3.17, the following definition is justified.

Definition 5.9.1. For self-adjoint bounded Fredholm operators $H_0, H_1 \in \mathbb{FB}_{sa}(\mathcal{H})$ with compact difference $H_1 - H_0 \in \mathbb{K}(\mathcal{H})$, the *relative Morse index* is defined by

$$\mu_{\mathrm{rel}}(H_0, H_1) = \mathrm{Ind}(P_0, P_1),$$

where $P_0 = \chi(H_0 < 0)$ and $P_1 = \chi(H_1 < 0)$.

Let us list the basic properties of the relative Morse index which are all directly inherited from properties of Fredholm pairs and their index. Hence even though there is little extra mathematical content, this allows summarizing all these properties in a compact form (moreover, in the language of relative Morse indices that may be more familiar to some readers).

Proposition 5.9.2. *Let $H_0, H_1, H_2 \in \mathbb{FB}_{sa}(\mathcal{H})$ be such that the differences $H_1 - H_0$ and $H_2 - H_1$ are compact.*
(i) *One has $\mu_{\mathrm{rel}}(H_0, H_1) = -\mu_{\mathrm{rel}}(H_1, H_0)$.*
(ii) *The relative Morse index is additive in the sense that*

$$\mu_{\mathrm{rel}}(H_0, H_2) = \mu_{\mathrm{rel}}(H_0, H_1) + \mu_{\mathrm{rel}}(H_1, H_2).$$

(iii) *Let $R \in \mathbb{B}(\mathcal{H})$ be invertible, then*

$$\mu_{\mathrm{rel}}(H_0, H_1) = \mu_{\mathrm{rel}}\big(R^* H_0 R, R^* H_1 R\big).$$

(iv) *If H_1 is positive semidefinite, then*

$$\mu_{\mathrm{rel}}(H_0, H_1) = \iota_-(H_0),$$

where the Morse index $\iota_-(H_1)$ is defined in (4.6).
(v) *Let $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{sa}(\mathcal{H})$ and $t \in [0,1] \mapsto H'_t \in \mathbb{FB}_{sa}(\mathcal{H})$ be norm-continuous paths of invertibles such that $H_t - H'_t \in \mathbb{K}(\mathcal{H})$ is compact for all $t \in [0,1]$. Then $t \in [0,1] \mapsto \mu_{\mathrm{rel}}(H'_t, H_t)$ is constant.*

Proof. The first claim follows from the remark after Lemma 5.3.3. Item (ii) is a direct consequence of Proposition 5.3.15 and (iv) follows from Definition 5.2.1. Claim (v) is implied by Proposition 5.2.7 because $t \in [0,1] \mapsto \chi(H_t < 0)$ and $t \in [0,1] \mapsto \chi(H'_t < 0)$ are norm-continuous paths of orthogonal projections with compact difference. It remains to show (iii). Let us set $R_t = U|R|^{1-t}$ where $U = R|R|^{-1}$ is the unitary phase of R, then $R_0 = R$, $R_1 = U$ and, moreover, $R_t^* H_0 R_t$ and $R_t^* H_1 R_t$ are Calkin equivalent for all $t \in [0,1]$.

Because $t \in [0,1] \mapsto \dim(\mathrm{Ker}(R_t^* H_0 R_t))$ and $t \in [0,1] \mapsto \dim(\mathrm{Ker}(R_t^* H_0 R_t))$ are constant $t \in [0,1] \mapsto \chi(R_t^* H_0 R_t < 0)$ and $t \in [0,1] \mapsto \chi(R_t^* H_1 R_t < 0)$ are norm-continuous paths of orthogonal projections with compact difference. Hence, by Proposition 5.2.7,

$$\mu_{\mathrm{rel}}(R^* H_0 R, R^* H_1 R) = \mu_{\mathrm{rel}}(U^* H_0 U, U^* H_1 U) = \mu_{\mathrm{rel}}(H_0, H_1),$$

where the last equality follows from the fact that the relative Morse index is invariant under conjugation by unitary operators by (5.14). □

The relative Morse index can be used to give an alternative description of the spectral flow in Theorem 5.9.6 below, as put forward in [84]. It is based on the following fact for which we provide an alternative proof.

Theorem 5.9.3. *Associated to* $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ *are norm-continuous paths of invertibles* $t \in [0,1] \mapsto M_t \in \mathbb{B}(\mathcal{H})$ *and self-adjoint compacts* $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ *such that*

$$M_t^* H_t M_t = Q + K_t, \tag{5.29}$$

where Q is a symmetry. If H_0 is invertible, one can choose $K_0 = 0$.

Proof. Suppose that the partition (4.2) and $a_m > 0$ are chosen such that the spectral projections $t \in [t_{m-1}, t_m] \mapsto P_{a_m,t} = \chi_{[-a_m,a_m]}(H_t)$ are norm-continuous and finite dimensional, see (4.3). Then let us set

$$H_{t,-} = P_{-\infty,-a_m,t} H_t, \quad H_{t,0} = P_{a_m,t} H_t, \quad H_{t,+} = P_{a_m,\infty,t} H_t,$$

where $P_{-\infty,-a_m,t} = \chi_{(-\infty,-a_m)}(H_t)$ and $P_{a_m,\infty,t} = \chi_{(a_m,\infty)}(H_t)$ are spectral projections of H_t. Let us note that all of these operators are not necessarily continuous at $t_1, \dots, t_M$, as there may be jumps in the dimension of the finite-dimensional projection. Nevertheless, for each $t \in [t_{m-1}, t_m]$, let us set

$$S_t = (H_{t,+})^{-\frac{1}{2}} + P_{a_m,t} + (H_{t,-})^{-\frac{1}{2}}.$$

Here $H_{t,\pm}$ are understood as invertible operators on their range. By construction, $t \in [t_{m-1}, t_m] \mapsto S_t$ is norm-continuous, self-adjoint, and invertible. Moreover, for $t \in [t_{m-1}, t_m]$,

$$S_t H_t S_t = -P_{-\infty,-a_m,t} + H_{t,0} + P_{a_m,\infty,t}.$$

Each summand on the right-hand side is continuous in $t \in [t_{m-1}, t_m]$. Moreover, the operator $-P_{-\infty,-a_m,t} + P_{a_m,\infty,t}$ differs from a symmetry only by a compact operator, which will be chosen to be $P_{a_m,t}$, notably let us set

$$Q_t = -P_{-\infty,-a_m,t} + P_{a_m,t} + P_{a_m,\infty,t}.$$

These symmetries can be continuously deformed by a path of unitaries U_t into a given one, say $Q_m = Q_{t_{m-1}}$. Hence there exists a continuous path $t \in [t_{m-1}, t_m] \mapsto N_t$ of invertible operators such that

$$N_t^* H_t N_t = Q_m + K_{m,t},$$

where $t \in [t_{m-1}, t_m] \mapsto K_{m,t}$ is norm-continuous and compact. This proves the statement locally in t. It remains to join the pieces in such a manner that the Q_m can be chosen to be equal. This will be achieved inductively in m after a finite number of steps. Hence let us assume that (5.29) already holds for $t \le t_{m-1}$. At t_{m-1}, one then has

$$\begin{aligned} H_{t_{m-1}} &= (M_{t_{m-1}}^*)^{-1}(Q + K_{t_{m-1}})(M_{t_{m-1}})^{-1} \\ &= (N_{t_{m-1}}^*)^{-1}(Q_m + K_{m,t_{m-1}})(N_{t_{m-1}})^{-1}. \end{aligned}$$

Thus set $A = (N_{t_{m-1}})^{-1} M_{t_{m-1}}$ and $M_t = N_t A$ for $t \in [t_{m-1}, t_m]$. It now follows that

$$A^* Q_m A = Q + K_{t_{m-1}} - M_{t_{m-1}}^* (N_{t_{m-1}}^*)^{-1} K_{m,t_{m-1}} (N_{t_{m-1}})^{-1} M_{t_{m-1}},$$

and so $A^* Q_m A = Q + K$ for a compact self-adjoint operator K. Hence, for $t \in [t_{m-1}, t_m]$,

$$\begin{aligned} M_t^* H_t M_t &= A^* N_t^* H_t N_t A \\ &= A^*(Q_m + K_{m,t})A \\ &= Q + K + A^* K_{m,t} A \\ &= Q + K_t, \end{aligned}$$

for the compact self-adjoint operators $K_t = K + A^* K_{m,t} A$. This finishes the proof. □

Remark 5.9.4. It is possible to reformulate Theorem 5.9.3. Because M_t is invertible, one can set $\hat{H}_t = (M_t^{-1})^* Q M_t^{-1}$ and $\hat{K}_t = (M_t^{-1})^* K_t M_t^{-1}$ and obtains

$$H_t = \hat{H}_t + \hat{K}_t. \tag{5.30}$$

Hence the path $t \mapsto H_t$ can be decomposed into a path $t \mapsto \hat{H}_t$ of invertibles and a compact perturbation $t \mapsto \hat{K}_t$ thereof. Let us stress that if $t \in [0,1] \mapsto H_t$ is a loop, namely $H_0 = H_1$, the two paths $t \mapsto \hat{H}_t$ and $t \mapsto \hat{K}_t$ are in general *not* closed.

Provided that H_0 is invertible (so that $K_0 = 0$), one can homotopically deform the time parameter in the two summands on the right-hand side of (5.30) to deduce the following: the nontrivial loop $t \in [0,1] \mapsto H_t$ is homotopic to the concatenation of two paths

$$\big(t \in [0,1] \mapsto \hat{H}_t\big) * \big(t \in [0,1] \mapsto \hat{H}_1 + \hat{K}_t\big).$$

The first of these paths is within the invertible operators and hence has no spectral flow, while the second is merely a compact perturbation of $\hat{H}_1 = H_0 - \hat{K}_1$. On this second part though, there is possibly a spectral flow given by

$$\mathrm{Sf}\big(t \in [0,1] \mapsto \hat{H}_1 + \hat{K}_t\big) = \mathrm{Sf}\big(t \in [0,1] \mapsto H_t\big).$$

Let us note that a particular case of this is the following: given two symmetries Q_0 and Q_1 with compact difference $Q_0 - Q_1$ and a given index $\mathrm{Ind}(Q_0, Q_1)$, first rotate Q_0 into Q_1 by a path of unitaries, then use the straight-line path to complete a nontrivial loop rooted in Q_0 (playing the role of H_0 in the above). Such a loop is constructed explicitly in Example 8.3.4, which is based on Example 5.7.4. In order to be even closer to this Example 8.3.4, the next result further specializes (5.30) to the case where H_t is a proper symmetry up to a compact perturbation. ◇

Corollary 5.9.5. *Let $t \in [0,1] \mapsto H_t$ be a path of essential proper symmetries, namely lying in the set*

$$\mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H}) = \big\{H \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H}) : \mathrm{spec}_{\mathrm{ess}}(H) = \{-1, 1\}\big\}.$$

Then there are norm-continuous paths of unitaries $t \in [0,1] \mapsto U_t \in \mathbb{U}(\mathcal{H})$ and self-adjoint compacts $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ such that

$$U_t^* H_t U_t = Q + K_t, \tag{5.31}$$

for some proper symmetry Q.

Proof. Let us start out from (5.30). As $H_t \in \mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$ and $\hat{K}_t \in \mathbb{K}(\mathcal{H})$, it follows that also $\hat{H}_t \in \mathbb{FB}^{*,\mathrm{C}}_{\mathrm{sa}}(\mathcal{H})$, due to the compact stability of the essential spectrum. As $\hat{H}_t$ is invertible, the proof of Proposition 3.6.5 implies that it can be decomposed as $\hat{H}_t = Q_t + \tilde{K}_t$ into a symmetry Q_t and a compact $\tilde{K}_t$. Moreover, this decomposition is continuous, see Remark 3.6.6. Then

$$H_t = Q_t + \hat{K}_t + \tilde{K}_t.$$

By Proposition 5.3.20, one can write $Q_t = U_t Q_0 U_t^*$ for some path of unitaries. Setting $Q = Q_0$ and $K_t = U_t^*(\hat{K}_t + \tilde{K}_t)U_t$ concludes the proof. □

Based on Theorem 5.9.3, one has the following formula for the spectral flow as Morse index.

Theorem 5.9.6. *For paths $t \in [0,1] \mapsto H_t \in \mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ and $t \in [0,1] \mapsto M_t^* H_t M_t = Q + K_t$ where as above Q is a symmetry, $t \in [0,1] \mapsto M_t \in \mathbb{B}(\mathcal{H})$ is a path of invertibles and $t \in [0,1] \mapsto K_t \in \mathbb{K}(\mathcal{H})$ is a path of compacts, the spectral flow of the path $t \in [0,1] \mapsto H_t$ satisfies*

$$\begin{aligned}\mathrm{Sf}\big(t\in[0,1]\mapsto H_t\big) &= \mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)\\ &= \frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\mu_{\mathrm{rel}}(Q+K_0,Q+K_1)-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big).\end{aligned}\tag{5.32}$$

Proof. For $t\in[0,1]$, let $U_t^*=M_t^*|M_t^*|^{-1}$ be the unitary phase of M_t^*. Then let us consider the continuous homotopy $h:[0,1]\times[0,1]\to\mathbb{FB}_{\mathrm{sa}}(\mathcal{H})$ defined by

$$h(t,s)=U_t^*|M_t^*|^s H_t|M_t^*|^s U_t.$$

By Theorem 4.2.2, one has

$$\begin{aligned}&\mathrm{Sf}\big(t\in[0,1]\mapsto h(t,1)\big)\\ &\quad=-\mathrm{Sf}\big(s\in[0,1]\mapsto h(0,s)\big)+\mathrm{Sf}\big(t\in[0,1]\mapsto h(t,0)\big)+\mathrm{Sf}\big(s\in[0,1]\mapsto h(1,s)\big).\end{aligned}$$

As $s\in[0,1]\mapsto\dim(\mathrm{Ker}(h(0,s)))$ and $s\in[0,1]\mapsto\dim(\mathrm{Ker}(h(1,s)))$ are constant, item (i) of Theorem 4.2.1 implies

$$\mathrm{Sf}\big(s\in[0,1]\mapsto h(0,s)\big)=\mathrm{Sf}\big(s\in[0,1]\mapsto h(1,s)\big)=0.$$

Therefore

$$\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)=\mathrm{Sf}\big(t\in[0,1]\mapsto U_t^*H_tU_t\big)=\mathrm{Sf}\big(t\in[0,1]\mapsto H_t\big),$$

where the last step follows from item (vi) of Theorem 4.2.1. This implies the first claim. The second holds because

$$\begin{aligned}\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)=\;&\frac{1}{2}\dim\big(\mathrm{Ker}(Q+K_0)\big)\\ &+\sum_{m=1}^{M}\mathrm{Ind}\big(P^{\geq}(Q+K_{t_m}),P^{\geq}(Q+K_{t_{m-1}})\big)\\ &-\frac{1}{2}\dim\big(\mathrm{Ker}(Q+K_1)\big)\end{aligned}$$

for a partition $0=t_0<t_1<\cdots<t_{M-1}<t_M=1$ as in Proposition 5.8.2. By definition,

$$\begin{aligned}\mathrm{Ind}\big(P^{\geq}(Q+K_{t_m}),P^{\geq}(Q+K_{t_{m-1}})\big)&=-\mu_{\mathrm{rel}}(Q+K_{t_m},Q+K_{t_{m-1}})\\ &=\mu_{\mathrm{rel}}(Q+K_{t_{m-1}},Q+K_{t_m}).\end{aligned}$$

Therefore and as $\dim(\mathrm{Ker}(Q+K_0))=\dim(\mathrm{Ker}(H_0))$ and $\dim(\mathrm{Ker}(Q+K_1))=\dim(\mathrm{Ker}(H_1))$,

$$\begin{aligned}&\mathrm{Sf}\big(t\in[0,1]\mapsto Q+K_t\big)\\ &\quad=\frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\sum_{m=1}^{M}\mu_{\mathrm{rel}}(Q+K_{t_{m-1}},Q+K_{t_m})-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big)\\ &\quad=\frac{1}{2}\dim\big(\mathrm{Ker}(H_0)\big)+\mu_{\mathrm{rel}}(Q+K_0,Q+K_1)-\frac{1}{2}\dim\big(\mathrm{Ker}(H_1)\big),\end{aligned}$$

where the last step follows from item (ii) in Proposition 5.9.2. □