

Contents

Preface — V

1 Overview of the basic problems — 1

- 1.1 Semilinear elliptic boundary value problems — 1
- 1.2 Eigenvalue problems — 2
 - 1.2.1 The problem of vibrating membranes — 2
 - 1.2.2 Buckling problem for a clamped plate — 3
- 1.3 Green's function, Hadamard's formula — 4

2 Basic concepts — 7

- 2.1 Domain perturbations — 7
- 2.2 Geometry of hypersurfaces — 9
 - 2.2.1 Preliminaries — 9
 - 2.2.2 Projections — 12
 - 2.2.3 Reduction to Hadamard perturbations — 13
- 2.3 First and second variation of the volume — 18
 - 2.3.1 Discussion of $\ddot{V}(0)$ — 19
 - 2.3.2 First and second variation of the surface area perimeter — 23
 - 2.3.3 Area variations for the surface of the ball — 27
 - 2.3.4 Hadamard perturbations — 30
 - 2.3.5 Tangential perturbations — 31
- 2.4 Intrinsic versus extrinsic — 32
 - 2.4.1 The distance function — 32
 - 2.4.2 Shape derivatives — 34
 - 2.4.3 Notes — 37

3 Spherical harmonics and eigenvalue problems — 38

- 3.1 Spherical harmonics — 38
- 3.2 The Steklov eigenvalue problem — 43
 - 3.2.1 The ball — 43
 - 3.2.2 The outer domain of a ball — 45
 - 3.2.3 A Steklov–Dirichlet eigenvalue problem in annular domains — 45
 - 3.2.4 Steklov eigenvalues in annular domains — 46
- 3.3 Membrane eigenvalue problems — 48
 - 3.3.1 The generalized Steklov problem — 49
- 3.4 The buckling plate — 49
 - 3.4.1 Fourth order Steklov eigenvalue problem — 50
- 3.5 Notes — 51

4 Variational formulas — 52

- 4.1 The change of variables method — 52
- 4.1.1 Introduction — 52
- 4.1.2 Volume integrals which depend on $\mathbf{u}$ only — 53
- 4.1.3 Boundary integrals which depend on u only — 55
- 4.1.4 The Dirichlet integral — 57
- 4.2 The moving surface method — 59
- 4.2.1 Second variation for special cases — 63
- 4.3 Notes — 69

5 Geometric inequalities, convolutions, cost functions — 71

- 5.1 Moment of inertia — 71
- 5.2 Isoperimetric inequalities in spaces of constant curvature — 75
- 5.2.1 Spherical space $\mathbb{S}^n$ — 75
- 5.2.2 Hyperbolic space $\mathbb{H}^n$ — 79
- 5.3 Convolutions — 81
- 5.3.1 The sign of $\check{C}(0)$ for the ball — 86
- 5.3.2 Gamov's liquid model — 89
- 5.4 An optimal control problem — 92
- 5.4.1 The torsion problem — 93
- 5.5 Notes — 97

6 Domain variations for energies — 99

- 6.1 Energies and critical points — 99
- 6.2 Elliptic boundary value problems — 100
- 6.2.1 General setting, change of variables — 100
- 6.2.2 Boundary value problem for the shape derivative — 101
- 6.3 Differentiation of the energy — 105
- 6.3.1 First and second domain variation — 105
- 6.4 The first domain variation — 107
- 6.4.1 The first domain variation for volume preserving perturbations — 108
- 6.5 The second domain variation — 109
- 6.5.1 The transformations of $\mathcal{F}_1(0)$, $\mathcal{F}_2(0)$, and $\mathcal{F}_3(0)$ — 109
- 6.5.2 Main results — 114
- 6.5.3 Notes — 118

7 Discussion of the main results — 119

- 7.1 Translations and rotations — 119
- 7.2 Hadamard perturbations — 121
- 7.3 Tangential perturbations $u = u^t$ — 123
- 7.4 Volume preserving perturbations — 125
- 7.4.1 Dirichlet energy — 125

7.4.2	Robin energy — 126
7.5	Neumann energy — 128
7.5.1	Notes — 130
8	General strategy and applications — 131
8.1	A generalized Steklov eigenvalue problem — 131
8.2	Nearly spherical domains — 135
8.2.1	Robin energy — 135
8.2.2	The torsion problem with $\alpha > 0$ — 143
8.2.3	Dirichlet energy — 144
8.3	Notes — 147
9	Eigenvalue problems — 148
9.1	Robin eigenvalue problem — 148
9.1.1	Domain variations for Robin eigenvalues with $\alpha > 0$ — 149
9.1.2	The discussion of $\dot{\lambda}(0)$ — 151
9.1.3	Discussion of $\ddot{\lambda}(0)$ for the ball — 153
9.2	Nonlinear eigenvalue problems — 157
9.2.1	Examples — 159
9.2.2	First variation of the turning point — 163
9.3	Notes — 166
10	Quantitative estimates — 167
10.1	Preliminary remark — 167
10.2	The Fraenkel asymmetry function — 168
10.2.1	Stability criterion — 169
10.3	Example — 170
10.3.1	The Robin energy for $\alpha > 0$ — 170
10.3.2	Notes — 172
11	The Robin eigenvalues for $\alpha < 0$ — 173
11.1	The interior eigenvalue problem — 173
11.2	The exterior eigenvalue problem — 178
11.3	Notes — 187
12	Problems with infinitely many positive and negative eigenvalues — 188
12.1	Eigenvalue problem with dynamical boundary conditions — 188
12.2	Known results — 189
12.3	Comparison with balls of the same volume — 191
12.4	Notes — 194

13 The torsion problem for $\alpha < 0$ — 195

- 13.1 General remarks — 195
- 13.2 Domain variations for nearly spherical domains — 197
- 13.2.1 Second variation — 197
- 13.3 An alternative approach: Robin versus Dirichlet torsion — 201
- 13.3.1 The functional $\mathcal{J}(\Omega)$ — 203
- 13.3.2 Domain variations of $\mathcal{J}(t)$ — 203
- 13.3.3 The second variation of $\mathcal{J}(\Omega_t)$ in nearly spherical domains — 204

14 Problems in annular domains — 206

- 14.1 An eigenvalue problem related to trace inequalities — 206
- 14.2 Bernoulli's problem — 209
- 14.3 The Robin torsion problem for $\alpha > 0$ — 213
- 14.4 Notes — 216

15 The first buckling eigenvalue of a clamped plate — 217

- 15.1 The first eigenvalue — 217
- 15.2 Shape derivatives — 218
- 15.3 The first domain variation — 220
- 15.4 The second domain variation — 222
- 15.5 Minimization of the second domain variation — 226
- 15.6 The optimal domain is a ball — 232
- 15.7 Notes — 234

16 A fourth order Steklov problem — 236

- 16.1 The Steklov eigenvalue — 236
- 16.2 Shape derivatives — 237
- 16.3 First domain variation — 239
- 16.4 First variation for the ball — 242
- 16.5 Second variation for the ball — 243
- 16.6 Notes — 251

A General remarks — 253

- A.1 Jacobi's formula for determinants — 253
- A.2 Hölder continuity — 253
- A.2.1 Whitney's extension theorem — 254

B Geometry — 257

- B.1 Some concepts from differential geometry — 257
- B.1.1 Curvilinear coordinates in $\mathbb{R}^n$ — 258
- B.1.2 Normal coordinates — 259
- B.2 Implicit function theorem — 261

C Sobolev spaces and inequalities — 263**D Bilinear forms — 265**D.1 Abstract setting — **265**D.2 Examples — **266**D.2.1 Eigenvalues on the sphere — **266**D.2.2 Robin eigenvalues — **267**D.2.3 Buckling eigenvalue — **267**D.2.4 Steklov eigenvalue of fourth order — **268****Notation — 269****Bibliography — 271****Index — 277**

