Preface

The description of optimal forms plays an important role in mathematics and applications. In many areas of mathematics, pure and applied, it has recently received a lot of attention.

In this book we consider domain functionals, like energies and eigenvalues of elliptic operators. Assuming the existence of an optimal domain, we are interested in its shape. Global methods like symmetrization and rearrangement inequalities are applicable in certain settings; however, for many interesting problems, no global approach is known.

In the absence of a global approach, one can instead study the effect of local perturbations. In the spirit of calculus one considers one-parameter families of perturbations of a fixed domain. The first derivative with respect to this parameter can be used to derive a necessary condition for a domain to be optimal. The second derivative provides additional information to help to determine local maxima or minima.

In general these derivatives are difficult to analyze. In particular the second derivative, which is crucial to understand stability, is complicated. We develop techniques that enable us to determine its sign in various cases of interest.

The structure of this book can briefly be summarized as follows.

We start with an overview of the main examples that will be treated in this book. They are selected in order to show the breadth of the applicability of our methods. Among them are energies of nonlinear boundary value problems, eigenvalues, and problems of fourth order.

We then describe the class of admissible domain perturbations. It should be emphasized that our technique requires smoothness of the domains. The arguments will in general fail in the case of nonsmooth domains. Nonsmooth analysis for domain variations is not yet available.

We pay special attention to volume and area preserving perturbations. In order to compare the family of perturbed domains, we introduce differential geometric tools. This allows us to compute variational formulas for volume and surface area.

Our functionals are domain and boundary integrals. We present two methods to derive variational formulas. The first is the change of variables method. All quantities will be mapped onto a fixed domain. This requires only the chain rule. The second approach, due to Reynolds, is the moving surface method. It captures the shape by means of a boundary displacement and generally requires additional regularity. The resulting variational formulas coincide and are illustrated with some examples. Among them are problems of optimal control, convolutions, and weighted isoperimetric inequalities.

It becomes apparent that there is a difference between variational formulas for purely geometrical functionals and for those functionals which depend on functions varying with the domain.

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The most involved variational formulas are derived for energies of semilinear elliptic problems. They depend only on the boundary data and the perturbations of the boundary. We discuss these formulas for different boundary value problems.

To find precise estimates for the second variation, we expand it in a suitable basis of functions consisting of eigenfunctions of elliptic eigenvalue problems. In many examples we are able to determine the sign. This method applies to any critical domain. For the ball this system of eigenfunctions restricted to the sphere corresponds to the expansion with respect to spherical harmonics.

The first and second variations allow us to derive some local isoperimetric inequalities. The well-known classical isoperimetric inequality states that among all domains of given volume the ball has the smallest perimeter. We will call an inequality *isoperimetric* if it relates geometrical or physical quantities defined on the same domain and if the equality sign is attained for some optimal domain. We extend this notion to a broader class of functionals such as energies and eigenvalues.

Fourth order problems are studied at the end of the book by means of the moving surface method. For the buckling eigenvalue, we are able to use the variational formulas to prove uniqueness of the optimal domain.

This book attempts to bridge the gap between analysis and geometry. Our exposition is example-driven and draws on problems that are the object of study in the current literature. We present them in a self-contained way, at times pointing out open questions that we hope will lead to further investigation and interesting discoveries.

Domain variation has a long history, which can be traced back to Hadamard (1908). More detailed historical comments are given at the end of the relevant chapters. The book is aimed at readers with basic knowledge of analysis and geometry.

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