Preface

When I entered my PhD program in the fall of 1991, I was pretty sure I would study differential geometry, even though I had spent a lot of time as an undergraduate learning general topology and a lot of algebra. My first semester did not clarify things for me–the standard first course in measure theory, more algebra, a topics course in algebraic geometry that was a bit advanced for me at the time. My mathematical trajectory changed forever that spring, however, when I was introduced to algebraic topology. Finally, here was a branch of mathematics that appealed to my topological interests and my affinity for algebraic calculation. I was hooked.

Our professor, Richard Hain (who would eventually become my PhD advisor), told very good stories. His geometric intuition was (and still is) otherworldly, but he brought it down to earth. I still have the course notes I took that semester, and they formed the basis for the notes I have used when I teach algebraic topology. They have expanded to include more material and topics that are especially interesting to me, and this book is the LaTeX'ed result.

My goal here is to present a streamlined introduction to what I view as the essential elements of algebraic topology, a toolkit as it were. This text is not intended to serve as an encyclopedic reference work, but to be a solid foundation and an invitation to further study. An ambitious instructor could, with judicious omissions, cover it in one semester. I prefer a more leisurely two-semester amble, but that is partially a function of my department's course requirements.

Readers familiar with standard texts in algebraic topology will notice some influences. That first course for me was based on the book of Greenberg & Harper [1]. It has much to recommend it, but it is quite terse (perhaps too much so). The current standard text is Hatcher's excellent book [2], and I have used it successfully in the past. Indeed, his direct calculation of the cohomology ring of projective space (rather than appealing to Poincaré duality) appears here with little modification; the extra technical work required to do it is actually quite illuminating. Hatcher's book is rather encyclopedic, however, and as mentioned above, I am aiming for a more pared-down version. Much of the material on cellular homology and spectral sequences is modeled on the place where I learned it, the book of Griffiths & Morgan [3]. Spectral sequences are an important tool that every topologist needs in their kit, and that is why they are included here. I will say that this presentation is analogous to what we show undergraduates in the introductory calculus sequence: a focus on *how to use* these techniques over the details of how they work. Other sources I consulted include Spanier's classic text [4], Munkres's algebraic topology book [5], Steenrod's treatise on fiber bundles [6], and Weibel's book on homological algebra [7].

The book contains four chapters. Chapter 1 focuses on preliminary material: continuous maps and homotopy, lifting properties and fibrations, basic category theory, and the fundamental group. Chapter 2 is about homology in all its forms–simplicial, singular,

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cellular–and the properties thereof. Manifolds and their homology are covered in detail. This chapter also includes a section about persistent homology and topological data analysis. This area has gained prominence over the last couple of decades, bringing what was once a very theoretical branch of mathematics into a world of applications. Chapter 3 is about cohomology and duality, including the cup product and cohomology ring. Poincaré duality appears here, of course. Finally, Chapter 4 introduces homotopy groups and the basics of the homotopy theory of cell complexes. The Leray–Serre spectral sequence is introduced and used to compute many interesting examples of cohomology rings and homotopy groups.

Here is one topic that I have omitted that may be controversial: a detailed study of covering spaces. While this theory is beautiful and interesting, every time I teach the course I find myself regretting spending so much time on it. It is true that covering spaces are needed from time to time later in the text, but the limited discussion here is sufficient. I have included an exploration of covering spaces as a project at the end of the first chapter, however, if an instructor wishes to cover it.

Each chapter concludes with one or two projects for students to explore topics in greater depth. Exercises are included throughout as well.

I hope the reader finds this book to be a clear and compelling introduction to a subject that continues to find new applications. If mathematicians, both emerging and established, find it useful, then I have been successful.

Gainesville, Florida, July 2024 K.P.K.