3 Conformal mapping

> Second Hypothesis: That small regions of the Earth should be displayed as similar figures in the plane.
>
> Leonhard Euler, “On the mapping of spherical surfaces onto the plane” (1777)

3.1 Motivation: classifying complex regions up to conformal equivalence

As we discussed in Chapter 1, the notion of a conformal mapping is a highly appealing geometric idea that can be explained to anyone without any requirement that they ever heard of complex analysis, let alone understand any of the mathematics underlying it. Anyone who can appreciate the art of M. C. Escher (see Fig. 1.2 on p. 8) will intuitively grasp that there is something special and beautiful about conformal maps.

Conformal maps are also an important tool in the toolkit of applied mathematicians. They have many applications for solving important partial differential equations that show up in physics, engineering, and in other areas as diverse as cartography [68] and medical imaging [37].

In this chapter, we will approach the area of conformal mapping from a purely complex-analytic direction. We will see that this side of the theory has a beauty all its own, which, while subtle and requiring patience and contemplation to appreciate, equals and perhaps surpasses the more obvious aspects appreciated by art lovers and equation solvers.

Let $\Omega \subset \mathbb{C}$ be a complex region. In complex analysis, we often wish to understand the classes of functions $\mathcal{H}(\Omega)$ and $\mathcal{M}(\Omega)$ of holomorphic and meromorphic functions on Ω, respectively. You might think that the structures of these classes of functions would depend in some highly sensitive way on the particular choice of the region Ω. As it turns out, this is largely untrue: although the structure of such a family does vary somewhat, there are large families of regions Ω for which the structure of $\mathcal{H}(\Omega)$ (respectively, $\mathcal{M}(\Omega)$) is the same across all members of a given family, so that it is in practice enough to understand what is happening in one representative region of each family. Moreover, the question of which family a particular region Ω belongs to can in many cases be answered using *topological* properties of Ω.

To make this idea precise, we define an equivalence relation on regions that captures the notion that for two regions Ω and Ω', $\mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega')$ “have the same structure.” This relation is called **biholomorphism** or **conformal equivalence**. We say that Ω and Ω' are **conformally equivalent** if there is a bijective holomorphic map $g : \Omega \to \Omega'$ whose inverse is also holomorphic. Such a map g is called a **biholomorphism**, **biholomorphic map**, or **conformal map**. Note that a conformal map must satisfy $g'(z) \neq 0$ for

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any $z \in \Omega$, by Corollary 1.58. It is trivial to check that the relation of conformal equivalence is, as its name suggests, an equivalence relation.[1]

If Ω and Ω' are conformally equivalent and related by a conformal map $g : \Omega \to \Omega'$, then each holomorphic function (respectively, meromorphic function) $f : \Omega \to \mathbb{C}$ can be used to define a holomorphic (respectively, meromorphic) function $\widetilde{f} : \Omega' \to \mathbb{C}$ by

$$\widetilde{f} = f \circ g^{-1}.$$

It is immediate to check that the correspondence $f \mapsto \widetilde{f}$ defines a bijection between $\mathcal{H}(\Omega)$ and $\mathcal{H}(\Omega')$ (respectively, between $\mathcal{M}(\Omega)$ and $\mathcal{M}(\Omega')$). Thus the conformal map allows us to translate any question about holomorphic or meromorphic functions on Ω' to a question about holomorphic or meromorphic functions on Ω. The definition of conformal equivalence therefore captures precisely the notion of equivalence we were interested in.

In many areas of mathematics, when we find an interesting equivalence relation, this immediately leads to a standard set of interesting questions: how do we determine equivalence? Can we describe all equivalence classes, or at least some particularly simple or important ones? Do there exist some canonical representatives in each of those equivalence classes? How can we construct a map demonstrating equivalence, and to what extent is it unique? And so on. Asking such questions for this particular equivalence relation turns out to be very fruitful and is what the area of conformal mapping is about.

Examples. Here are some regions that seem worth thinking about from the point of view of conformal mapping, both theoretically and because they arise in applications (for example, in the study of Laplace's equation in mathematical physics, electrostatics, hydrodynamics, etc):
1. the complex plane $\mathbb{C}$
2. the punctured plane $\mathbb{C} \setminus \{0\}$
3. the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
4. the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$
5. the Riemann sphere[2] $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

1 In this chapter, we use the term "conformal map" with a slightly different meaning than the sense in which this term was used in Subsection 1.3.4. That subsection was concerned with understanding the property of being conformal as a *local* property; here we develop the conceptually much richer set of ideas related to understanding maps that are *globally* conformal—that is, conformal everywhere in the local sense but also bijective. Moreover, the conformal maps from Subsection 1.3.4 were not assumed to be orientation preserving. Here we focus on conformal maps that are holomorphic, which in particular means that they are orientation preserving (see (1.25)).

2 The Riemann sphere is not quite a complex region in the usual sense; technically, it is a Riemann surface, but we will still count it and trust that you understand how the various definitions apply in that situation; refer to Section 1.11. Actually, the same classification questions we are addressing in the context

6. the slit plane $\mathbb{C} \setminus (-\infty, 0]$
7. a strip $S(x_1, x_2) = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$
8. a rectangle $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1, a < \mathrm{Im}(z) < b\}$
9. an annulus $A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$
10. a quadrant $\{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$
11. an ellipse $\{z = x + iy : (\frac{x}{A})^2 + (\frac{y}{B})^2 < 1\}$
12. the plane with an interval removed, $\mathbb{C} \setminus [-1, 1]$
13. the upper half-plane with an interval removed, $\mathbb{H} \setminus [0, i]$
14. a “blob” (Fig. 3.1)

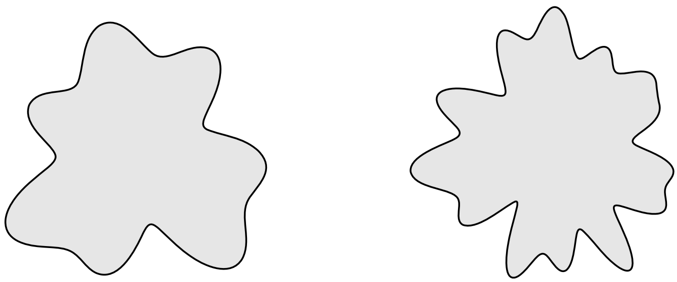

Figure 3.1: Two blob-shaped regions. Are they conformally equivalent?

Can you guess what is the correct grouping of these regions according to conformal equivalence? (Note: in example 9 of the annulus, we in fact have a *family* of regions, which may not all be conformally equivalent to each other.) By the end of this chapter, you will know the answers.

Since conformal maps are continuous, the relation of conformal equivalence is a stronger notion of equivalence than topological equivalence (a. k. a. homeomorphism). We record this obvious but important fact as a lemma.

Lemma 3.1. *If regions Ω and Ω' are conformally equivalent, then they are homeomorphic.*

Next, if regions Ω and Ω' are conformally equivalent, with the conformal map $g : \Omega \to \Omega'$ relating them, then is g unique? If not, can the extent to which it is not unique be made precise? The answer to these questions is described in terms of the **automorphism group** of a complex region. More precisely, if $\tilde{g} : \Omega \to \Omega'$ is another conformal map, then the map $h : \Omega \to \Omega$ defined by

$$h = g^{-1} \circ \tilde{g}$$

of conformal equivalence apply more generally in the theory of Riemann surfaces. We will encounter an interesting example of the classification of a class of Riemann surfaces up to conformal equivalence in Chapters 4 and 5; see Sections 4.15, 5.5, and 5.11.

is a conformal equivalence map between Ω and itself. We call such a map a (conformal) **automorphism** of Ω. Conversely, if $g : \Omega \to \Omega'$ is a conformal map and $h : \Omega \to \Omega$ is a conformal automorphism, then $\widetilde{g} : \Omega \to \Omega'$ defined by

$$\widetilde{g} = g \circ h$$

is also a conformal map from Ω to Ω', and clearly every conformal map $\widetilde{g} : \Omega \to \Omega'$ can be represented in such a way for some automorphism $h : \Omega \to \Omega$ (just define h as above). Thus the family of automorphisms of Ω precisely measures the extent of the nonuniqueness of the conformal map $g : \Omega \to \Omega'$ for any Ω' that is conformally equivalent to Ω. This family has the algebraic structure of a group, with the group operation being composition of maps, and is thus referred to as the automorphism group of Ω. We denote this group by $\mathrm{Aut}(\Omega)$. We will seek to give explicit descriptions of automorphism groups whenever this is possible.

To conclude this general discussion, we note one additional useful fact about conformal maps.

Lemma 3.2. *In the definition of conformal equivalence, the condition that g^{-1} is holomorphic can be dropped, that is, if $g : \Omega \to \Omega'$ is holomorphic and bijective, then g^{-1} is automatically holomorphic.*

Proof. Since g satisfies $g'(z_0) \neq 0$ for any $z_0 \in \Omega$, the inverse function theorem (Theorem 1.56) implies that the inverse map g^{-1} exists locally in a neighborhood of $g(z_0)$ as a holomorphic function for any $z_0 \in \Omega$. Since g is a bijection, the inverse function exists globally (in the sense of set theory) as a function $g^{-1} : \Omega' \to \Omega$. The fact that g^{-1} is locally holomorphic implies that the global inverse function g^{-1} is holomorphic, which is the claim of the lemma. □

In the next few sections, we begin to classify some of the main conformal equivalence classes that every complex analyst should be familiar with. The most important classification result in this chapter is the Riemann mapping theorem, which is formulated in Section 3.4.

Suggested exercises for Section 3.1. 3.1.

3.2 First singleton conformal equivalence class: the complex plane

The first conformal equivalence class we discuss contains just a single element, the complex plane. This is explained by the following theorem.

Theorem 3.3. *Let $g : \mathbb{C} \to \Omega$ be a conformal map between $\mathbb{C}$ and a region Ω. Then $\Omega = \mathbb{C}$, $g(z)$ is a conformal automorphism, and $g(z)$ has the form*

$$g(z) = az + b$$

for some complex numbers a, b with $a \neq 0$.

Proof. Let $g : \mathbb{C} \to \Omega$ be a conformal equivalence map. We will prove that $g(z)$ is of the form $g(z) = az + b$ with $a \neq 0$ just based on the assumption that it is an entire function and that it is injective; the additional claims that $\Omega = \mathbb{C}$ and $g(z)$ is an automorphism will then follow.

Since $g(z)$ is an entire function, it is either a polynomial, or it is not. We treat each of those two cases separately (proving that $g(z)$ is of the desired form in the first case and proving that the second case cannot occur).

If $g(z)$ is a polynomial, it cannot be a constant since those certainly are not injective maps. We claim that it also cannot be a polynomial of degree $k \geq 2$, which if true would leave only the option of a linear function $g(z) = az + b$ with $a \neq 0$. The fact that polynomials of degree higher than 1 are not injective is easy to see: a polynomial of degree k has k roots counting with multiplicity, which means that either there are at least two distinct zeros (contradicting the assumption of injectivity), or there is a single zero of multiplicity k, which means that the polynomial is of the form $g(z) = c(z - a)^k$. This polynomial is clearly also not injective since in that case the equation $g(z) = 1$ has k distinct solutions.

It remains to consider the other possibility of an entire function that is not a polynomial. In that scenario, we claim that $g(z)$ has an essential singularity at $z = \infty$. For otherwise, by our classification of singularities (Section 1.12), $g(z)$ must have a pole of some order k at infinity. However, having such a pole implies that the rate of growth of $|g(z)|$ is restricted by the order of the pole; specifically, $g(z)$ satisfies a bound of the form $|g(z)| \leq A + B|z|^k$ for all z, where A and B are positive real constants. Now a well-known argument from basic complex analysis (Exercise 1.25) implies that $g(z)$ is actually a polynomial of degree at most k, which is a contradiction.

We are now in a good position to apply the Casorati–Weierstrass theorem (Theorem 1.46) about the behavior of functions near an essential singularity. Denote $w_0 = g(0)$. Since $g(z)$ is an open mapping by the open mapping theorem (Theorem 1.50), the image $g(\mathbb{D})$ of the unit disc under $g(z)$ contains an open neighborhood E of w_0. But by the Casorati–Weierstrass theorem the image $g(\mathbb{C} \setminus D_{\leq R}(0))$ of the complement of any closed disc around 0 (i. e., any neighborhood of ∞) is dense in $\mathbb{C}$ and therefore has a nonempty intersection with E. This intersection means that there exist points $z_1 \in \mathbb{D}$ and $z_2 \in \mathbb{C} \setminus D_R(0)$ for which

$$g(z_1) = g(z_2).$$

Now if $R > 1$, then $z_1 \neq z_2$. We have therefore shown that $g(z)$ is not injective, which contradicts our initial assumption. Thus the scenario of a conformal map on $\mathbb{C}$ that is not a polynomial is impossible, and the proof is complete. □

By Theorem 3.3 the group of conformal automorphisms of $\mathbb{C}$ is

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \, : \, a, b \in \mathbb{C}, a \neq 0\}.$$

3.3 Second singleton conformal equivalence class: the Riemann sphere

There is a second conformal equivalence class that is a singleton, the Riemann sphere. The following result is the analogue of Theorem 3.3 for $\widehat{\mathbb{C}}$.

Theorem 3.4. *If $g : \widehat{\mathbb{C}} \to \Omega$ is a conformal map between $\widehat{\mathbb{C}}$ and a region Ω, then $\Omega = \widehat{\mathbb{C}}$, $g(z)$ is a conformal automorphism, and $g(z)$ has the form*

$$g(z) = \frac{az+b}{cz+d} \tag{3.1}$$

for some complex numbers a, b, c, d with $ad - bc \neq 0$.

Proof of Theorem 3.4. We start by proving that $\Omega = \widehat{\mathbb{C}}$. Assume that this is not the case, i. e., that there is at least one point $w \in \widehat{\mathbb{C}}$ that is not in the image $g(\widehat{\mathbb{C}})$. We can assume without loss of generality that $w = \infty$; otherwise, replace the map $g(z)$ with $\tilde{g}(z) = \frac{1}{g(z)-w}$. Once $\tilde{g}(z)$ is shown to be of the desired form (3.1), solving the equation $\tilde{g}(z) = \frac{1}{g(z)-w}$ for $g(z)$ shows that $g(z)$ is of that form as well.

Since $g(z)$ does not take the value ∞, it also cannot *approach* infinity, that is, there does not exist a sequence $(z_n)_{n=1}^{\infty}$ of points in $\widehat{\mathbb{C}}$ for which $g(z_n) \to \infty$. If such a sequence existed, we could use the fact that $\widehat{\mathbb{C}}$ is compact to extract a convergent subsequence $z_{n_k} \to Z \in \widehat{\mathbb{C}}$, whence it would follow, since $g(z)$ is a continuous function, that $g(Z) = \infty$, which cannot happen since ∞ is not in the image of $g(z)$.

The fact that $g(z)$ does not approach ∞ means simply that $g(z)$ is a bounded function and a holomorphic one at that (our a priori assumption that allows Ω to contain the point ∞ only means it is meromorphic). Thus it is a bounded entire function and hence constant by Liouville's theorem, a contradiction.

Having established that $\Omega = \widehat{\mathbb{C}}$, we now know that $g(z)$ is a genuine automorphism of $\widehat{\mathbb{C}}$. Denote $w = g(\infty)$. Once again, we can assume without loss of generality that $w = \infty$; otherwise, replace the map $g(z)$ with $\tilde{g}(z) = \frac{1}{g(z)-w}$ as before. Under this assumption, the restriction of $g(z)$ to $\mathbb{C}$ is a conformal automorphism of $\mathbb{C}$, so from the discussion in the previous section we know that $g(z)$ is of the form $az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. □

By Theorem 3.4 the group of conformal automorphisms of $\widehat{\mathbb{C}}$ is

$$\operatorname{Aut}(\widehat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \; : \; a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}. \tag{3.2}$$

The elements of this group are known as **Möbius transformations**. An important and easy-to-check property of such transformations is that they act as 2×2 linear transformations; more precisely, given two Möbius transformations

$$T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}, \tag{3.3}$$

their composition is given by

$$(T_1 \circ T_2)(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{3.4}$$

where $\alpha, \beta, \gamma, \delta$ are the entries of the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{3.5}$$

For this reason, Möbius transformations are also known as **fractional linear transformations**.

The group (3.2) is also sometimes referred to as the **projective linear group** (of order 2 over the complex numbers) and denoted $\mathrm{PSL}(2, \mathbb{C})$. The reason for this terminology is as follows. If we define the **special linear group** (of order 2 over the complex numbers) by

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc = 1 \right\},$$

then we can easily check that the association mapping a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ to the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ is a surjective group homomorphism, which has the subgroup $\{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ as its kernel. Thus, by the first isomorphism theorem in group theory, the group $\mathrm{Aut}(\widehat{\mathbb{C}})$ can be identified with the quotient group

$$\mathrm{SL}(2, \mathbb{C}) / \{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}.$$

The quotienting operation in this context is often referred to as *projectivization*, which leads to the name projective linear group both for the quotient group and the occasional use of the same name and notation for the group of Möbius transformations.

The group $\mathrm{PSL}(2, \mathbb{C})$ is an important group in mathematics and even has interesting connections to physics; see the box overleaf.

Suggested exercises for Section 3.3. 3.2.

3.4 The Riemann mapping theorem

We have seen two conformal equivalence classes consisting of a single element each. Obviously, if all other equivalence classes were also singletons, the situation would be extremely boring, and the notion of conformal equivalence would not even deserve its own name. It is easy to see however that the true situation is, at least, more complicated than this simplistic scenario (see Exercise 3.3).

The group $\mathrm{PSL}(2,\mathbb{C})$ and the night sky of a relativistically moving observer

Suppose you get into a spaceship and speed away from Earth, reaching a velocity of αc, where c is the speed of light, and the fraction α is substantial (say, higher than 5 %). We know from science fiction movies that your view of the stars as you peer through the spaceship window will appear distorted. But how, exactly? This problem has a delightful connection to complex analysis and the automorphism group $\mathrm{PSL}(2,\mathbb{C})$ of the Riemann sphere. In fact, your view of the celestial sphere of stars gets transformed by a Möbius transformation acting on the celestial sphere precisely as if it were the Riemann sphere.

Mathematically, the connection is roughly as follows: it is well known from the theory of special relativity that an observer moving at relativistic velocity $\mathbf{v}$ relative to the Earth (which for the sake of discussion we assume is an inertial frame of reference) will have their time and space coordinates transformed from the Earth's time and space coordinate system according to a type of linear transformation known as a **proper, orthochronous Lorentz transformation**. The group of such transformations can be represented as the group of 4×4 real matrices

$$L_+^\uparrow = \left\{T \in \mathrm{Mat}_{4\times4}(\mathbb{R}) \,:\, \det(T) = 1,\ T_{1,1} < 0,\ T^\top X T = X\right\},$$

where X is the 4×4 diagonal matrix with diagonal entries $-1, 1, 1, 1$. In fact, it can be shown that $L_+^\uparrow$ is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$ and that the isomorphism $\rho : L_+^\uparrow \to \mathrm{PSL}(2,\mathbb{C})$ is such that for the moving observer with a given associated Lorentz transformation T, the distortion of the moving observer's celestial sphere relative to the celestial sphere of the static frame of reference is described precisely by the Möbius transformation $\rho(T)$, under the obvious identification between the celestial sphere and the Riemann sphere. See [53, Appendix B] and [55, Ch. 1] for the details of this surprising result.

On this optimistic note, it looks like there ought to be some interesting phenomena for us to explore. This brings us to one of the most fundamental results on conformal mapping, the **Riemann mapping theorem**, which identifies the first nontrivial conformal equivalence class and the one that undoubtedly plays the most central role in complex analysis.

Theorem 3.5 (Riemann mapping theorem: simple version). *Let $\Omega, \Omega' \subset \mathbb{C}$ be simply connected complex regions with $\Omega, \Omega' \neq \mathbb{C}$. Then Ω and Ω' are conformally equivalent.*

As an immediate corollary, we get an interesting result in topology, an illustration of the principle that the often symbiotic relationship between complex analysis and topology involves a flow of ideas in both directions.

Corollary 3.6. *Any two simply connected regions in the plane are homeomorphic.*

This well-known result can also be proved without the use of complex analysis. See [W17] for a related discussion.

To prove Theorem 3.5, we will need to develop some new theoretical ideas (which are also interesting in their own right and are of broader applicability). A more precise version of the theorem is stated in Section 3.7.

Tangentially to that effort, we also wish to understand the structure of the automorphism groups $\mathrm{Aut}(\Omega)$ for regions Ω belonging to the conformal equivalence class described by the theorem. By Exercise 3.1 all such groups are isomorphic in such a way that the isomorphism between any two can be described in terms of conformal equivalence maps $g : \Omega \to \Omega'$ relating different class members. Thus, to understand the automorphism groups, it is in fact sufficient to classify the automorphisms for just one representative member of the class. There are two fairly canonical choices for such a member, the unit disc $\mathbb{D}$ and the upper half-plane $\mathbb{H}$ (and those two are easy to relate to each other, though doing so is still interesting). We discuss these regions in the next two sections.

Suggested exercises for Section 3.4. 3.3.

3.5 The unit disc and its automorphisms

The next result, known as the Schwarz lemma, is a simple yet powerful result about holomorphic functions from the unit disc to itself that keep the origin fixed. It is an important tool on the path to characterizing the automorphisms of the unit disc.

If $g : \mathbb{D} \to \mathbb{D}$, then we say that $g(z)$ is a **rotation map**, or simply a **rotation**, if it is of the form $g(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$.

Lemma 3.7 (The Schwarz lemma). *Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic function that satisfies $g(0) = 0$. Then:*
1. $|g(z)| \le |z|$ *for all* $z \in \mathbb{D}$.
2. *If* $|g(z)| = |z|$ *for some* $z \neq 0$, *then* $g(z)$ *is a rotation.*
3. $|g'(0)| \le 1$.
4. *If* $|g'(0)| = 1$, *then* $g(z)$ *is a rotation.*

Proof. Since $g(z)$ has a zero at $z = 0$, we know that it satisfies $|g(z)| \le C|z|$ for some $C > 0$ and all z in some neighborhood of 0. This is a weaker inequality than the one we are trying to prove, but in fact it is a helpful observation, as it can be restated as the claim that $h(z) = g(z)/z$ satisfies $|h(z)| \le C$ for all $z \in \mathbb{D} \setminus \{0\}$; that is, $h(z)$ is *bounded* in a punctured neighborhood of 0 and of course holomorphic there. By Riemann's removable singularity theorem (Theorem 1.38), $h(z)$ therefore has a removable singularity at 0 and can be extended to a holomorphic function on all of $\mathbb{D}$ (which we still denote $h(z)$, as per the usual convention when talking about analytic continuation). Now let $z \in \mathbb{D} \setminus \{0\}$, and let r be a real number with $|z| < r < 1$. By the maximum modulus principle (Theorem 1.51) the maximum modulus of $h(z)$ in the closed disc of radius r around 0 is attained at the boundary of that disc. Therefore we have that

$$\left|\frac{g(z)}{z}\right| = |h(z)| \le \max_{|w| \le r} |h(w)| \le \max_{0 \le t < 2\pi} |h(re^{it})| = \max_{0 \le t < 2\pi} \frac{|g(re^{it})|}{r} \le \frac{1}{r}.$$

(In the last step, we used the fact that $g(z)$ maps $\mathbb{D}$ into itself, so $|g(w)| \le 1$ for all $w \in \mathbb{D}$.) Since this is true for all $|z| < r < 1$, we then have that

$$\left|\frac{g(z)}{z}\right| \le \inf_{|z|<r<1} \frac{1}{r} = 1,$$

that is, $|g(z)| \le |z|$, which was the first claim of the lemma. Now claim 3 also follows by taking an additional limit of these inequalities as $z \to 0$, since $|g'(0)| = |\lim_{z\to 0} \frac{g(z)-g(0)}{z}| = \lim_{z\to 0} |\frac{g(z)}{z}|$.

Now, for the claim 2, note that an equality for some $z \in \mathbb{D}$ in the bound $|h(z)| \le 1$ means that $|h(z)|$ attains its maximal value in the interior of the disc. By the condition for equality in the maximum modulus principle, $h(z)$ must be a constant, which is of unit magnitude (since we know that $|h(z)| = 1$ for some z). That is, we have shown that $h(z) \equiv e^{i\theta}$ for some θ or, equivalently, that $g(z)$ is a rotation, giving claim 2.

Similarly, for the fourth claim, if $1 = |g'(0)| = \lim_{z\to 0} |\frac{g(z)}{z}| = \lim_{z\to 0} |h(z)| = |h(0)|$, then again we see that $|h(z)|$ attains its maximum value in the interior of the disc (in this case at $z = 0$) and infer using the same argument as above that $g(z)$ is a rotation. □

Corollary 3.8 (Automorphisms of the unit disc that fix 0). *The automorphisms $g : \mathbb{D} \to \mathbb{D}$ of the unit disc that fix* 0 (*that is, satisfy* $g(0) = 0$) *are precisely the rotations.*

Proof. Obviously, a rotation is a conformal automorphism of $\mathbb{D}$ that fixes 0. Conversely, let $g : \mathbb{D} \to \mathbb{D}$ be an automorphism that fixes 0. Then both $g(z)$ and its inverse function $g^{-1}(z)$ satisfy the assumptions of the Schwarz lemma. It follows that $|g(z)| \le z$ and $|g^{-1}(w)| \le w$ for all $z, w \in \mathbb{D}$; or, setting $w = g(z)$ for an arbitrary $z \in \mathbb{D}$ in the second inequality,

$$\left|g(z)\right| \le z \text{ and } |z| \le \left|g(z)\right| \quad \Longrightarrow \quad \left|g(z)\right| = |z|$$

for all $z \in \mathbb{D}$. By part 2 of the Schwarz lemma, $g(z)$ is a rotation. □

We can now exhibit a more general two-parameter family of automorphisms of $\mathbb{D}$, which are obtained by composing rotations with an additional family of automorphisms that *do not* fix 0. As a first step, for $w \in \mathbb{D}$, we define the Möbius transformation

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}. \tag{3.6}$$

Lemma 3.9. *The transformation φ_w is an automorphism of* $\mathbb{D}$. *Moreover, it has the following properties:* (a) $\varphi_w(0) = w$; (b) $\varphi_w(w) = 0$; (c) $\varphi_w^{-1} = \varphi_w$.

Proof. Properties (a)–(c) are trivial to check through a direct calculation, which I leave as an exercise. For the claim that φ_w is an automorphism, note that if $|z| = 1$, then

$$\left|\varphi_w(z)\right| = \frac{|w-z|}{|1-\overline{w}z|} = \frac{|w-z|}{|1-\overline{w}z|\cdot|\overline{z}|} = \frac{|w-z|}{|\overline{z}-\overline{w}z\overline{z}|} = \frac{|w-z|}{|\overline{z-w}|} = 1.$$

Thus φ_w maps the unit circle into itself. It is also injective (as a meromorphic function on $\mathbb{C}$) since it is a Möbius transformation. Therefore either φ maps the unit disc $\mathbb{D}$ into itself and maps the complement $\widetilde{\mathbb{D}} = \{|z| > 1\}$ of the closed unit disc into itself, or φ_w

maps $\mathbb{D}$ into $\widetilde{\mathbb{D}}$ and maps $\widetilde{\mathbb{D}}$ into $\mathbb{D}$. However, we know that $\varphi_w(0) = w$ and $w \in \mathbb{D}$, so that rules out the latter possibility. Finally, since we have established that $\varphi_w(\mathbb{D}) \subset \mathbb{D}$, and we know that $\varphi_w^{-1} = \varphi_w$, the mapping of $\mathbb{D}$ into itself by φ is bijective, and φ_w is a conformal equivalence. □

The composition of an arbitrary member of the family of rotations (specified by a real-valued parameter $\theta \in [0, 2\pi)$) and an arbitrary member of the family φ_w, specified by the point $w \in \mathbb{D}$, is a map of the form

$$z \mapsto e^{i\theta} \frac{w - z}{1 - \overline{w}z}.$$

It turns out that all automorphisms of the unit disc are of this form. This is the well-known characterization of the automorphism group $\mathrm{Aut}(\mathbb{D})$, given in the following theorem.

Theorem 3.10 (Automorphisms of the unit disc). *A function $g : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$ if and only if it is of the form*

$$g(z) = e^{i\theta} \frac{w - z}{1 - \overline{w}z} \tag{3.7}$$

for some $\theta \in [0, 2\pi)$ and $w \in \mathbb{D}$. The pair (θ, w) in this representation is unique.

Proof. The "if" part was already explained above. To prove the "only if" claim, let $g : \mathbb{D} \to \mathbb{D}$ be an automorphism. Denote $w = g^{-1}(0) \in \mathbb{D}$, and let $h = g \circ \varphi_w$. As the composition of two automorphisms of $\mathbb{D}$, $h(z)$ is itself an automorphism of $\mathbb{D}$. It also leaves $z = 0$ fixed. By Corollary 3.8 it is a rotation and can be expressed as $h(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$. Therefore $g(z) = (h \circ \varphi_w)(z)$ is of the desired form (3.7).

For the uniqueness claim, note that (3.7) implies that $w = g^{-1}(0)$, which determines w uniquely for a given automorphism g. Now if $w \neq 0$, then we have $g(0) = e^{i\theta}w$, which can be written as $e^{i\theta} = g(0)/w$, and thus θ is also determined uniquely from the map g. In the second case where $w = 0$, we are back to the scenario of an automorphism that fixes 0, which we have seen must be a rotation $g(z) = e^{i\theta}z$, with θ again clearly being uniquely determined. □

An alternative, but less frequently used, characterization of the automorphisms of the unit disc is given in the next result. The proof is left as an exercise (Exercise 3.4).

Theorem 3.11 (Automorphisms of the unit disc: alternative representation). *A function $g : \mathbb{D} \to \mathbb{D}$ is an automorphism of $\mathbb{D}$ if and only if it is of the form*

$$g(z) = \frac{\mu z + \nu}{\overline{\nu}z + \overline{\mu}} \tag{3.8}$$

for some $\mu, \nu \in \mathbb{C}$ satisfying $|\mu|^2 - |\nu|^2 = 1$. The pair (μ, ν) is unique.

The explicit description of the automorphisms of $\mathbb{D}$ in terms of the representations (3.7)–(3.8), involving formulas that one rarely encounters outside of complex analysis, masks the fact that the group of such automorphisms bears a close relationship with a standard matrix group you may be familiar with from linear algebra, the theory of Lie groups, topology, and other areas. As we will see in the next section, the connection becomes apparent when we switch from the unit disc to its "conformal sibling," the upper half-plane.

Suggested exercises for Section 3.5. 3.4.

3.6 The upper half-plane and its automorphisms

Lemma 3.12. *The unit disc $\mathbb{D}$ and the upper half-plane $\mathbb{H}$ are conformally equivalent. The pair of maps $\Phi : \mathbb{H} \to \mathbb{D}$ and $\Psi : \mathbb{D} \to \mathbb{H}$ given by*

$$\Phi(z) = \frac{z-i}{z+i} \quad \text{and} \quad \Psi(z) = -i\frac{z+1}{z-1} \tag{3.9}$$

give an explicit pair of mutually inverse conformal maps mapping each of the regions onto the other.

Proof. Note that if $z = x + iy$, then $|\Phi(z)|^2 = \frac{|z-i|^2}{|z+i|^2} = \frac{x^2+(y-1)^2}{x^2+(y+1)^2}$, which is < 1 if and only if $\mathrm{Im}(z) = y > 0$ (the geometric meaning of this statement is simply that $\Phi(z)$ is the ratio of the distances of z to i and $-i$, and the upper half-plane is precisely the locus of points that are closer to i than to $-i$). Thus Φ maps $\mathbb{H}$ into $\mathbb{D}$ and the complement of $\mathbb{H}$ into the complement of $\mathbb{D}$. Since we know that Φ is a conformal map when regarded as a map from $\widehat{\mathbb{C}}$ to itself, this is enough to imply that it maps $\mathbb{H}$ surjectively and conformally onto $\mathbb{D}$. Finally, it is trivial to verify by direct calculation that the inverse map to $\Phi(z)$ is given by the formula defining $\Psi(z)$. □

Theorem 3.13 (Conformal automorphisms of the upper half-plane). *A function $g : \mathbb{H} \to \mathbb{H}$ is a conformal automorphism if and only if it is of the form*

$$g(z) = \frac{az+b}{cz+d} \tag{3.10}$$

for real numbers a, b, c, d satisfying $ad - bc = 1$. The numbers a, b, c, d in this representation are unique up to a single choice of sign, in the sense that if a, b, c, d and a', b', c', d' are coefficients in two distinct representations, then $(a', b', c', d') = \pm(a, b, c, d)$.

Proof. "If": assume that $g(z)$ has the stated form (3.10) with a, b, c, d real and $ad - bc = 1$. As we already know from Theorem 3.4, $g(z)$ is a conformal automorphism of $\widehat{\mathbb{C}}$. Moreover, since $a, b, c, d \in \mathbb{R}$, we have

$$\begin{aligned}\operatorname{Im}\left(\frac{az+b}{cz+d}\right) &= \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right)\\ &= \frac{1}{|cz+d|^2}\operatorname{Im}\big(ac|z|^2+bd+adz+bc\bar{z}\big) = \frac{ad-bc}{|cz+d|^2}\operatorname{Im}(z).\end{aligned} \tag{3.11}$$

This immediately implies that $\operatorname{Im}(g(z)) > 0$ if and only if $\operatorname{Im}(z) > 0$, that is, g is an automorphism of $\mathbb{H}$.

"Only if": assume that $g \in \operatorname{Aut}(\mathbb{H})$. Then $f = \Phi \circ g \circ \Psi$ is an automorphism of the unit disc, where Φ and Ψ are given in (3.9). By Theorem 3.11, f can be expressed as

$$f(z) = \frac{\mu z+\nu}{\bar{\nu}z+\bar{\mu}}$$

for some $\mu, \nu \in \mathbb{C}$ with $|\mu|^2 - |\nu|^2 = 1$. To calculate what this means for $g = \Psi \circ f \circ \Phi$, we switch to the notation of matrix multiplication, which, as we know from (3.3)–(3.5), is a way to represent the action of Möbius transformations. The matrices associated with the action of Φ, Ψ, and f are

$$\Phi = \begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}, \quad \Psi = \begin{pmatrix}-i & -i\\ 1 & -1\end{pmatrix}, \quad f = \begin{pmatrix}\mu & \nu\\ \bar{\nu} & \bar{\mu}\end{pmatrix}.$$

Therefore the map $\Psi \circ f \circ \Phi$ is represented by the matrix product

$$\Psi f \Phi = \begin{pmatrix}-i & -i\\ 1 & -1\end{pmatrix}\begin{pmatrix}\mu & \nu\\ \bar{\nu} & \bar{\mu}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}.$$

More explicitly, if we denote $\mu = x + iy$ and $\nu = u + iv$ to represent μ, ν in terms of their real and imaginary parts, then this matrix product is

$$\begin{aligned}\Psi f \Phi &= \begin{pmatrix}-i & -i\\ 1 & -1\end{pmatrix}\begin{pmatrix}x+iy & u+iv\\ u-iv & x-iy\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\\ &= 2i\begin{pmatrix}-x-u & -y+v\\ y+v & -x+u\end{pmatrix} =: 2i\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\end{aligned}$$

The numbers a, b, c, d thus defined are real, and moreover it is easy to check that $ad-bc = 1$ (hint: determinants). Note that the scalar factor $2i$ multiplying the matrix is irrelevant when we go back to considering g as a Möbius transformation instead of a matrix, that is, we see that $g(z)$ is indeed of the form $\frac{az+b}{cz+d}$ with a, b, c, d as claimed in the theorem. □

The automorphism group

$$\operatorname{Aut}(\mathbb{H}) = \left\{z \mapsto \frac{az+b}{cz+d} \;:\; a, b, c, d \in \mathbb{R},\ ad-bc = 1\right\}$$

is known as the **projective special linear group** (of order 2 over the real numbers) and sometimes denoted $\operatorname{PSL}(2,\mathbb{R})$. By the natural association between 2×2 matrices and

Möbius transformations discussed in Section 3.3, it can be identified with the quotient group

$$\mathrm{SL}(2,\mathbb{R})/\{\pm I\},$$

where $\mathrm{SL}(2,\mathbb{R})$ is the special linear group of order 2 over $\mathbb{R}$ (the group of invertible 2×2 real matrices with determinant 1), and $\{\pm I\}$ is its subgroup with two elements containing the identity matrix and its negation.

3.7 The Riemann mapping theorem: a more precise formulation

We formulated in Section 3.4 a version of the Riemann mapping theorem that identifies an interesting conformal equivalence class of complex regions. Conceptually, this is what I regard as the main content of the theorem. Note that this formulation is carefully "neutral" in the sense of not singling out any member of the equivalence class as being more important or worthy of attention than others. However, in practice, we already discussed the fact that the unit disc and upper half-plane are each in their own way somewhat canonical members of the class. By contrast, other member regions such as, say, the unit square, seldom play a particularly important role in the theory, although from a purely geometric point of view, they may be just as natural, and they may appear in specific applications.

Furthermore, as we inch our way toward a proof of the theorem, it does in fact become convenient to fix a specific member of the class—the unit disc—as the target region for the conformal maps we will construct. Another small conceptual advance is to add more information about the conformal map mapping a given region Ω to $\mathbb{D}$ so as to ensure uniqueness. This leads us to the following more detailed version of the theorem.

Theorem 3.14 (Riemann mapping theorem: detailed version). *Let $\Omega \subset \mathbb{C}$ be a simply connected complex region with $\Omega \neq \mathbb{C}$, and let $z_0 \in \Omega$. Then there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ with the property that*

1. $F(z_0) = 0$
2. *$F'(z_0)$ is a positive real number.*

Proof of uniqueness. Let F_1 and F_2 be two biholomorphisms with the properties described in the theorem. Then the conformal map $\Phi = F_2 \circ F_1^{-1}$ is an automorphism of $\mathbb{D}$ that fixes 0, so by Corollary 3.8 it is a rotation, that is, of the form $\Phi(z) = az$ for some a with $|a| = 1$. On the other hand, the constant a can be expressed as

$$a = \Phi'(0) = F_2'(F_1^{-1}(0))(F_1^{-1})'(0) = \frac{F_2'(z_0)}{F_1'(z_0)},$$

which shows that it is a positive real number. It follows that $a = 1$ and $\Phi(z) \equiv z$, that is, $F_1 \equiv F_2$. □

The history of the Riemann mapping theorem

The Riemann mapping theorem was formulated by the great Bernhard Riemann in 1851 as part of his PhD thesis. Riemann stated the result for regions with a piecewise smooth boundary and gave a proof that contained useful ideas but was later realized to be flawed. Later nineteenth-century mathematicians worked hard to fill in the gaps in Riemann's argument, with varying levels of success. The first proof considered to be fully correct by modern standards was given by Osgood in 1900. Osgood's proof, like others before it, relied on the "potential-theoretic" approach (related to Dirichlet's principle and the study of Laplace's equation) advocated by Riemann rather than on ideas of a more conceptually complex-analytic nature. This approach, while interesting, has since fallen out of fashion as an approach to proving the Riemann mapping theorem because of various technical shortcomings it has.

The proof of the theorem we present in Sections 3.8–3.9 is described in Walsh's historical survey [72] as the "standard modern proof." You will find it described in most complex analysis textbooks, as it appears to be the simplest proof known today. For additional details on the interesting history of Riemann's famous theorem and the ideas developed out of it, see the historical reviews [33, 72].

The more difficult part of Theorem 3.14 is the existence claim. As we will see, the key insight needed for the proof is that the problem of mapping Ω conformally to $\mathbb{D}$ can be formulated as a maximization problem for a certain functional. Specifically, in the class $\mathcal{F}$ consisting of all the *injective* maps from Ω into $\mathbb{D}$ that map z_0 to 0 and for which $F'(z_0)$ is a positive real number, we will see that the one map that is also surjective (and thus establishes the required conformal equivalence of Ω to $\mathbb{D}$) is the one for which the number $F'(z_0)$ is maximal. This will be shown in a somewhat constructive way by arguing that if $F(z)$ is not surjective, then we can exploit the point that is "missing" from the image to produce a new conformal map $G : \Omega \to \mathbb{D}$ with a larger value of $G'(z_0)$. Although the basic idea of how this is done is fairly simple (see Lemma 3.21), there are a few technical issues that need to be addressed to turn it into a complete proof, namely showing that the class $\mathcal{F}$ is nonempty, that the functional $F \mapsto F'(z_0)$ attains a maximum, and so on. The details are given in the next two sections.

3.8 Proof of the Riemann mapping theorem, part I: technical background

In this section, we prove a few auxiliary results needed for the proof of the Riemann mapping theorem. Two of the results, Montel's and Hurwitz's theorems, are theorems in complex analysis. The third, the Arzelà–Ascoli theorem, is a theorem in real analysis.

Let $\mathcal{F}$ be a family of complex-valued continuous functions on a complex region Ω. We say that $\mathcal{F}$ is **locally uniformly bounded** if for any compact set $K \subset \Omega$, we have

$$\sup_{f\in\mathcal{F},\, z\in K} |f(z)| < \infty. \tag{3.12}$$

We say that $\mathcal{F}$ is **locally uniformly equicontinuous** if for any compact $K \subset \Omega$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } z_1, z_2 \in K \text{ and } |z_1 - z_2| < \delta, \quad \text{then} \quad \sup_{f \in \mathcal{F}} \left| f(z_1) - f(z_2) \right| < \varepsilon. \tag{3.13}$$

The following is a version of the well-known Arzelà–Ascoli theorem, a staple of real and functional analysis, slightly adapted to our setting.

Theorem 3.15 (Arzelà–Ascoli theorem). *Let $\mathcal{F}$ be a family of continuous complex-valued functions on Ω. Assume that the family is locally uniformly equicontinuous and locally uniformly bounded. Then any sequence $(f_n)_{n=1}^{\infty}$ of functions in $\mathcal{F}$ has a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges uniformly on compacts in Ω to some continuous function f.*

Proof. Let $Q = (z_m)_{m=1}^{\infty}$ be a dense countable set of points in Ω (ordered as a sequence according to some arbitrary enumeration). The sequence $(f_n(z_1))_{n=1}^{\infty}$ is a sequence of complex numbers taking values in a compact set $\{|z| \le M_1\}$, where we denote $M_1 = \sup_{f \in \mathcal{F}} |f(z_1)| < \infty$ (guaranteed to be finite by (3.12)). By compactness this sequence therefore has a convergent sequence, which we denote by $(f_n^{(1)}(z_1))_{n=1}^{\infty}$ (instead of the more traditional subsequence notation $f_{n_k}(z_1)$). That is, $f_n^{(1)}$ is the notation for the nth function in the extracted *subsequence* of the original sequence of functions $(f_n(z))_n$.

Now we extract a further subsequence of this subsequence, noting that the sequence $(f_n^{(1)}(z_2))_{n=1}^{\infty}$ is a sequence of complex numbers taking values in a compact set $\{|z| \le M_2\}$, where

$$M_2 = \sup_{f \in \mathcal{F},\ z \in \{z_1, z_2\}} \left| f(z) \right|.$$

(Again, the local uniform boundedness assumption guarantees that $M_2 < \infty$.) So again by compactness, this sequence has a convergent sequence, which we denote by $(f_n^{(2)}(z_1))_{n=1}^{\infty}$.

Continuing in this way, we proceed to successively extract nested subsequences $(f_n^{(3)})_{n=1}^{\infty}, (f_n^{(4)})_{n=1}^{\infty}, \ldots$ of the original sequence of functions, where each subsequence is extracted as a further subsequence of the previous one. These subsequences have the property that for each $j \ge 1$, the jth sequence $(f_n^{(j)})_{n=1}^{\infty}$ is a subsequence of the original sequence $(f_n)_n$ for which $f_n^{(j)}(z_m)$ converges to a limit as $n \to \infty$ for $m = 1, 2, \ldots, j$.

Now consider the "diagonal" sequence in this nested sequence of subsequences: we let $g_n = f_n^{(n)}$. Then $(g_n)_{n=1}^{\infty}$ is a subsequence of $(f_n)_n$ with the property that $g_n(z_m)$ converges to a limit as $n \to \infty$ for *all* $m \ge 1$.

We claim that the sequence of functions $(g_n(z))_{n=1}^{\infty}$ converges uniformly on compacts in Ω. Let $K \subset \Omega$ be compact, and let $\varepsilon > 0$. Let $\delta > 0$ be a number, guaranteed to exist by the assumption of local uniform equicontinuity, with the property that

$$\text{if } z_1, z_2 \in K \text{ and } |z_1 - z_2| < \delta, \quad \text{then} \quad \sup_{f \in \mathcal{F}} \left| f(z_1) - f(z_2) \right| < \frac{\varepsilon}{3}.$$

(Compare with (3.13): we merely replaced ε there with $\varepsilon/3$, with the usual goal in mind that some other bound later will end up smaller than ε.) The containment

$K \subset \cup_{\xi\in K} D_{\delta/2}(\xi)$ gives an open covering of K, which by compactness has a finite subcovering $(D_{\delta/2}(\xi_j))_{j=1}^q$. Select a point z_{ν_j} of the countable dense set Q from each of the subcovering discs $D_{\delta/2}(\xi_j)$. For any $1 \le j \le q$, $(g_k(z_{\nu_j}))_{k=1}^\infty$ is a convergent sequence or, equivalently, is a Cauchy sequence; therefore there exists an index $N_j \ge 1$ such that

$$|g_\ell(z_{\nu_j}) - g_k(z_{\nu_j})| < \frac{\varepsilon}{3}$$

whenever $k, \ell \ge N_j$. Set $N = \max(N_1, N_2, \dots, N_q)$. Then for any $w \in K$, we have that $w \in D_{\delta/2}(\xi_j) \subset D_\delta(z_{\nu_j})$ for some $1 \le j \le q$. It follows that, for $k, \ell \ge N$,

$$\begin{aligned} |g_\ell(w) - g_k(w)| &\le |g_\ell(w) - g_\ell(z_{\nu_j})| + |g_\ell(z_{\nu_j}) - g_k(z_{\nu_j})| \\ &\quad + |g_k(z_{\nu_j}) - g_k(w)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This establishes that $(g_k(z))_{k=1}^\infty$ is a Cauchy sequence uniformly on K and hence (by a standard fact from real analysis) converges uniformly on K. The compact K was arbitrary, so we proved the existence of a subsequence that converges uniformly on compacts; the fact that the limiting function must be continuous is standard, and the proof of the theorem is complete. □

Returning to the realm of complex analysis, we now introduce the concept of a **normal family** of functions. Let Ω be a complex region as before. A family $\mathcal{F}$ of holomorphic functions on Ω is called **normal**, or a **normal family**, if every sequence $(f_n)_{n=1}^\infty$ in the family has a subsequence $(f_{n_k})_{k=1}^\infty$ such that f_{n_k} converges uniformly on compacts to a holomorphic function g.

Theorem 3.16 (Montel's theorem). *Let $\mathcal{F}$ be a family of holomorphic functions on a region Ω that is locally uniformly bounded. Then $\mathcal{F}$ is a normal family.*

Proof. We claim that the added assumption of holomorphicity of the members of $\mathcal{F}$, together with local uniform boundedness, implies that the family is uniformly locally equicontinuous. Once we show this, the Arzelà–Ascoli theorem will imply that every sequence $(F_n)_{n=1}^\infty$ of elements in the family has a subsequence F_{n_k} that converges uniformly on compacts to a limiting function F. Then it would follow that F is holomorphic by standard properties of uniform convergence on compacts (Theorem 1.39 on p. 45), and we would be done.

We start by showing a weaker version of the required property that does not include uniformity over compact subsets. Fix a point $a \in \Omega$ and a radius $\rho > 0$ such that $D_{2\rho}(a) \subset \Omega$. Later we will need to emphasize the dependence of ρ on a, so we will then denote it by $\rho(a)$. If $z_1, z_2 \in D_\rho(a)$, then by Cauchy's integral formula we have, uniformly over all $f \in \mathcal{F}$,

$$|f(z_1) - f(z_2)| = \left| \frac{1}{2\pi i} \oint_{|w-a|=2\rho} f(w) \left(\frac{1}{w - z_1} - \frac{1}{w - z_2} \right) dw \right|$$

$$= \left| \frac{z_1 - z_2}{2\pi i} \oint_{|w-a|=2\rho} \frac{f(w)}{(w - z_1)(w - z_2)}\, dw \right|$$

$$\leq \frac{1}{2\pi} |z_1 - z_2| \cdot \sup_{|w-a|=2\rho} |f(w)| \cdot 2\pi(2\rho) \frac{1}{\rho^2} \leq \frac{2M}{\rho} |z_1 - z_2|, \tag{3.14}$$

where we denote $M = \sup_{f \in \mathcal{F},\, |w-a|=2\rho} |f(w)|$, a finite number by the local uniform boundedness assumption.

Now fix a number $\varepsilon > 0$. If we define the number

$$\eta = \min\left(\rho, \frac{\rho\varepsilon}{4M}\right) > 0,$$

then by (3.14) we have the property that

$$\text{if } z_1, z_2 \in D_\eta(a), \quad \text{then } \sup_{f \in \mathcal{F}} \left| f(z_1) - f(z_2) \right| < \varepsilon. \tag{3.15}$$

This is the nonuniform local equicontinuity property alluded to above. Note that the parameter η depends on the point a, so we will now redenote it by $\eta(a)$ to emphasize this dependence. (η also depends on ε, but the value of ε will remain fixed throughout the discussion.)

Finally, we can derive the uniform-over-compacts version of local equicontinuity. Let $K \subset \Omega$ be a compact set, and let $\varepsilon > 0$ be the same as above. Consider the covering of K by open sets given by

$$K \subset \bigcup_{a \in K} D_{\eta(a)/2}(a).$$

By compactness there exists a finite subcovering

$$K \subset \bigcup_{j=1}^{n} D_{\eta(a_j)/2}(a_j)$$

for some points $a_1, \ldots, a_n \in K$. Denote $\delta = \frac{1}{2} \min(\eta(a_1), \ldots, \eta(a_n))$. Then we claim that for all $z_1, z_2 \in K$ such that $|z_1 - z_2| < \delta$,

$$\sup_{f \in \mathcal{F}} \left| f(z_1) - f(z_2) \right| < \varepsilon. \tag{3.16}$$

Indeed, z_1 must belong to $D_{\eta(a_j)/2}(a_j)$ for some $1 \leq j \leq n$ by the defining property of the subcovering. This also implies that

$$|z_2 - a_j| \leq |z_2 - z_1| + |z_1 - a_j| < \delta + \frac{\eta(a_j)}{2} \leq \frac{\eta(a_j)}{2} + \frac{\eta(a_j)}{2} = \eta(a_j),$$

so altogether we see that both z_1, z_2 are in $D_{\eta(a_j)}(a_j)$. Relation (3.16) therefore follows from (3.15). To summarize, we proved that for any compact set $K \subset \Omega$ and $\varepsilon > 0$, (3.13) is satisfied without choice of δ as defined above; this proves that the family $\mathcal{F}$ is locally uniformly equicontinuous and concludes the proof of the theorem. □

Theorem 3.17 (Hurwitz's theorem). *Let $\Omega \subset \mathbb{C}$ be a region, and let $(f_n(z))_{n=1}^{\infty}$ and $g(z)$ be holomorphic functions on Ω such that $f_n(z) \to g(z)$ uniformly on compacts in Ω as $n \to \infty$, where $g(z)$ is not the zero function. If $z_0 \in \Omega$ is a zero of $g(z)$ of order $k \geq 0$, and $D_r(z_0) \subset \Omega$ is a disc centered at z_0 such that the punctured closed disc $D_{\leq r}(z_0) \setminus \{z_0\}$ contains no zeros of $g(z)$, then for any large enough n, $f_n(z)$ has precisely k zeros in $D_r(z_0)$ counting multiplicities.*

Proof. Recall that by the argument principle the order k of the zero of $g(z)$ at z_0 can be expressed as the contour integral

$$k = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{g'(z)}{g(z)}\, dz. \tag{3.17}$$

Denote by κ_n the number of zeros of $f_n(z)$ in $D_r(z_0)$ counting multiplicities. We wish to express κ_n similarly as a contour integral over the same circle. This can be done but requires first checking that $f_n(z)$ does not have any zeros on the circle, which is indeed true for large n. Let $M = \inf_{|z-z_0|=r} |g(z)|$ and note that $M > 0$ by the assumption that $g(z)$ has no zeros in the punctured disc $D_{\leq r}(z_0) \setminus \{z_0\}$ and, in particular, on the circle. By the uniform convergence of $f_n(z)$ to $g(z)$ on the circle there exists an index $N \geq 1$ such that for all $n \geq N$, $\inf_{|z-z_0|=r} |f_n(z)| \geq M/2$, so that, in particular, $f_n(z)$ also does not have any zeros on the circle $|z - z_0| = r$ as we wanted to show. Thus we have the expression

$$\kappa_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f_n'(z)}{f_n(z)}\, dz \tag{3.18}$$

for all $n \geq N$.

Note also that on the circle $|z-z_0|$ we have not only the uniform convergence $f_n(z) \to g(z)$, but also that of the derivatives $f_n'(z) \to g'(z)$ (recall Theorem 1.39). Combining those facts, we deduce also that

$$\frac{f_n'(z)}{f_n(z)} \xrightarrow[n\to\infty]{} \frac{g'(z)}{g(z)}$$

uniformly on the circle $|z - z_0| = r$. Finally, this, together with (3.17) and (3.18), implies that

$$\kappa_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f_n'(z)}{f_n(z)}\, dz \xrightarrow[n\to\infty]{} \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{g'(z)}{g(z)}\, dz = k$$

Since k and κ_n are all integers, it follows that $\kappa_n = k$ for all sufficiently large n. □

Corollary 3.18. *Let $\Omega \subset \mathbb{C}$ be a region, and as in Hurwitz's theorem, let $(f_n(z))_{n=1}^{\infty}$ and $g(z)$ be holomorphic functions on Ω such that $f_n(z) \to g(z)$ uniformly on compacts in Ω. If the functions $f_n(z)$ are all injective, then $g(z)$ is either injective or a constant.*

Proof. Assume by contradiction that $g(z)$ is not injective and also not a constant function. Then there exist distinct points $a, b, \in \Omega$ for which $g(a) = g(b)$. We have the convergence $f_n(a) \to g(a)$, and so, if we define functions $\psi(z)$ and $\varphi_n(z)$, $n = 1, 2, \ldots$, by

$$\psi(z) = g(z) - g(a), \quad \varphi_n(z) = f_n(z) - f_n(a),$$

then $\varphi_n(z) \to \psi(z)$ uniformly on compacts in Ω. Moreover, $\psi(z)$ is not the zero function. Therefore we are in a position to apply Hurwitz's theorem. Specifically, note that $\psi(b) = 0$, and denote the order of the zero at b by $k \geq 1$. Let $r > 0$ be such that the punctured closed disc $D_{\leq r}(b) \setminus \{b\}$ does not contain any other zeros of $\psi(z)$ (so, in particular, it does not contain the point $z = a$). Applying Hurwitz's theorem, we conclude that for all sufficiently large n, $\varphi_n(z)$ has at least one zero in the disc $D_r(b)$. However, this is impossible, since $\varphi_n(z)$ already has one zero at $z = a$ and was assumed to be an injective function. We have reached a contradiction, and the proof is complete. □

Suggested exercises for Section 3.8. 3.5, 3.6.

3.9 Proof of the Riemann mapping theorem, part II: the main construction

From now on, let Ω be a simply connected complex region with $\Omega \neq \mathbb{C}$ and $z_0 \in \Omega$, as in the statement of Theorem 3.14.

Lemma 3.19. *There exists an injective holomorphic function $G : \Omega \to \mathbb{D}$.*

Proof. We know that Ω is not the entire complex plane, so take some point $\alpha \in \mathbb{C} \setminus \Omega$. The function $z \mapsto z - \alpha$ has no zeros on Ω, so, since Ω is simply connected, by Theorem 1.53 there exists a branch of the logarithm function of $z - \alpha$ on it, that is, a holomorphic function $h(z)$ such that $e^{h(z)} = z - \alpha$ for all $z \in \Omega$.

Fix an arbitrary point $\beta \in \Omega$, and define a function $G : \Omega \to \mathbb{C}$ by

$$G(z) = \frac{1}{h(z) - h(\beta) - 2\pi i}. \tag{3.19}$$

We claim that $G(z)$ is holomorphic, injective, and bounded on Ω; this would imply that its scaled version $F(z) = cG(z)$ is injective and maps into $\mathbb{D}$ if c is a small enough positive constant, which would prove the result.

To establish these properties of $G(z)$, note first that $h(z)$ is injective, since $h(z) = h(w)$ implies $z - \alpha = e^{h(z)} = e^{h(w)} = w - \alpha$, so $z = w$. Clearly, $G(z) = G(w)$ also implies $h(z) = h(w)$, so similarly implies $z = w$, which shows that $G(z)$ is injective.

Now the claim that $G(z)$ is bounded is equivalent to the claim that

$$\inf_{z\in\Omega}\big|h(z) - \big(h(\beta) + 2\pi i\big)\big| > 0.$$

Assume by contradiction that this is not true. Then there is a sequence $(z_n)_{n=1}^{\infty}$ of points in Ω such that $h(z_n) \xrightarrow[n\to\infty]{} h(\beta) + 2\pi i$. Exponentiating, we get that

$$z_n - \alpha = e^{h(z_n)} \xrightarrow[n\to\infty]{} e^{h(\beta)+2\pi i} = e^{h(\beta)} = \beta - \alpha.$$

In other words, z_n converges to β as $n \to \infty$. However, then we would have that $h(z_n)$ converges to $h(\beta)$ and not to $h(\beta) + 2\pi i$. This gives a contradiction and finishes the proof. □

Now define the family of functions

$$\mathcal{F} = \{F : \Omega \to \mathbb{D} \,:\, F(z) \text{ is holomorphic and injective, } F(z_0) = 0\}.$$

The family $\mathcal{F}$ is not empty: if $G(z)$ is an injective holomorphic function $G : \Omega \to \mathbb{D}$ guaranteed to exist by Lemma 3.19, then clearly $F(z) = c(G(z) - G(z_0))$ is an element of $\mathcal{F}$ if c is a small enough positive number. Define the number $\lambda \in [0, \infty]$ by

$$\lambda = \sup_{F\in\mathcal{F}}\big|F'(z_0)\big|.$$

Lemma 3.20. $0 < \lambda < \infty$.

Proof. Let $F \in \mathcal{F}$. To bound $|F'(z_0)|$ from above, observe that, by the Cauchy integral formula, if $r > 0$ is a number for which the closed disc $D_{\le r}(z_0)$ is contained in Ω, then

$$\big|F'(z_0)\big| = \left|\frac{1}{2\pi i}\oint_{|w-z_0|=r}\frac{F(w)}{(w-z_0)^2}\,dw\right| \le \frac{1}{2\pi}(2\pi r)\frac{1}{r^2}\sup_{w\in\Omega}\big|F(w)\big| \le \frac{1}{r},$$

since F maps into the unit disc. Since this is true for all $F \in \mathcal{F}$, we get that $\lambda \le \frac{1}{r}$. On the other hand, we claim that $|F'(z_0)| > 0$, which would show that $\lambda > 0$. Indeed, if $F'(z_0) = 0$, then $F(z)$ has a zero of order at least 2 in z_0. By Corollary 1.58, $F(z)$ is not locally injective in any neighborhood of z_0, in contradiction to the fact that F is injective. Thus $|F'(z_0)|$ must be positive. □

We now come to the most important lemma of this section, which contains the key idea behind our proof of the Riemann mapping theorem.

Lemma 3.21. *Given $F \in \mathcal{F}$, if $F(\Omega) \subsetneq \mathbb{D}$ (that is, the image of Ω under F does not cover all of $\mathbb{D}$), then there exists $G \in \mathcal{F}$ for which $|G'(z_0)| > |F'(z_0)|$.*

Proof. Take some $w \in \mathbb{D} \setminus F(\Omega)$, known to exist by the assumption. Since w is not in the image of Ω under F, the point 0 is not in the image of the composed map $\varphi_w \circ F : \Omega \to \mathbb{D}$, where (recall from (3.6) and Lemma 3.9) $\varphi_w(z) = \frac{w-z}{1-\overline{w}z}$ is the standard automorphism of $\mathbb{D}$ mapping 0 and w to each other. Since $\varphi_w \circ F$ does not take the value 0 and is defined on a simply connected region, by the construction of nth root functions described in Section 1.15 there exists a holomorphic branch of its square root, that is, a holomorphic function $S : \Omega \to \mathbb{D}$ satisfying

$$S(z)^2 = (\varphi_w \circ F)(z). \tag{3.20}$$

Now define $G : \Omega \to \mathbb{D}$ by the composition

$$G(z) = (\varphi_{S(z_0)} \circ S)(z). \tag{3.21}$$

We claim that $G(z)$ has the properties claimed by the lemma. First,

$$G(z_0) = (\varphi_{S(z_0)} \circ S)(z_0) = \varphi_{S(z_0)}\big(S(z_0)\big) = 0.$$

Second, note that $S(z)$ is injective since its square is injective as a composition of two injective maps. Therefore $G(z)$ is also injective. Both of those facts together show that $G \in \mathcal{F}$.

Third and crucially, we wish to show that $|G'(z_0)| > |F'(z_0)|$. To this end, note that by (3.20) and (3.21), $F(z)$ can be represented in terms of $G(z)$ as

$$F(z) = \varphi_w\big((\varphi_{S(z_0)} \circ G)(z)^2\big). \tag{3.22}$$

(This is a key relation that deserves to be digested properly. Take a minute or two to unwrap all the horrible notation and convince yourself that this relation is correct, and see if you can find some deeper meaning here.) Alternatively, if we define the function $W : \mathbb{D} \to \mathbb{D}$ by

$$W(z) = \varphi_w\big(\varphi_{S(z_0)}(z)^2\big),$$

then (3.22) can be rewritten as

$$F(z) = (W \circ G)(z). \tag{3.23}$$

Note that

$$W(0) = \varphi_w\big(\varphi_{S(z_0)}(0)^2\big) = \varphi_w\big(S(z_0)^2\big) = \varphi_w\big(\varphi_w\big(F(0)\big)\big) = F(0) = 0.$$

Thus $W(z)$ satisfies the assumptions of Schwarz's lemma, and we conclude that $|W'(0)| \leq 1$, and in fact the strict inequality $|W'(0)| < 1$ holds, since $W(z)$ is clearly not a rotation. This is what we want, since by (3.23)

$$|F'(z_0)| = |W'(G(z_0))G'(z_0)| = |W'(0)| \cdot |G'(z_0)|,$$

which gives the desired conclusion that $|G'(z_0)| > |F'(z_0)|$. □

Lemma 3.22. *The family $\mathcal{F}$ is a normal family.*

Proof. The functions in $\mathcal{F}$ all map into the unit disc, so they are uniformly bounded, and a fortiriori locally uniformly bounded. By Montel's theorem, $\mathcal{F}$ is normal. □

Lemma 3.23. *There exists an element $F \in \mathcal{F}$ for which $|F'(z_0)| = \lambda$, that is, the functional $G \mapsto |G'(z_0)|$ attains a maximum in the family $\mathcal{F}$.*

Proof. Let $(F_n)_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{F}$ such that we have the convergence $|F_n'(z_0)| \to \lambda$. By Lemma 3.22 there is a subsequence $(F_{n_k})_{k=1}^{\infty}$ that converges uniformly on compacts in Ω to some limiting function $F : \Omega \to \mathbb{C}$, which moreover satisfies $F(z_0) = 0$, since $F_n(z_0) = 0$ for all n. Since uniform convergence on compacts implies convergence of the derivatives, we have that $|F'(z_0)| = \lambda$. Since the F_n are all injective, by Hurwitz's theorem, F either is a constant function or is injective, but we know from Lemma 3.20 that $|F'(z_0)| = \lambda > 0$, and hence F is not a constant and is therefore injective.

Let $z \in \Omega$. We know that $|F(z)| \leq 1$, since it is the limit of functions whose modulus is bounded by 1. However, F is holomorphic, and hence by the open mapping theorem, $F(\Omega)$ is an open set contained in the closed disc $\{z : |z| \leq 1\}$ and therefore is contained in the open disc $\mathbb{D}$. Thus we have shown that F is an element of $\mathcal{F}$, and the proof is complete. □

Proof of existence in Theorem 3.14. Take the element $F \in \mathcal{F}$, guaranteed to exist by Lemma 3.23, for which $|F'(z_0)| = \lambda$. By composing F with a rotation if necessary, we may assume that $F'(z_0)$ is real and positive. By Lemma 3.21, $F(z)$ must be surjective, which, together with the positivity of $F'(z_0)$ and the properties implied by belonging to $\mathcal{F}$, gives that $F(z)$ is the biholomorphism whose existence was claimed. □

Summarizing, we proved the uniqueness claim from Theorem 3.14 in Section 3.7, and the existence claim was proved above. This finishes the proof of the Riemann mapping theorem.

3.10 Annuli and doubly connected regions

The topic of conformal mapping does not end with the consideration of simply connected regions, where the problem of classifying complex regions up to conformal equivalence is now essentially settled (at least in principle) by the Riemann mapping theorem. To conclude this chapter, we give a brief taste of some of the interesting phenomena that arise when we try to classify conformal equivalence classes of regions that are *not* simply connected, starting with the next simplest case of regions that are

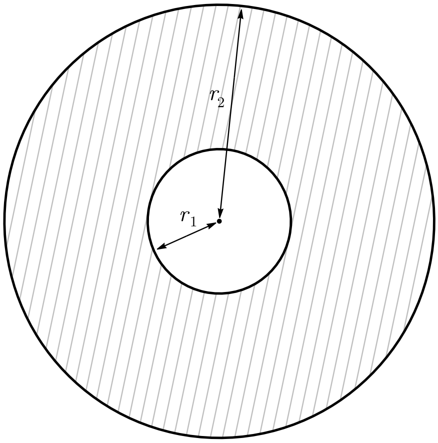

Figure 3.2: An annulus $A(r_1, r_2)$.

doubly connected. A region Ω is called doubly connected if the complement $\mathbb{C} \setminus \Omega$ has two connected components.[3]

One important class of doubly connected regions are the **annuli**. For $0 < r_1 < r_2$, we denote

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\},$$

an open annulus centered at 0 with internal radius r_1 and external radius r_2 (Fig. 3.2). It turns out that unlike the situation for simply connected regions, these annuli are not all in a single conformal equivalence class, despite being homeomorphic. The precise classification is given in the next result, sometimes known as Schottky's theorem.

Theorem 3.24 (Conformal classification of annuli). *Let $0 < r_1 < r_2$ and $0 < \rho_1 < \rho_2$. The annuli $A(r_1, r_2)$ and $A(\rho_1, \rho_2)$ are conformally equivalent if and only if*

$$\frac{r_1}{r_2} = \frac{\rho_1}{\rho_2}.$$

Proof. "If": assume that $\frac{r_1}{r_2} = \frac{\rho_1}{\rho_2}$. Then the map $z \mapsto \frac{\rho_1}{r_1} z = \frac{\rho_2}{r_2} z$ is a conformal equivalence between $A(r_1, r_2)$ and $A(\rho_1, \rho_2)$.

"Only if": this is the nontrivial direction. Assume that $A(r_1, r_2)$ and $A(\rho_1, \rho_2)$ are conformally equivalent. We start with a normalization that fixes the two inner radii at 1 to simplify things a bit: denote $\mu = r_2/r_1$ and $\nu = \rho_2/\rho_1$. Then $A(1, \mu)$ is conformally equivalent to $A(r_1, r_2)$ (by the scaling transformation mentioned in the "if" part), and

3 More generally, Ω is called k-**connected** if $\mathbb{C} \setminus \Omega$ has k connected components and **finitely connected** if it is k-connected for some $k \geq 1$.

similarly $A(1,\nu)$ is conformally equivalent to $A(\rho_1,\rho_2)$. Therefore $A(1,\mu)$ and $A(1,\nu)$ are conformally equivalent to each other. Let $f : A(1,\mu) \to A(1,\nu)$ be a conformal map. We can assume without loss of generality that f maps the inner boundary circle $|z| = 1$ to itself and maps the outer boundary circle $|z| = \mu$ of $A(1,\nu)$ to its counterpart $|z| = \nu$ in $A(1,\nu)$; otherwise, f maps the inner circle of $A(1,\mu)$ to the outer circle of $A(1,\nu)$ and vice versa, and in that case, we can get a conformal map that maps the inner circle to itself by replacing f by $f(\mu/z)$ (the composition of f with the inversion $z \mapsto \mu/z$, which is a conformal automorphism of $A(1,\mu)$).

For each $1 < r < \mu$, let γ_r denote the circular contour $\{|z| = r\}$, and let $\Gamma_r = f \circ \gamma_r$ denote its image under the map f. The curve Γ_r is a simple closed curve and hence encloses a well-defined region (see Theorem 1.26 and the discussion following it in Section 1.8), which we denote by Ω_r. The area enclosed by γ_r is, of course, πr^2. The area of Ω_r is a continuous increasing function of r, which we denote $\alpha(r)$. Two important observations about $\alpha(r)$ are that

$$\lambda_- := \lim_{r\searrow 1} \alpha(r) = \pi \quad \text{and} \quad \lambda_+ := \lim_{r\nearrow \mu} \alpha(r) = \pi\nu^2,$$

since λ_- and λ_+ are simply the areas enclosed by the inner and outer boundary circles of $A(1,\nu)$, respectively.

Now we claim that

$$\alpha(r) \geq \pi r^2 \quad \text{for all } 1 < r < \mu. \tag{3.24}$$

This would imply, by taking the limit as $r \nearrow \mu$, that $\pi\nu^2 = \lambda_+ \geq \pi\mu^2$, so we would get that $\nu \geq \mu$. Reversing the roles of the two annuli would imply the reverse inequality $\nu \leq \mu$, and we would get that $\mu = \nu$, which is the claim we wanted, and the proof would be done.

To prove (3.24), we note that $\alpha(r)$ can be evaluated as a contour integral using a complex-analytic version of Green's theorem from calculus. Specifically, appealing to the result of Exercise 3.7, we see that

$$\alpha(r) = \frac{1}{2i} \oint_{\Gamma_r} \overline{z}\, dz = \frac{1}{2i} \int_0^{2\pi} \overline{f(re^{it})} \frac{d}{dt}(f(re^{it}))\, dt = \frac{r}{2} \int_0^{2\pi} \overline{f(re^{it})} f'(re^{it}) e^{it}\, dt. \tag{3.25}$$

Now let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{3.26}$$

be the Laurent expansion of f, which converges uniformly on compacts in the annulus $1 < |z| < \mu$ where f is holomorphic (see Theorem 1.65). Substituting (3.26) into (3.25), we get that

$$a(r) = \frac{1}{2}\int_0^{2\pi}\Big(\sum_n \overline{c_n} r^n e^{-int}\Big)\Big(\sum_m m c_m r^m e^{i(m-1)t}\Big) r e^{it}\, dt$$
$$= \frac{1}{2}\sum_{n,m} m c_m \overline{c_n} r^{n+m}\int_0^{2\pi} e^{i(m-n)t}\, dt = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n}.$$

Taking the limit as $r \searrow 1$ gives that

$$\sum_{n=-\infty}^{\infty} n|c_n|^2 = 1.$$

Now it follows that

$$a(r) - \pi r^2 = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 r^{2n} - \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 (r^{2n} - 1).$$

Since each summand in this last expression is nonnegative, we have that $a(r) - \pi r^2 \geq 0$, as claimed. □

Having classified the annuli up to conformal equivalence, we state without proof an additional result that explains why the family of annuli plays a role in the theory of conformal mapping of doubly connected regions that parallels the role of the unit disc in the case of simply connected regions. For the proof, see [2, 6].

Theorem 3.25 (Conformal classification of doubly connected regions). *The annuli* $A(1, \rho)$, $\rho > 1$, *form a complete set of conformal equivalence representatives for doubly connected complex regions. That is, if* $\Omega \subset \mathbb{C}$ *is a doubly connected region, then* Ω *is conformally equivalent to* $A(1, \Lambda)$ *for precisely one value of* $\Lambda > 1$.

The number $m_\Omega = \frac{1}{2\pi}\log(\Lambda)$, where Λ is the outer radius of the annulus to which Ω maps, is called the **conformal modulus of** Ω. Theorem 3.24 guarantees that if such a number exists, then it is unique, and the much stronger Theorem 3.25 guarantees that it exists. Thus m_Ω is an important example of what is known as a **conformal invariant**. Much more can be said about m_Ω, including a more direct way to define it that is intrinsic to Ω and does not rely on the idea of conformally mapping Ω to an annulus; consult the references mentioned above for details.

The final component in the discussion of conformal equivalence classes of doubly connected regions is the identification of the conformal automorphisms of such a region.

Theorem 3.26 (Conformal automorphisms of an annulus). *The conformal automorphism group of the annulus* $A(r_1, r_2)$ *is*

$$\mathrm{Aut}(A(r_1, r_2)) = \{z \mapsto e^{i\theta} z \;:\; 0 \leq \theta < 2\pi\} \cup \Big\{z \mapsto e^{i\theta}\frac{r_1 r_2}{z} \;:\; 0 \leq \theta < 2\pi\Big\}.$$

That is, the automorphisms consist of the rotations $z \mapsto e^{i\theta}z$, together with the compositions of the inversion map $z \mapsto \frac{r_1 r_2}{z}$ with a rotation.

Proof. Exercise 3.9. □

Suggested exercises for Section 3.10. 3.7, 3.8, 3.9.

Exercises for Chapter 3

3.1 If Ω and Ω' are conformally equivalent with a conformal map $g : \Omega \to \Omega'$, then describe an explicit group isomorphism between $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(\Omega')$.

3.2 Let $z_1, z_2, z_3, w_1, w_2, w_3$ be elements of $\widehat{\mathbb{C}}$. Prove that there is a unique Möbius transformation mapping z_j to w_j for $j = 1, 2, 3$.

3.3 Prove that besides the singleton conformal equivalence classes $\{\mathbb{C}\}$ and $\{\widehat{\mathbb{C}}\}$ described above, any other conformal equivalence class $\mathcal{K}$ is infinite and in fact contains an infinity of regions any two of which are not images of each other under an affine transformation $z \mapsto az + b$.

3.4 Prove Theorem 3.11.

3.5 Show that the assumption of holomorphicity in Montel's theorem (Theorem 3.16) cannot be removed; that is, the result properly belongs in complex analysis and does not have a real analysis analogue (at least not an obvious one).

3.6 Show that the real analysis analogue of Hurwitz's theorem is not true.

3.7 The complex-analytic version of Green's formula from multivariate calculus states that if γ is a simple closed contour in the plane, then the area A enclosed inside γ is given by

$$A = \frac{1}{2i} \oint_{\gamma} \bar{z}\, dz.$$

Show that this follows from the usual Green's theorem in real-variable calculus.

3.8 Prove that the statement of Theorem 3.24 is also correct under the relaxed assumption $0 \le r_1 < r_2$ and $0 \le \rho_1 < \rho_2$, which addresses also the case of "degenerate" annuli with an inner radius of 0 (that is, punctured discs).

3.9 Prove Theorem 3.26.