Preface

Mathematical logic is a particular branch of mathematics that applies mathematical tools to investigate the nature of mathematics. Consequently, in this book our attention will be focused on the language of mathematics. This will be done by discussing formal languages, first-order logic, model-theoretic semantics, deductive systems, and their mathematical properties and relations. Such a focus radiates a bright and direct light on mathematics itself.

Aristotle was the first formal logician who identified several key principles of correct reasoning, namely, the syllogisms. On the other hand, modern mathematical logic is based on the late nineteenth- and early twentieth-century innovative work of Boole, Frege, Peano, Russell, Whitehead, Hilbert, Skolem, Gödel, Tarski, Cantor, and their followers. The material presented in this text is the result of these pioneers in mathematical logic.

This textbook delivers an upper-division undergraduate course in mathematical logic. This subject typically attracts students with various backgrounds, some of whom may be quite familiar with logical notation and mathematical proof while others may not be as familiar. The book strives to address the needs of such a course. My primary goal was to write a book that would be accessible to all readers having a fundamental background in mathematics. Thus, I have made an effort to write clear and complete proofs throughout the text. In addition, these proofs favor detail over brevity. This approach should be to the benefit of all students, readers, and instructors.

Topics covered

The book presents the fundamental topics in mathematical logic that will lead to the statements and coherent proofs of Gödel's completeness and incompleteness theorems. The basics of logic and elementary set theory are first discussed in Chapter 1. Since students, typically, are acquainted with these topics, one should not necessarily begin the book by starting with this chapter. However, Section 1.1.5 and Theorem 1.1.27 should definitely be discussed. In particular, Theorem 1.1.27 is a recursion theorem that justifies many of the key definitions and proofs that are presented in the text. Few books in mathematical logic explicitly state and prove this often applied result. Such books usually justify their definitions and proofs with the expression "by recursion;" however, we in such cases will cite and apply Theorem 1.1.27. Understanding the statement of this theorem is more important than reading and understanding its proof.

Chapter 2 introduces the syntax and semantics of propositional logic. The chapter also carefully discusses an induction principle that is illustrated by correctly proving fundamental results about the language of propositional logic. This is followed by establishing the completeness of the logical connectives, the compactness theorem, and the deduction theorem. We also state and prove the associated soundness and completeness

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theorems. Most students who are familiar with propositional logic have not yet seen a mathematical development of this logic. Therefore, these topics offer an important prerequisite for the development of first-order logic.

Chapter 3 discusses the syntax and semantics of first-order languages. First-order logic is quite a bit more subtle than propositional logic, although they do share some common characteristics. The chapter also introduces structures, which can be viewed as vehicles for interpreting a given first-order language. Tarski's definition of satisfaction gives a precise meaning that yields a method for interpreting a first-order language in a given structure. This is then followed by the notion of a deduction (formal proof) in a first-order language.

The main goal of Chapter 4 is to present and prove the soundness and completeness theorems of first-order logic. For each of these proofs, I have respectively isolated the technical lemmas (Sections 4.1.1 and 4.2.1) that support the proofs of these important theorems. The compactness theorem is then presented with a proof. The chapter ends with several applications of the soundness, completeness, and compactness theorems.

In Gödel's proof of the incompleteness theorem, he encodes formulas into natural numbers using (primitive) recursive functions. In preparation for the proof of Gödel's theorem, Chapter 5 covers (primitive) recursive functions and relations. Since Gödel's encoding techniques created a link between logic and computing, we begin the chapter with an introduction to abstract computing machines and partial recursive functions. Today, computability theory in mathematical logic is closely related to the theory of computation in computer science.

In Chapter 6 the focus is on the language of elementary number theory $\mathcal{L}$ and the standard model $\mathcal{N}$ for number theory. The chapter begins with the question: Is there a decidable set of $\mathcal{L}$-sentences, that hold in $\mathcal{N}$, from which one can deduce all the sentences that are true in $\mathcal{N}$? This is followed by an introduction to the Ω-axioms. These axioms allow one to deduce some of the basic statements that are true in the standard model. Then representable relations and functions are discussed. Eventually it is established that a function is representable if and only if the function is recursive. Then a technique is presented that allows one to perform a Gödel encoding of all of the formulas in the language $\mathcal{L}$. This is followed by a proof of the fixed-point lemma and two results of Gödel, namely, the first and second incompleteness theorems. These two theorems are among the most important results in mathematical logic.

How to use the book

It is strongly recommended that the reader be familiar with the basics of sets, functions, relations, logic, and mathematical induction. These topics are typically introduced in a "techniques of proof" course (for example, see [1]). As the emphasis will be on theorems and their proofs, the reader should be comfortable reading and writing mathematical proofs.

If time is short or an instructor would like to end a one-semester course by covering Gödel's incompleteness theorems, certain topics can be bypassed. In particular, the following sections can be omitted without loss of continuity:

– 3.2.5 Classes of structures
– 3.2.6 Homomorphisms
– 4.3.1 Nonstandard models
– 4.3.4 Prenex normal form
– 5.1 The informal concept
– 5.2.1 Turing machines
– 5.2.2 Register machines

Furthermore, an instructor could focus on the statements of the technical lemmas in Sections 4.1.1 and 4.2.1 rather than on the proofs of these lemmas. These proofs could then be given as assigned reading. These technical lemmas support the respective proofs of the soundness and completeness theorems in Chapter 4. Similarly, in Sections 5.3 and 5.4 one could focus attention on the results rather than on the proofs. Of course, the material in Chapter 6 is more interesting than the proofs presented in these two sections. Perhaps, after seeing the theorems presented in Chapter 6, one would be more interested in the meticulous proofs presented in Sections 5.3 and 5.4.

Exercises are given at the end of each section in a chapter. An exercise marked with an asterisk $*$ is one that is cited, or referenced, elsewhere in the book. Suggestions are also provided for those exercises that a newcomer to upper-division mathematics may find more challenging.