Introduction

Fractional calculus has attracted the attention of mathematicians and engineers for a long time. The concept of fractional (more precisely, noninteger) differentiation dates back to a famous correspondence between L'Hospital and Leibniz in 1695. Many mathematicians contributed to the development of this branch of mathematical analysis with the pioneer work owing to Euler, Laplace, Abel, Liouville and Riemann. Factional calculus was mainly regarded as a purely theoretical topic for centuries, with a little connection to the practical problems of physics and engineering. During the past three decades, this subject served as an effective modeling methodology for researchers. The tools of fractional calculus are found to be of great interest in improving the mathematical modeling of several phenomena occurring in engineering and sciences such as physics, mechanics, electricity, chemistry, biology, economics, etc. Now the research on fractional differential equations has become an active and popular area of investigation around the world.

In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives; see the monographs of Miller et al. [163], Podlubny [174], Hilfer [124], Kilbas et al. [133], Diethelm [84], Zhou [225, 226] and the references therein. Fractional evolution equations provide a unifying framework to investigate well-posedness of complex systems of various types describing the time evolution of concrete systems.

A strong motivation for investigating this class of equations owes to the fact that differential models involving fractional derivative operators provide an excellent instrument for describing memory and hereditary properties of the associated physical phenomena. The fractional-order models of real systems are considered to be more adequate than their classical counterparts, since the description of some systems is more accurate when the fractional derivative is used. The advantages of fractional derivative becomes evident in modeling mechanical and electrical properties of real materials, description of rheological properties of rocks and various other applications. Such models are interesting not only for engineers and physicists but also for pure mathematicians.

This monograph is devoted to the theoretical analysis of fractional evolution equations. In particular, we are interested in the existence, attractivity, stability, periodicity and control theory for fractional evolution equations. The development of such a basic theory provides a useful platform for further research concerning the dynamics, numerical analysis and applications of fractional differential equations.

This monograph is arranged and organized as follows:

In order to make the book self-contained, we devote the first chapter to a description of general information on fractional calculus, special functions, semigroups, Laplace and Fourier transforms, measure of noncompactness, fixed-point theorems and stochastic process.

https://doi.org/10.1515/9783110769272-202

The second chapter deals with existences of mild solutions, integral solutions and globally attractive solutions for fractional evolution equations of order $\alpha \in (0,1)$. In Section 2.1, we study fractional evolution equations with the Hilfer fractional derivative, which is a generalization of both Riemann–Liouville and Caputo fractional derivatives. By the noncompact measure method, we obtain some sufficient conditions to ensure the existence of mild solutions. In Section 2.2, we discuss the existence of integral solutions for two classes of fractional-order evolution equations with nondensely defined linear operators. First, we consider the nonhomogeneous fractional-order evolution equation and obtain its integral solution by Laplace transform and probability density function. Subsequently, based on the form of integral solution for nonhomogeneous fractional-order evolution equations, we investigate the existence of integral solution for nonlinear fractional-order evolution equations by the method of noncompact measure. In Section 2.3, the terminal value problem for a class of nonlinear fractional evolution equations with Liouville–Weyl derivative is considered. By using Fourier transform, such a problem is converted into a singular integral equation on an infinite interval. Some sufficient conditions are obtained to ensure the existence of mild solutions when the semigroup is compact as well as noncompact. In Section 2.4, we discuss the question of the attractivity of solutions for fractional evolution equations with almost sectorial operators. We establish sufficient conditions for the existence of globally attractive solutions for the Cauchy problems when the semigroup is compact as well as noncompact. Our results essentially reveal certain characteristics of solutions for fractional evolution equations, which are not possessed by integer-order evolution equations.

The third chapter is devoted to the study of fractional control systems of order $\alpha \in (0,1)$. In Section 3.1, we obtain the existence of mild solutions and controllability for fractional evolution inclusions in Banach spaces, avoiding hypotheses of compactness on the semigroup generated by the linear part and any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness. Finally, two examples are given to illustrate the theoretical results. In Section 3.2, we investigate a class of fractional evolution equations and optimal controls in Hilbert spaces. The strategy of this section is establishing low dimensional approximations for this type of equations by using approximation methods. We derive three kinds of convergence results of mild solutions under appropriate assumptions. Then the convergence result holds for cost functionals as well. Later, error estimates of cost functionals and optimal controls are obtained. Finally, the proposed procedure is illustrated by an example.

The fourth chapter is mainly concerned with the investigation of fractional evolution equations and inclusions with order $\alpha \in (1,2)$. First, we obtain some interesting results for mild solutions and controllability to fractional evolution systems with order $\alpha \in (1,2)$ in Banach spaces. By using Laplace transform and a Wright-type function, we deduce a new representation of solution operators and introduce a new concept of mild solutions for the objective equations. Then we proceed to establish a new compact result for the solution operators when the sine family is compact. The controllability of mild solutions is also discussed. Second, we study the existence of mild solutions and compactness for the set of mild solutions to a nonlocal problem of fractional evolution inclusions of order $\alpha \in (1,2)$. The main tools of our study include fractional calculus, multivalued analysis, cosine family, measure of noncompactness and a fixed-point theorem. Finally, we demonstrate the application of the obtained results to a control problem.

The fifth chapter is devoted to the study of neutral evolution equations and inclusions. In Section 5.1, by using the fractional power of operators and some fixed-point theorems, we discuss a class of fractional neutral evolution equations with nonlocal conditions. We give various criteria on the existence and uniqueness of mild solutions and an example to illustrate the applications of the abstract results. In Section 5.2, we investigate the topological structure for the solution set of Caputo-type neutral fractional stochastic evolution inclusions in Hilbert spaces. We introduce the concept of mild solutions for factional neutral stochastic inclusions and show that the solution set is nonempty compact and R_δ-set, which means that the solution set may not be a singleton, but from the point of view of algebraic topology, it is equivalent to a point in the sense that it has the same homology group as a one-point space. Finally, we illustrate the obtained theory with the aid of an example.

In the sixth chapter, we investigate fractional evolution equations with the Liouville–Weyl fractional derivative on whole real axis. In Section 6.1, we give appropriate definitions of mild solutions with the aid of Fourier transform. Then we accurately estimate the spectral radius of resolvent operator and obtain the existence and uniqueness of periodic solutions, S-asymptotically periodic solutions and other types of bounded solutions. In Section 6.2, combining the fixed-point theorem due to Krasnoselskii and a decomposition technique, we give some sufficient conditions to ensure the existence of asymptotically almost periodic mild solutions. An example is also presented as an application to illustrate the feasibility of the abstract result.

The seventh chapter deals with discrete-time fractional evolution equations involving the Riemann–Liouville-like difference operator. Based on the relationship between C_0-semigroups and a distinguished class of sequences of operators, we discuss the structure of the solutions for the inhomogenous Cauchy problem of abstract fractional difference equations. The criteria for the existence and uniqueness of solutions for the semilinear Cauchy problem are established. Further, we show the existence of stable solutions for the nonlinear Cauchy problem by means of a fixed-point technique and the compact method. Moreover, we establish the Ulam–Hyers–Rassias stability of the proposed problem. Two examples are presented to explain the main results.

In summary, this monograph serves as an excellent source of knowledge for the one who is interested in the theory of fractional differential equations and their applications in science and engineering.